ON BREZIS-NIRENBERG PROBLEM ON $\mathbf{S}^3$ AND A CONJECTURE OF BANDLE-BENGURIA

WENYI CHEN AND JUNCHENG WEI

Abstract. We consider the following Brezis-Nirenberg problem on $\mathbf{S}^3$

$$-\Delta_{\mathbf{S}^3} u = \lambda u + u^5 \text{ in } D, \quad u > 0 \text{ in } D \text{ and } u = 0 \text{ on } \partial D,$$

where D is a geodesic ball on $\mathbf{S}^3$ with geodesic radius θ_1, and $\Delta_{\mathbf{S}^3}$ is the Laplace-Beltrami operator on $\mathbf{S}^3$. We prove that for any $\lambda < -\frac{3}{4}$ and for every $\theta_1 < \pi$ with $\pi - \theta_1$ sufficiently small (depending on λ), there exists bubbling solution to the above problem. This solves a conjecture raised by Bandle-Benguria [1] and Brezis-Peletier [4].

Sur l'Équation de Brezis-Nirenberg sur $\mathbf{S}^3$ et une conjecture de Bandle-Benguria

Résumé. Nous considérons le problème de Brezis-Nirenberg suivant sur $\mathbf{S}^3$

$$-\Delta_{\mathbf{S}^3} u = \lambda u + u^5 \text{ dans } D, \quad u > 0 \text{ dans } D \text{ et } u = 0 \text{ sur } \partial D,$$

où D est une boule géodésique sur $\mathbf{S}^3$ de rayon géodésique θ_1, et $-\Delta_{\mathbf{S}^3}$ est l'opérateur de Laplace-Beltrami sur S^3. Nous montrons que pour tout $\lambda < -\frac{3}{4}$ et tout $\theta_1 < \pi$ avec $\pi - \theta_1$ suffisamment petit (dependant de λ), il existe des solutions pour le problème précédent. Ce résultat répond à une conjecture de Bandle-Benguria [1] et de Brezis-Peletier [4].

1. Introduction

We consider the following problem

$$-\Delta_{\mathbf{S}^3} u = \lambda u + u^5, \quad u > 0 \text{ in } D \text{ and } u = 0 \text{ on } \partial D, \tag{1.1}$$

where $\Delta_{\mathbf{S}^3}$ is the Laplace-Beltrami operator on $\mathbf{S}^3$ and D is the geodesic ball centered at the North Pole with geodesic radius θ_1. Of particular interest is the case of $\theta_1 \in (\frac{\pi}{2}, \pi)$. (Note that when $\theta_1 = \frac{\pi}{2}$, this corresponds to the upper half sphere; while when $\theta_1 = \pi$, this is the full sphere.)

The analogous problem in $\mathbb{R}^N$

$$-\Delta u = \lambda u + u^5, \quad u > 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega, \tag{1.2}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, was first studied in a celebrated paper by Brezis and Nirenberg [3]. In particular, they proved that if $\Omega = B_R(0)$ is a ball of radius R, the solutions to (1.2) exist only if $\lambda \in (0, \lambda_1)$ for $N \geq 4$ and $\lambda \in (\frac{\lambda_1}{4}, \lambda_1)$ when $N = 3$. Since then, there is a vast literature on many extensions of the problem considered by Brezis and Nirenberg (see, e.g. [9], Chapter 3 and the references therein).

In recent papers by Bandle-Benguria [1] and Bandle-Peletier [2], it was shown that on the sphere $\mathbf{S}^3$ the situation is quite different. In particular, they showed that in the range of $\lambda > -\frac{3}{4}$, there is a solution if and only if

$$\frac{\pi^2 - 4\theta_1^2}{4\theta_1^2} < \lambda < \frac{\pi^2 - \theta_1^2}{\theta_1^2}.$$

For $\lambda \leq -\frac{3}{4}$, it was shown in [1], by means of a Pohozaev type identity, that there exist no solutions if $\theta_1 \leq \frac{\pi}{2}$. Then they conjectured (see a more general conjecture in [4]):

Conjecture: For every $\lambda < -\frac{3}{4}$ and every $\theta_1 < \pi$ with $\pi - \theta_1$ sufficiently small, there exists a solution to (1.1).

