Vortex-peak interaction and lattice shape in rotating two-component Bose-Einstein condensates

Amandine Aftalion[1], Peter Mason[1], Juncheng Wei[2]
[1] *CNRS & Université Versailles-Saint-Quentin-en-Yvelines, Laboratoire de Mathématiques de Versailles, CNRS UMR 8100, 45 avenue des États-Unis, 78035 Versailles Cédex, France*
[2] *Department of Mathematics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong.*
(Dated: November 28, 2011)

When a two component Bose-Einstein condensate is placed into rotation, a lattice of vortices and cores appear. The geometry of this lattice (triangular or square) varies according to the rotational value and the intercomponent coupling strengths. In this paper, assuming a Thomas-Fermi regime, we derive a point energy which allows us to determine for which values of the parameters the lattice goes from triangular to square. It turns out that the separating curve in the phase diagram agrees fully with the complete numerical simulations of the Gross-Pitaevskii equations.

PACS numbers:

INTRODUCTION

When a two component condensate is set into rotation, topological defects of both order parameters are created, which leads to more exotic defects than in a single component condensate. Experiments on two component condensates have shown how the condensates can exhibit either triangular or square vortex lattices [1]. According to the values of the interaction strengths, the defect patterns can vary a lot, as illustrated in the numerical simulations [2, 3]. One specific feature is the appearance of coreless vortices: the existence of a vortex in component-1 corresponds to a peak in component-2 and vice-versa. The interaction between vortices and peaks leads to changes in the geometry of the vortex lattice. We are interested in determining the equations governing this vortex peak behaviour in the Thomas Fermi regime and estimating the interaction energy between the lattices of the two components. Indeed, for a single condensate, the vortex lattice is triangular, while for a two component, the vortex-peak interaction can lead to a square lattice. In [4], an asymptotic interaction between two half quantized vortices is derived for two component homogeneous condensates. In this paper, we want to take into account the nonhomogeneity of the condensate due to the trapping potential and estimate the vortex-peak energy according to the parameters of the system. We derive an energy depending on the location of vortices and peaks and determine for which values of the experimental parameters, the lattice goes from triangular to square. These critical values agree well with the ones found from the numerical computations of the full Gross Pitaevskii equations of [3]. We note that in the rapid rotation regime, using the lowest Landau level approximation, several papers [5, 6] (see [2, 7] for review) have analyzed the transition between triangular to square lattices.

We first review the results for a single condensate, before moving to the derivation of homogeneous equations and the computation of the interaction term in two-component condensates.

For a single component condensate, the wave function minimizes the energy

$$E_{g,\Omega}(\psi) = \int \frac{1}{2}|\nabla\psi - i\Omega\times r\psi|^2 + \frac{1}{2}(V(r)-\Omega^2 r^2)|\psi|^2 + \frac{g}{2}|\psi|^4 \tag{1}$$

under $\int |\psi|^2 = 1$, where $\Omega = \Omega e_z$ is the rotation, $V(r)$ is the trapping potential and in most cases $V(r) = r^2$. We will denote by ∇_Ω the operator $\nabla - i\Omega \times r$. For g large, at $\Omega = 0$, the ground state η of $E_{g,0}$ approaches the inverted parabola

$$\frac{1}{2g}(\lambda - r^2)$$

in the disk of radius $R^2 = \lambda = 2\sqrt{g/\pi}$, and goes to 0 elsewhere. If the problem is rescaled on a disk of size 1, then the analysis of the vortex cores leads to a vortex of size $1/\sqrt{g}$ and, close to the core, the wave function behaves like $f(r)e^{i\theta}$ where f is the solution tending to 1 at infinity of

$$f'' + \frac{f'}{r} - \frac{f}{r^2} + f(1-f^2) = 0. \tag{2}$$

This is the equation of a vortex core in a uniform system. In the case of a single condensate, from the equation of the vortex core, one can estimate the energy of vortices, the critical velocity for the nucleation of the first vortex and the interaction energy between vortices [8–10] which is

$$-\sum_{i\neq j} \log|p_i - p_j| + \sum_i |p_i|^2 \tag{3}$$

where p_i are the location of the vortex cores. Numerically, the minimization of (3) yields an almost triangular

lattice for a large number of vortex points.

