

# Self-similar and solitary wave solutions with ring profiles of two-component nonlinear Schrödinger systems

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## Abstract

Blowup ring profiles have been investigated by finding self-similar non-vortex solutions of nonlinear Schrödinger equations (NLSE) (cf. [4] and [5]). However, those solutions have infinite  $L^2$  norm so one may not maintain the ring profile all the way up to the singularity. To find self-similar  $H^1$  non-vortex solutions with ring profiles, we study self-similar solutions of two-component systems of NLSE with nonlinear coefficients  $\beta$  and  $\nu_j$ ,  $j = 1, 2$ . When  $\beta < 0$  and  $\nu_1 \gg \nu_2 > 0$ , the two-component system can be transformed into a multi-scale system with fast and slow variables which may produce self-similar  $H^1$  solutions with non-vortex ring profiles. We use the localized energy method with symmetry reduction to construct these solutions rigorously. On the other hand, these solutions may describe steady non-vortex bright ring solitons. Various types of ring profiles including  $m$ -ring and ring-ring profiles are presented by numerical solutions.

**Keywords:** self-similar, solitary wave, ring profile, two-component systems of NLSE

## 1 Introduction

Self-similar solutions of nonlinear Schrödinger equations (NLSE) may describe nonlinear wave collapse which is universal to many areas of physics including nonlinear optics (cf. [9]), plasma physics (cf. [17]), and Bose-Einstein condensates (BEC) (cf. [18]). The spatial profile of a collapsing wave may evolve into a universal, self-similar, circularly symmetric shape with a single peak known as the Townes profile which has been observed experimentally by amplified laser beams (cf. [14]). Theoretically, one may find the Townes profile by investigating self-similar solutions of self-focusing cubic NLSE as follows:

$$\begin{cases} i\partial_t \Psi + \Delta \Psi + \nu |\Psi|^2 \Psi = 0, \\ \Psi = \Psi(x, t) \in \mathbb{C}, \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad t > 0, \\ \Psi(\cdot, t) \in H^1(\mathbb{R}^2), \quad t > 0, \end{cases} \quad (1.1)$$

where  $\nu$  is a positive constant. It is well-known that the equation (1.1) has self-similar  $H^1$  solutions with the Townes profile to express finite-time blowup behavior and have the singularity at  $t = T$  (i.e.  $\|\Psi(\cdot, t)\|_{L^\infty} < \infty$  for  $0 < t < T$  and  $\|\Psi(\cdot, t)\|_{L^\infty} \rightarrow \infty$  as  $t \uparrow T < \infty$ ) (cf. [19]). Hence the Townes profile can be maintained all the way up to the singularity.

In high-power laser beams, different collapsing behaviors may develop self-similar ring profiles which break into filaments with multi-Townes profiles under the effect of noise (cf. [8]). It would be naive to think that ring profiles can be obtained by finding self-similar solutions of the equation (1.1). One may find self-similar solutions of (1.1) with ring profiles in [4] and [5]. However, those solutions have infinite  $L^2$  norm so one may not maintain the ring profile all the way up to the singularity. Recently, self-similar  $H^1$  vortex solutions with ring profiles have been found (cf. [6]). However, until now, it is still an open issue whether there exist self-similar  $H^1$  non-vortex solutions with ring profiles.

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In order to find self-similar  $H^1$  non-vortex solutions with ring profiles, we study two-component systems of self-focusing cubic NLSE given by

$$\begin{cases} i\partial_t\Phi + \Delta\Phi + \nu_1|\Phi|^2\Phi + \beta|\Psi|^2\Phi = 0, \\ i\partial_t\Psi + \Delta\Psi + \nu_2|\Psi|^2\Psi + \beta|\Phi|^2\Psi = 0, \\ \Phi = \Phi(x, t), \Psi = \Psi(x, t) \in \mathbb{C}, \quad x = (x_1, x_2) \in \mathbb{R}^2, t > 0, \\ \Phi(\cdot, t), \Psi(\cdot, t) \in H^1(\mathbb{R}^2), t > 0, \end{cases} \quad (1.2)$$

under the condition  $\nu_1 \gg \nu_2 > 0$ , where  $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$ ,  $\nu_j$ 's are positive constants and  $\beta < 0$  is a coupling constant. The system (1.2) is a well-known model for photorefractive media in nonlinear optics (cf. [1]). Besides, the system (1.2) may also describe two-component BEC in the limit of strong transverse confinement (cf. [7]). Physically, the coefficients  $\nu_j$ 's and  $\beta$  satisfy  $\nu_j \sim -a_{jj}$ ,  $j = 1, 2$ , and  $\beta \sim -a_{12}$ , where  $a_{ij}$ 's are the scattering lengths. Due to Feshbach resonance,  $a_{ij}$ 's can be tuned over a very large range by adjusting the externally applied magnetic field (cf. [10]). Consequently, we may let  $a_{jj} < 0$  i.e.  $\nu_j > 0$ ,  $j = 1, 2$ , and  $a_{12} > 0$  i.e.  $\beta < 0$ . Recently, a small and negative scattering length has been achieved by experiments (cf. [16]) so we may assume  $0 < -a_{22} \ll -a_{11}$  i.e.  $0 < \nu_2 \ll \nu_1$ .

To study the system (1.2) with  $\nu_1 \gg \nu_2 > 0$ , we may set  $\nu_1 = h\mu_1$ ,  $\nu_2 = h^{-1}\mu_2$ ,  $\Phi(x, t) = \phi(x, t)$ , and  $\Psi(x, t) = \sqrt{h}\psi(x, t)$ , where  $\mu_j$ 's positive constants and  $h \sim \sqrt{\nu_1/\nu_2} \gg 1$  a large parameter. Then the system (1.2) can be transformed into the following system

$$\begin{cases} i\varepsilon^2\partial_t\phi + \varepsilon^2\Delta\phi + \mu_1|\phi|^2\phi + \beta|\psi|^2\phi = 0, \\ i\partial_t\psi + \Delta\psi + \mu_2|\psi|^2\psi + \beta|\phi|^2\psi = 0, \\ \phi = \phi(x, t), \psi = \psi(x, t) \in \mathbb{C}, \quad x = (x_1, x_2) \in \mathbb{R}^2, t > 0, \\ \phi(\cdot, t), \psi(\cdot, t) \in H^1(\mathbb{R}^2), t > 0, \end{cases} \quad (1.3)$$

where  $\varepsilon = h^{-1/2} > 0$  a small parameter and  $\beta < 0$  a coupling constant. Similar systems of NLSE with trap potentials and different dispersion coefficients can be found in [15]. Note that due to the small parameter  $\varepsilon$ , the system (1.3) can be regarded as a multi-scale system having fast and slow variables. In this paper, we want to prove that the system (1.3) may have self-similar  $H^1$  solutions with blowup ring profiles.

To get self-similar solutions of the system (1.3), as for [11], we may set

$$\phi(x, t) = A_1(x, t)e^{i\theta_1(x, t)}, \quad \psi(x, t) = A_2(x, t)e^{i\theta_2(x, t)}, \quad (1.4)$$

where

$$A_1(x, t) = u(\xi) \exp\left(-\int_0^t a(\tau)d\tau\right), \quad A_2(x, t) = v(\xi) \exp\left(-\int_0^t a(\tau)d\tau\right), \quad (1.5)$$

$$\theta_j(x, t) = a(t)\frac{|x|^2}{4} + \gamma_j(t), \quad j = 1, 2, \quad (1.6)$$

and

$$\gamma_1'(t) = \frac{\lambda_1}{\varepsilon^2} \exp\left(-2\int_0^t a(\tau)d\tau\right), \quad \gamma_2'(t) = \lambda_2 \exp\left(-2\int_0^t a(\tau)d\tau\right). \quad (1.7)$$

Here  $u$  and  $v$  are real-valued functions,  $\lambda_j$ 's are positive constants,  $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$  is defined by

$$\xi = x \exp\left(-\int_0^t a(\tau)d\tau\right), \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad (1.8)$$

and  $a(\cdot)$  is defined by an ordinary differential equation given by

$$a'(t) + a^2(t) = 0, \quad \forall t > 0, \quad (1.9)$$

with initial data

$$a(0) = a_0 < 0. \quad (1.10)$$

By (1.4)-(1.9), we may transform the system (1.3) into

$$\begin{cases} \varepsilon^2 \Delta u - \lambda_1 u + \mu_1 u^3 + \beta v^2 u = 0 & \text{in } \mathbb{R}^2, \\ \Delta v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0 & \text{in } \mathbb{R}^2, \\ u, v \in H^1(\mathbb{R}^2), \quad u, v > 0 & \text{in } \mathbb{R}^2, \end{cases} \quad (1.11)$$

where  $0 < \varepsilon \ll 1$  is a small parameter,  $\lambda_j$ 's and  $\mu_j$ 's are positive constants, and  $\beta$  is a negative constant. Here  $\Delta$  is the Laplacian corresponding to  $\xi$ -coordinates denoted as  $\Delta = \sum_{j=1}^2 \partial_{\xi_j}^2$ . To get non-vortex solutions, we only consider positive solutions of (1.11) i.e.  $u, v > 0$  in  $\mathbb{R}^2$ . Moreover, (1.9) and (1.10) imply

$$a(t) = \frac{a_0}{a_0 t + 1} \rightarrow -\infty \quad \text{as } t \uparrow T = -1/a_0, \quad (1.12)$$

and then both  $|\phi|$  and  $|\psi|$  blowup at the same time  $T = -1/a_0$  by (1.4) and (1.5).

Blowup profiles for self-similar solutions of (1.2) and (1.3) are governed by the system (1.11). Here we prove that as  $\varepsilon > 0$  sufficiently small, there are two kinds of  $H^1$  positive solutions  $(u_\varepsilon, v_\varepsilon)$ 's of (1.11) having different asymptotic behaviors. One is that  $u_\varepsilon$  concentrates at vertices of a regular  $k$ -polygon (for any  $k \geq 2$ ) and  $v_\varepsilon$  concentrates at the origin (see Theorem 2.1 in Section 2). The other is that  $u_\varepsilon$  concentrates on a circle away from the origin and  $v_\varepsilon$  concentrates at the origin (see Theorem 2.2 in Section 2). Now we fix  $\varepsilon > 0$  as a small enough constant. Then the graph of  $u_\varepsilon$  may approach to a single ring profile without any vortex. Hereafter, the single ring profile is defined as the graph of a positive function  $f = f(r)$  ( $r = |x|$  is the radial variable for  $x \in \mathbb{R}^2$ ) such that  $f(\infty) = 0$ , and  $f$  is increasing on  $(0, r_1)$  but decreasing on  $(r_1, \infty)$ , where  $r_1$  a positive constant. Hence by (1.4), (1.5) and (1.12), we may obtain self-similar  $H^1$  non-vortex solutions  $(\phi, \psi)$ 's of (1.3) i.e.  $(\Phi, \Psi)$ 's of (1.2) blowing up at  $T = -1/a_0$ , and the blowup profile of  $\Phi$  is of ring profiles. This may provide self-similar non-vortex ring profiles which can be maintained all the way up to the singularity.

Another motivation of the system (1.11) may come from bright ring solitons which exist as stationary localized states observed in self-focusing Kerr media modelled by NLSE (cf. [20]). One may find quantized vortices corresponding to bright ring solitons by solving vortex solutions of the equation (1.1) (cf. [3]). However, until now, steady non-vortex bright ring solitary wave solutions of (1.1) have not yet been found. To learn steady non-vortex bright ring solitons, we study steady solitary wave solutions of the system (1.3) by setting  $\phi(x, t) = e^{i\lambda_1 t/\varepsilon^2} u(x)$  and  $\psi(x, t) = e^{i\lambda_2 t} v(x)$  for  $x = (x_1, x_2) \in \mathbb{R}^2, t > 0$ , where both  $u$  and  $v$  are positive functions. Then the system (1.3) can be transformed into (1.11) with  $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$ . Hence Theorem 2.2 may also provide steady solitary wave solutions of the system (1.3) to describe non-vortex bright ring solitons.