In this paper, we solve the conjecture affirmatively. To state our result, we introduce the corresponding equation on $\mathbb{R}^3$. By using stereo-graphic projection at the North Pole, equation (1.1) can be transformed to the following ODE:

$$\Delta u - p(r)u + 3u^5 = 0, u = u(r) > 0,\ r \geq \varepsilon,\ \ u(\varepsilon) = 0,\ u(r) = O(\frac{1}{r})\ \text{ as } r \to +\infty \tag{1.3}$$

where $p(r) = \frac{-\frac{3}{4}-\lambda}{(1+r^2)^2}$ and $\varepsilon = \frac{\sin\theta_1}{1-\cos\theta_1}$.

Let $U_\Lambda(r) = (\frac{\Lambda}{\Lambda^2+r^2})^{\frac{1}{2}}$ be the unique radial solution of $\Delta u + 3u^5 = 0, u = u(r) > 0$. Our main result in this paper is the following.

Theorem 1.1. *Let $\lambda < -\frac{3}{4}$ be a fixed number. Then there exists an $\varepsilon_0 = \varepsilon_0(\lambda) > 0$ such that for each $0 < \varepsilon < \varepsilon_0$, problem (1.3) has a solution $u_\varepsilon(r)$ with the following form*

$$u_\varepsilon(r) - U_{\sqrt{\varepsilon}\Lambda_\varepsilon}(r) = O(\frac{\varepsilon^{3/4}}{r}),\ \textit{for } r \geq \varepsilon,\quad \textit{where } \Lambda_\varepsilon \to \Lambda_0 > 0. \tag{1.4}$$

We remark that equation (1.1) with $\lambda \to -\infty$ is also studied in [4] and [11]. There it is shown that more and more peaked solutions arise when $|\lambda| \to +\infty$.

The proof of Theorem 1.1 mainly relies upon a finite dimensional reduction procedure. Such a method has been used successfully in many papers, see e.g. [5], [6], [7], [8], [10]. In particular, we shall follow the one used in [10].

By the scaling $r \to \sqrt{\varepsilon} r$, problem (1.3) is reduced to the following ODE

$$\Delta u - \varepsilon p(\sqrt{\varepsilon} r)u + 3u^5 = 0, u = u(r) > 0, r \geq \sqrt{\varepsilon},\quad u(\sqrt{\varepsilon}) = 0, u(r) = O(\frac{1}{r}) \text{ as } r \to +\infty. \tag{1.5}$$

From now on, we shall work with (1.5).

Throughout this paper, unless otherwise stated, the letter C will always denote various generic constants which are independent of ε, for ε sufficiently small. The notation $A_\varepsilon = O(B_\varepsilon)$ means that $|\frac{A_\varepsilon}{B_\varepsilon}| \leq C$, while $A_\varepsilon = o(B_\varepsilon)$ means that $\lim_{\varepsilon\to 0} \frac{|A_\varepsilon|}{|B_\varepsilon|} = 0$.

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2. Approximate Solutions and Some Estimates

In this section, we introduce a family of approximate solutions to (1.5) and derive some useful estimates.

Let $\Lambda > 0$ be a fixed positive constant such that $\frac{1}{C} < \Lambda < C$ for some large constant $C > 0$. We define $V_{\varepsilon,\Lambda}$ to be the unique solution satisfying

$$\Delta v - \varepsilon p(\sqrt{\varepsilon} r)v + 3U_\Lambda^5 = 0, r \geq \sqrt{\varepsilon},\ v(\sqrt{\varepsilon}) = 0,\ v(r) \to 0 \text{ as } r \to +\infty. \tag{2.1}$$

To analyze $V_{\varepsilon,\Lambda}$, we introduce two functions: let $\psi_{\varepsilon,\Lambda}$ be the unique solution of

$$\Delta v - p(r)v + p(r)U_{\sqrt{\varepsilon}\Lambda} = 0,\ v'(0) = 0,\ v(r) \to 0 \text{ as } r \to +\infty, \tag{2.2}$$

and $G(r)$ be the Green's function satisfying

$$\Delta v - p(r)v + 4\pi\delta_0 = 0, v(r) \to 0 \text{ as } r \to +\infty. \tag{2.3}$$