The aim of this paper is to describe the equivalent of (2)-(3) in the case of two-component condensates. We define g_i to be the intra-component coupling strength for component i and g_{12} to be the inter-component coupling strength. For simplicity, we assume equal masses for the atoms in each component and equal trapping potential, but a general case could be handled. The ground state of a two-component condensate is given by the infimum of

$$E_{g_1,g_2,g_{12},\Omega}(\psi_1,\psi_2) = E_{g_1,\Omega}(\psi_1) + E_{g_2,\Omega}(\psi_2) + g_{12}\int |\psi_1|^2|\psi_2|^2 \quad (4)$$

under $\int|\psi_1|^2 = N_1$, $\int|\psi_2|^2 = N_2$. We set $g_1 = \alpha_1 g$, $g_2 = \alpha_2 g$, $g_{12} = \alpha_0 g$ where g is large, so that $\varepsilon = 1/\sqrt{g}$ is small. We change wave functions to $\psi_1(x,y) = \sqrt{\varepsilon}u_1(x\sqrt{\varepsilon}, y\sqrt{\varepsilon})$, $\psi_2(x,y) = \sqrt{\varepsilon}u_2(x\sqrt{\varepsilon}, y\sqrt{\varepsilon})$. Calling $\alpha = (\alpha_0, \alpha_1, \alpha_2)$, the energy we want to minimize is

$$\begin{aligned} E_{\alpha,\Omega}(u_1,u_2) = \int & \frac{\varepsilon^2}{2}|\nabla u_1|^2 + \frac{1}{2}r^2|u_1|^2 + \frac{\alpha_1}{2}|u_1|^4 \\ & - \varepsilon\Omega \times r(iu_1, \nabla u_1) \\ & + \frac{\varepsilon^2}{2}|\nabla u_2|^2 + \frac{1}{2}r^2|u_2|^2 + \frac{\alpha_2}{2}|u_2|^4 \\ & - \varepsilon\Omega \times r(iu_2, \nabla u_2) \\ & + \alpha_0|u_1|^2|u_2|^2 \end{aligned} \quad (5)$$

where $(iu, \nabla u) = iu\nabla\bar{u} - i\bar{u}\nabla u$. For $\Omega = 0$, the ground state is real valued and we denote it by (η_1, η_2). It is a solution of

$$-\varepsilon^2\Delta\eta_1 + r^2\eta_1 + 2\alpha_1\eta_1^3 + 2\alpha_0\eta_2^2\eta_1 = \mu_1\eta_1 \quad (6)$$

$$-\varepsilon^2\Delta\eta_2 + r^2\eta_2 + 2\alpha_2\eta_2^3 + 2\alpha_0\eta_1^2\eta_2 = \mu_2\eta_2. \quad (7)$$

The shape of the ground state varies according to α and when $\alpha_0^2 - \alpha_1\alpha_2 \le 0$, can be either 2 disks or a disk and an annulus, as we will see below.

REDUCTION TO THE CORE EQUATIONS

We consider (u_1, u_2) a ground state of $E_{\alpha,\Omega}$ and call (f_1, f_2) such that $u_1 = \eta_1 f_1$ and $u_2 = \eta_2 f_2$. We expect η_i to include the slow varying profile and f_i to include the vortex or peak contribution, so that f_i is 1 almost everywhere except close to the vortex and peak cores. We want to write the energy of (u_1, u_2) as the energy of (η_1, η_2) plus a rest, which is the energy that we are going to study. This follows a trick introduced in [11], and used for single Bose Einstein condensates in [8]. We multiply (6) by $\eta_1(|f_1|^2 - 1)$ and (7) by $\eta_2(|f_2|^2 - 1)$, and integrate and add the two equations, which yields the identity

$$\begin{aligned} & \int \frac{\varepsilon^2}{2}|\nabla\eta_1|^2(|f_1|^2-1) + \varepsilon^2\eta_1 f_1\nabla\eta_1\cdot\nabla f_1 \\ & + \frac{1}{2}r^2\eta_1^2(|f_1|^2-1) + \alpha_1\eta_1^4(|f_1|^2-1) + \alpha_0\eta_2^2\eta_1^2(|f_1|^2-1) \\ & + \frac{\varepsilon^2}{2}|\nabla\eta_2|^2(|f_2|^2-1) + \varepsilon^2\eta_2 f_2\nabla\eta_2\cdot\nabla f_2 \\ & + \frac{1}{2}r^2\eta_2^2(|f_2|^2-1) + \alpha_2\eta_2^4(|f_2|^2-1) + \alpha_0\eta_2^2\eta_1^2(|f_2|^2-1) = 0. \end{aligned} \quad (8)$$

