ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A BIHARMONIC DIRICHLET PROBLEM WITH LARGE EXPONENTS

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ABSTRACT. We analyze the blow up phenomena of bounded integrable solutions of a semilinear fourth order elliptic problem with a large exponent under Dirichlet boundary conditions. We extend the results obtained by Ren-Wei in [25] and [26] to the biharmonic case.

1. INTRODUCTION

The study of asymptotic behavior of fourth order elliptic equations is of considerable interest. Let (M, g) be a smooth four-dimensional Riemannian manifold, and let us consider the so-called *Paneitz* operator [24] on the manifold M, as

$$P_g\psi = \Delta_g^2\psi - div_g(\frac{2}{3}S_g - 2Ric_g)d\psi$$

where div_g denotes the divergence, d the de-Rham differential and S_g, Ric_g denote the scalar and Ricci curvature of the metric g respectively and $\Delta_g\psi = -div_g(\nabla\psi)$ is the laplacian with respect to g. Under a conformal change of metric $\tilde{g} = e^{2\varphi}g$, the Paneitz operator P_g is related to $P_{\tilde{g}}$ in the following way

$$P_{\tilde{g}}\psi = e^{-4\varphi}P_g(\psi),\ \forall\ \psi \in C^\infty(M).$$

Moreover, the scalar curvatures of g and $\tilde{g}$ are related by the equation

$$P_g\varphi + Q_g = Q_{\tilde{g}}e^{4\varphi} \tag{1.1}$$

where Q_g is the $Q-$ curvature of the metric g, and $Q_{\tilde{g}}$ is the $Q-$ curvature of the new metric $\tilde{g}$ and

$$Q_g = \frac{1}{6}(\Delta_g S_g + S_g^2 - 3|Ric_g|^2) \tag{1.2}$$

is associated with the Paneitz operator. Integrating (1.1) over M, one obtains

$$k_g = \int_M Q_g = \int_M Q_{\tilde{g}}e^{4\varphi}$$

where k_g is conformally-invariant. Hence we can write (1.1)

$$P_g\varphi + Q_g = k_g\frac{Q_{\tilde{g}}e^{4\varphi}}{\int_M Q_{\tilde{g}}e^{4\varphi}}. \tag{1.3}$$

When the manifold is the Euclidean space, then (1.3) transforms to

$$\Delta^2\varphi = \rho\frac{h(x)e^{4\varphi}}{\int_\Omega h(x)e^{4\varphi}}. \tag{1.4}$$

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This type of problem arises in statistical mechanics and differential geometry and has been extensively studied by Adimurthi-Robert-Struwe [3], Barakat-Dammak-Ouni-Pakard [5], Chang-Yang [10], Djadli-Malchiodi [13], del Pino-Kowalczyk-Musso [12], Lin-Wei [21], [22] , Hebey-Robert [17] and many other authors.

Let $\Omega \subset \mathbb{R}^4$ be a bounded $C^{4,\beta}$ domain. In this paper, we study the asymptotic behavior of a sequence of solutions of the following nonlinear equation

$$\begin{cases} \Delta^2 u = (u^+)^p & \text{in } \Omega \\ u \not\equiv 0 & \text{in } \Omega \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.5}$$

as the parameter $p \to +\infty$ under the assumption that u satisfies,

$$\int_\Omega (u^+)^{p+1} \leq \frac{C}{p} \tag{1.6}$$

for some $C > 0$ independent of p, where $u^\pm = \max\{\pm u, 0\}$. Note that the least energy solution to (1.5) satisfy the assumption (1.6). See [6]. The equation has a very close relationship to (1.4). In course of this paper, we will introduce a blow-up solution for u, which along a subsequence converges to an entire solution of (1.4).

In two dimensions, an analogous problem was studied by Ren–Wei in [25] and [26] in a star-shaped domain for the least energy solutions. Adimurthi and Grossi [2] extended the result to a general two-dimensional domains and obtained more precise asymptotic behavior for the least energy solutions. Recently, Esposito-Musso-Pistoia [14] proved for any $m \in \mathbb{N}$, u_p exhibits the asymptotic behavior

$$p\int_\Omega |\nabla u_p|^2 \xrightarrow{p\to\infty} 8\pi m e$$

under some topological assumption on the domain.
For the biharmonic case the problem was studied by Takahashi [29], [30], with the convexity of the domain Ω and for positive solutions in the Navier boundary case. Ben Ayed-El Mehdi-Grossi [6] extended to non-convex domains and proved the single point condensation for least energy solutions, again for positive solutions in the Navier boundary case.

In this paper we study the asymptotic behavior of all solutions of (1.5) satisfying the integral bound (1.6). Note that neither we have assumed the convexity of the domain Ω nor the positivity of the solution.
Define

$$v_p := pu_p\ .$$

Then we call $\mathcal{S}$ a blow-up set of a sequence v_{p_n} if
(1.7)

$$\mathcal{S} = \{x \in \overline{\Omega} : \exists \text{ a subsequence of } v_{p_n} \text{and } \ x_n \in \Omega \text{ such that } \ x_n \to x \text{ and } \ v_{p_n}(x_n) \to +\infty\}.$$

Consider the functional $I_p : H_0^2(\Omega) \to \mathbb{R}$

$$I_p(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 dx - \frac{1}{p+1}\int_\Omega (u^+)^{p+1} dx.$$

Any solution to (1.5) is in fact a critical point of the above functional and by regularity all solutions u of (1.5) are $C^\infty(\Omega) \cap C^4(\overline{\Omega})$. (See Lemma B.3 of [31].)

Our first theorem characterize the limit of u_p in L^∞ norm.

Theorem 1.1. *If u_p is a solution of (1.5) satisfying (1.6), then as $p \to +\infty$,*

$$\lim_{p \to +\infty} \|u_p\|_\infty = \sqrt{e}. \tag{1.8}$$

We note that this type of result is proved in [2] and [6] but only for u_p being a *least energy solution.* We prove the result in a more general setting and it also covers *finite Morse index solutions* satisfying (1.6).

Next we analyze the asymptotic behavior of $v_p = pu_p$.

Theorem 1.2. *Let u_p be a family of solutions of (1.5) satisfying the bound (1.6). There exists a subsequence v_{p_n} such that*
(f_1) $p_n \int_\Omega (u_{p_n}^+)^{p_n} \to 64\pi^2 N\sqrt{e}$ for some positive integer N.
(f_2) v_{p_n} has exactly $N-$ blow up points. Let the blow up set $\mathcal{S} = \{x_1, \cdots x_N\}$. Then $\mathcal{S} \subset \Omega$ and $v_{p_n} \to v$ for every compact subset of $\overline{\Omega} \setminus \mathcal{S}$ where

$$v(x) = 64\pi^2\sqrt{e}\sum_{i=1}^{N} G(x, x_j)$$

and G is a Green's function of Δ^2 under Dirichlet boundary conditions. That is,

$$\begin{cases} \Delta^2 G(x,y) = \delta(x-y) & \text{in } \Omega \\ G(x,y) = \dfrac{\partial G}{\partial \nu}(x,y) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

(f_3) Furthermore, the blow-up points $x_j \in \Omega$, $i \le j \le N$ satisfy the following relation

$$\nabla_x H(x_j, x_j) + \sum_{l \neq j} \nabla_x G(x_j, x_l) = 0 \tag{1.10}$$

where

$$H(x,y) = G(x,y) + \frac{log|x-y|}{8\pi^2} \tag{1.11}$$

is the regular part of the Green's function G.

Result similar to Theorem 1.2 is proved in [6] for the least energy solutions of (1.5) under Navier boundary condition. Our result is more general in this context, as we precisely study the asymptotic of the blow-up solution in order to derive the result.

Corollary 1.3. *Let u_p be a least energy solution to* (1.5). *If $\mathcal{S} = \{x_0\}$, then up to a subsequence we have*
(f_1) $p_n \int_\Omega (u_{p_n}^+)^p \to 64\pi^2\sqrt{e}$;
(f_2) $v_{p_n} \to v$ for every compact subset of $\overline{\Omega} \setminus \{x_0\}$ and

$$v(x) = 64\pi^2\sqrt{e}G(x, x_0)$$

where G is a Green's function of Δ^2 under Dirichlet boundary conditions that is where x_0 is a critical point of $R(x) = H(x,x)$.

We mention the main difficulties and the main ideas in this paper.

The first difficulty in working with fourth order equations in a general domain, is the absence of maximum principle in the Dirichlet case. See [11] and [15]. More precisely, the Dirichlet Green's function may become negative in some domains. It is important to note that for the Laplacian case, the method of moving planes has been used to show that the blow-up points are away from the boundary as in [25], and the process was extended by Lin and Wei for biharmonic problems with Navier boundary conditions [21]. In the Dirichlet case, we cannot apply *the method of moving planes in order to exclude boundary blow-up* as in Ren–Wei [25] and Ben Ayed-El Mehdi-Grossi[6]. To overcome this difficulty, we use Pohozaev identity and strong pointwise estimates for blowing up solutions of (1.5) as in Robert–Wei [27]. (Note that by Boggio's principle [7], the Green function in a unit ball with Dirichlet boundary conditions is positive and explicitly given by this formula,

$$G(x,y) = \frac{1}{8\pi^2}\int_1^{\frac{[x,y]}{|x-y|}} \frac{v^2-1}{v^3}dv$$

where $[x,y] = \sqrt{|x-y|^2 + (1-|x|^2)(1-|y|^2)}$, see [15]. In the case of a ball, positive solutions of (1.5) are radially symmetric which was proved in [8].)

The second difficulty is to establish the estimate (1.8) for *general solutions.* Even in the second order case [2], (1.8) is proved only for *least energy solutions.* It turns out that we need more refined $(sup + inf)$ estimates to establish (1.8). Our results are *new* even for the second order case.

In course of the paper we will only prove Theorems 1.1 and 1.2. Our method can be used to study the above problem with polyharmonic operators.

Our paper is organized as follows. In Section 2, we collect three useful lemmas. In Section 3 we give a preliminary estimate of the solutions. We prove that the blow-up points are isolated and lie inside the domain in Section 4. More refined estimates as well as the proof of Theorem 1.1 are given in Section 5. Finally Theorem 1.2 is proved in Section 6.

Notations: Throughout this paper, the constant C will denote various constants which are independent of p: the value of C might change from one line to the other, and even in the same line. The equality $B = O(A)$ means that there exists $C > 0$ such that $|B| \leq CA$. All the convergence results are stated up to the extraction of a subsequence of p.