Note that $G(r) = \frac{1}{r} + O(1)$ for $r << 1$ and $\psi_{\varepsilon,\Lambda} = \varepsilon^{1/4}\Lambda^{1/2}\psi_0(r) + o(\varepsilon^{1/4}(1+r)^{-1})$, where ψ_0 is the unique solution of

$$\Delta v - p(r)v + p(r)\frac{1}{r} = 0, v^{'}(0) = 0, \ v(r) \to 0 \text{ as } r \to +\infty. \tag{2.4}$$

Indeed, the solution of (2.4) exists and $\psi_0(0) > 0$ whenever $p(r) > 0$ and $p(r) \le C(1+r^2)^{-2}$. It is then easy to see that

$$V_{\varepsilon,\Lambda}(r) = U_\Lambda(r) - \varepsilon^{1/4}\left[\psi_{\varepsilon,\Lambda}(\sqrt{\varepsilon}r) + \beta_{\varepsilon,\Lambda}G(\sqrt{\varepsilon}r)\right], \tag{2.5}$$

where

$$\beta_{\varepsilon,\Lambda} = \frac{U_{\sqrt{\varepsilon}\Lambda}(\varepsilon) - \psi_{\varepsilon,\Lambda}(\varepsilon)}{G(\varepsilon)} = \varepsilon^{3/4}\Lambda^{-1/2}(1+o(1)). \tag{2.6}$$

Substituting (2.6) into (2.5) gives us

$$V_{\varepsilon,\Lambda}(r) = U_\Lambda(r) + O(\frac{\varepsilon^{1/2}}{r}). \tag{2.7}$$

Let

$$I_\varepsilon = [\sqrt{\varepsilon}, +\infty), \quad \text{and } S_\varepsilon[u] = \Delta u - \varepsilon p(\sqrt{\varepsilon}r)u + 3u_+^5, \text{ where } u_+ = \max(u, 0).$$

To estimate $S_\varepsilon[V_{\varepsilon,\Lambda}]$, we define two norms

$$\|\phi\|_* = \sup_{r\in I_\varepsilon}(1+r^2)^{1/2}|\phi(r)|, \ \|f\|_{**} = \sup_{r\in I_\varepsilon}(r(1+r^2)^{5/4}|f(r)|). \tag{2.8}$$

The reason for defining these two norms lies behind the following lemma:

Lemma 2.1. *It holds*

$$\|\phi\|_* \le C\|\Delta\phi - \varepsilon p(\sqrt{\varepsilon}r)\phi\|_{**} \quad \textit{where} \quad \phi(\sqrt{\varepsilon}) = \phi(+\infty) = 0. \tag{2.9}$$

Proof: Let ϕ_0 be the unique solution of

$$\Delta\phi_0 + |\Delta\phi - \varepsilon p(\sqrt{\varepsilon}r)\phi| = 0, \ r \ge \sqrt{\varepsilon}, \ \phi_0(\sqrt{\varepsilon}) = \phi_0(+\infty) = 0. \tag{2.10}$$

$\phi_0(r)$ can be computed explicitly:

$$\phi_0(r) = \int_{\sqrt{\varepsilon}}^{+\infty} \frac{s}{r}|\Delta\phi(s) - \varepsilon p(\sqrt{\varepsilon}s)\phi(s)|(\min(r,s) - \sqrt{\varepsilon})ds. \tag{2.11}$$

Then it is easy to see that

$$(1+r^2)^{1/2}\phi_0(r) \le C\|\Delta\phi - \varepsilon p(\sqrt{\varepsilon}r)\phi\|_{**}. \tag{2.12}$$

By the Maximum Principle, we deduce

$$(1+r^2)^{1/2}|\phi(r)| \le (1+r^2)^{1/2}\phi_0(r) \le C\|\Delta\phi - \varepsilon p(\sqrt{\varepsilon}r)\phi\|_{**} \tag{2.13}$$

which proves (2.9). □

Since $S_\varepsilon[V_{\varepsilon,\Lambda}] = 3V_{\varepsilon,\Lambda}^5 - 3U_\Lambda^5$, by (2.7), it is not difficult to see that

$$\|S_\varepsilon[V_{\varepsilon,\Lambda}]\|_{**} \le C\varepsilon^{1/2}. \tag{2.14}$$