In addition, non-vortex ring profiles can be obtained by numerical simulations on the system (1.11) with  $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$ . Setting  $u = u(r), v = v(r)$ , and  $r = |x|$  for  $x \in \mathbb{R}^2$ , we may rewrite the system (1.11) as follows:

$$\begin{cases} \varepsilon^2 (u'' + \frac{1}{r}u') - \lambda_1 u + \mu_1 u^3 + \beta v^2 u = 0, & \text{for } r > 0, \\ (v'' + \frac{1}{r}v') - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0, & \text{for } r > 0, \\ u, v > 0, & \text{for } r > 0, \\ u'(0) = v'(0) = 0, \quad u(\infty) = v(\infty) = 0. \end{cases} \quad (1.13)$$

We may use a singular boundary value problem solver BVP4C in MATLAB to solve (1.13) and obtain numerical solutions with ring profiles as described in Theorem 2.2 and Remark 3 (see also Figs 1-3 in Section 5). Our numerical scheme is reliable since it produces numerical solutions of (1.13) with computational errors of order  $\mathcal{O}(10^{-15})$  (see, e.g., Fig 1(c)). Besides, numerical solutions with  $m$ -ring and ring-ring profiles can be shown in Fig 4 and Fig 5, respectively. Here the  $m$ -ring profile is the graph of a positive function  $g = g(r)$  ( $r = |x|$  is the radial variable for  $x \in \mathbb{R}^2$ ) with  $m$  bumps. The ring-ring profile means that both  $u$  and  $v$  have ring profiles. Until now, we have no theoretical result to support the existence of the ring-ring profile.

The rest of this paper is organized as follows: We state Theorem 2.1 and 2.2 in Section 2. In Section 3 and 4, we give rigorous arguments to prove Theorem 2.1 and 2.2 using the localized energy method with symmetry reduction. In Section 5, various numerical solutions of (1.13) are given.

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## 2 Main Results

Let  $\omega_j$  be the unique positive solution of

$$\begin{cases} \Delta\omega_j - \lambda_j\omega_j + \mu_j\omega_j^3 = 0, & \text{in } \mathbb{R}^2, \\ \omega_j = \omega_j(r) > 0 & \text{for } r = |x| > 0, \\ \lim_{r \rightarrow \infty} \omega_j(r) = 0, & j = 1, 2. \end{cases} \quad (2.1)$$

Then it is easy to check that  $\omega_j = \omega_j(r) = \sqrt{\frac{\lambda_j}{\mu_j}} \omega(\sqrt{\lambda_j}r)$ , where  $\omega = \omega(r)$  is the unique positive solution of  $\Delta\omega - \omega + \omega^3 = 0$  in  $\mathbb{R}^2$ . Our first result is stated as follows:

**THEOREM 2.1.** *Let  $k \in \mathbb{N}$ ,  $k \geq 2$ . Then for  $\varepsilon$  sufficiently small, problem (1.11) has a solution  $(u_\varepsilon, v_\varepsilon)$  with the following properties*

- (a)  $v_\varepsilon(x) = \omega_2(|x|)(1 + o_\varepsilon(1))$ ,
- (b)  $u_\varepsilon(x) = \sum_{l=1}^k \omega_1\left(\left|\frac{x - P_{\varepsilon,l}}{\varepsilon}\right|\right) (1 + o_\varepsilon(1))$ ,

where  $P_{\varepsilon,l}$ 's are spike centers of  $u_\varepsilon$  satisfying

$$|P_{\varepsilon,i} - P_{\varepsilon,j}| \sim \varepsilon \log \frac{1}{\varepsilon}, \quad P_{\varepsilon,i} = o_\varepsilon(1), \quad (2.2)$$

for  $i, j = 1, \dots, k$  and  $i \neq j$ . Moreover,  $P_{\varepsilon,l}$ 's are located at vertices of a regular  $k$ -polygon in  $\mathbb{R}^2$ .

In this paper, we use the notation  $\sim$  to denote  $A_\varepsilon \sim B_\varepsilon$  which means  $C_1 B_\varepsilon \leq A_\varepsilon \leq C_2 B_\varepsilon$  as  $\varepsilon \rightarrow 0$ , where  $C_j$ 's are positive constant independent of  $\varepsilon$ . Besides,  $o_\varepsilon(1)$  is a small quantity tending to zero as  $\varepsilon$  goes to zero.

**REMARK 1.** We can also prove the existence of solutions with more complex structures: for example,  $u_\varepsilon$  may have spikes at concentric polygons. For the different case that  $\varepsilon = 1$  and  $-\beta$  is a positive but small parameter, one may refer to [11] to get multi-Townes profiles far away from the origin.

In Theorem 2.1, we rigorously prove that as  $\varepsilon$  sufficiently small, there exist  $(u_\varepsilon, v_\varepsilon)$ 's solutions of (1.11) with  $u_\varepsilon$  concentrating at  $P_{\varepsilon,l}, l = 1, \dots, k$  vertices of a regular  $k$ -polygon near the origin and  $v_\varepsilon$  concentrating at the origin. Now we fix  $\varepsilon > 0$  as a sufficiently small constant. By (1.4)-(1.8) and (1.12), the associated solution  $(\Phi, \Psi)$  of (1.2) satisfies  $|\Phi|(x, t) = \frac{T}{T-t} u_\varepsilon\left(\frac{Tx}{T-t}\right) \sim \frac{T}{T-t} \sum_{l=1}^k \omega_1\left(\left|\frac{Tx}{\varepsilon(T-t)} - \varepsilon^{-1} P_{\varepsilon,l}\right|\right)$  and  $|\Psi|(x, t) = \varepsilon^{-1} \frac{T}{T-t} v_\varepsilon\left(\frac{Tx}{T-t}\right) \sim \varepsilon^{-1} \frac{T}{T-t} \omega_2\left(\frac{T|x|}{T-t}\right)$ . Thus  $|\Phi|$  has the  $k$ -fold Townes profile with  $k$  peaks at  $\frac{T-t}{T} P_{\varepsilon,l}$ 's and  $|\Psi|$  has the Townes profile with a single peak at the origin such that as  $t \uparrow T$ ,  $|\Phi|(x, t) \rightarrow \infty$  for  $x = \frac{T-t}{T} P_{\varepsilon,l}, l = 1, \dots, k$  and  $|\Psi|(0, t) \rightarrow \infty$ . Note that both  $|\Phi|$  and  $|\Psi|$  blow up at the same time  $T$ . Moreover, both  $u_\varepsilon$  and  $v_\varepsilon$  have finite  $H^1$  norms which may imply that the the  $k$ -fold Townes profile of  $|\Phi|$  and the Townes profile of  $|\Psi|$  can be maintained all the way up to the singularity.

Next theorem shows that there exist solutions  $(u, v)$ 's with  $u$  concentrating on a circle away from the origin and  $v$  concentrating at the origin. To state the result, we need to introduce some functions. Let  $U$  be the unique homoclinic solution of

$$U'' - U + \mu_1 U^3 = 0, \quad U(y) = U(-y), \quad U > 0, \quad U \rightarrow 0 \text{ at } \infty. \quad (2.3)$$

Let

$$M(r) = r^{2/3} V(r) \quad \text{and} \quad V(r) = \lambda_1 - \beta \omega_2^2(r). \quad (2.4)$$

Then  $r^{-2/3} M'(r) = \frac{2}{3r} V(r) + V'(r)$ . Due to  $\beta < 0$ , it is obvious that  $V(r) > 0$  and  $V'(r) < 0$  for  $r > 0$ . Moreover,  $M'(r) > 0$  for  $r$  sufficiently close to zero or infinity. Suppose

$$\max_{r>0} \frac{|rV'(r)|}{V(r)} > \frac{2}{3}. \quad (2.5)$$

Then  $M'(r_0) < 0$  for some  $r_0 > 0$ . Hence the function  $M$  may have two critical points  $r_j, j = 1, 2$  such that  $0 < r_1 < r_0 < r_2 < \infty$ .

**REMARK 2.** To fulfill the condition (2.5), we remark that due to  $\beta < 0$ , we have

$$V'(r) = -2\beta\omega_2(r)\omega_2'(r) = 2|\beta|\frac{\lambda_2^{3/2}}{\mu_2}\omega(\sqrt{\lambda_2 r})\omega'(\sqrt{\lambda_2 r}),$$

and hence

$$r \frac{V'(r)}{V(r)} = \frac{t\omega(t)\omega'(t)}{A + \omega^2(t)}, \quad (2.6)$$

where  $t = \sqrt{\lambda_2}r$  and  $A = \frac{\lambda_1\mu_2}{|\beta|\lambda_2}$ . Now we set  $f(a) = \max_{t>0} \frac{|t\omega(t)\omega'(t)|}{a + \omega^2(t)}$  for  $a > 0$ . Then it is obvious that  $f$  is monotone decreasing in  $a$ ,  $\lim_{a \rightarrow 0} f(a) = \infty$  and  $\lim_{a \rightarrow \infty} f(a) = 0$ . Consequently, there exists a unique  $A_0 > 0$  such that  $f(A_0) = \frac{2}{3}$  i.e.  $\max_{r>0} \frac{|rV'(r)|}{V(r)} = \frac{2}{3}$  if  $A = A_0$ . Therefore the condition (2.5) can be replaced by  $A < A_0$  i.e.

$$-\beta > \frac{\lambda_1\mu_2}{\lambda_2 A_0} > 0. \quad (2.7)$$

Now we state another main theorem as follows:

**THEOREM 2.2.** *Assume (2.7) holds. Then the problem (1.11) has two solutions  $(u_{\varepsilon,1}, v_{\varepsilon,1})$  and  $(u_{\varepsilon,2}, v_{\varepsilon,2})$  such that*

$$(a) \quad u_{\varepsilon,i}(r) \sim \sqrt{V(r_{\varepsilon,i})} U \left( \sqrt{V(r_{\varepsilon,i})} \frac{|r - r_{\varepsilon,i}|}{\varepsilon} \right), \quad i = 1, 2,$$

$$(b) \quad v_{\varepsilon,i}(r) \sim \omega_2(r), \quad i = 1, 2,$$

where  $r_{\varepsilon,i} \rightarrow r_i$  as  $\varepsilon \rightarrow 0+$ , and  $r_1 < r_2$  are two critical points of  $M(r)$ .

**REMARK 3.** Following the proof of [13], we can also show the existence of clustered ring solutions, i.e.  $u_{\varepsilon,i}(r) \sim \sum_{j=1}^K \sqrt{V(r_{\varepsilon,i}^j)} U \left( \sqrt{V(r_{\varepsilon,i}^j)} \frac{|r - r_{\varepsilon,i}^j|}{\varepsilon} \right)$ , where  $r_{\varepsilon,i}^j \rightarrow r_i, j = 1, \dots, K$ .

In Theorem 2.2, we rigorously prove that as  $\varepsilon$  sufficiently small, there exist  $(u_{\varepsilon,i}, v_{\varepsilon,i})$ 's solutions of (1.11) with  $u_{\varepsilon,i}$  concentrating on a circle (with a center at the origin and radius  $r_i$ ) away from the origin and  $v_{\varepsilon,i}$  concentrating at the origin. Now we fix  $\varepsilon > 0$  as a small enough constant. By (1.4)-(1.8), (1.12) and Theorem 2.2, the associated solution  $(\Phi, \psi)$  satisfies  $|\Phi|(x, t) = \frac{T}{T-t} u_{\varepsilon,i} \left( \frac{T|x|}{T-t} \right) \sim \frac{T}{T-t} \sqrt{V(r_{\varepsilon,i})} U \left( \sqrt{V(r_{\varepsilon,i})} \left| \frac{T|x|}{\varepsilon(T-t)} - \varepsilon^{-1} r_{\varepsilon,i} \right| \right)$  and  $|\Psi|(x, t) = \varepsilon^{-1} \frac{T}{T-t} v_{\varepsilon,i} \left( \frac{T|x|}{T-t} \right) \sim \varepsilon^{-1} \frac{T}{T-t} \omega_2 \left( \frac{T|x|}{T-t} \right)$ . Thus as  $t \uparrow T$ ,  $|\Phi|(x, t) \rightarrow \infty$  for  $x \in \Gamma_t^i$  and  $|\Psi|(0, t) \rightarrow \infty$ , where  $\Gamma_t^i = \{x \in \mathbb{R}^2 : |x| = \frac{T-t}{T} r_{\varepsilon,i}\}$  is a circle shrinking to the origin as  $t$  goes to  $T$ . Note that both  $|\Phi|$  and  $|\Psi|$  blow up at the same time  $T$ . Furthermore, both  $u_\varepsilon$  and  $v_\varepsilon$  have finite  $H^1$  norms which may imply that the ring profile of  $|\Phi|$  and the Townes profile of  $|\Psi|$  can be maintained all the way up to the singularity.

### 3 Proof of Theorem 2.1

In this section, we use the *method of localized energy with symmetry reduction* to prove Theorem 2.1. For an overview on localized energy method, we refer to Chapter 2 of [21]. A combination of localized energy method and symmetry reduction has been used in [11], which we follow closely.