2. Preliminary Lemmas

We state three known results in this section. The first one concerns the properties of the Green's function (1.9). The second one is the Pohozaev identity. The third is the classification result of a fourth order Liouville problem.

Lemma 2.1. *There exists $C > 0$ such that for all $x, y \in \Omega$, $x \neq y$, we have*

$$|G(x,y)| \leq C \log\left(1 + \frac{1}{|x-y|}\right) \tag{2.1}$$

and

$$|\nabla^i G(x,y)| \leq \frac{C}{|x-y|^i} \tag{2.2}$$

for $1 \leq i \leq 3$. Moreover, there exists a constant $C > 0$ depending on Ω such that

$$G(x,y) \geq -C. \tag{2.3}$$

Proof. The first two estimates are due to Krasovskiĭ [18]. We also refer Dall'Acqua-Sweers [11]. The third result (2.3) is due to Grunau-Robert [16] and in fact it tells us that negative part of the Green's function is bounded. □

Now we state a Pohozaev identity for fourth order equations.

Lemma 2.2. *Suppose $u \in C^4(\overline{\Omega})$ be a solution of $\Delta^2 u = f(u)$. Let $F(u) = \int_0^u f(t)dt$. Then*

$$
\begin{aligned}
4\int_\Omega F(u) &= \int_{\partial\Omega} \langle x-y,\nu\rangle F(u)ds + \frac{1}{2}v^2\langle x-y,\nu\rangle ds + 2\frac{\partial u}{\partial \nu}v ds \\
&+ \int_{\partial\Omega}\left(\frac{\partial v}{\partial \nu}\langle x-y, Du\rangle + \frac{\partial u}{\partial \nu}\langle x-y, Dv\rangle - \langle Du, Dv\rangle\langle x-y,\nu\rangle\right)ds
\end{aligned} \tag{2.4}
$$

where $-\Delta u = v$ and ν denotes the outward normal derivative of x on $\partial\Omega$. In particular, we have

$$
\int_{\partial\Omega}\nu F(u)ds + \frac{1}{2}\int_{\partial\Omega} v^2\nu ds + \int_{\partial\Omega}\left\{\frac{\partial u}{\partial \nu}Dv + \frac{\partial v}{\partial \nu}Du - \langle Du, Dv\rangle \nu\right\}ds = 0. \tag{2.5}
$$

Proof. This identity follows from [23]. □

Our last lemma concerns the classification result of Lin [20] for the following Liouville equation

$$
\Delta^2 W + e^W = 0 \text{ in } \mathbb{R}^4, \quad \int_{\mathbb{R}^4} e^W < +\infty. \tag{2.6}
$$

Lemma 2.3. *(Theorem 1.1 and 1.2 of* [20]*) Suppose W is a solution to (2.6). Then the following statements hold.*
(i) After an orthogonal transformation, $W(x)$ can be represented by

$$
\begin{aligned}
W(x) &= \frac{1}{8\pi^2}\int_{\mathbb{R}^4} e^{W(y)}\log\frac{|y|}{|x-y|}dy - \sum_{j=1}^4 a_j(x_j - x_j^0)^2 + c_0 \\
&= -\sum_{j=1}^4 a_j(x_j - x_j^0)^2 - \alpha\log|x| + c_0 + o(1),
\end{aligned} \tag{2.7}
$$

as $|x| \to \infty$. Here $a_j \geq 0$, c_0 are constants and $x^0 = (x_1^0 \ldots, x_4^0) \in \mathbb{R}^4$. The function W satisfies

$$
\Delta W(x) = -\frac{1}{4\pi^2}\int_{\mathbb{R}^4}\frac{e^{W(y)}}{|x-y|^2}dy - 2\sum_{j=1}^4 a_j. \tag{2.8}
$$

Moreover, if $a_j \neq 0$ for all j, then W is symmetric with respect to the hyperplane $\{x \mid x_j = x_j^0\}$. If $a_1 = a_2 = a_3 = a_4 = 0$, then W is radially symmetric with respect to x^0.
(ii) The total integration

$$
\alpha = \frac{1}{32\pi^2}\int_{\mathbb{R}^4} e^{W(y)}dy \leq 2.
$$

If $\alpha = 2$, then all a_j are zero and W has the following form:

$$
W(x) = 4\log\frac{2\lambda}{1+\lambda^2|x-x^0|^2} + \log 24, \quad \text{with} \quad \lambda > 0. \tag{2.9}
$$

(iii) If $W(x) = o(|x|^2)$ at ∞, then $\alpha = 2$.

3. Preliminary Asymptotic Analysis

Let u_p be a family of solutions of problem (1.5) such that there exists constant $C>0$ such that

$$\int_\Omega (u_p^+)^{p+1} \le \frac{C}{p}. \tag{3.1}$$

Recall that $v_p = pu_p$. In this section, we give a preliminary estimate on the asymptotic behavior of unbounded v_p. First we claim the following.

Lemma 3.1. *Let u_p be a solution of (1.5).*

$$u_p \ge -\frac{C}{p}. \tag{3.2}$$

Moreover, if $x_p \in \Omega$ is a maximum point of u_p, then $u_p(x_p) \ge \frac{1}{2}$ and in particular

$$\lim_{p\to\infty} p\|u_p\|^{p-1}_{L^\infty(\Omega)} = +\infty. \tag{3.3}$$

Proof. By Hölder's inequality and $p \gg 1$ we have

$$\int_\Omega (u_p^+)^p \le \left(\int_\Omega (u_p^+)^{p+1}\right)^{\frac{p}{p+1}} \left(\int_\Omega 1^{p+1}\right)^{\frac{1}{p+1}} = \left(\int_\Omega (u_p^+)^{p+1}\right)^{\frac{p}{p+1}} |\Omega|^{\frac{1}{p+1}} \le \frac{C}{p}.$$

Moreover, by Green's representation and (2.3), we get

$$u_p(x) = \int_\Omega G(x,y)(u_p^+(y))^p dy \ge -C\int_\Omega (u_p^+(y))^p dy \ge -\frac{C}{p}$$

which proves (3.2). This also implies that the points of blow-up of v_p are precisely the point of positive maxima of u_p.

To prove (3.3), we let λ_1 be the first eigenvalue of Δ^2 in $H_0^2(\Omega)$, i.e.,

$$\lambda_1 = \inf_{\phi\in H_0^2(\Omega)} \frac{\int_\Omega |\Delta\phi|^2}{\int_\Omega \phi^2}. \tag{3.4}$$

By Krein-Rutman theorem the first eigenvalue λ_1 is positive. For any solution u_p of (1.5) we have

$$\int_\Omega |\Delta u_p|^2 = \int_\Omega (u_p^+)^{p+1}.$$

Also from (3.4) we have $\lambda_1 \int_\Omega |u_p|^2 \le \int_\Omega |\Delta u_p|^2 = \int_\Omega (u_p^+)^{p+1}$ which implies that $\lambda_1 \int_\Omega |u_p|^2 \le \int_\Omega |u_p|^{p+1}$. Hence we have $\max_\Omega |u_p(x)|^{p-1} \int_\Omega (u_p)^2 \ge \lambda_1 \int_\Omega u_p^2$ which implies that there exists $x_p \in \Omega$ such that $|u_p(x_p)|^{p-1} \ge \lambda_1$ and hence $\|u_p\|_{L^\infty} \ge \lambda_1^{\frac{1}{p-1}} \to 1$ as $p\to\infty$. Also from (3.2) we must have $u_p(x_p) > 0$. As a result, we get $\lim_{p\to\infty} p\|u_p\|^{p-1}_{L^\infty(\Omega)} = +\infty$. ∎

Let x_p be a point of maxima of u_p in Ω and

$$\varepsilon_p^4 p u_p^{p-1}(x_p) = 1. \tag{3.5}$$

Then by Lemma 3.1, $\varepsilon_p \to 0$ as $p\to\infty$. Let $\Omega_p := \frac{\Omega - x_p}{\varepsilon_p}$. In order to prove Theorem 1.1, we have to study the blow-up sequence. Using an idea of Adimurth-Struwe [4] and Adimurthi-Grossi [2], we set

$$W_p(x) := p\frac{u_p(x_p+\varepsilon_p x) - u_p(x_p)}{u_p(x_p)} \tag{3.6}$$

for $x \in \Omega_p$. Then W_p satisfies the problem

$$
\begin{cases} \Delta^2 W_p = \left(1 + \dfrac{W_p}{p}\right)_+^p & \text{in } \Omega_p \\ W_p = -p, \dfrac{\partial W_p}{\partial \nu} = 0 & \text{on } \partial\Omega_p \end{cases} \tag{3.7}
$$

We will show that W_p converges W in $C^4_{loc}(\mathbb{R}^4)$ such that W satisfies

$$
\begin{cases} \Delta^2 W = e^W & \text{in } \mathbb{R}^4 \\ \displaystyle\int_{\mathbb{R}^4} e^W = 64\pi^2. \end{cases} \tag{3.8}
$$

But first we need to show that the problem (3.8) is not a half space problem. The following lemma asserts that the blow-up point must have some distance from the boundary.