Finally, we define two functions which are important in linearized analysis: Let $Z_\Lambda = U_\Lambda^4(\frac{\partial U_\Lambda}{\partial \Lambda})$ and $z_{\varepsilon,\Lambda}$ be the unique solution of

$$\Delta v - \varepsilon p(\sqrt{\varepsilon} r) v + 15 U_\Lambda^4(\frac{\partial U_\Lambda}{\partial \Lambda}) = 0,\ v(\sqrt{\varepsilon}) = v(+\infty) = 0,\ r \geq \sqrt{\varepsilon}. \tag{2.15}$$

It is easy to see from (2.7) that

$$z_{\varepsilon,\Lambda} = \frac{\partial U_\Lambda}{\partial \Lambda} + O(\varepsilon^{1/4}\frac{1}{r}). \tag{2.16}$$

3. Reduction Process

In this section, we perform a finite-dimensional reduction procedure which is similar to that of [10].

We first consider the following linear problem: Given $h = h(r)$, find a pair (ϕ, c) satisfying

$$\begin{cases} L_{\varepsilon,\Lambda}[\phi] := \Delta\phi - \varepsilon p(\sqrt{\varepsilon} r)\phi + 15 V_{\varepsilon,\Lambda}^4 \phi = h + cZ_\Lambda, r \geq \sqrt{\varepsilon}, \\ \phi(\sqrt{\varepsilon}) = \phi(+\infty) = 0,\ \int_{I_\varepsilon} \phi Z_\Lambda r^2 dr = 0. \end{cases} \tag{3.1}$$

We have the following a priori estimates.

Lemma 3.1. *Let (ϕ, c) satisfy (3.1). Then for ε sufficiently small, there holds*

$$\|\phi\|_* \leq C\|h\|_{**}. \tag{3.2}$$

Proof. The proof of this Lemma is similar to that of Proposition 3.1 of [10]. For the sake of completeness, we include a proof here.

Arguing by contradiction, assume that there exists a subsequence $\varepsilon_k \to 0$ and $(\phi_{\varepsilon_k}, c_{\varepsilon_k}, h_{\varepsilon_k})$ which satisfy (3.1) such that

$$\|\phi_{\varepsilon_k}\|_* = 1; \qquad \|h_{\varepsilon_k}\|_{**} = o(1) \text{ as } \varepsilon_k \to 0. \tag{3.3}$$

We suppress the dependence on the index k for the sake of simplicity.

Multiplying (3.1) by $r^2 z_{\varepsilon,\Lambda}$ and integrating over I_ε, we obtain

$$c_\varepsilon(\Lambda)\int_{I_\varepsilon} z_{\varepsilon,\Lambda} Z_\Lambda r^2 dr = -\int_{I_\varepsilon} h_\varepsilon z_{\varepsilon,\Lambda} r^2 dr + \int_{I_\varepsilon}\left[\Delta\phi_\varepsilon - \varepsilon p(\sqrt{\varepsilon} r)\phi_\varepsilon + 15 V_{\varepsilon,\Lambda}^4\phi_\varepsilon\right] z_{\varepsilon,\Lambda} r^2 dr. \tag{3.4}$$

It is easy to see that

$$\int_{I_\varepsilon} h_\varepsilon z_{\varepsilon,\Lambda} r^2 dr = O(\|h_\varepsilon\|_{**}) = o(1). \tag{3.5}$$

Moreover, integrating by parts, we deduce

$$\int_{I_\varepsilon}\left[\Delta\phi_\varepsilon - \varepsilon p(\sqrt{\varepsilon} r)\phi_\varepsilon + 15 V_{\varepsilon,\Lambda}^4\phi_\varepsilon\right] z_{\varepsilon,\Lambda} r^2 dr = \int_{I_\varepsilon} 15[V_{\varepsilon,\Lambda}^4 - U_\Lambda^4] z_{\varepsilon,\Lambda}\phi_\varepsilon r^2 dr + o(1) = o(1). \tag{3.6}$$

On the other hand

$$\int_{I_\varepsilon} z_{\varepsilon,\Lambda} Z_\Lambda r^2 dr = \int_0^\infty U_\Lambda^4(\frac{\partial U_\Lambda}{\partial \Lambda})^2 r^2 dr + o(1). \tag{3.7}$$