#### 3.1 Symmetry Class

For  $k \geq 2$ , we define a class of functions with the symmetry property as follows:

$$\Sigma_1 = \left\{ u \left( \tilde{r} \cos \left( \tilde{\theta} + \frac{2\pi}{k} \right), \tilde{r} \sin \left( \tilde{\theta} + \frac{2\pi}{k} \right) \right) = u \left( \tilde{r} \cos \tilde{\theta}, \tilde{r} \sin \tilde{\theta} \right) \right\}. \quad (3.1)$$

Then we have

**LEMMA 3.1.** *If  $\phi \in H^2(\mathbb{R}^2) \cap \Sigma_1$  and*

$$L_2\phi := \Delta\phi - \lambda_2\phi + 3\mu_2\omega_2^2\phi = 0, \quad (3.2)$$

*then  $\phi \equiv 0$ . Moreover,  $\|\phi\|_{H^2} \leq C\|L_2\phi\|_{L^2}$  for  $\phi \in H^2(\mathbb{R}^2) \cap \Sigma_1$ , where  $C$  is a positive constant independent of  $\phi$ .*

The proof of Lemma 3.1 is quite standard so we may omit it here. (See [11].) As a consequence of Lemma 3.1, we have

**LEMMA 3.2.** *There exists  $\delta > 0$  such that if*

$$\|g\|_{L^2(\mathbb{R}^2)} < \delta, \quad (3.3)$$

*then the equation*

$$\Delta v - \lambda_2 v + \mu_2 v^3 + gv = 0 \quad \text{in } \mathbb{R}^2 \quad (3.4)$$

*has a unique solution  $v \in H^2(\mathbb{R}^2) \cap \Sigma_1$  satisfying*

$$\|v - \omega_2\|_{H^2(\mathbb{R}^2)} < c\delta, \quad (3.5)$$

*where  $c$  is a positive constant independent of  $\delta$ .*

*Proof.* We use contraction mapping theorem to prove the lemma. Suppose  $v = \omega_2 + \psi$  is the solution of (3.4), where  $\psi \in H^2(\mathbb{R}^2) \cap \Sigma_1$ . Then  $\psi$  satisfies

$$\Delta\psi - \lambda_2\psi + 3\mu_2\omega_2^2\psi + N[\psi] + g\omega_2 + g\psi = 0,$$

which is equivalent to

$$\psi = -L_2^{-1} [N[\psi] + g\psi + g\omega_2] \equiv \mathcal{A}[\psi],$$

where  $N[\psi] = \mu_2(3\omega_2\psi^2 + \psi^3)$ . Note that by Lemma 3.1,

$$\|L_2^{-1}[g\omega_2]\|_{H^2} \leq C\|g\omega_2\|_{L^2} \leq C\|g\|_{L^2}.$$

Now let

$$B = \{\psi \in H^2 \cap \Sigma_1 : \|\psi\|_{H^2} < C_1\delta\}.$$

Then by choosing  $C_1$  sufficiently large and  $\delta$  small enough, the map  $\mathcal{A}$  may become a contraction mapping from  $B$  to  $B$ . Hence a unique solution in  $B$  is guaranteed.  $\square$

Given  $g$ , let us denote the function  $v$  in Lemma 3.2 as  $T[g] := v$ . Next we introduce a framework to solve the first equation (i.e.  $u$ ) of (1.11). Let  $P_0 = (\varepsilon l, 0)$ ,  $P_i = R_i(\varepsilon l, 0)$ ,  $i = 1, \dots, k$ , where

$$R_i = \begin{pmatrix} \cos\left(\frac{2\pi}{k}(i-1)\right) & -\sin\left(\frac{2\pi}{k}(i-1)\right) \\ \sin\left(\frac{2\pi}{k}(i-1)\right) & \cos\left(\frac{2\pi}{k}(i-1)\right) \end{pmatrix} \quad (3.6)$$

and  $l$  satisfies  $\alpha \log \frac{1}{\varepsilon} \leq l \leq \gamma \log \frac{1}{\varepsilon}$ ,  $\alpha > 1$  and  $\gamma$  will be chosen later. Let  $\tilde{\omega}$  be the unique positive solution of

$$\Delta \omega - (\lambda_1 - \beta \omega_2^2(P_0)) \omega + \mu_1 \omega^3 = 0 \quad \text{in } \mathbb{R}^2, \quad (3.7)$$

with  $\lim_{|x| \rightarrow \infty} \omega(|x|) = 0$ . Note that  $\omega_2^2(P_0) = \omega_2^2(P_i)$  for  $i = 1, \dots, k$ . Let  $\tilde{\omega}_i(x) = \tilde{\omega}\left(\frac{|x - P_i|}{\varepsilon}\right)$  and  $W(x) = \sum_{i=1}^k \tilde{\omega}_i(x)$ . Of course,  $W \in \Sigma_1$  so we may choose  $W$  as an ansatz to approximate the  $u$ -component solution. Now we rescale the spatial variables by  $\varepsilon$  i.e.  $y = x/\varepsilon$ , and consider the following operator

$$S[u] = \Delta u - \lambda_1 u + \mu_1 u^3 + \beta(T[\beta u^2](\varepsilon y))^2 u \quad (3.8)$$

on  $H^2(\mathbb{R}^2)$  with norms given by

$$\|\cdot\|_{**} = \left[ \int_{\mathbb{R}^2} u^2(y) dy \right]^{1/2}, \quad \|\cdot\|_* = \|\cdot\|_{H^2(\mathbb{R}^2)}.$$

Now let us estimate the error introduced by  $W$ .

### 3.2 Error Estimate

Let us compute the error  $E = S[W]$ . From Lemma 3.2, we have

$$\begin{aligned} T[\beta W^2](\varepsilon y) &= \omega_2(\varepsilon y) + O(\|W^2\|_{L^2}) \\ &= \omega_2(\varepsilon y) + O(\varepsilon). \end{aligned} \quad (3.9)$$

Here we have used the fact that  $\alpha \log \frac{1}{\varepsilon} \leq l \leq \gamma \log \frac{1}{\varepsilon}$  and  $\alpha > 1$ . Hence

$$\begin{aligned} E = S[W] &= \Delta W - \lambda_1 W + \mu_1 W^3 + \beta(T[\beta W^2](\varepsilon y))^2 W \\ &= \Delta W - (\lambda_1 - \beta \omega_2^2) W + \mu_1 W^3 + \beta [(T[\beta W^2](\varepsilon y))^2 - \omega_2^2(\varepsilon y)] W \\ &= E_1 + E_2, \end{aligned}$$

where  $E_1 = \Delta W - (\lambda_1 - \beta \omega_2^2) W + \mu_1 W^3$  and  $E_2 = \beta [(T[\beta W^2](\varepsilon y))^2 - \omega_2^2(\varepsilon y)] W$ . It is easy to check that

$$\begin{aligned} E_1 &= \Delta W - (\lambda_1 - \beta \omega_2^2) W + \mu_1 W^3 \\ &= (\beta \omega_2^2(\varepsilon y) - \beta \omega_2^2(P_0)) W + \mu_1 \left[ \left( \sum_{i=1}^k \tilde{\omega}_i \right)^3 - \sum_{i=1}^k \tilde{\omega}_i^3 \right] \\ &= E_{11} + E_{12}, \end{aligned}$$

where  $E_{11} = (\beta \omega_2^2(\varepsilon y) - \beta \omega_2^2(P_0)) W$  and  $E_{12} = \mu_1 \left[ \left( \sum_{i=1}^k \tilde{\omega}_i \right)^3 - \sum_{i=1}^k \tilde{\omega}_i^3 \right]$ .

For  $E_{12}$ , we have

$$\begin{aligned} \int_{\mathbb{R}^2} |E_{12}|^2 &\leq C \int_{\mathbb{R}^n} \sum_{k \neq l} \tilde{\omega}_k^4 \tilde{\omega}_l^2 \\ &\leq C \sum_{k \neq l} \tilde{\omega}^2 \left( \frac{|P_k - P_l|}{\varepsilon} \right) \\ &\leq C \tilde{\omega}^2 \left( 2l \sin \frac{\pi}{k} \right), \end{aligned} \quad (3.10)$$

where  $C$  is a positive constant independent of  $\varepsilon$ . For  $E_{11}$ , we have

$$E_{11} = \beta (\omega_2^2(|\varepsilon y|) - \omega_2^2(|P_0|)) W.$$

Let  $\varepsilon y = P_0 + \varepsilon z$ . Then we obtain

$$\begin{aligned} |E_{11}| &\leq CW \cdot [|\omega_2'(|P_0|)| \times (|P_0 + \varepsilon z| - |P_0|) + O((|P_0 + \varepsilon z| - |P_0|)^2)] \\ &\leq CW \cdot (\varepsilon |P_0| |z| + \varepsilon^2 |z|^2), \end{aligned} \quad (3.11)$$

and hence

$$\int_{\mathbb{R}^2} |E_{11}|^2 \leq C(\varepsilon^2 |P_0|^2 + \varepsilon^4) \leq C\varepsilon^4 \log^2 \frac{1}{\varepsilon}, \quad (3.12)$$

where  $C$  is a positive constant independent of  $\varepsilon$ . For  $E_2$ , we may use (3.9) to get

$$\int_{\mathbb{R}^2} |E_2|^2 \leq C\varepsilon^2 \int_{\mathbb{R}^2} |W|^2 \leq C\varepsilon^2. \quad (3.13)$$

Combining the estimates in (3.10)–(3.13), we obtain the following error estimates.

**LEMMA 3.3.** *The error  $E = S[W]$  satisfies*

$$\|E\|_{**} \leq C \left( \varepsilon + \tilde{w} \left( 2l \sin \frac{\pi}{k} \right) \right), \quad (3.14)$$

where  $\|\cdot\|_{**} = \|\cdot\|_{L^2(\mathbb{R}^2)}$ .

### 3.3 Linear Theory

We consider the following linear problem

$$\begin{cases} \Delta \phi - \lambda_1 \phi + 3\mu_1 W^2 \phi + \beta \omega_2^2 \phi = h + c \frac{\partial W}{\partial l}, \\ \int_{\mathbb{R}^2} \phi \frac{\partial W}{\partial l} = 0, \quad \phi \in \Sigma_1, \end{cases} \quad (3.15)$$

with the solution  $(\phi, c)$ , where  $h \in L^2(\mathbb{R}^2) \cap \Sigma_1$ . Then we may derive apriori estimates as follows:

**LEMMA 3.4.** *For  $\varepsilon$  sufficiently small, given  $\|h\|_{**} < \infty$ , problem (3.15) has a unique solution  $(\phi, c)$  such that*

$$\|\phi\|_* + |c| \leq C \|h\|_{**}, \quad (3.16)$$

where  $\|\cdot\|_* = \|\cdot\|_{H^2(\mathbb{R}^2)}$  and  $\|\cdot\|_{**} = \|\cdot\|_{L^2(\mathbb{R}^2)}$ .

*Proof.* Firstly, we prove (3.16). We note that

$$\begin{aligned} \frac{\partial W}{\partial l} &= \sum_{i=1}^k \tilde{\omega}' \left( \frac{|\varepsilon y - P_i|}{\varepsilon} \right) \cdot \frac{1}{\varepsilon} \cdot \frac{-(\varepsilon y - P_i) \cdot \varepsilon (R_i e_1)}{|\varepsilon y - P_i|} \\ &= \sum_{i=1}^k \tilde{\omega}' \left( \frac{|\varepsilon y - P_i|}{\varepsilon} \right) \frac{-(\varepsilon y - P_i) \cdot (R_i e_1)}{|\varepsilon y - P_i|}, \end{aligned} \quad (3.17)$$

where  $e_1 = (1, 0)$  and  $R_i$ 's are defined in (3.6). Multiplying (3.15) by  $\frac{\partial W}{\partial l}$  and integrating over  $\mathbb{R}^2$ , we obtain

$$\begin{aligned} |c| &\leq \left| \int_{\mathbb{R}^2} \left( \Delta \frac{\partial W}{\partial l} - \tilde{\lambda}_1 \frac{\partial W}{\partial l} + 3\mu_1 W^2 \frac{\partial W}{\partial l} \right) \phi \right| + \left| \int_{\mathbb{R}^2} (\beta \omega_2^2 - \beta \omega_2^2(P_0)) \phi \frac{\partial W}{\partial l} \right| + \|h\|_{**} \\ &\leq o(\|\phi\|_{**}) + \|h\|_{**}, \end{aligned} \quad (3.18)$$

where  $\tilde{\lambda}_1 = \lambda_1 - \beta \omega_2^2(P_0)$  and  $o(1)$  is a small quantity tending to zero as  $\varepsilon$  goes to zero. Here we have used the inequality (3.11) to deal with the second integral of (3.18). To get (3.16), it is enough to show that  $\|\phi\|_* \leq C \|h\|_{**}$ . In fact, we can prove it by contradiction using a similar argument to Lemma 4.1 of [11] so we omit the details here. Therefore by (3.16), Lemma 8, Proposition 1 and Lemma 10 of [12], we may complete the proof of Lemma 3.4.  $\square$