Lemma 3.2. *Then* $\displaystyle\lim_{p\to\infty} \frac{d(x_p, \partial\Omega)}{\varepsilon_p} = +\infty.$

Proof. We prove it by contradiction. Assume that $d(x_p, \partial\Omega) = O(\varepsilon_p)$. Then up to a rotation, we may assume that $\Omega_p \to (\beta, +\infty) \times \mathbb{R}^3$ where $\beta = \lim_{p\to+\infty} \frac{d(x_p,\partial\Omega)}{\varepsilon_p}$. Let $R > 0$ and $x \in B_R(0) \cap \Omega_p$. Then we have from the Green's representation and Lemma 2.1 that for $1 \le i \le 3$

$$
\begin{aligned}
|\nabla^i W_p(x)| &= \frac{p}{u_p(x_p)} |\varepsilon_p^i \nabla^i u_p(x_p + \varepsilon_p x)| \\
&= \frac{p\varepsilon_p^i}{u_p(x_p)} \left| \int_\Omega \nabla_x^i G(x_p + \varepsilon_p x, y)(u_p^+)^p dy \right| \\
&\le C \frac{p\varepsilon_p^i}{u_p(x_p)} \int_{B_{2\varepsilon_p R}(x_p)} \frac{1}{|x_p + \varepsilon_p x - y|^i} (u_p^+)^p dy \\
&+ C \frac{p\varepsilon_p^i}{u_p(x_p)} \int_{\Omega \setminus B_{2\varepsilon_p R}(x_p)} \frac{1}{|x_p + \varepsilon_p x - y|^i} (u_p^+)^p dy.
\end{aligned}
$$

Now in $\Omega \setminus B_{2\varepsilon_p R}(x_p)$ we have $|x_p + \varepsilon_p x - y| \ge |y - x_p| - \varepsilon_p |x| \ge R\varepsilon_p$, and $(u_p(y)^+)^p \le u(x_p)^p$. Hence we have by definition of ε_p,

$$
\begin{aligned}
|\nabla^i W_p(x)| &\le Cpu_p^{p-1}(x_p)\varepsilon_p^i \int_{B_{2\varepsilon_p R}(x_p)} \frac{1}{|x_p + \varepsilon_p x - y|^i} dy + Cp \int_\Omega (u_p^+)^p \\
&= C\varepsilon_p^{i-4} \int_{B_{2\varepsilon_p R}(x_p)} \frac{1}{|x_p + \varepsilon_p x - y|^i} dy + Cp \int_\Omega (u_p^+)^p \\
&= I_1 + I_2
\end{aligned}
$$

where

$$
I_1 = C\varepsilon_p^{i-4} \int_{B_{2\varepsilon_p R}(x_p)} \frac{1}{|x_p + \varepsilon_p x - y|^i} dy
$$

and

$$
I_2 = Cp \int_\Omega (u_p^+)^p.
$$

Since $y \in B_{2\varepsilon_p R}(x_p)$ we have $y - x_p = 2\varepsilon_p z$ where $|z| \le 2R$ and $dy = \varepsilon_p^4 dz$. And

$$\begin{aligned} I_1 &= C\varepsilon_p^{i-4}\int_{B_{2R}(0)} \frac{\varepsilon_p^4}{\varepsilon_p^i |x-z|^i} dz = C\int_{B_{2R}(0)} \frac{1}{|x-z|^i} dz \\ &= C\int_{B_{2R}(x)} \frac{1}{|z|^i} dz \le C|z|^{4-i}\Big|_{|x|}^{2R} = O(R). \end{aligned}$$

Hence we have from (1.6)

$$I_1 + I_2 = O(R) + O(1)$$

and as a result

$$|\nabla^i W_p(x)| = O(1).$$

Moreover, for $x \in \Omega_p \cap B_R(0)$ we have

$$|W_p(x) - W_p(0)| \le C.$$

But $W_p(0) = 0$ and hence W_p is uniformly bounded in a neighborhood of $\partial\Omega_p$. Choose $\tilde{x}_p \in \partial\Omega_p$ such that $|W_p(\tilde{x}_p)| \le C$. Then we have $p \le C$, a contradiction to the fact that $p \to +\infty$. □

The following lemma concerns the first bubble.

Lemma 3.3. *Then $W_p \to W$ as $p \to \infty$ in $C^4_{loc}(\mathbb{R}^4)$ where W satisfies (3.8). Moreover, $W(x) = -4\log(1 + \frac{|x|^2}{8\sqrt{6}})$.*

Proof. As $\Omega_p \to \mathbb{R}^4$ as $p \to \infty$ and by previous lemma we have $|\nabla^i W_p| \le C$ for $x \in B_R(0)$ for $i = 1, 2, 3$. By standard elliptic estimate we can conclude that $W_p \to W$ as $p \to \infty$ in $C^4_{loc}(\mathbb{R}^4)$ where W satisfies

$$\Delta^2 W = e^W, \; W(0) = 0.$$

Also note that as $\left(1 + \frac{W_p}{p}\right)_+^p \to e^W$ as $p \to \infty$ and hence by Fatou's lemma, we have

$$\int_{\mathbb{R}^4} e^W \le \liminf_{p\to\infty} \int_{\Omega_p} \left(1 + \frac{W_p}{p}\right)_+^p = \liminf_{p\to\infty} \frac{1}{\varepsilon_p^4}\int_\Omega \frac{(u_p^+)^p}{\|u_p\|_\infty^p} \le C < +\infty.$$

Now we show that in fact $W(x) = -4\log(1 + \frac{|x|^2}{8\sqrt{6}})$.

To this end, we first show that

$$\int_{B_R(0)} |\Delta W| \le CR^2$$

for any $R > 0$. In fact, by Green's representation, we have

$$\begin{aligned} |\Delta W_p(x)| &= \frac{p}{u_p(x_p)} |\varepsilon_p^2 \Delta u_p(x_p + \varepsilon_p x)| \\ &= \frac{p\varepsilon_p^2}{u_p(x_p)} \left| \int_\Omega \Delta_x G(x_p + \varepsilon_p x, y)(u_p^+)^p dy \right|. \end{aligned}$$

Hence we have by similar estimates as in Lemma 3.2

$$\int_{B_R(0)} |\Delta W_p(x)| dx \le Cp\int_\Omega \frac{(u_p^+(y))^p}{\|u_p\|_\infty}\left(\varepsilon_p^2 \int_{B_R(0)} \frac{dx}{|x_p + \varepsilon_p x - y|^2}\right) dy \le CR^2.$$

Letting $p \to \infty$ implies that for any $R > 0$ we have

$$\int_{B_R(0)} |\Delta W(x)| dx \le CR^2. \tag{3.9}$$

By Lemma 2.3, there exist nonnegative constants $a_i \ge 0, \alpha > 0$ and a point $x^0 = (x_1^0, ..., x_4^0) \in \mathbb{R}^4$ such that $W(x) = -\sum_{i=1}^4 a_i(x_i - x_i^0)^2 - \alpha \log|x| + c_0 + o(1)$ as $|x| \to +\infty$. Thus we obtain $-\Delta W(x) = \sum_{i=1}^4 2a_i$ as $|x| \to +\infty$. From (3.9), we get that $\sum_{i=1}^4 a_i = 0$ and hence $a_i = 0, i = 1, ..., 4$ since $a_i \ge 0$. Hence we have $W(x) = -\alpha \log|x| + c_0 + o(1)$ as $|x| \to +\infty$. And as a result $W(x) = o(|x|^2)$ as $|x| \to +\infty$. By Lemma 2.3, $\alpha = 2$. Hence W is radially symmetric around some point x^0 and $\int_{\mathbb{R}^4} e^W = 64\pi^2$. Since by definition, $W(0) = 0, \nabla W(0) = 0$, we conclude from (2.9) and Lemma 2.3 that $W(x) = -4\log(1 + \frac{|x|^2}{8\sqrt{6}})$. Moreover, using the fact that $\lim_{R\to\infty} \int_{B_R} \left(1 + \frac{|x|^2}{8\sqrt{6}}\right)^{-4} dx = 32|\mathbb{S}^3| = 64\pi^2$, we obtain that

$$\lim_{R\to\infty} \lim_{p\to\infty} \int_{B_{R(0)}} \left(1 + \frac{W_p}{p}\right)_+^p = 64\pi^2.$$

□

The next lemma gives a lower bound for the energy.

Lemma 3.4. *Let u_p be a solution of (1.5), then $p\int_\Omega (u_p^+)^{p+1} \ge C$ where $C > 0$ is a constant independent of p.*

Proof. The proof follows from [25]. For the sake of completeness we prove the result. Due to the Adams-Moser-Trudinger inequality [1], we obtain $\forall\ u \in H_0^2(\Omega)$

$$\int_\Omega e^{\beta\left(\frac{u}{\|\Delta u\|_{L^2}}\right)^2} dx \le C|\Omega| \tag{3.10}$$

whenever $\beta \le 32\pi^2$. Using Gamma function $\frac{x^s}{\Gamma(s+1)} \le e^x$ for all $x \ge 0, s \ge 0$ where Γ is a gamma function we obtain

$$\begin{aligned}
\frac{1}{\Gamma(\frac{s}{2}+1)} \int_\Omega |u|^s &\le \frac{1}{\Gamma(\frac{s}{2}+1)} \int_\Omega \left[32\pi^2\left(\frac{u}{\|\Delta u\|_{L^2}}\right)^2\right]^{\frac{s}{2}} dx \times (32\pi^2)^{-\frac{s}{2}} \|\Delta u\|_{L^2}^{s} \\
&\le \int_\Omega e^{32\pi^2\left(\frac{u}{\|\Delta u\|_{L^2}}\right)^2} dx \times (32\pi^2)^{-\frac{s}{2}} \|\Delta u\|_{L^2}^{s} \\
&\le C|\Omega|(32\pi^2)^{-\frac{s}{2}} \|\Delta u\|_{L^2}^{s}. \qquad (3.11)
\end{aligned}$$

Then we have

$$\|u\|_{L^s(\Omega)} \le (C|\Omega|)^{\frac{1}{s}} \Gamma\left(\frac{s}{2}+1\right)^{\frac{1}{s}} (32\pi^2)^{-\frac{1}{2}} \|\Delta u\|_{L^2}.$$

Hence we have

$$\|u\|_{L^s(\Omega)} \le D_s s^{\frac{1}{2}} \|\Delta u\|_{L^2(\Omega)} \tag{3.12}$$

where

$$D_s = (C|\Omega|)^{\frac{1}{s}}\Gamma\left(\frac{s}{2}+1\right)^{\frac{1}{s}}(32\pi^2)^{-\frac{1}{2}}s^{-\frac{1}{2}} \tag{3.13}$$

and by Stirling's formula we obtain

$$\Gamma\left(\frac{s}{2}+1\right)^{\frac{1}{s}} \sim (\frac{1}{\sqrt{2e}})\sqrt{s} \text{ as } s \to \infty$$

and hence $\lim_{s\to\infty} D_s = (\frac{1}{64\pi^2 e})^{\frac{1}{2}}$. Moreover, plugging $s = p+1$ in (3.12), any solution of (1.5) yields

$$\liminf_{p\to\infty} \frac{\int_\Omega p|\Delta u_p|^2}{(\int_\Omega (u_p^+)^{p+1})^{\frac{2}{p+1}}} \geq 64\pi^2 e$$

hence

$$\liminf_{p\to\infty} p\left(\int_\Omega (u_p^+)^{p+1}\right)^{\frac{p-1}{p+1}} \geq 64\pi^2 e$$

which in fact implies that

$$\liminf_{p\to\infty} p\left(\int_\Omega (u_p^+)^{p+1}\right) \geq 64\pi^2 e. \tag{3.14}$$

□

The next lemma says that u_p is uniformly bounded.