Substituting (3.5), (3.6) and (3.7) into (3.4), we obtain that $c_\varepsilon = o(1)$. Also, since we are assuming that $\|h_\varepsilon\|_{**} = o(1)$ and since $\|Z_\Lambda\|_{**} = O(1)$, there holds

$$\|h_\varepsilon + c_\varepsilon Z_\Lambda\|_{**} = o(1). \tag{3.8}$$

Thus (3.1) yields

$$\|\Delta\phi_\varepsilon - \varepsilon p(\sqrt{\varepsilon} r)\phi_\varepsilon + 15 V_{\varepsilon,\Lambda}^4\phi_\varepsilon\|_{**} = o(1). \tag{3.9}$$

We show that (3.9) is incompatible with our assumption $\|\phi_\varepsilon\|_* = 1$. First, we claim that, for a fixed R, $\phi_\varepsilon(r) \to 0$ for $r \le R$. In fact, suppose not, then $\phi_\varepsilon \to \phi_0$ in $H^1_{loc}(\mathbb{R}^3)$, where ϕ_0 satisfies

$$\Delta\phi_0 + 15U_\Lambda^4\phi_0 = 0, \|\phi_0\|_* \le 1. \tag{3.10}$$

Hence $\phi_0 = \alpha\frac{\partial U_\Lambda}{\partial\Lambda}$ for some constant α. On the other hand, $\int_{I_\varepsilon}\phi_\varepsilon Z_\Lambda r^2dr = 0$ implies $\alpha\int_0^\infty U_\Lambda^4(\frac{\partial U_\Lambda}{\partial\Lambda})^2r^2dr = 0$ and hence $\alpha = 0$. Thus $\phi_\varepsilon(r) \to 0$ in $L^\infty_{loc}((0,+\infty))$. By Lemma 2.1, this yields

$$\|\phi_\varepsilon\|_* \le C\|\Delta\phi_\varepsilon - \varepsilon p(\sqrt{\varepsilon}r)\phi_\varepsilon\|_{**} \le C\|V_{\varepsilon,\Lambda}^4\phi_\varepsilon\|_{**} + C\|h_\varepsilon + c_\varepsilon Z_\Lambda\|_{**} = o(1) \tag{3.11}$$

which is a contradiction to the assumption $\|\phi_\varepsilon\|_* = 1$.

□

Once we have Lemma 3.1, the following lemma can be proved along the same ideas of Proposition 3.2 of [10], using the estimate (2.14). We omit the details.

Lemma 3.2. *For ε sufficiently small, there exists a unique pair $(\phi_{\varepsilon,\Lambda}, c_\varepsilon(\Lambda))$ satisfying*

$$S_\varepsilon[V_{\varepsilon,\Lambda} + \phi_{\varepsilon,\Lambda}] = c_\varepsilon(\Lambda)Z_\Lambda, \quad \int_{I_\varepsilon}\phi_{\varepsilon,\Lambda}Z_\Lambda r^2dr = 0. \tag{3.12}$$

Moreover, we also have that

$$\|\phi_{\varepsilon,\Lambda}\|_* \le C\varepsilon^{1/2} \tag{3.13}$$

and that the map $\Lambda \to c_\varepsilon(\Lambda)$ is continuous.

□

4. Proof of Theorem 1.1

From (3.12), we see that, to prove Theorem 1.1, it is enough to find a zero of function $c_\varepsilon(\Lambda)$. To this end, let us expand $c_\varepsilon(\Lambda)$.

Multiplying equation (3.12) by $r^2z_{\varepsilon,\Lambda}(r)$, we obtain, using Lemma 3.2,

$$c_\varepsilon\int_{I_\varepsilon}z_{\varepsilon,\Lambda}Z_\Lambda r^2dr = \int_{I_\varepsilon}S_\varepsilon[V_{\varepsilon,\Lambda}]z_{\varepsilon,\Lambda}r^2dr + \int_{I_\varepsilon}L_{\varepsilon,\Lambda}[\phi_{\varepsilon,\Lambda}]z_{\varepsilon,\Lambda}r^2dr + o(\varepsilon^{1/2}). \tag{4.1}$$