### 3.4 Nonlinear reduction

From Lemma 3.4, we deduce the following Lemma

**LEMMA 3.5.** *For  $\varepsilon$  sufficiently small, there exist a unique solution  $(\phi_l, c_l)$  such that*

$$S[W + \phi_l] = c_l \frac{\partial W}{\partial l}, \quad \int_{\mathbb{R}^2} \phi_l \frac{\partial W}{\partial l} = 0, \quad (3.19)$$

and

$$\|\phi_l\|_* \leq C \left( \varepsilon + \tilde{\omega} \left( 2l \sin \frac{\pi}{k} \right) \right). \quad (3.20)$$

*Proof.* Let

$$B = \left\{ \phi \in H^2 \cap \Sigma_1 : \|\phi\|_* \leq \rho \left( \varepsilon + \tilde{\omega} \left( 2l \sin \frac{\pi}{k} \right) \right) \right\},$$

where  $\rho$  is a suitable positive constant. Then by (3.8), we have

$$\begin{aligned} S[W + \phi] = & S[W] + \Delta\phi - \lambda_1\phi + 3\mu_1 W^2\phi + \beta\omega_2^2\phi + \beta (T[\beta(W + \phi)^2]^2 - \omega_2^2) \phi \\ & + N[\phi] + \beta (T[\beta(W + \phi)^2]^2 - T[\beta W^2]^2) W, \end{aligned} \quad (3.21)$$

where  $N[\phi] = \mu_1 (3W\phi^2 + \phi^3)$ . By (3.9), we may calculate

$$\begin{aligned} \|\beta (T[\beta(W + \phi)^2]^2 - \omega_2^2) \phi\|_{**} & \leq C\varepsilon\|\phi\|_*, \\ \|N[\phi]\|_{**} & \leq C\|\phi\|_*^2, \\ \|\beta (T[\beta(W + \phi)^2]^2 - T[\beta W^2]^2) W\|_{**} & \leq C\varepsilon. \end{aligned}$$

The rest of the proof follows from standard contraction mapping theorem. One may refer to [11] for the details.  $\square$

### 3.5 Expansion of $c_l$

Let us now expand  $c_l$  as follows: Multiply the first equation in (3.19) by  $\frac{\partial W}{\partial l}$  and integrate over  $\mathbb{R}^2$ . Then we may use (3.21) to get

$$\begin{aligned} c_l \int_{\mathbb{R}^2} \left( \frac{\partial W}{\partial l} \right)^2 dy & = \int_{\mathbb{R}^2} E \frac{\partial W}{\partial l} dy + \int_{\mathbb{R}^2} [\Delta\phi - \lambda_1\phi + 3\mu_1 W^2\phi + \beta\omega_2^2\phi] \frac{\partial W}{\partial l} dy \\ & \quad + \int_{\mathbb{R}^2} \beta (T[\beta(W + \phi)^2]^2 - \omega_2^2) \phi \frac{\partial W}{\partial l} dy \\ & \quad + \int_{\mathbb{R}^2} N[\phi] \frac{\partial W}{\partial l} dy + \int_{\mathbb{R}^2} \beta (T[\beta(W + \phi)^2]^2 - T[\beta W^2]^2) W \frac{\partial W}{\partial l} dy \\ & = I_1 + I_2 + I_3 + I_4 + I_5, \end{aligned}$$

where  $x = P_0 + \varepsilon y$ ,  $E = S[W]$ ,  $\phi = \phi_l$ , and

$$\begin{aligned} I_1 & = \int_{\mathbb{R}^2} E \frac{\partial W}{\partial l} dy, \\ I_2 & = \int_{\mathbb{R}^2} [\Delta\phi - \lambda_1\phi + 3\mu_1 W^2\phi + \beta\omega_2^2\phi] \frac{\partial W}{\partial l} dy, \\ I_3 & = \int_{\mathbb{R}^2} \beta (T[\beta(W + \phi)^2]^2 - \omega_2^2) \phi \frac{\partial W}{\partial l} dy, \\ I_4 & = \int_{\mathbb{R}^2} N[\phi] \frac{\partial W}{\partial l} dy, \\ I_5 & = \int_{\mathbb{R}^2} \beta (T[\beta(W + \phi)^2]^2 - T[\beta W^2]^2) W \frac{\partial W}{\partial l} dy \end{aligned}$$

Using (3.9) and (3.20), it is obvious that

$$I_3 = \int_{\mathbb{R}^2} \beta (T[\beta(W + \phi)^2] - \omega_2^2) \phi \frac{\partial W}{\partial l} dy = O\left(\varepsilon^2 + \tilde{\omega}^2 \left(2l \sin \frac{\pi}{k}\right)\right), \quad (3.22)$$

and

$$I_4 = \int_{\mathbb{R}^2} N[\phi] \frac{\partial W}{\partial l} dy = O\left(\varepsilon^2 + \tilde{\omega}^2 \left(2l \sin \frac{\pi}{k}\right)\right). \quad (3.23)$$

To estimate  $I_5$ , we set  $\psi = T[\beta(W + \phi)^2] - T[\beta W^2]$ . Then  $\psi$  satisfies

$$\Delta \psi - \lambda_2 \psi + 3\mu_2 (T[\beta W^2])^2 \psi + \beta W^2 \psi + O(\psi^2) + O(|(2\phi W + \phi^2) T[\beta(W + \phi)^2]|) = 0. \quad (3.24)$$

By (3.9) and (3.20), it is easy to check that

$$\|(2\phi W + \phi^2) T[\beta(W + \phi)^2]\|_{L^2(\mathbb{R}^2)} \leq C\varepsilon^2, \quad (3.25)$$

and

$$\|\psi^2\|_{L^2(\mathbb{R}^2)} \leq C\varepsilon^2. \quad (3.26)$$

Due to  $\beta < 0$  and (3.9), the linear operator  $L_*$  defined by

$$-L_* \psi = \Delta \psi - \lambda_2 \psi + 3\mu_2 (T[\beta W^2])^2 \psi + \beta W^2 \psi$$

is invertible. Consequently, as for Lemma 3.1,

$$\|\psi\|_{H^2(\mathbb{R}^2)} \leq C \|L_* \psi\|_{L^2(\mathbb{R}^2)}. \quad (3.27)$$

Moreover, (3.24)-(3.27) may give

$$\|\psi\|_{H^2(\mathbb{R}^2)} \leq C\varepsilon^2, \quad (3.28)$$

and hence by Sobolev embedding,

$$\|\psi\|_{L^\infty(\mathbb{R}^2)} \leq C\varepsilon^2.$$

Consequently,

$$I_5 \leq C\varepsilon^2.$$

For  $I_2$ , we may use integration by parts to get

$$\begin{aligned} I_2 &= \int_{\mathbb{R}^2} \left[ \Delta \frac{\partial W}{\partial l} - \lambda_1 \frac{\partial W}{\partial l} + 3\mu_1 W^2 \frac{\partial W}{\partial l} + \beta \omega_2^2 \frac{\partial W}{\partial l} \right] \phi dy \\ &= \int_{\mathbb{R}^2} \left[ \Delta \frac{\partial W}{\partial l} - \tilde{\lambda}_1 \frac{\partial W}{\partial l} + 3\mu_1 W^2 \frac{\partial W}{\partial l} \right] \phi dy + \int_{\mathbb{R}^2} \beta (\omega_2^2 - \omega_2^2(P_0)) \frac{\partial W}{\partial l} \phi dy, \end{aligned}$$

where  $\tilde{\lambda}_1 = \lambda_1 - \beta \omega_2^2(P_0)$ . By (3.17) and (3.20), it is obvious that

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \beta (\omega_2^2 - \omega_2^2(P_0)) \frac{\partial W}{\partial l} \phi dy \right| &\leq C |\omega_2'(P_0)| \int_{\mathbb{R}^2} |\varepsilon y - P_0| \left| \frac{\partial W}{\partial l} \right| |\phi| dy \\ &\leq C\varepsilon^2, \end{aligned} \quad (3.29)$$

and

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \left[ \Delta \frac{\partial W}{\partial l} - \tilde{\lambda}_1 \frac{\partial W}{\partial l} + 3\mu_1 W^2 \frac{\partial W}{\partial l} \right] \phi dy \right| &\leq C \left( \varepsilon + \tilde{\omega} \left(2l \sin \frac{\pi}{k}\right) \right) \varepsilon \\ &\leq C\varepsilon^2 + C\varepsilon \tilde{\omega} \left(2l \sin \frac{\pi}{k}\right). \end{aligned} \quad (3.30)$$

Here we have used the fact that

$$\Delta \frac{\partial W}{\partial l} - \tilde{\lambda}_1 \frac{\partial W}{\partial l} + 3\mu_1 \sum_{i=1}^k \tilde{\omega}_i^2 \frac{\partial \tilde{\omega}_i}{\partial l} - \frac{\partial \tilde{\lambda}_1}{\partial l} W = 0 \quad \text{in } \mathbb{R}^2,$$

and  $\frac{\partial \tilde{\lambda}_1}{\partial l} = O(\varepsilon)$  since  $W = \sum_{i=1}^k \tilde{\omega}_i$  and  $\tilde{\omega}_i$ 's satisfy (3.7). Consequently,

$$I_2 = O\left(\varepsilon^2 + \tilde{\omega}^2 \left(2l \sin \frac{\pi}{k}\right)\right). \quad (3.31)$$

Now it remains to compute  $I_1$ :

$$I_1 = \int_{\mathbb{R}^2} [\Delta W - (\lambda_1 - \beta\omega_2^2) W + \mu_1 W^3] \frac{\partial W}{\partial l} dy + \int_{\mathbb{R}^2} \beta[(T[\beta W^2])^2 - \omega_2^2] W \frac{\partial W}{\partial l} dy.$$

Note that

$$\|T[\beta W^2] - \omega_2\|_{H^2(\mathbb{R}^2)} \leq C\varepsilon, \quad (3.32)$$

and then by Sobolev embedding,

$$\|T[\beta W^2] - \omega_2\|_{C^1(\mathbb{R}^2)} \leq C\varepsilon. \quad (3.33)$$

Hence

$$\begin{aligned} & \int_{\mathbb{R}^2} \beta[(T[\beta W^2])^2 - \omega_2^2] W \frac{\partial W}{\partial l} dy \\ &= \int_{\mathbb{R}^2} \beta[(T[\beta W^2])^2(P_0) - \omega_2^2(P_0)] W \frac{\partial W}{\partial l} dy \\ & \quad + \int_{\mathbb{R}^2} \beta \{ [(T[\beta W^2])^2(x) - \omega_2^2(x)] - [(T[\beta W^2])^2(P_0) - \omega_2^2(P_0)] \} W \frac{\partial W}{\partial l} dy \\ &= O(\varepsilon) \int_{\mathbb{R}^2} |x - P_0| W \left| \frac{\partial W}{\partial l} \right| dy \\ &= O(\varepsilon^2), \end{aligned} \quad (3.34)$$

where we have used (3.33),  $x = P_0 + \varepsilon y$ ,  $P_0 = (\varepsilon l, 0)$  and  $\int_{\mathbb{R}^2} W \frac{\partial W}{\partial l} dy = 0$ . Finally, by (3.7), we obtain

$$\begin{aligned} & \int_{\mathbb{R}^2} [\Delta W - (\lambda_1 - \beta\omega_2^2) W + \mu_1 W^3] \frac{\partial W}{\partial l} dy \\ &= \int_{\mathbb{R}^2} \left\{ [\beta\omega_2^2 - \beta\omega_2^2(P_0)] W dy + \mu_1 \left[ \left( \sum_{i=1}^k \tilde{\omega}_i \right)^3 - \sum_{i=1}^k \tilde{\omega}_i^3 \right] \right\} \frac{\partial W}{\partial l} dy \\ &= I_{11} + I_{22}, \end{aligned}$$

where

$$\begin{aligned} I_{11} &= \beta \int_{\mathbb{R}^2} [\omega_2^2 - \omega_2^2(P_0)] W \frac{\partial W}{\partial l} dy, \\ I_{12} &= \int_{\mathbb{R}^2} \mu_1 \left[ \left( \sum_{i=1}^k \tilde{\omega}_i \right)^3 - \sum_{i=1}^k \tilde{\omega}_i^3 \right] \frac{\partial W}{\partial l} dy. \end{aligned}$$