Lemma 3.5. *Let u_p be a solution of (1.5) satisfying (1.6). Then $\|u_p\|_{L^\infty(\Omega)} \leq C$ where C is a constant independent of p and hence there exist $c > 0, C > 0$ such that $c \leq p\int_\Omega (u_p^+)^p \leq C$ for $p >> 1$.*

Proof. Using the definition of ε_p and Fatou's lemma we have

$$\begin{aligned}
\liminf_{p\to\infty} p\int_\Omega (u_p^+)^{p+1} &= \liminf_{p\to\infty} p\|u_p\|_\infty^{p+1}\varepsilon_p^4 \int_{\Omega_p}\left(1+\frac{W_p}{p}\right)_+^{p+1} dx \\
&= \liminf_{p\to\infty} \|u_p\|_\infty^2 \int_{\Omega_p}\left(1+\frac{W_p}{p}\right)_+^{p+1} \\
&\geq \liminf_{p\to\infty} \|u_p\|_\infty^2 \int_{\mathbb{R}^4} e^W dx \\
&= 64\pi^2 \liminf_{p\to\infty} \|u_p\|_\infty^2. \qquad (3.15)
\end{aligned}$$

On the other hand, we have

$$p\int_\Omega (u_p^+)^{p+1} \leq \|u_p\|_{L^\infty} p\int_\Omega (u_p^+)^p.$$

Using (1.6), we deduce that $\|u_p\|_\infty \leq C$ and $p\int_\Omega (u_p^+)^p \geq C$. □

4. Analysis of Blow-up Set

The purpose of this section is to prove that the blow-up points of v_p are isolated, finite and has quite a distance from the boundary.

Recall that $v_p = pu_p$ satisfies

$$\begin{cases} \Delta^2 v_p = p^{-(p-1)}(v_p^+)^p & \text{in } \Omega \\ v_p = 0 = \dfrac{\partial v_p}{\partial \nu} & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

and the blow-up set $\mathcal{S}$ is defined at (1.7).

Let $\varepsilon_0 > 0$ be fixed small number. We define another blow-up set

$$\Lambda = \left\{ x \in \overline{\Omega} : \forall\ r_0 > 0 \text{ and } \forall\ p_0 > 1;\ \exists p > p_0 \ \text{ such that } p \int_{B(x,r_0)\cap\Omega} (u_p^+)^p \geq \varepsilon_0 \right\}.$$

In this section, we are going to prove that in fact $\mathcal{S} = \Lambda$ (Lemma 4.1), Λ is finite (Lemma 4.1), and $\mathcal{S} \subset \Omega$ (Lemma 4.3).

Let $f_p = p(u_p^+)^p$. By Lemma 3.1 and 3.2, $v_p \geq -C$. Hence v_p has only one-sided (positive) blow-up.

Let $x_{p,i} \in \Omega$ and we define

$$p\varepsilon_{p,i}^4 u_p^{p-1}(x_{p,i}) = 1$$

and

$$W_{p,i}(x) = \frac{pu_p(x_{p,i} + \varepsilon_{p,i}x) - pu_p(x_{p,i})}{u_p(x_{p,i})}.$$

We say that the property $\mathcal{H}_k$ holds if there exists $(x_{p,1}, \cdots x_{p,k}) \in \Omega^k$ such that

(i) $\lim_{p\to+\infty} \frac{|x_{p,i} - x_{p,j}|}{\varepsilon_{p,i}} = +\infty,\ i \neq j$

(ii) $\lim_{p\to+\infty} \frac{d(x_{p,i}, \partial\Omega)}{\varepsilon_{p,i}} = +\infty$

(iii) $\lim_{p\to+\infty} W_{p,i} = -4\log(1 + \frac{|x|^2}{8\sqrt{6}})$ in $C^4_{loc}(\mathbb{R}^4)$; $\forall i \in \{1, 2, \cdots k\}$.

By Lemma 3.1–Lemma 3.3, $\mathcal{H}_1$ holds.

The first lemma in this section shows that $\mathcal{S} = \Lambda$ and that Λ is finite.

Lemma 4.1. *(a) Assume that $\mathcal{H}_k$ holds. Then either $\mathcal{H}_{k+1}$ holds or there exists a $C > 0$ such that*

$$\inf_{i=1,2\cdots k} \{|x - x_{p,i}|^4\} f_p \leq C \ \forall x \in \Omega. \tag{4.2}$$

(b) Then there exists N and $C > 0$ such that $\mathcal{H}_N$ holds and

$$\inf_{i=1,2\cdots k} \{|x - x_{p,i}|^4\} f_p \leq C \ \forall x \in \Omega. \tag{4.3}$$

As a consequence, $\Lambda = \{\lim_{p\to+\infty} x_{p,i}\}$.

(c) For $j = 1, 2, 3$ there exists a $C > 0$ such that

$$\inf_{i=1,2\cdots k} \{|x - x_{p,i}|^j\} |\nabla^j v_p| \leq C \ \forall x \in \Omega \tag{4.4}$$

and hence for any compact set of $K \subset \overline{\Omega} \setminus \Lambda$ we have

$$\|\nabla^j v_p\|_{L^\infty(K)} \leq C \ \forall j = 0, 1, 2, 3.$$

(d) In particular, $\mathcal{S} = \Lambda$.

Proof. (a) Let $w_p(x) = \inf_{i=1,2\cdots k}\{|x-x_{p,i}|^4\} f_p(x)$. Assume that $y_p \in \Omega$ is such that $0 < w_p(y_p) = \|w_p\|_\infty \to \infty$ when $p \to \infty$. Define

$$A_p(x) = \frac{pu_p(y_p+\alpha_p x) - pu_p(y_p)}{u_p(y_p)}$$

where $p\alpha_p^4 u_p^{p-1}(y_p) = 1$. Then $\Delta^2 A_p = (1+\frac{A_p}{p})_+^p$ and we have

$$\begin{aligned} w_p(y_p) &= \inf_{i=1,\cdots k} p|y_p - x_{p,i}|^4 u_p^p(y_p) \\ &= \inf_{i=1,\cdots k} \frac{|y_p-x_{p,i}|^4}{\alpha_p^4}|u_p(y_p)| \\ &\leq C \inf_{i=1,\cdots k} \frac{|y_p-x_{p,i}|^4}{\alpha_p^4} \end{aligned}$$

which implies that $\frac{|y_p-x_{p,i}|}{\alpha_p} \to +\infty$ for all $i = 1,2\cdots k$ as $p \to \infty$. Assume that there exist a k_0 such that $y_p - x_{p,k_0} = O(\varepsilon_{p,k_0})$. Then $y_p - x_{p,k_0} = \theta_{p,k_0}\varepsilon_{p,k_0}$ where θ_{p,k_0} is uniformly bounded and we have by assumption (iii)

$$\begin{aligned} |y_p - x_{p,k_0}|^4 f_p(y_p) &= |\theta_{p,k_0}|^4 \varepsilon_{p,k_0}^4 p u_p(x_{p,k_0} + \varepsilon_{p,k_0}\theta_{p,k_0})^p \\ &= |\theta_{p,k_0}|^4 p\varepsilon_{p,k_0}^4 u_p(x_{p,k_0})^p \left(1 + \frac{W_{p,k_0}}{p}\right)_+^p \\ &\leq Ce^{|W_{p,k_0}(\theta_{p,k_0})|} \leq C \end{aligned}$$

which implies that $\lim_{p\to\infty} w_p(y_p)$ is finite, a contradiction. Hence $\frac{|y_p-x_{p,i}|}{\varepsilon_{p,i}} \to +\infty$. Now we know that $w_p(y_p+\alpha_p x) \leq w_p(y_p)$ and hence we have,

$$\frac{f_p(y_p+\alpha_p x)}{f_p(y_p)} \leq \frac{\inf_{1,2\cdots k}|y_p - x_{p,i}|^4}{\inf_{1,2\cdots k}|y_p - x_{p,i} + \alpha_p x|^4}.$$

Let $x \in B_R(0)$. Let $\eta \in (0,1)$. Let $p \geq p(R)$ such that $\frac{|y_p-x_{p,i}|}{\alpha_p} \geq \frac{R}{\eta}$ for all $i = 1,\cdots k$. We then have $|y_p - x_{p,i} + \alpha_p x| \geq (1-\eta)|y_p - x_{p,i}|$ and hence $\inf_{i=1,2\cdots k}|y_p - x_{p,i} + \alpha_p x|^4 \geq (1-\eta)^4 \inf_{i=1,2\cdots k}|y_p - x_{p,i}|^4$. This again implies that

$$\frac{(u_p^+(y_p+\alpha_p x))^p}{u_p^+(y_p)^p} \leq \frac{1}{(1-\eta)^4}$$

that is

$$\left(1+\frac{A_p}{p}\right)_+^p \leq \frac{1}{(1-\eta)^4}. \tag{4.5}$$

Hence using similar techniques as Lemma 3.2, Lemma 3.3 and (4.5) we can prove that

$$\frac{d(y_p,\partial\Omega)}{\alpha_p} \to \infty;\ A_p \to -4\log\left(1+\frac{|x|^2}{8\sqrt{6}}\right)$$

in $C^4_{loc}(\mathbb{R}^4)$ as $p \to +\infty$. Letting $x_{p,k+1} := y_p$ and $A_p = W_{p,k+1}$, then $\mathcal{H}_{k+1}$ holds.

(b) Let $\mathcal{H}_k$ hold for some k. Then we choose $R>0$ such that $B_{\varepsilon_{p,i}R}(x_{p,i}) \cap B_{\varepsilon_{p,j}R}(x_{p,j}) = \emptyset$ for all $i \neq j$. Then by Lemma 3.5

$$\begin{aligned}
C_1 &\geq p\int_\Omega (u_p^+)^p \geq \int_{\cup_{i=1}^k B_{\varepsilon_{p,i}R}(x_{p,i})} \frac{u_p^p(x_{p,i})}{\varepsilon_{p,i}^4}\frac{(u_p^+)^p}{u_p^p(x_{p,i})} \\
&\geq \sum_{i=1}^k \int_{B_{\varepsilon_{p,i}R}(x_{p,i})} \frac{(u_{p,i}^+)^p}{u_p^p(x_{p,i})}\frac{u_p^p(x_{p,i})}{\varepsilon_{p,i}^4} = \sum_{i=1}^k |u_p(x_{p,i})| \int_{B_R(0)} \left(1+\frac{W_{p,i}}{p}\right)_+^p \\
&= \sum_{i=1}^k |u_p(x_{p,i})| \int_{B_R(0)} (e^W + o(1)) \\
&\geq Ck \min_i u_p(x_{p,i}) \geq Ck
\end{aligned}$$

and hence k is bounded. As a result, $\Lambda = \{\lim_{p\to+\infty} x_{p,i}\}$. Moreover, (4.3) follows from (4.2).