By (2.15) and integrating by parts, the second term on the right hand side of (4.1) can be estimated as follows:

$$\int_{I_\varepsilon}L_{\varepsilon,\Lambda}[\phi_{\varepsilon,\Lambda}]z_{\varepsilon,\Lambda}r^2dr = \int_{I_\varepsilon}L_{\varepsilon,\Lambda}[z_{\varepsilon,\Lambda}]\phi_{\varepsilon,\Lambda}r^2dr = \int_{I_\varepsilon}15[V_{\varepsilon,\Lambda}^4 - U_\Lambda^4]z_{\varepsilon,\Lambda}\phi_{\varepsilon,\Lambda}r^2dr + o(\varepsilon^{1/2}) = o(\varepsilon^{1/2}).$$

It remains to compute the first term in the right hand side of (4.1):

$$\int_{I_\varepsilon}S_\varepsilon[V_{\varepsilon,\Lambda}]z_\varepsilon r^2dr = \int_{I_\varepsilon}3[V_{\varepsilon,\Lambda}^5 - U_\Lambda^5]z_{\varepsilon,\Lambda}r^2dr$$

$$= -15\varepsilon^{\frac14}\int_{I_\varepsilon}U_\Lambda^4\left[\psi_{\varepsilon,\Lambda}(\sqrt{\varepsilon}r) + \beta_{\varepsilon,\Lambda}G(\sqrt{\varepsilon}r)\right](\frac{\partial U_\Lambda}{\partial\Lambda})r^2dr + o(\sqrt{\varepsilon})$$

$$= -15\varepsilon^{1/2}\Lambda^{1/2}\psi_0(0)\int_0^{+\infty}(U_\Lambda^4\frac{\partial U_\Lambda}{\partial\Lambda})r^2dr - 15\varepsilon^{-1/4}\beta_{\varepsilon,\Lambda}\int_0^{+\infty}(U_\Lambda^4\frac{\partial U_\Lambda}{\partial\Lambda})rdr + o(\sqrt{\varepsilon}). \tag{4.2}$$

By direct computations, we have

$$\int_0^{+\infty}(U_\Lambda^4\frac{\partial U_\Lambda}{\partial\Lambda})r^2dr = \frac{1}{5}\frac{d}{d\Lambda}(\int_0^{+\infty}U_\Lambda^5r^2dr) = \frac{1}{10}(\int_0^\infty U_1^5r^2dr)\Lambda^{-1/2}, \tag{4.3}$$

$$\int_0^{+\infty} (U_\Lambda^4 \frac{\partial U_\Lambda}{\partial \Lambda}) r dr = \frac{1}{5} \frac{d}{d\Lambda} (\int_0^{+\infty} U_\Lambda^5 r dr) = -\frac{1}{10} (\int_0^{\infty} U_1^5 r dr) \Lambda^{-3/2}. \tag{4.4}$$

Substituting (2.6), (4.3) and (4.4) into (4.2), we arrive at

$$\int_{I_\varepsilon} S_\varepsilon[V_{\varepsilon,\Lambda}] z_{\varepsilon,\Lambda} r^2 dr = \varepsilon^{1/2}(-\gamma_0 + \gamma_1 \Lambda^{-2}) + o(\varepsilon^{1/2}) \tag{4.5}$$

where γ_0, γ_1 are two generic positive constants.

We obtain from (3.7), (4.1) and (4.5) that

$$c_\varepsilon(\Lambda) = c_0 \varepsilon^{1/2}(\gamma_0 - \gamma_1 \Lambda^{-2}) + o(\varepsilon^{1/2}) \quad \text{for some} \quad c_0 \neq 0. \tag{4.6}$$

Theorem 1.1 now follows from (4.6): in fact, (4.6) implies $c_\varepsilon(\Lambda_0 - \delta) c_\varepsilon(\Lambda_0 + \delta) < 0$ where $\Lambda_0 = \sqrt{\frac{\gamma_1}{\gamma_0}}$ and δ small. By the continuity of $c_\varepsilon(\Lambda)$, a zero of $c_\varepsilon(\Lambda)$, denoted by $\Lambda_\varepsilon \in (\Lambda_0 - \delta, \Lambda_0 + \delta)$, is guaranteed. Then $u_\varepsilon = V_{\varepsilon,\Lambda_\varepsilon} + \phi_{\varepsilon,\Lambda_\varepsilon}$ is a solution to (1.5). This proves Theorem 1.1.

□

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Department of Mathematics, Wuhan University, Wuhan, Hubei 430072, P.R. China

E-mail address: wychencn@whu.edu.cn

Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong

E-mail address: wei@math.cuhk.edu.hk