By symmetry,

$$\begin{aligned} I_{11} &= \beta \int_{\mathbb{R}^2} [\omega_2^2(x) - \omega_2^2(P_0)] W \frac{\partial W}{\partial l} dy \\ &= \beta k \int_{\Gamma_0} [\omega_2^2(x) - \omega_2^2(P_0)] W \frac{\partial W}{\partial l} dy, \\ I_{12} &= \mu_1 k \int_{\Gamma_0} \left[ \left( \sum_{i=1}^k \tilde{\omega}_i \right)^3 - \sum_{i=1}^k \tilde{\omega}_i^3 \right] \frac{\partial W}{\partial l} dy, \end{aligned}$$

where

$$\Gamma_0 = \left\{ (r \cos \theta, r \sin \theta) : r \geq 0, -\frac{\pi}{k} < \theta < \frac{\pi}{k} \right\}.$$

Thus

$$\begin{aligned} I_{11} &= \beta k \int_{\Gamma_0} 2\omega_2(P_0)\omega_2'(P_0) \left[ \frac{P_0}{|P_0|} \cdot (x - P_0) \right] W \frac{\partial W}{\partial l} dy + O(\varepsilon^2) \\ &= \beta k \int_{\Gamma_0} 2\omega_2(0)\omega_2''(0)|P_0|\varepsilon y_1 \tilde{\omega} \left( -\frac{\partial \tilde{\omega}}{\partial y_1} \right) dy + O(\varepsilon^2) \\ &= \beta k \left( C_0 \int_{\mathbb{R}^2} \tilde{\omega}^2 dy \right) \omega_2(0)\omega_2''(0)\varepsilon^2 l + O(\varepsilon^2) := C_1 \varepsilon^2 l + O(\varepsilon^2), \end{aligned} \quad (3.35)$$

where  $x = P_0 + \varepsilon y$ ,  $P_0 = (\varepsilon l, 0, 0)$ ,  $C_1 := \beta k (c_0 \int_{\mathbb{R}^n} \tilde{\omega}^2) \omega_2(0)\omega_2''(0) > 0$ , and  $C_0$  is a positive constant.

To estimate  $I_{12}$ , we observe that

$$\begin{aligned} I_{12} &= \mu_1 k \int_{\Gamma_0} 3\tilde{\omega}_1^2 \sum_{i=2}^k \tilde{\omega}_i \left( -\frac{\partial \tilde{\omega}_1}{\partial y_1} \right) + O(\varepsilon^2) \\ &= \mu_1 k \int_{\Gamma_0} 3\tilde{\omega}_1^2 (\tilde{\omega}_2 + \tilde{\omega}_k) \left( -\frac{\partial \tilde{\omega}_1}{\partial y_1} \right) + O(\varepsilon^2). \end{aligned}$$

In particular,

$$\int_{\Gamma_0} 3\tilde{\omega}_1^2 \tilde{\omega}_2 \left( -\frac{\partial \tilde{\omega}_1}{\partial y_1} \right) dy = \int_{\mathbb{R}^2} \tilde{\omega}_1^3 \frac{\partial \tilde{\omega}_2}{\partial y_1} dy + O(\varepsilon^2).$$

Note that

$$\begin{aligned} \tilde{\omega}_1 &= \tilde{\omega} \left( \frac{|x - P_1|}{\varepsilon} \right) = \tilde{\omega} \left( \frac{|\varepsilon y + P_0 - P_1|}{\varepsilon} \right) = \tilde{\omega}(|y|), \\ \frac{\partial \tilde{\omega}_2}{\partial y_1} &= \tilde{\omega}' \left( \left| y + \frac{P_0 - P_2}{\varepsilon} \right| \right) \cdot \frac{y_1 + \frac{\langle P_0 - P_2, e_1 \rangle}{\varepsilon}}{\left| y + \frac{P_0 - P_2}{\varepsilon} \right|} \\ &= \tilde{\omega}' \left( \frac{|P_0 - P_2|}{\varepsilon} \right) \left( \frac{\langle P_0 - P_2, e_1 \rangle}{|P_0 - P_2|} \right) \cdot (1 + O(\varepsilon|y|^2)) e^{-\frac{\langle P_0 - P_2, y \rangle}{|P_0 - P_2|}}. \end{aligned}$$

Consequently,

$$\begin{aligned} \int_{\mathbb{R}^2} \tilde{\omega}_1^3 \frac{\partial \tilde{\omega}_2}{\partial y_1} dy &= \left( \int_{\mathbb{R}^2} \tilde{\omega}^3(|y|) e^{-\frac{\langle P_0 - P_2, y \rangle}{|P_0 - P_2|}} dy \right) \cdot \tilde{\omega}' \left( \frac{|P_0 - P_2|}{\varepsilon} \right) \frac{\langle P_0 - P_2, e_1 \rangle}{|P_0 - P_2|} \\ &= C_2 \tilde{\omega}' \left( 2l \sin \frac{\pi}{2k} \right) + O(\varepsilon^2), \end{aligned} \quad (3.36)$$

where

$$C_2 = \left( \int_{\mathbb{R}^2} \tilde{\omega}^3(|y|) e^{-\frac{\langle P_0 - P_2, y \rangle}{|P_0 - P_2|}} dy \right) \cdot \frac{\langle P_0 - P_2, e_1 \rangle}{|P_0 - P_2|} > 0. \quad (3.37)$$

Hence

$$I_{12} = \hat{C}_2 \tilde{\omega}' \left( 2l \sin \frac{\pi}{2k} \right) + O(\varepsilon^2), \quad (3.38)$$

where  $\hat{C}_2$  is a positive constant independent of  $\varepsilon$ . Therefore by (3.35) and (3.38), we have

$$I_{11} + I_{12} = C_1 \varepsilon^2 l + \hat{C}_2 \tilde{\omega}' \left( 2l \sin \frac{\pi}{2k} \right) + O(\varepsilon^2). \quad (3.39)$$

In summary, we have

$$c_l = \tilde{C}_1 \varepsilon^2 l + \tilde{C}_2 \tilde{\omega}' \left( 2l \sin \frac{\pi}{2k} \right) + O(\varepsilon^2), \quad (3.40)$$

where  $\tilde{C}_1, \tilde{C}_2 > 0$  are positive generic constants independent of  $\varepsilon$ .

### 3.6 Proof of Theorem 2.1

We prove Theorem 2.1 by a continuity argument. Note that

$$\tilde{\omega}'(2l \sin \frac{\pi}{2k}) = -A_0 \left(2l \sin \frac{\pi}{2k}\right)^{-\frac{1}{2}} e^{-2l \sin \frac{\pi}{2k}} \left(1 + O\left(\frac{1}{l}\right)\right), \quad (3.41)$$

where  $A_0 > 0$  is a constant independent of  $\varepsilon$ . Let  $\alpha = (1 - \eta)/\sin \frac{\pi}{2k}$ , and  $\gamma = (1 + \eta)/\sin \frac{\pi}{2k}$ , where  $0 < \eta \ll 1$  is a small constant independent of  $\varepsilon$ . Then by (3.40), we have

$$\begin{aligned} c_l &= \tilde{C}_1 \varepsilon^2 l - \tilde{C}_2 A_0 \left[2\alpha \left(\sin \frac{\pi}{2k}\right) \cdot \log \frac{1}{\varepsilon}\right]^{-\frac{1}{2}} e^{-2\alpha \sin \frac{\pi}{2k} \cdot \log \frac{1}{\varepsilon}} \\ &\leq -\varepsilon^{2-\eta} < 0, \end{aligned} \quad (3.42)$$

provided  $l = \alpha \log \frac{1}{\varepsilon}$  and  $\varepsilon > 0$  is small enough. Here we have used the fact that  $\alpha = (1 - \eta)/\sin \frac{\pi}{2k}$ . On the other hand, if  $l = \gamma \log \frac{1}{\varepsilon}$ , then by (3.41) and  $\gamma = (1 + \eta)/\sin \frac{\pi}{2k}$ , we obtain

$$\tilde{\omega}'(2l \sin \frac{\pi}{2k}) = O\left(\varepsilon^{2(1+\eta)}\right).$$

Moreover, (3.40) may give

$$c_l \geq \frac{1}{2} \tilde{C}_1 \varepsilon^2 l > 0,$$

as  $\varepsilon > 0$  is sufficiently small. Since  $c_l$  is continuous to  $l$ , there exists  $l_\varepsilon \in (\alpha \log \frac{1}{\varepsilon}, \gamma \log \frac{1}{\varepsilon})$  such that  $c_{l_\varepsilon} = 0$ , which implies that  $S[W + \phi_{l_\varepsilon}] = 0$ . Therefore by setting  $u_\varepsilon = W + \phi_{l_\varepsilon}$  and  $v_\varepsilon = T[\beta u_\varepsilon^2]$ , we may complete the proof of Theorem 2.1.

## 4 Proof of Theorem 2.2

In this section, we prove Theorem 2.2. Let us explain the main ideas as follows: Suppose the solution  $(u_\varepsilon, v_\varepsilon)$  of the system (1.11) formally having

$$v_\varepsilon \sim \omega_2(r). \quad (4.1)$$

Then substituting (4.1) into the equation of  $u$  in (1.11), we find  $u_\varepsilon$  satisfies (formally)

$$\varepsilon^2 \Delta u - V(r)u + \mu_1 u^3 = 0 \quad \text{in } \mathbb{R}^2, \quad (4.2)$$

where  $V(r) = \lambda_1 - \beta \omega_2^2(r)$ . For the equation (4.2), Ambrosetti, Malchiodi and Ni [2] have showed that as long as  $M(r) = r^{2/3}V(r)$  has a zero, then there exists a positive solution concentrating on a circle. The main problem here is to control the error induced by  $v_\varepsilon$ .

### 4.1 Solving $v_\varepsilon$ first

As for the proof of Theorem 2.1, we consider  $\Sigma_2 = \{u = u(r)\}$  the class of all radial functions and we have

**LEMMA 4.1.** *Let  $g = g(r)$  be such that  $\|g\|_{L^p(\mathbb{R}^2)} < \delta$ , where  $1 < p < 2$ . Then there exists a unique solution  $v = v(r) \equiv T_2[g]$  of*

$$\Delta v - \lambda_2 v + \mu_2 v^3 + gv = 0 \quad \text{in } \mathbb{R}^2, \quad \text{and } v \in \Sigma_2$$

satisfying

$$\|v - \omega_2\|_{W^{2,p}(\mathbb{R}^2)} \leq C \|g\|_{L^p(\mathbb{R}^2)} < C\delta,$$

where  $C$  is a positive constant independent of  $\delta$ .

## 4.2 Approximate Solutions

For  $t > 0$ , let

$$U_t(r) = \left( \sqrt{V(t)} U \left( \sqrt{V(t)} \frac{|r-t|}{\varepsilon} \right) \right) \eta(r), \quad \forall r > 0,$$

where  $U$  is defined in (2.3),  $V(r) = \lambda_1 - \beta\omega_2^2(r)$  and  $\eta(r) = 1$  for  $r \in [\alpha, \gamma]$  and  $\eta(r) = 0$  for  $r \in [0, \alpha/2] \cup [2\gamma, +\infty)$ . Here  $\alpha$  and  $\gamma$  are positive constants such that  $0 < \alpha < t < \gamma$ . Note that by (2.3),  $U_t$  satisfies

$$\varepsilon^2 U_t'' - V(t)U_t + \mu_1 U_t^3 = 0, \quad \forall r \in [\alpha, \gamma], \quad (4.3)$$

and for  $r \notin [\alpha, \gamma]$ ,  $U_t$  decays to zero exponentially as  $\varepsilon$  goes to zero. For  $t > 0$ , let

$$Z_t(r) := 3U^2 U' \left( \sqrt{V(t)} \frac{|r-t|}{\varepsilon} \right) \eta(r), \quad \forall r > 0.$$

## 4.3 Linear and Nonlinear Reductions

Let

$$\|u\|_{*,2} = \left\| e^{\sigma|r-t|/\varepsilon} u \right\|_{L^\infty(\mathbb{R}^2)},$$

where  $0 < \sigma < 1$  is a small number independent of  $\varepsilon$ . Then we have

**LEMMA 4.2.** *There exists a unique solution  $(\phi_t(r), d_t)$  such that*

$$\begin{cases} S_2[U_t + \phi_t] = d_t Z_t, \\ \int_{\mathbb{R}^2} U_t Z_t = 0, \end{cases} \quad (4.4)$$

where

$$S_2[u] = \varepsilon^2 \left( u'' + \frac{1}{r} u' \right) - \lambda_1 u + \mu_1 u^3 + \beta (T_2[\beta u^2])^2 u. \quad (4.5)$$