(c) We have from Green's representation that for $1 \leq i \leq 3$

$$\begin{aligned}
|\nabla^i v_p(x)| &= p\left|\int_\Omega \nabla_x^i G(x,y)(u_p^+(y))^p dy\right| \\
&\leq p\int_\Omega |\nabla_x^i G(x,y)|(u_p^+(y))^p dy \\
&\leq Cp\int_\Omega |x-y|^{-i}(u_p^+(y))^p dy. \qquad (4.6)
\end{aligned}$$

Let $R_p(x) := \inf_{i=1,\cdots N} |x-x_{p,i}|$ and $\Omega_{p,i} = \{x \in \Omega : |x-x_{p,i}| = R_p(x)\}$. Then we have

$$\begin{aligned}
p\int_{\Omega_{p,i}} |x-y|^{-j}(u_p^+)^p dy &= p\int_{\Omega_{p,i}\cap B_{\frac{|x-x_{p,i}|}{2}}(x_{p,i})} |x-y|^{-j}(u_p^+)^p dy \\
&+ p\int_{\Omega_{p,i}\setminus B_{\frac{|x-x_{p,i}|}{2}}(x_{p,i})} |x-y|^{-j}(u_p^+)^p dy.
\end{aligned}$$

Note that for $y \in \Omega_{p,i} \setminus B_{\frac{|x-x_{p,i}|}{2}}(x_{p,i})$, we have

$$p|x-y|^{-j}(u_p^+)^p \leq \frac{C}{|x-y|^j|x-x_{p,i}|^4}$$

and hence

$$p\int_{\Omega_{p,i}\setminus B_{\frac{|x-x_{p,i}|}{2}}(x_{p,i})} |x-y|^{-j}(u_p^+)^p \leq \int_{\Omega_{p,i}\setminus B_{\frac{|x-x_{p,i}|}{2}}(x_{p,i})} \frac{1}{|x-y|^j|x-x_{p,i}|^4} \leq \frac{C}{|x-x_{p,i}|^j}$$

When $\Omega_{p,i} \cap B_{\frac{|x-x_{p,i}|}{2}}(x_{p,i})$, we have $|x-y| \geq |x-x_{p,i}| - |y-x_{p,i}| \geq \frac{1}{2}|x-x_{p,i}|$ and

$$p\int_{\Omega_{p,i}\cap B_{\frac{|x-x_{p,i}|}{2}}(x_{p,i})} |x-y|^{-j}(u_p^+)^p dy \leq \frac{C}{|x-x_{p,i}|^j}.$$

Hence for any compact set $K \subset \overline{\Omega} \setminus \Lambda$ we have $\|v_p\|_{L^\infty(K)} \leq C$.

(d) Now we prove that $\mathcal{S} = \Lambda$. Suppose $x_0 \notin \Lambda$, then from (c) we have v_p is uniformly bounded in $L^\infty(K)$ for some compact set K containing x_0 and hence

$x_0 \notin \mathcal{S}$. Hence $\mathcal{S} \subset \Lambda$.
Let $x_0 \in \Lambda$, then every compact set K containing x_0, $\|v_p\|_{L^\infty(K)} \to +\infty$ as $p \to \infty$, otherwise there exists $r > 0$ such that $\|v_p\|_{L^\infty(B_r(x_0))} \leq C$ but $f_p = p^{1-p}(v_p^+)^p$, hence $f_p \to 0$ as $p \to \infty$ uniformly in $B_r(x_0)$ and this implies a contradiction as

$$p \int_{B_r(x_0)\cap\Omega} (u_p^+)^p \to 0 \text{ as } r \to 0$$

implying that $x_0 \notin \Lambda$. □

It remains to show that $\Lambda = \mathcal{S}$ lies inside Ω. To this end, we first analyze the behavior of v_p outside the blow-up set.

Lemma 4.2. *Let $x_j = \lim_{p\to\infty} x_{p,j}$; $j = 1, 2, \cdots N$. Then there exists $\gamma_j > 0$ such that*

$$\lim_{p\to\infty} v_p(x) = \sum_{j=1}^{N} \gamma_j G(., x_j) \text{ in } C^4_{loc}(\overline{\Omega}\backslash\mathcal{S}).$$

Proof. Since x_i's are isolated, there exist a $R > 0$ such that $\Omega' = \Omega \setminus \cup_{i=1}^{N} \overline{B_R}(x_i)$ is connected. Then $|v_p| + |\nabla v_p| \leq C$ for all $x \in \Omega'$ by Lemma 4.1. Let $x' \in \partial\Omega \cap \partial\Omega'$, then $|v_p(x) - v_p(x')| \leq C$ for all $x \in \Omega'$. But this implies v_p is uniformly bounded in $\Omega' \cap \overline{\Omega}$. By standard regularity we have $v_p \to v$ as $p \to \infty$ in $C^4_{loc}(\overline{\Omega} \setminus \mathcal{S})$. Choose $r > 0$ small such that $u_p^p(y) > 0$ in $B_r(x_j)$. Then we have

$$v_p(x) = p \int_\Omega G(x, y)(u_p^+(y))^p dy = \sum_{j=1}^{N} p \int_{B_r(x_j)} G(x, y) u_p^p(y) dy + o(1).$$

Note that we are using the decay estimate of u_p in $\Omega \setminus B_r(x_j)$ from (4.3) of Lemma 4.1: $\inf_{i=1,2\cdots N}\{|x - x_i|^4\} p u_p^p(x) \leq C \ \forall x \in \Omega\backslash\mathcal{S}$.

Furthermore $G(x, .)$ is continuous in $\overline{\Omega} \setminus \{x\}$, we obtain

$$v_p(x) = \sum_{j=1}^{N} p \int_{B_r(x_j)} G(x, y) u_p^p(y) dy + o(1)$$

where

$$\gamma_j = \lim_{r\to 0} \lim_{p\to+\infty} p \int_{B_r(x_j)} u_p^p(y) dy \geq 64\pi^2 \liminf_{p\to\infty} \|u_p\|^2_{L^\infty(B_r(x_j))} \tag{4.7}$$

by Lemma 3.5. □

Finally we prove that no boundary blow-up occurs.

Lemma 4.3. *[No boundary blow-up] In particular, $\mathcal{S} \cap \partial\Omega = \emptyset$.*

Proof. We argue by contradiction. Suppose that $x_0 \in \partial\Omega \cap \Lambda$. Then since $\mathcal{S} = \Lambda$, we have for all $r_1 > 0$ and for all $p_0 > 1$; there exists a $p > p_0$ such that

$$p \int_{\Omega\cap B_{r_1}(x_0)} (u_p^+)^p \geq \varepsilon_0. \tag{4.8}$$

Choose $r_1 > 0$, such that $\Lambda \cap B_{r_1}(x_0) = \{x_0\}$. Let $y_p = x_0 + \rho_{p,r}\nu(x_0)$ where

$$\rho_{p,r} = \frac{\int_{\partial\Omega\cap B_r(x_0)} \langle x - x_0, \nu\rangle (\Delta v_p)^2}{\int_{\partial\Omega\cap B_r(x_0)} \langle \nu(x_0), \nu\rangle (\Delta v_p)^2}$$

where $r << r_1$ such that $\frac{1}{2} \le \langle \nu(x_0), \nu \rangle \le 1$ for $x \in \overline{B}_r(x_0) \cap \Omega$. Here $\nu(x)$ is an outer normal vector to $T_{x_0}\partial\Omega$ at x. Then it follows that $|\rho_{p,r}| \le 2r$ and

$$\int_{\partial\Omega\cap B_r(x_0)} \langle x - y_p, \nu\rangle (\Delta v_p)^2 dx = 0 \tag{4.9}$$

and hence

$$\int_{\partial\Omega\cap B_r(x_0)} \langle x - x_0 - \rho_{p,r}\nu(x_0), \nu\rangle (\Delta v_p)^2 dx = 0. \tag{4.10}$$

Now applying the Pohozaev identity on $\Omega \cap B_r(x_0)$ with $y = y_p$, we obtain

$$\begin{aligned}\frac{4p^2}{(p+1)} \int_{\Omega\cap B_r(x_0)} (u_p^+)^{p+1} dx &= \frac{p^2}{(p+1)} \int_{\Omega\cap\partial B_r(x_0)} \langle x - y_p, \nu\rangle (u_p^+)^{p+1} ds \\ &+ 2\int_{\Omega\cap\partial B_r(x_0)} \frac{\partial v_p}{\partial\nu}\Delta v_p ds + \frac{1}{2}\int_{\Omega\cap\partial B_r(x_0)} \langle x - y_p, \nu\rangle (\Delta v_p)^2 ds \\ &+ \int_{\Omega\cap\partial B_r(x_0)} \langle x - y_p, \nabla v_p\rangle \frac{\partial\Delta v_p}{\partial\nu} ds + \int_{\Omega\cap\partial B_r(x_0)} \langle x - y_p, \nabla\Delta v_p\rangle \frac{\partial v_p}{\partial\nu} ds \\ &- \int_{\Omega\cap\partial B_r(x_0)} \langle x - y_p, \nu\rangle \langle \nabla v_p, \nabla\Delta v_p\rangle ds.\end{aligned} \tag{4.11}$$

As $v_p(x) \to \sum_{i=1}^N \gamma_i G(x, x_i)$ in $C^3_{loc}(\overline{\Omega}\setminus\mathcal{S})$ follows from the previous lemma. Again by the boundary values $G(x, x_0) = 0$. Also note that last five terms in right hand side $O(r^3)$ and hence we have

$$\begin{aligned}\frac{4p^2}{(p+1)} \int_{\Omega\cap B_r(x_0)} (u_p^+)^{p+1} dx &= \frac{p^2}{(p+1)} \int_{\Omega\cap\partial B_r(x_0)} \langle x - y_p, \nu\rangle (u_p^+)^{p+1} ds \\ &+ O(r^3) = O(r^3)\end{aligned}$$

as $\frac{p}{p+1} |\int_{\Omega\cap\partial B_r(x_0)} \langle x - y_p, \nu\rangle (u_p^+)^{p+1} ds| \le o_p(1) r^4$. Hence

$$\lim_{r\to 0}\lim_{p\to\infty} \frac{p^2}{(p+1)} \int_{\Omega\cap B_r(x_0)} (u_p^+)^{p+1} dx = 0.$$

By Hölder's inequality we have

$$p\int_{\Omega\cap B_r(x_0)} (u_p^+)^p dx \le C\left(\frac{p^2}{(p+1)} \int_{\Omega\cap B_r(x_0)} (u_p^+)^{p+1}\right)^{\frac{p}{p+1}} \left(p^{-\frac{p-1}{(p+1)}} (p+1)^{\frac{p}{p+1}}\right)$$

and hence

$$\lim_{r\to 0}\lim_{p\to\infty} p\int_{\Omega\cap B_r(x_0)} (u_p^+)^p dx = 0$$

a contradiction to (4.8). □

The remaining problem is to estimate $\|u_p\|_{L^\infty(\Omega)}$ and γ_j. This will be done in the next two sections.