Furthermore,

$$\|\phi_t\|_* \leq C\varepsilon^{1/p}. \quad (4.6)$$

*Proof.* Let  $r = t + \varepsilon y \in [\alpha, \gamma]$ . Then it is easy to compute that

$$S_2[U_t] = U''V(t)^{3/2} - \lambda_1 V(t)^{1/2}U + \mu_1 V(t)^{3/2}U^3 + \frac{\varepsilon}{t + \varepsilon y} V(t)U' + \beta V(t)^{1/2} (T_2[\beta U_t^2])^2 U.$$

Hence by (2.3), we have

$$S_2[U_t] = \frac{\varepsilon}{t + \varepsilon y} V(t)U' + \beta V(t)^{1/2} (T_2[\beta U_t^2])^2 (t + \varepsilon y) - \omega_2^2(t) U. \quad (4.7)$$

By Lemma 4.1,

$$\begin{aligned} T_2[\beta U_t^2](t + \varepsilon y) &= \omega_2(t + \varepsilon y) + O(\|U_t^2\|_{L^p(\mathbb{R}^2)}) \\ &= \omega_2(t + \varepsilon y) + O(\varepsilon^{1/p}) \\ &= \omega_2(t) + O\left(\varepsilon^{1/p} + \frac{|\varepsilon y|}{1 + |\varepsilon y|}\right). \end{aligned} \quad (4.8)$$

Thus (4.7) and (4.8) give

$$\begin{aligned} S_2[U_t] &= \frac{\varepsilon}{t + \varepsilon y} V(t)U' + O(\varepsilon^{1/p}U) \\ &= O(\varepsilon U' + \varepsilon^{1/p}U), \end{aligned} \quad (4.9)$$

which implies that

$$\|S_2[U_t]\|_* \leq C\varepsilon^{1/p}. \quad (4.10)$$

The rest of the proof is similar to [13] so we omit the details here.  $\square$

#### 4.4 Expansion of $d_t$

Let  $\tilde{U}(y) = V(t)U'(\sqrt{V(t)}y)$  and set  $r = t + \varepsilon y$ . Then  $\frac{d}{dy}U_t(r) = \tilde{U}(y)$  for  $r = t + \varepsilon y \in [\alpha, \gamma]$ . Hence (4.3) implies

$$\tilde{U}'' - V(t)\tilde{U} + 3\mu_1 U_t^2 \tilde{U} = 0, \quad \forall r = t + \varepsilon y \in [\alpha, \gamma]. \quad (4.11)$$

We may multiply (4.4) by  $\tilde{U}(y)\eta(r)$  and integrate it over  $\mathbb{R}^2$  with respect to  $y$  variable. It is easy to calculate that

$$\begin{aligned} d_t \int_{\mathbb{R}^2} Z_t \tilde{U} \eta &= \int_{\mathbb{R}^2} S_2[U_t] \tilde{U} \eta + \int_{\mathbb{R}^2} \left( \varepsilon^2 \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) \phi - \lambda_1 \phi + 3\mu_1 U_t^2 \phi + \beta(T_2[\beta U_t^2])^2 \phi \right) \tilde{U} \eta \\ &\quad + \int_{\mathbb{R}^2} N[\phi] \tilde{U} \eta + \int_{\mathbb{R}^2} \beta (T_2[\beta(U_t + \phi)^2]^2 - T_2[\beta U_t^2]^2) (U_t + \phi) \tilde{U} \eta \\ &:= J_1 + J_2 + J_3 + J_4, \end{aligned} \quad (4.12)$$

where  $\phi = \phi_t(r)$  defined in Lemma 4.2,  $N[\phi] = \mu_1 (3U_t \phi^2 + \phi^3)$ ,

$$\begin{aligned} J_1 &= \int_{\mathbb{R}^2} S_2[U_t] \tilde{U} \eta, \\ J_2 &= \int_{\mathbb{R}^2} \left( \varepsilon^2 \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) \phi - \lambda_1 \phi + 3\mu_1 U_t^2 \phi + \beta(T_2[\beta U_t^2])^2 \phi \right) \tilde{U} \eta, \\ J_3 &= \int_{\mathbb{R}^2} N[\phi] \tilde{U} \eta, \\ J_4 &= \int_{\mathbb{R}^2} \beta (T_2[\beta(U_t + \phi)^2]^2 - T_2[\beta U_t^2]^2) (U_t + \phi) \tilde{U} \eta. \end{aligned}$$

By (4.6), we have

$$J_3 = O(\varepsilon^{2/p}). \quad (4.13)$$

As for the proof of (3.28), we may obtain

$$\|T_2[\beta(U_t + \phi)^2] - T_2[\beta U_t^2]\|_{L^\infty([\alpha, \gamma])} \leq C\varepsilon^{2/p},$$

and hence

$$J_4 = O(\varepsilon^{2/p}). \quad (4.14)$$

For  $J_2$ , we may use integration by parts to get

$$\begin{aligned} J_2 &= \int_{\mathbb{R}^2} \left[ \varepsilon^2 \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) (\tilde{U} \eta) - \lambda_1 \tilde{U} \eta + 3\mu_1 U_t^2 \tilde{U} \eta + \beta(T_2[\beta U_t^2])^2 \tilde{U} \eta \right] \phi \\ &= \int_{\mathbb{R}^2} \left[ \varepsilon^2 \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) (\tilde{U} \eta) - \lambda_1 \tilde{U} \eta + 3\mu_1 U_t^2 \tilde{U} \eta + \beta\omega_2^2(t) \tilde{U} \eta \right] \phi + O(\varepsilon^{2/p}) \\ &= \int_{\mathbb{R}^2} \left[ \tilde{U}'' \eta + \frac{\varepsilon}{t + \varepsilon y} \tilde{U}' \eta - V(t) \tilde{U} \eta + 3\mu_1 U_t^2 \tilde{U} \eta \right] \phi + O(\varepsilon^{2/p}) \\ &= O(\varepsilon^{2/p}). \end{aligned} \quad (4.15)$$

Here we have used (4.8) and (4.11). Now it remains to estimate  $J_1$ . We may use (4.7) to get

$$\begin{aligned} J_1 &= \int_{\mathbb{R}^2} S_2[U_t] \tilde{U} \eta \\ &= \int_{\mathbb{R}^2} \left[ \frac{\varepsilon}{t + \varepsilon y} V(t) U' + \beta V(t)^{1/2} (T_2[\beta U_t^2]^2(t + \varepsilon y) - \omega_2^2(t)) U \right] \tilde{U} \eta \\ &= \int_{\mathbb{R}^2} \left[ \frac{\varepsilon}{t + \varepsilon y} V(t) U' - (V(t + \varepsilon y) - V(t)) V^{1/2}(t) U \right] \tilde{U} \eta \\ &\quad + \int_{\mathbb{R}^2} \beta V(t)^{1/2} (T_2[\beta U_t^2]^2 - \omega_2^2) (t + \varepsilon y) U \tilde{U} \eta \\ &:= J_{11} + J_{12}, \end{aligned}$$

where

$$\begin{aligned} J_{11} &= \int_{\mathbb{R}^2} \left[ \frac{\varepsilon}{t + \varepsilon y} V(t) U' - (V(t + \varepsilon y) - V(t)) V^{1/2}(t) U \right] \tilde{U} \eta, \\ J_{12} &= \int_{\mathbb{R}^2} \beta V(t)^{1/2} (T_2[\beta U_t^2]^2 - \omega_2^2) (t + \varepsilon y) U \tilde{U} \eta. \end{aligned}$$

For  $J_{12}$ , we have

$$\begin{aligned} J_{12} &= \int_{\mathbb{R}^2} \beta V(t)^{1/2} \{ (T_2[\beta U_t^2]^2 - \omega_2^2)(t + \varepsilon y) - (T_2[\beta U_t^2]^2 - \omega_2^2)(t) \} U \tilde{U} \eta + O(\varepsilon^2) \\ &= O \left( \|T_2[\beta U_t^2] - \omega_2\|_{W^{2,p}(\mathbb{R}^2)} \int_{\mathbb{R}^2} \varepsilon |y| |U \tilde{U} \eta| \right) + O(\varepsilon^2) \\ &= O(\varepsilon^{2/p}). \end{aligned} \tag{4.16}$$

For  $J_{11}$ , we have

$$\begin{aligned} J_{11} &= \int_{\mathbb{R}^2} \left[ \frac{\varepsilon}{t + \varepsilon y} V(t) U' - (V(t + \varepsilon y) - V(t)) V^{1/2}(t) U \right] \tilde{U} \eta \\ &= \varepsilon V(t) \left[ \frac{1}{t} V(t) \int_{\mathbb{R}^2} (U'(\sqrt{V(t)} y))^2 dy - V'(t) V^{1/2}(t) \int_{\mathbb{R}^2} U(\sqrt{V(t)} y) U'(\sqrt{V(t)} y) y dy \right] + O(\varepsilon^{2/p}) \\ &= \varepsilon \sqrt{V(t)} \left[ \frac{1}{t} V(t) \int_0^\infty U'(z)^2 dz - V'(t) \int_0^\infty U U'(z) z dz \right] + O(\varepsilon^{2/p}) \\ &= \varepsilon \sqrt{V(t)} \left[ \frac{V(t)}{t} \int_0^\infty U'(z)^2 dz + \frac{1}{2} V'(t) \int_0^\infty U^2(z) dz \right] + O(\varepsilon^{2/p}) \\ &= \varepsilon c_0 \sqrt{V(t)} t^{-2/3} M'(t) + o(\varepsilon), \end{aligned} \tag{4.17}$$

where  $c_0 = \frac{1}{2} \int_0^\infty U^2(z) dz > 0$  and  $M(t) = t^{2/3} V(t)$ . Here we have used the following identity:

$$\int_0^\infty (U'(z))^2 dz = \frac{1}{3} \int_0^\infty U^2(z) dz. \tag{4.18}$$

Combining (4.12)-(4.17), we may obtain

$$d_t = \varepsilon \tilde{c}_0 t^{-2/3} M'(t) + o(\varepsilon), \tag{4.19}$$

where  $\tilde{c}_0 \neq 0$  and  $o(1)$  is a small quantity tending to zero as  $\varepsilon$  goes to zero.

## 4.5 Proof of Theorem 2.2

Let  $H(t) = t^{-2/3} M'(t)$  for  $t > 0$ . Then it is obvious that

$$\lim_{t \rightarrow 0^+} H(t) > 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} H(t) > 0.$$

By (2.5) (see Remark 1), there exists  $r_0 > 0$  such that  $H(r_0) < 0$ . Hence there exists  $[\alpha_2, \beta_2] \subset (0, r_0)$  such that  $H(\alpha_2) > 0 > H(\beta_2)$ . By (4.19) and the continuity of  $H(t)$ , there exists  $\gamma_{\varepsilon,1} \in (0, r_0)$  such that  $d_{\gamma_{\varepsilon,1}} = 0$ . Thus

$$S_2[U_{\gamma_{\varepsilon,1}} + \phi_{\gamma_{\varepsilon,1}}] = 0$$

and  $(U_{\gamma_{\varepsilon,1}} + \phi_{\gamma_{\varepsilon,1}}, T_2[\beta(U_{\gamma_{\varepsilon,1}} + \phi_{\gamma_{\varepsilon,1}})^2])$  satisfies the properties of Theorem 2.2. Similarly, we can find  $\gamma_{\varepsilon,2} \in (r_0, +\infty)$  such that  $d_{\gamma_{\varepsilon,2}} = 0$  and  $(U_{\gamma_{\varepsilon,2}} + \phi_{\gamma_{\varepsilon,2}}, T_2[\beta_j(U_{\gamma_{\varepsilon,2}} + \phi_{\gamma_{\varepsilon,2}})^2])$  becomes the second solution. Therefore we may complete the proof of Theorem 2.2.

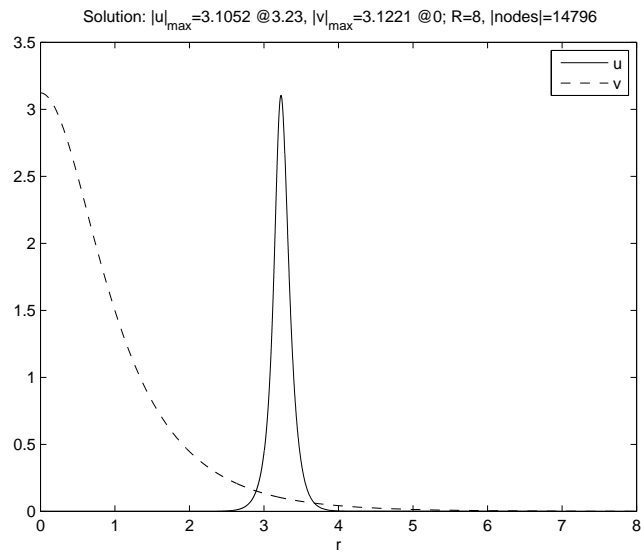
## 5 Numerical Investigations

We use the solver BVP4C in MATLAB to find solutions of (1.13) with ring profiles including a single ring profile, a double ring profile and  $m$ -ring profiles for  $m \geq 3$ . A single ring profile is the graph of a positive function  $f = f(r)$  ( $r = |x|$  is the radial variable for  $x \in \mathbb{R}^2$ ) with  $f(\infty) = 0$  and one bump which means that  $f$  is increasing on  $(0, r_1)$  but decreasing on  $(r_1, \infty)$ , where  $r_1$  is a positive constant. A double ring profile is the graph of a positive function  $g = g(r)$  with  $g(\infty) = 0$  and two bumps which means that  $g$  is increasing on  $(0, r_2) \cup (r_3, r_4)$  but decreasing on  $(r_2, r_3) \cup (r_4, \infty)$ , for some positive constants  $r_j, j = 2, 3, 4$  with  $r_2 < r_3 < r_4$ . Similarly, the  $m$ -ring profile is the graph of a positive function  $h = h(r)$  with  $h(\infty) = 0$  and  $m$  bumps for  $m \geq 3$ .

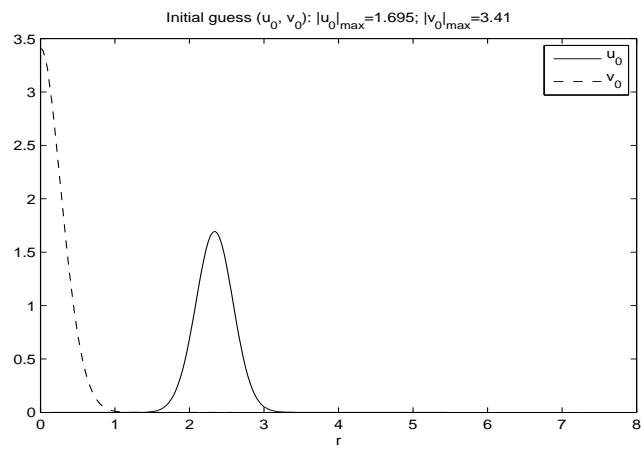
For notation convenience, we may denote the solution of (1.13) as  $(u, v) = (u(r), v(r))$  for  $r \in [0, \infty)$ . Due to the limitation of numerical computations, we can only approximate solutions of (1.13) on a bounded interval  $[0, R]$  ( $R > 0$ ). To implement the solver, we firstly need to transform (1.13) into a first-order ODE system by setting  $\zeta = u'$  and  $\eta = v'$ . We want to find the positive solutions  $(u, v)$ 's (i.e.  $u(r), v(r) > 0$  for  $r \geq 0$ ) with a ring profile, i.e., either  $u$  or  $v$  has a ring profile. It is necessary to have "good" initial guesses in order to obtain solutions as desired. Otherwise, the solver may generate either an unwanted solution (e.g., a solution  $(u^*, v^*)$  with  $u^* \equiv 0$ ) or no solution if an initial guess is not "good" enough. To obtain "good" initial guesses, we firstly choose the initial guess  $(u_0, v_0)$  so that  $u_0 = u_0(r)$  has a single ring profile away from the origin and  $v_0 = v_0(r)$  has a single peak at the origin (see Fig. 1(b)). If  $(u_0, v_0)$  is not "good" enough, then we may replace  $u_0(r)$  and  $v_0(r)$  by  $C_1 r^2 e^{-k_1(r - a r_{max})^2} u_0(r)$  and  $C_2 e^{-k_2 r^2} v_0(r)$ , respectively, where  $a, C_i$ 's and  $k_j$ 's are positive constants, and  $r_{max} = \arg \max(u_0)$  is the maximum point of  $u_0$ . With  $\varepsilon^2 = 0.02, \lambda_1 = 2, \lambda_2 = 1, \mu_1 = \mu_2 = 0.5, \beta = -0.05$ , we may adjust  $a, C_i$ 's and  $k_j$ 's to get the numerical solution  $(u_1, v_1)$  of (1.13) with a single ring profile of  $u_1$  and the Townes profile of  $v_1$  (see Fig. 1(a)). Similarly, we may set another "good" initial guess to find the numerical solution  $(u_2, v_2)$  with a single ring profile of  $u_2$  and the Townes profile of  $v_2$  (see Fig. 2(a)). Moreover, the numerical solution  $(u_3, v_3)$  with a double ring profile of  $u_3$  and the Townes profile of  $v_3$  (see Fig. 2(c)) can be obtained such that  $u_3 \approx u_1 + u_2$  and the profiles of  $v_i$ 's ( $i = 1, 2, 3$ ) are indistinguishable. Our numerical experiments may support Theorem 2.2 and Remark 3. On the other hand, the solver BVP4C also provides the first and second derivatives of numerical solutions which can be substituted into (1.13) to check the computational errors of order  $\mathcal{O}(10^{-15})$  (see Fig. 1(c)). This may assure the reliability of our numerical scheme so we may use it to produce further solutions beyond those of Theorem 2.2 and Remark 3.

Besides solutions  $(u_j, v_j), j = 1, 2, 3$ , we may find the solution  $(u_4, v_4)$  with a single ring profile of  $u_4$  and the Townes profile of  $v_4$  (see Fig. 3(a)) under the same numerical parameters as those of  $(u_j, v_j), j = 1, 2, 3$ . We also obtain the solution  $(u_5, v_5)$  with a double ring profile of  $u_5$  and the Townes profile of  $v_5$  (see Fig. 3(b)). The ring profile of  $u_4$  may almost fit the outer ring profile of  $u_5$ , and the profiles of  $v_4$  and  $v_5$  are indistinguishable (see Fig. 3(c)). Hence there exist at least two solutions  $(u, v)$ 's of (1.13) with a double ring profile of  $u$  and the Townes profile of  $v$ . Such a result of nonuniqueness can not be obtained from Theorem 2.2 and Remark 3. Further numerical solutions  $(u, v)$ 's with  $m$ -ring profiles of  $u$  and Townes profiles of  $v$  are sketched in Fig 4, wherein the same numerical parameters are used as those in Figs. 1-3 except  $\varepsilon^2 = 0.01$ .

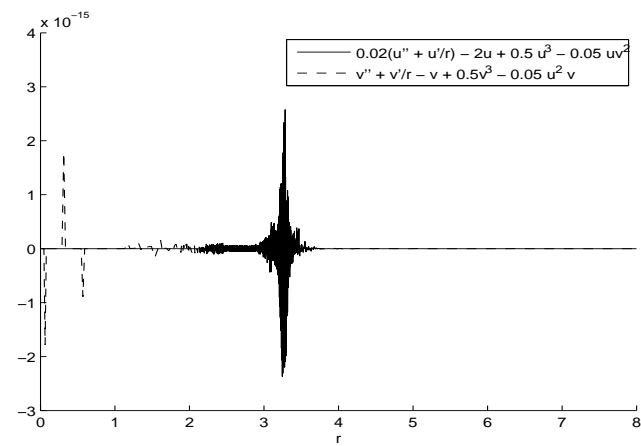
Finally, a new type of numerical solution  $(u, v)$  to (1.13) with ring-ring profiles (i.e. the graphs of both  $u$  and  $v$  are of ring profiles) on the interval  $[0, 20]$  is shown in Fig. 5 with  $\varepsilon^2 = 0.05, \lambda_1 = 2, \lambda_2 = 1, \mu_1 = \mu_2 = 0.5, \beta = -1$ . The ring profile of  $u$  concentrates in a narrow region due to the small  $\varepsilon$ . However, the ring profile of  $v$  spreads on a much wider region than that of  $u$ . That would make it very difficult to find solutions of (1.13) with ring-ring profiles on the interval  $[0, 8]$ . On the other hand, until now, there is no theoretical argument to prove the existence of solutions with ring-ring profiles. It would be a nice problem to study in the future.



(a)

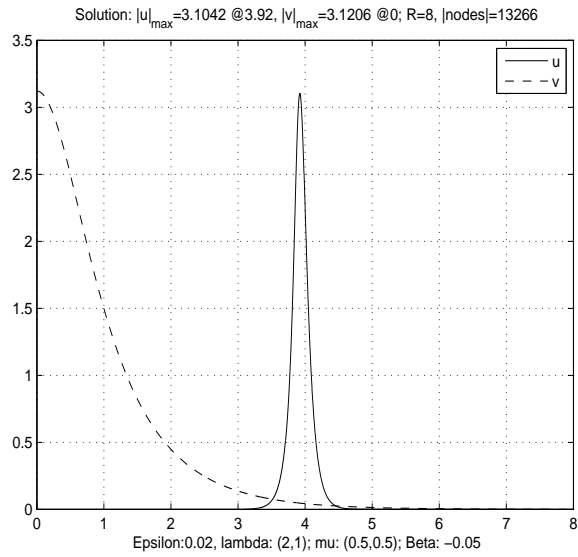


(b)

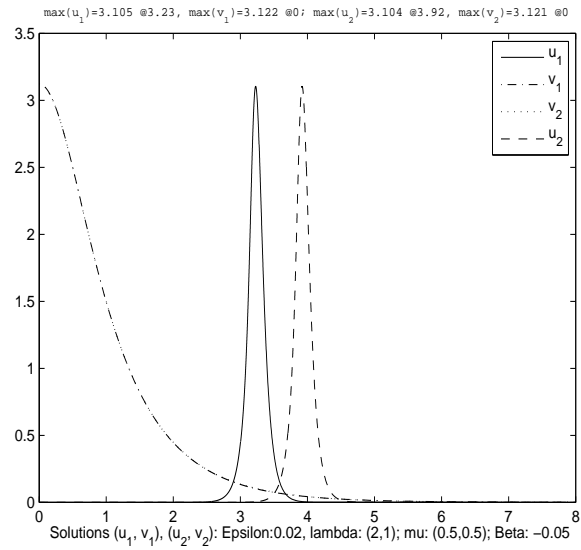


(c)

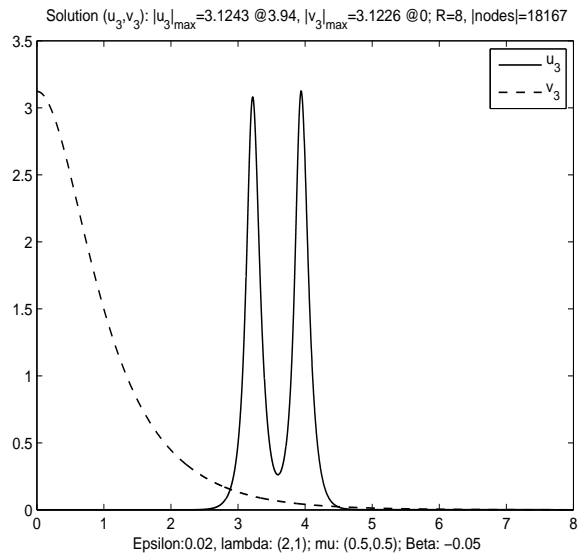
Figure 1: (a) The graph of  $u_1$  and  $v_1$  on  $[0, 8]$  with  $\varepsilon^2 = 0.02$ ,  $\lambda_1 = 2$ ,  $\lambda_2 = 1$ ,  $\mu_1 = \mu_2 = 0.5$ ,  $\beta = -0.05$ . (b) Initial guess. (c) Computational errors for  $(u_1, v_1)$ .



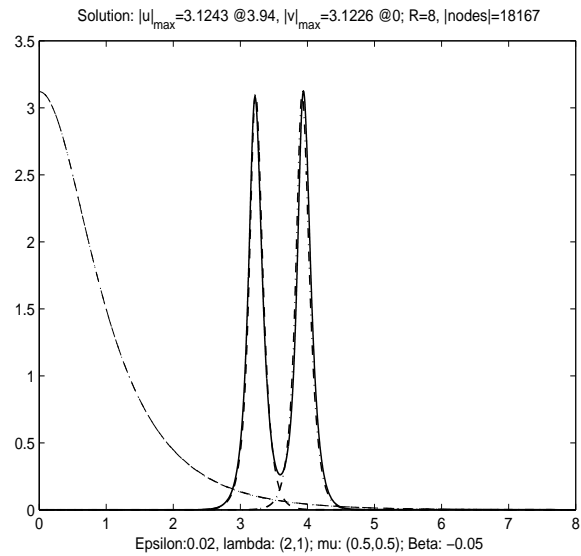
(a)



(b)



(c)



(d)

Figure 2: (a) The graph of  $u_2$  and  $v_2$  on  $[0, 8]$  with the same numerical parameters as used in Fig. 1(a). (b) Plot two solutions  $(u_i, v_i)$  ( $i = 1, 2$ ) in Fig. 1(a) and Fig. 2(a) together, where  $v_1$  and  $v_2$  are indistinguishable. (c) The graph of  $u_3$  and  $v_3$  on  $[0, 8]$  with the same numerical parameters as Fig. 1(a). (d) Plot three solutions  $(u_i, v_i)$  ( $i = 1, 2, 3$ ) in Fig. 1(a) and Fig. 2(a)&(c) together, where  $v_i$ 's are indistinguishable and  $u_3 \approx u_1 + u_2$ .