5. Refined Estimates of the blow-up solution and Proof of Theorem 1.1

In this section, we give a precise estimate of $\|u_p\|_{L^\infty(\Omega)}$ and prove (1.8) for Theorem 1.1. To this end, we need to do some refined analysis of W_p (defined at (3.6)). This kind of estimates are called *sup+inf* estimates, which was first initiated by Brezis-Li-Shafrir [9], Li-Shafrir [19] for second order equation, using the method of moving planes. Lin-Wei [21] used potential analysis to give the sup+inf estimates for fourth order equation.
Since the estimate is local, throughout this section, we may assume that $B_1(0) \subset \Omega$ and $u_p(0) = \max_{B_1(0)} u_p(x)$. For small $r > 0$ such that $u_p > 0$ in $B_r(0)$ we define

$$\alpha_p(r) := \frac{p}{u_p(0)} \int_{B_r(0)} u_p^p dx. \tag{5.1}$$

The first lemma computes the values of α_p.

Lemma 5.1. *We have*

$$\lim_{r\to 0}\lim_{p\to\infty} \alpha_p(r) := \alpha = 64\pi^2. \tag{5.2}$$

Proof. By Lemma 4.2, we have

$$\lim_{p\to\infty} v_p(x) = \sum_{j=1}^{N} \gamma_j G(x, x_j)$$

in $\Omega \setminus \mathcal{S}$. We may assume that $N = 1$ and $x_j = 0$. The Green function at the origin is

$$\begin{cases} \Delta^2 G(x,0) = 0 & x \in \Omega, \\ G(x,0) = \dfrac{\partial G(x,0)}{\partial \nu} = 0 & x \in \partial\Omega. \end{cases} \tag{5.3}$$

The Green's function can be decomposed into singular part S (fundamental solution of the biharmonic) and regular part H as

$$\begin{aligned} G(x,0) &= S(x,0) + H(x,0) \\ &= \frac{1}{8\pi^2} \log \frac{1}{|x|} + H(x,0) \end{aligned} \tag{5.4}$$

where

$$\begin{cases} \Delta^2 H(x,0) = 0 & x \in \Omega, \\ H(x,0) = \dfrac{1}{8\pi^2} \log \dfrac{1}{|x|} & x \in \partial\Omega, \\ \dfrac{\partial H(x,0)}{\partial\nu} = \dfrac{1}{8\pi^2} \dfrac{\partial}{\partial\nu}\left(\log \dfrac{1}{|x|}\right) & x \in \partial\Omega \end{cases} \tag{5.5}$$

and hence near the origin can be written as

$$G(x,0) = \frac{1}{8\pi^2}\left(\log \frac{1}{|x|} + O(1)\right).$$

Let $r \in (0,1)$ such that $v_p > 0$ in $B_r(0)$. Since $v_p \to \gamma_1 G(x,0)$ in $C^4_{loc}(\bar\Omega \setminus 0)$ (Lemma 4.2) where

$$\gamma_1 = \alpha \lim_{p\to+\infty} u_p(0),$$

we have

$$v_p = \gamma_1(G(x,0) + h_p),\ z_p = -\Delta v_p = \gamma_1(-\Delta G(x,0) - \Delta h_p) \text{ on } \partial B_r$$

where

$$-\Delta G(x,0) = -\frac{1}{8\pi^2}\left(\frac{2}{|x|^2} + O(1)\right)$$

and $|\nabla^i h_p| \le C$ for $i = 1,2,3$. We will use a local Pohozaev identity from Lemma 2.2 in $B_r(0)$ on v_p. We have

$$\begin{aligned}
&\frac{4p^2}{(p+1)}\int_{B_r(0)}(u_p^+)^{p+1}dx = \frac{p^2}{(p+1)}\int_{\partial B_r}\langle x,\nu\rangle(u_p^+)^{p+1}ds\\
+\quad & \frac{1}{2}\int_{\partial B_r}\langle x,\nu\rangle z_p^2 ds + 2\int_{\partial B_r}\frac{\partial v_p}{\partial \nu}z_p ds\\
+\quad & \int_{\partial B_r}\left\{\frac{\partial z_p}{\partial\nu}\langle x, Dv_p\rangle ds + \frac{\partial v_p}{\partial\nu}\langle x, Dz_p\rangle - \langle Dv_p, Dz_p\rangle\langle x,\nu\rangle\right\}ds
\end{aligned}$$

where $z_p = -\Delta v_p$. Using uniform convergence of v_p and its derivatives on compact sets we have

$$\begin{aligned}
&\frac{p^2}{(p+1)}\int_{\partial B_r}\langle x,\nu\rangle(u_p^+)^{p+1}ds = o(1),\\
&\frac{1}{2}\int_{\partial B_r}\langle x,\nu\rangle z_p^2 ds \to \frac{\gamma_1^2}{16\pi^2} + o(1),\\
&2\int_{\partial B_r}\frac{\partial v_p}{\partial\nu}z_p \to -\frac{\gamma_1^2}{8\pi^2} + o(1),\\
&\int_{\partial B_r}\frac{\partial z_p}{\partial\nu}\langle x, Dv_p\rangle ds \to \frac{\gamma_1^2}{8\pi^2} + o(1),\\
&\int_{\partial B_r}\frac{\partial v_p}{\partial\nu}\langle x, Dz_p\rangle \to \frac{\gamma_1^2}{8\pi^2} + o(1),\\
&\int_{\partial B_r}\langle Dv_p, Dz_p\rangle\langle x,\nu\rangle \to \frac{\gamma_1^2}{8\pi^2} + o(1).
\end{aligned}$$

This implies that

$$\frac{p^2}{(p+1)}\int_{B_r}u_p^{p+1}dx = \frac{\gamma_1^2}{64\pi^2} + o(1).$$

As a result we have

$$\frac{p^2}{p+1}\int_{B_r}u_p^{p+1} = \frac{1}{64\pi^2}\left(p\int_{B_r}u_p^p\right)^2 + o(1). \tag{5.6}$$

By Fatou's Lemma, we obtain

$$\begin{aligned}
p\int_{B_r}u_p^p &= pu_p(0)^p\varepsilon_p^4\int_{B_{\frac{r}{\varepsilon_p}}(0)}\left(1+\frac{W_p}{p}\right)^p dx\\
&= u_p(0)\int_{B_{\frac{r}{\varepsilon_p}}(0)}\left(1+\frac{W_p}{p}\right)^p\\
&\ge u_p(0)\left(\int_{\mathbb{R}^4}e^W dx + o(1)\right) \ge 64\pi^2 u_p(0) + o(1).
\end{aligned} \tag{5.7}$$

On the other hand,

$$\frac{p^2}{p+1}u_p(0)\int_{B_r(0)}(u_p^+)^p \geq \frac{p^2}{p+1}\int_{B_r(0)}(u_p^+)^{p+1}. \tag{5.8}$$

Hence we derive from (5.6) and (5.8) that

$$\lim_{r\to 0}\limsup_{p\to\infty} p\int_{B_r} u_p^p dx \leq 64\pi^2 \lim_{p\to\infty} u_p(0). \tag{5.9}$$

Combining (5.7) and (5.9), we obtain (5.2). ☐

Define

$$\tilde{v}_p(x) = \frac{pu_p(x)}{u_p(0)}.$$

We now derive an important decay estimate for the blow-up solution $\tilde{v}_p$. In order to do so we first deduce the decay estimate for solution W_p defined by

$$W_p(x) := p\frac{u_p(\varepsilon_p x) - u_p(0)}{u_p(0)}, \quad x \in B_{\frac{1}{\varepsilon_p}}(0). \tag{5.10}$$

Further we define

$$\beta_p := \frac{p}{u_p(0)}\int_{B_1(0)} u_p^p dx \tag{5.11}$$

which can be written as

$$\beta_p = \int_{B_{\frac{1}{\varepsilon_p}}(0)} \left(1+\frac{W_p}{p}\right)_+^p dx \tag{5.12}$$

and

$$\beta_p = \alpha_p(r) + \frac{p}{u_p(0)}\int_{B_1(0)\setminus B_r(0)} u_p^p dx. \tag{5.13}$$

Note that the second term goes to zero as $p \to \infty$ (by Lemma 4.1) and hence we have

$$\beta_p \to 64\pi^2 \tag{5.14}$$

as $p \to \infty$.

The next lemma gives the decay estimates of W_p which will be needed later.

Lemma 5.2. *For any $\delta > 0$, there exist $r_\delta > 0$ and $p_0(\delta) \in \mathbb{N}$ such that*

$$|W_p(x)| \leq -\left(\frac{\beta_p}{8\pi^2} - \delta\right)\log|x| + C_\delta \tag{5.15}$$

for some $C_\delta > 0$ provided $|x| \geq 2r_\delta$ and $p \geq p_0$.

Proof. Using (5.14), given $\delta > 0$ we can choose $r_\delta > 1$ and $p_0 \in \mathbb{N}$ sufficiently large such that

$$\int_{|y|\leq r_\delta} \left(1+\frac{W_p}{p}\right)_+^p > 64\pi^2 - \frac{\delta}{2} \text{ for all } p \geq p_0. \tag{5.16}$$

Without loss of generality let $W_p(x) > 0$ in $B_{\frac{1}{\varepsilon_p}}(0)$. By Green's function representation for $|x| \geq 2r_\delta$, we have

$$
\begin{aligned}
W_p(x) &= \frac{1}{8\pi^2}\int_{B_{\frac{1}{\varepsilon_p}}} \log\frac{|y|}{|x-y|}\left(1+\frac{W_p}{p}\right)_+^p dy + O(1)\\
&= \frac{1}{8\pi^2}\int_{|y|\leq r_\delta} \log\frac{|y|}{|x||1-\frac{|y|}{|x|}|}\left(1+\frac{W_p}{p}\right)_+^p dy\\
&+ \frac{1}{8\pi^2}\int\limits_{B_{\frac{1}{\varepsilon_p}}\cap\{r_\delta\leq|y|\leq 2|x-y|\}} \log\frac{|y|}{|x-y|}\left(1+\frac{W_p}{p}\right)_+^p dy\\
&+ \frac{1}{8\pi^2}\int\limits_{B_{\frac{1}{\varepsilon_p}}\cap\{|y|\geq r_\delta,|y|\geq 2|x-y|\}} \log\frac{|y|}{|x-y|}\left(1+\frac{W_p}{p}\right)_+^p dy + O(1)\\
&\leq \frac{1}{8\pi^2}\log\frac{2r_\delta}{|x|}\int_{|y|\leq r_\delta}\left(1+\frac{W_p}{p}\right)_+^p dy + \frac{1}{8\pi^2}(\log 2)\int_{B_{\frac{1}{\varepsilon_p}}\cap\{|y|\geq r_\delta\}}\left(1+\frac{W_p}{p}\right)_+^p dy\\
&+ \frac{1}{8\pi^2}\int_{B_{\frac{1}{\varepsilon_p}}\cap\{|y|\geq r_\delta,|y|\geq 2|x-y|\}} (\log|y|)\left(1+\frac{W_p}{p}\right)_+^p dy\\
&+ \frac{1}{8\pi^2}\int_{B_{\frac{1}{\varepsilon_p}}\cap\{|y|\geq r_\delta,|y|\geq 2|x-y|\}} \log\frac{1}{|x-y|}\left(1+\frac{W_p}{p}\right)_+^p + O(1)\\
&\leq \frac{1}{8\pi^2}\log\frac{2r_\delta}{|x|}\int_{|y|\leq r_\delta}\left(1+\frac{W_p}{p}\right)_+^p + \frac{1}{8\pi^2}\log 2|x|\left(64\pi^2-\beta_p+\frac{\delta}{2}\right) + O(1).
\end{aligned}
$$

Hence for $x \in B_{\frac{1}{\varepsilon_p}}(0)$ with $|x| \geq 2r_\delta$

$$
\begin{aligned}
W_p(x) &\leq \frac{\beta_p}{8\pi^2}\log\frac{2r_\delta}{|x|} + \frac{\delta}{8\pi^2}\log 2|x| + C_\delta\\
&\leq -\left(\frac{\beta_p}{8\pi^2}-\delta\right)\log|x| + C_\delta. \qquad (5.17)
\end{aligned}
$$

as $\log\frac{2r_\delta}{|x|} \leq 0$. □

Remark 5.3. Let $r = r_\delta$. For $\log\frac{1}{\varepsilon_p} \leq |x| \leq \frac{1}{\varepsilon_p}$, let us define a radial function

$$
\tilde{\beta}_p(|x|) = \int_{|y|\leq r_0|x|}\left(1+\frac{W_p(y)}{p}\right)_+^p dy
$$

where $r_0 \leq \frac{1}{2}$. Note that by Lemma 5.2 as $\beta_p \to 64\pi^2$ we can consider $|W_p(x)| \leq 7\log\frac{1}{|x|}$ whenever $|x| \geq 2r$. Using this decay estimate we have

$$
\begin{aligned}
|\tilde{\beta}_p(|x|) - \beta_p| &\leq \int_{|y|\geq r_0|x|}\left(1+\frac{W_p(y)}{p}\right)_+^p dy \leq \int_{|y|\geq r_0|x|} e^{|W_p|}\\
&\leq C\int_{|y|\geq r_0|x|}\frac{1}{|y|^7}dy = O\left(\frac{1}{|x|^3}\right) \qquad (5.18)
\end{aligned}
$$

for $|x| \geq \log\frac{1}{\varepsilon_p}$.

Remark 5.4. The Pohozaev identity in Lemma 2.2 on a ball centered at the origin of radius r can be written as

$$
\begin{aligned}
4\int_{B_r} F(u) &= \int_{\partial B_r} |x|F(u)ds - \int_{\partial B_r} r\left[\frac{(\Delta u)^2}{2} + \frac{\partial u}{\partial r}\frac{\partial \Delta u}{\partial r}\right]ds \\
&+ \int_{\partial B_r} \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\Delta u ds. \qquad (5.19)
\end{aligned}
$$

Lemma 5.5. *For* $|x| = \log\frac{1}{\varepsilon_p}$, *we have*

$$\left|W_p - \frac{\beta_p}{8\pi^2}\log\frac{1}{|x|}\right| \le C, \tag{5.20}$$

$$\left|\frac{\partial W_p}{\partial r} + \frac{\beta_p}{8\pi^2}\frac{1}{|x|}\right| \le O\left(\left(\log\frac{1}{\varepsilon_p}\right)^{-1}|x|^{-1}\right), \tag{5.21}$$

$$\left|\Delta W_p + \frac{\beta_p}{4\pi^2}\frac{1}{|x|^2}\right| \le O\left(\left(\log\frac{1}{\varepsilon_p}\right)^{-1}|x|^{-2}\right), \tag{5.22}$$

$$\left|\frac{\partial}{\partial r}\left(r\frac{\partial W_p}{\partial r}\right)\right| \le O\left(|x|^{-2}\right) \tag{5.23}$$

and

$$\left|\frac{\partial \Delta W_p}{\partial r} - \frac{\beta_p}{2\pi^2}\frac{1}{|x|^3}\right| \le O\left(\left(\log\frac{1}{\varepsilon_p}\right)^{-1}|x|^{-3}\right). \tag{5.24}$$

Moreover, (5.20) holds for $\log\frac{1}{\varepsilon_p} \le |x| \le \frac{1}{\varepsilon_p}$.

Proof. For $\log\frac{1}{\varepsilon_p} \le |x| \le \varepsilon_p^{-1}$ and using the fact that $\tilde{\beta}_p(|x|) - \beta_p = O(\frac{1}{|x|^3})$ we have

$$
\begin{aligned}
W_p(x) &= \frac{1}{8\pi^2}\int_{|y|\le\frac{1}{\varepsilon_p}} \log\frac{|y|}{|x-y|}\left(1 + \frac{W_p(y)}{p}\right)_+^p + O(1) \\
&= \frac{1}{8\pi^2}\int_{|y|\le r_0|x|} \log\frac{|y|}{|x-y|}\left(1 + \frac{W_p(y)}{p}\right)_+^p + O(1) \\
&= \frac{\tilde{\beta}_p}{8\pi^2}\log\frac{1}{|x|} + O(1) \\
&= \frac{\beta_p}{8\pi^2}\log\frac{1}{|x|} + O(1).
\end{aligned}
$$

$$\begin{aligned}
\frac{\partial W_p(x)}{\partial r} &= -\frac{1}{8\pi^2}\int_{|y|\le\frac{1}{\varepsilon_p}} \frac{\partial}{\partial r}\left(\frac{\langle x-y,x\rangle}{|x||x-y|^2}\right)\left(1+\frac{W_p(y)}{p}\right)_+^p dy + O(\varepsilon_p)\\
&= -\frac{1}{8\pi^2}\int_{|y|\le r_0|x|} \frac{\partial}{\partial r}\left(\frac{\langle x-y,x\rangle}{|x-y|^2}\right)\left(1+\frac{W_p(y)}{p}\right)_+^p dy + O(\varepsilon_p+|x|^{-4})\\
&= -\frac{1}{8\pi^2}\frac{1}{|x|}\int_{|y|\le r_0|x|}\left(1+\frac{W_p(y)}{p}\right)_+^p dy + O(|x|^{-2})\\
&= -\frac{\tilde\beta_p}{8\pi^2}\frac{1}{|x|} + O(\frac{1}{|x|^2})\\
&= -\frac{\beta_p}{8\pi^2}\frac{1}{|x|} + O\left(|x|^{-1}\left(\log\frac{1}{\varepsilon_p}\right)^{-1}\right).
\end{aligned}$$

$$\begin{aligned}
\Delta W_p(x) &= -\frac{1}{4\pi^2}\int_{|y|\le r_0|x|}\frac{1}{|x-y|^2}\left(1+\frac{W_p(y)}{p}\right)_+^p + O\left(|x|^{-5}\right)\\
&= -\frac{1}{4\pi^2}\int_{|y|\le r_0|x|}\frac{1}{|x|^2}\left(1+\frac{W_p(y)}{p}\right)_+^p dy\\
&- \frac{1}{2\pi^2}\int_{|y|\le r_0|x|}\frac{\langle x,y\rangle}{|x|^4}\left(1+\frac{W_p(y)}{p}\right)_+^p dy + O\left(|x|^{-4}\right)\\
&= -\frac{\tilde\beta_p}{4\pi^2}\frac{1}{|x|^2} + O\left(|x|^{-3}\right)\\
&= -\frac{\beta_p}{4\pi^2}\frac{1}{|x|^2} + O(|x|^{-3})\\
&= -\frac{\beta_p}{4\pi^2}\frac{1}{|x|^2} + O\left(|x|^{-2}\left(\log\frac{1}{\varepsilon_p}\right)^{-1}\right).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\frac{\partial}{\partial r}\left(r\frac{\partial W_p(x)}{\partial r}\right) &= -\frac{1}{8\pi^2}\int_{|y|\le\frac{1}{\varepsilon_p}} \frac{\partial}{\partial r}\left(\frac{\langle x-y,x\rangle}{|x-y|^2}\right)\left(1+\frac{W_p(y)}{p}\right)_+^p dy + O(\varepsilon_p)\\
&= -\frac{1}{8\pi^2}\int_{|y|\le r_0|x|} \frac{\partial}{\partial r}\left(\frac{\langle x-y,x\rangle}{|x-y|^2}\right)\left(1+\frac{W_p(y)}{p}\right)_+^p dy + O(\varepsilon_p+|x|^{-4})\\
&= \frac{O(1)}{|x|^2}\int_{|y|\le r_0|x|}|y|(1+|y|)^{-6}dy + O(\varepsilon_p+|x|^{-4})\\
&= O(|x|^{-2}).
\end{aligned}$$

The proof of (5.24) follows similarly as above. □

Using the fact that

$$W_p(\varepsilon_p^{-1}x) = \tilde v_p(x) - p$$

and (5.20) we have

$$\tilde v_p(x) = -\frac{\beta_p}{8\pi^2}\log|x| + p + \frac{\beta_p}{8\pi^2}\log\varepsilon_p + O(1). \tag{5.25}$$

Corollary 5.6. *As a result, we have*

$$\left|\frac{\partial\tilde v_p}{\partial r} + \frac{\beta_p}{8\pi^2}\frac{1}{|x|}\right| \le O\left(\left(\log\frac{1}{\varepsilon_p}\right)^{-1}|x|^{-1}\right), \tag{5.26}$$

$$\left|\Delta \tilde{v}_p + \frac{\beta_p}{4\pi^2}\frac{1}{|x|^2}\right| \leq O\left(\left(\log \frac{1}{\varepsilon_p}\right)^{-1}|x|^{-2}\right), \tag{5.27}$$

$$\left|\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{v}_p}{\partial r}\right)\right| \leq O\left(|x|^{-2}\right) \tag{5.28}$$

and

$$\left|\frac{\partial \Delta \tilde{v}_p}{\partial r} - \frac{\beta_p}{2\pi^2}\frac{1}{|x|^3}\right| \leq O\left(\left(\log \frac{1}{\varepsilon_p}\right)^{-1}|x|^{-3}\right) \tag{5.29}$$

whenever $|x| = \varepsilon_p \log \frac{1}{\varepsilon_p}$.