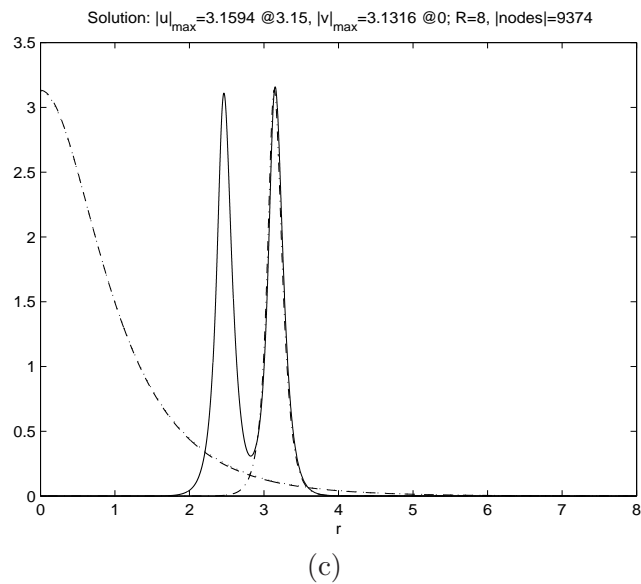
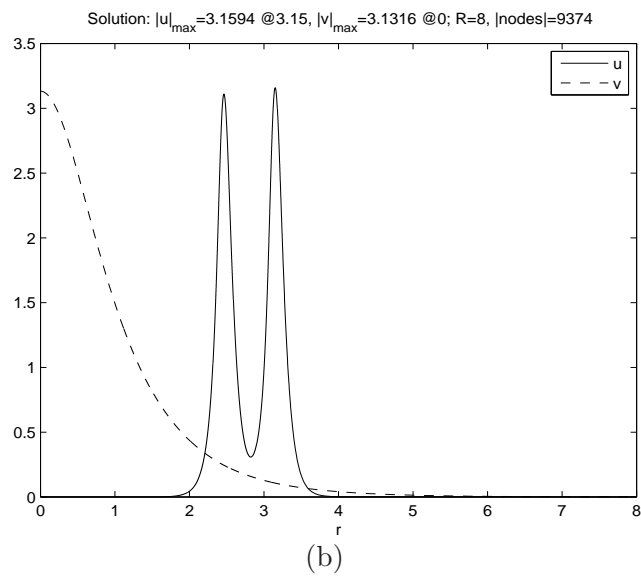
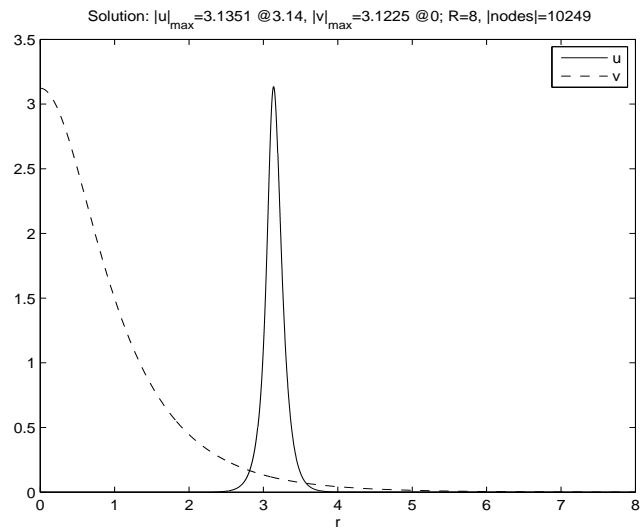


Figure 3: (a) The graph of  $u_4$  and  $v_4$  (b) The graph of  $u_5$  and  $v_5$  (c) Plot (a)&(b) together, where  $v_i$ 's are indistinguishable, and the ring profile of  $u_4$  may almost fit the outer ring profile of  $u_5$ . Same numerical parameters as those in Fig. 1(a).

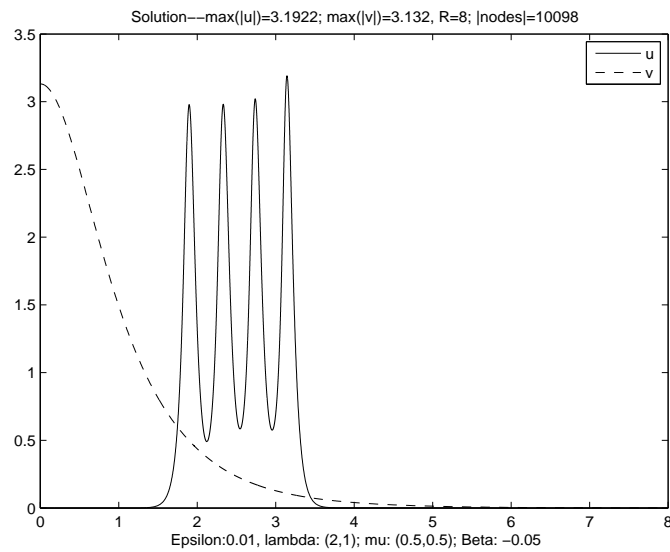
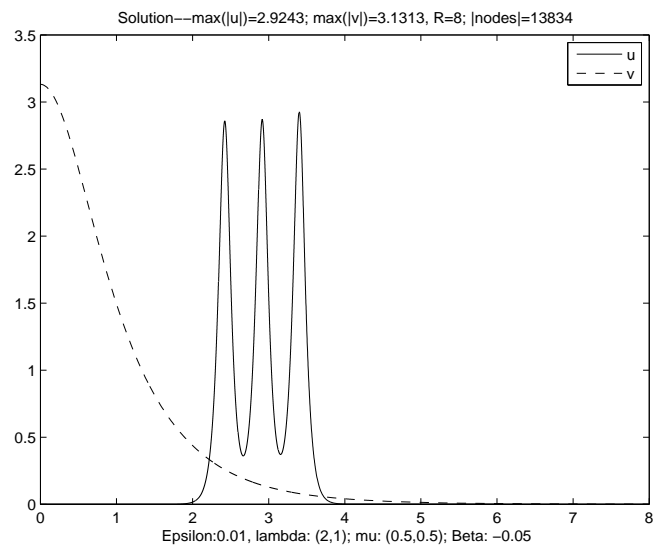
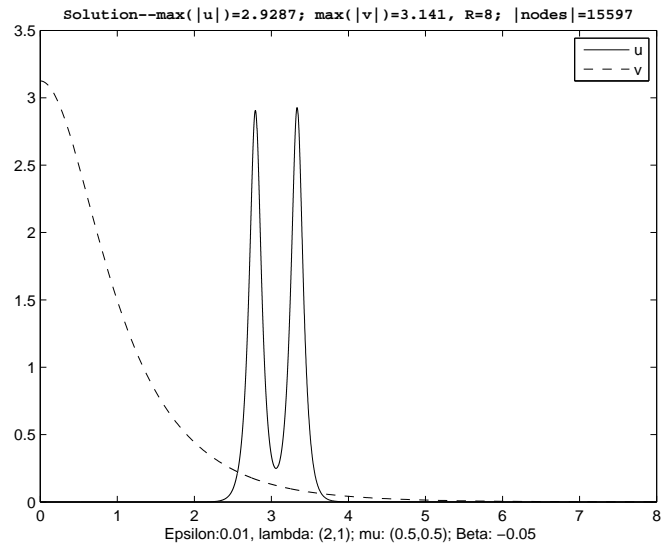


Figure 4: Sketch  $m$ -ring profiles on  $[0, 8]$  with  $\varepsilon^2 = 0.01$ ,  $\lambda_1 = 2$ ,  $\lambda_2 = 1$ ,  $\mu_1 = \mu_2 = 0.5$ ,  $\beta = -0.05$ .

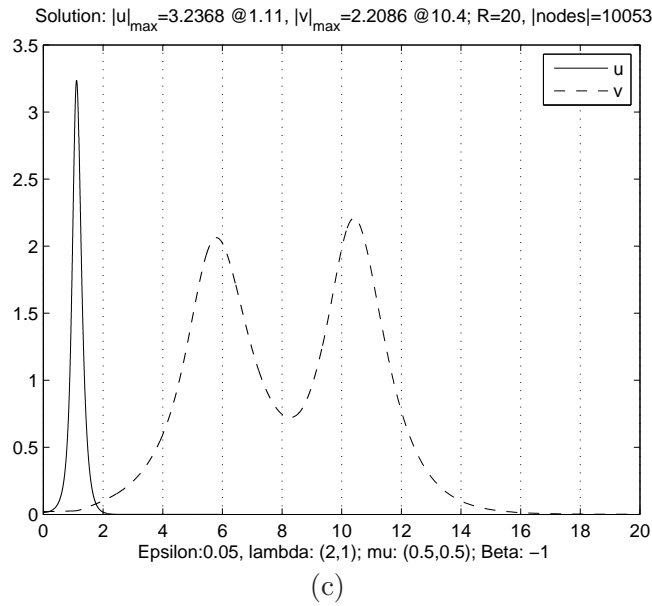
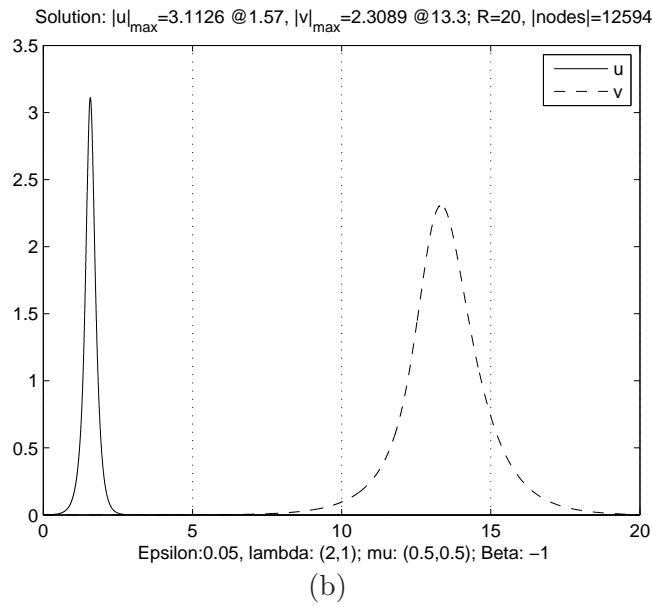
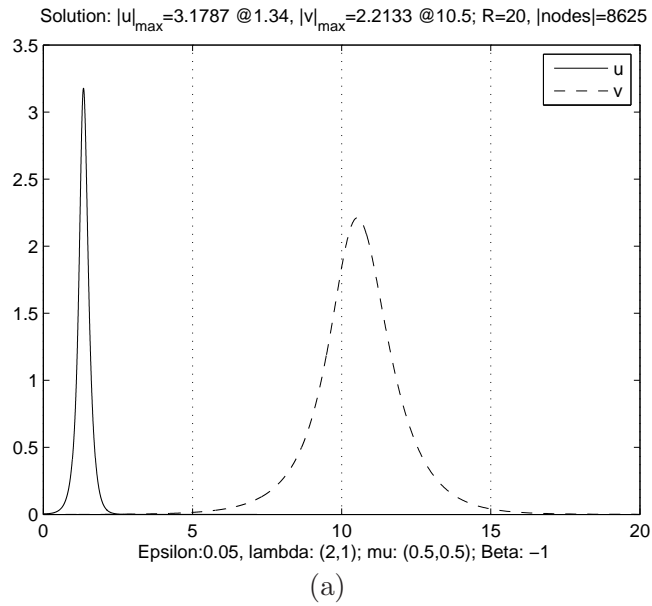


Figure 5: Sketch ring profiles of  $u$  and  $v$  on  $[0, 20]$  with  $\varepsilon^2 = 0.05$ ,  $\lambda_1 = 2$ ,  $\lambda_2 = 1$ ,  $\mu_1 = \mu_2 = 0.5$ ,  $\beta = -1$ .

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