Proof of Theorem 1.1. Multiplying (1.5) by u_p and integrating over B_1, we obtain

$$\begin{aligned} p\int_{B_1}|\Delta u_p|^2 dx &= p\int_{B_1}(u_p^+)^{p+1} - p\int_{\partial B_1} u_p \frac{\partial \Delta u_p}{\partial \nu} + p\int_{\partial B_1}\Delta u_p \frac{\partial u_p}{\partial \nu} \\ &= p\int_{B_1}(u_p^+)^{p+1} + O\left(\frac{1}{p}\right). \end{aligned} \tag{5.30}$$

Let us now use Corollary 5.6 to estimate the two integrals in (5.30).

Note that from definition of ε_p we have

$$\log p + 4\log \varepsilon_p + (p-1)\log u_p(0) = 0. \tag{5.31}$$

We will calculate

$$\int_{\varepsilon_p A}|\Delta u_p|^2 dx. \tag{5.32}$$

This is the same as estimating ΔW_p in $A = \{x \in \Omega_p : \log\frac{1}{\varepsilon_p} \leq |x| \leq \frac{1}{\varepsilon_p}\}$. By definition, $u_p(0)\Delta W_p = p\varepsilon_p^2 \Delta u_p(\varepsilon_p x)$. Using Lemma 5.5 and integrating both sides of (5.22) we obtain

$$\begin{aligned} \int_{\varepsilon_p A}|\Delta u_p|^2 dx &= \frac{\beta_p^2}{(4\pi^2)^2}\frac{(u_p(0))^2}{p^2}\omega_4 \log\frac{1}{\varepsilon_p} + O(p^{-2}) \\ &= 512\pi^2\frac{(u_p(0))^2}{p^2}\log\frac{1}{\varepsilon_p} + O(p^{-2}) \\ &= 128\pi^2\frac{(u_p(0))^2}{p}\log u_p(0) + O(p^{-2}) \end{aligned}$$

where $\omega_4 = 2\pi^2$ is the volume of unit sphere in $\mathbb{R}^4$. This implies that

$$p\int_{\varepsilon_p A}|\Delta u_p|^2 dx = 128\pi^2 (u_p(0))^2 \log u_p(0) + o(1). \tag{5.33}$$

We use Lemma 5.2 and dominated convergence theorem to conclude

$$\begin{aligned} p\int_{B_1}(u_p)_+^{p+1}dx &= (u_p(0))^2\int_{B_{\frac{1}{\varepsilon_p}}}\left(1+\frac{W_p}{p}\right)_+^p \\ &= (64\pi^2 + o(1))(u_p(0))^2. \end{aligned} \tag{5.34}$$

Substituting (5.33) and (5.34) into (5.30), we obtain that

$$\log u_p(0) \to \frac{1}{2}$$

which implies that $u_p(0) \to \sqrt{e}$ as $p \to +\infty$. $\square$

6. Proof of Theorem 1.2

In this section, we complete the proof of Theorem 1.2.
By Lemma 4.2 and Lemma 5.1, we have that $v_p \to \sum_{j=1}^{N} \gamma_j G(x, x_j)$ in $\Omega \backslash \mathcal{S}$, where we can compute

$$\gamma_j = \alpha \lim_{p \to +\infty} u_p(x_{p,j}) = 64\pi^2 \sqrt{e} \tag{6.1}$$

where $u_p(x_{p,j}) = \max_{B_r(x_j)} u_p(x)$.

This proves $(f_1) - (f_2)$ of Theorem 1.2.

Now we prove the identity in (f_3). Let $r > 0$ be a small number such that $B_r(x_i) \subset \overline{\Omega}$ and $B_r(x_i) \cap B_r(x_j) = \emptyset$, $i \neq j$. It is enough to prove the identity for $i = 1$.

Applying Lemma 2.2 to v_p on the domain $\Omega \setminus B_r(x_1)$, we have

$$\int\limits_{\partial(\Omega \backslash B_r(x_1))} \nu F(v_p) + \frac{1}{2} \int\limits_{\partial(\Omega \backslash B_r(x_1))} z_p^2 \nu + \int\limits_{\partial(\Omega \backslash B_r(x_1))} \left\{ \frac{\partial z_p}{\partial \nu} D v_p + \frac{\partial v_p}{\partial \nu} D z_p - \langle D z_p, D v_p \rangle \nu \right\} ds = 0$$

where $z_p = -\Delta v_p$, where $F(v_p) = \frac{p^2}{(p+1)} (u_p^+)^{p+1}$.

Since $u_p = \nabla u_p = 0$ on $\partial\Omega$, we obtain

$$\int\limits_{\partial B_r(x_1)} \nu F(v_p) + \frac{1}{2} \int\limits_{\partial(\Omega \backslash B_r(x_1))} z_p^2 \nu + \int\limits_{\partial B_r(x_1)} \left\{ \frac{\partial z_p}{\partial \nu} D v_p + \frac{\partial v_p}{\partial \nu} D z_p - \langle D z_p, D v_p \rangle \nu \right\} ds = 0.$$

Letting $p \to \infty$, we obtain

$$\frac{1}{2} \int\limits_{\partial(\Omega \backslash B_r(x_1))} z^2 \nu + \int\limits_{\partial B_r(x_1)} \left\{ \frac{\partial z}{\partial \nu} D v + \frac{\partial v}{\partial \nu} D z - \langle D z, D v \rangle \nu \right\} ds = 0$$

where $v = \lim_{p \to +\infty} v_p$, $z = -\Delta v$. Note that $v(x) = 64\pi^2 \sqrt{e} \sum_{j=1}^{N} G(x, x_j)$ and $-\Delta v(x) = 64\pi^2 \sqrt{e} \sum_{j=1}^{N} (-\Delta G(x, x_j))$. But from (1.11), we have

$$\frac{v}{64\pi^2 \sqrt{e}} = H(x, x_1) + \sum_{j=2}^{N} G(x, x_j) - \frac{1}{8\pi^2} \log |x - x_1|, \tag{6.2}$$

$$\frac{z}{64\pi^2 \sqrt{e}} = (-\Delta) H(x, x_1) + \sum_{j=2}^{N} (-\Delta) G(x, x_j) + \frac{1}{4\pi^2} \frac{1}{|x - x_1|^2}. \tag{6.3}$$

Hence we have

$$\frac{1}{(64\pi^2 \sqrt{e})^2} \int\limits_{\partial(\Omega \backslash B_r(x_1))} z^2 \nu = O(r^2).$$

By using mean value theorem [28],

$$\begin{aligned}
\frac{1}{(64\pi^2\sqrt{e})^2}\int\limits_{\partial B_r(x_1)}\frac{\partial z}{\partial\nu}Dv &= \int\limits_{\partial B_r(x_1)}(-\frac{1}{2\pi^2 r^3}+O(1))\Big(\nabla_x(H(x,x_1)+\sum_{j=2}^N\nabla_x G(x,x_j)-\frac{x}{8\pi^2r^2}\Big)\\
&= -\frac{1}{2\pi^2 r^4}\int\limits_{\partial B_r(x_1)}\Big(\nabla_x H(x,x_1)+\sum_{j=2}^N\nabla_x G(x,x_j)\Big)+O(r)\\
&= -\Big(\nabla_x H(x_1^\star,x_1)+\sum_{j=2}^N\nabla_x G(x_1^\star,x_j)\Big)+O(r).
\end{aligned}$$

$$\begin{aligned}
\frac{1}{(64\pi^2\sqrt{e})^2}\int\limits_{\partial B_r(x_1)}\frac{\partial v}{\partial\nu}Dz &= \int\limits_{\partial B_r(x_1)}(-\frac{x}{2\pi^2 r^4}+O(1))\Big(\nabla_x H(x,x_1)+\sum_{j=2}^N\nabla_x G(x,x_j)-\frac{x}{8\pi^2r^2}\Big)\\
&= -\frac{1}{2\pi^2 r^4}\int\limits_{\partial B_r(x_1)}\Big(\nabla_x H(x,x_1)+\sum_{j=2}^N\nabla_x G(x,x_j)\Big)+O(r)\\
&= -\Big(\nabla_x H(x_2^\star,x_2)+\sum_{j=2}^N\nabla_x G(x_2^\star,x_j)\Big)+O(r).
\end{aligned}$$

$$\begin{aligned}
\frac{1}{(64\pi^2\sqrt{e})^2}\int\limits_{\partial B_r(x_1)}\langle Dz,Dv\rangle\nu &= \Big\langle(-\frac{x}{2\pi^2 r^4}+O(1)),\Big(\nabla_x H(x,x_1)+\sum_{j=2}^N\nabla_x G(x,x_j)-\frac{x}{8\pi^2r^2}\Big)\Big\rangle\\
&= -\Big(\nabla_x H(x_3^\star,x_1)+\sum_{j=2}^N\nabla_x G(x_3^\star,x_j)\Big)+O(r)
\end{aligned}$$

where $x_i^\star\in B_r(x_1)$; $i=1,2,3$. Letting $r\to 0$, we have $x_i^\star\to x_1$ and hence we have

$$\nabla_x H(x_1,x_1)+\sum_{j=2}^N\nabla_x G(x_1,x_j)=0.$$

We can obtain the other identities in a similar way.

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