Note that the Lagrange multiplier term has disappeared because u_i and η_i are normalized similarly. We replace (u_1, u_2) by $(f_1\eta_1, f_2\eta_2)$ into the energy, use the identity (8) and find

$$\begin{aligned} & E_{\alpha,\Omega}(u_1,u_2) = E_{\alpha,0}(\eta_1,\eta_2) + F_{\alpha,\Omega}(f_1,f_2) \text{ where} \\ & F_{\alpha,\Omega}(f_1,f_2) = \int \frac{\varepsilon^2}{2}\eta_1^2|\nabla f_1|^2 - \varepsilon\eta_1^2\Omega\times r(if_1,\nabla f_1) \\ & + \frac{1}{2}\alpha_1\eta_1^4(|f_1|^2-1)^2 + \alpha_0\eta_1^2\eta_2^2(1-|f_1|^2)(1-|f_2|^2) \\ & + \frac{\varepsilon^2}{2}\eta_2^2|\nabla f_2|^2 - \varepsilon\eta_2^2\Omega\times r(if_2,\nabla f_2) + \frac{1}{2}\alpha_2\eta_2^4(|f_2|^2-1)^2. \end{aligned} \quad (9)$$

This splitting of energy does not assume anything about the scales of energy: it is an exact identity. We point out that as soon as $\alpha_1\alpha_2 - \alpha_0^2 \ge 0$, then the energy $F_{\alpha,\Omega}$ is positive and minimizing $E_{\alpha,\Omega}$ in (u_1, u_2) amounts to minimizing $F_{\alpha,\Omega}$ in (f_1, f_2).

Now we assume that we scale everything close to a point p where $\eta_1^2 = \rho_1$, $\eta_2^2 = \rho_2$, and f_1, f_2 can be written as functions of $p + |r-p|/\varepsilon$. Then, in the new variable $\tilde{r} = |r-p|/\varepsilon$, the functions f_1, f_2 are a ground state of

$$\begin{aligned} & \mathcal{F}_{\alpha,\Omega}(f_1,f_2) = \int \frac{1}{2}\rho_1|\nabla f_1|^2 - \varepsilon\rho_1\Omega\times r(if_1,\nabla f_1) \\ & + \frac{1}{2}\alpha_1\rho_1^2(|f_1|^2-1)^2 + \alpha_0\rho_1\rho_2(1-|f_1|^2)(1-|f_2|^2) \\ & + \frac{1}{2}\rho_2|\nabla f_2|^2 - \varepsilon\rho_2\Omega\times r(if_2,\nabla f_2) + \frac{1}{2}\alpha_2\rho_2^2(|f_2|^2-1)^2 \end{aligned} \quad (10)$$

and solve the system

$$\begin{aligned} & -\rho_1\Delta f_1 - i\varepsilon\Omega\times r\rho_1\nabla f_1 + 2\alpha_1\rho_1^2(|f_1|^2-1)f_1 \\ & \quad + 2\alpha_0\rho_1\rho_2 f_1(|f_2|^2-1) = \tilde{\lambda}_1 f_1 \\ & -\rho_2\Delta f_2 - i\varepsilon\Omega\times r\rho_2\nabla f_2 + 2\alpha_2\rho_2^2(|f_2|^2-1)f_2 \\ & \quad + 2\alpha_0\rho_1\rho_2 f_2(|f_1|^2-1) = \tilde{\lambda}_2 f_2. \end{aligned} \quad (11)$$

This is exactly the system studied in [4, 12] for a homogeneous condensate. The splitting of energy has allowed us to reach a homogeneous system. Assuming a vortex

in component-1 and a spike in component-2, we have $f_1 = v_1(r)e^{i\theta}$ and $f_2 = v_2(r)$. We expect that v_1, v_2 tend to 1 at infinity so that $\tilde{\lambda}_1 = \varepsilon\Omega$ and $\tilde{\lambda}_2 = 0$. This yields the following system

$$-v_1'' - \frac{v_1'}{r} + \frac{v_1}{r^2} + 2\alpha_1\rho_1(v_1^2-1)v_1 + 2\alpha_0\rho_2 v_1(v_2^2-1) = 0 \quad (12)$$

$$-v_2'' - \frac{v_2'}{r} + 2\alpha_2\rho_2(v_2^2-1)v_2 + 2\alpha_0\rho_1 v_2(v_1^2-1) = 0 \quad (13)$$

From this system, asymptotic expansions can be obtained for v_1 and v_2 at infinity: $v_1(r) - 1 \sim -\gamma_1/r^2$ and $v_2(r) - 1 \sim \gamma_2/r^2$ for some constants γ_1 and γ_2. Equation (12)-(13) at infinity imply that $\alpha_2\rho_2\gamma_2 = \alpha_0\rho_1\gamma_1$ and $1 - 4\alpha_1\rho_1\gamma_1 + 4\alpha_0\rho_2\gamma_2 = 0$, thus

$$\gamma_1 = \frac{1}{4\rho_1\alpha_1\Gamma_{12}} \text{ and } \gamma_2 = \frac{\alpha_0}{4\rho_2\alpha_1\alpha_2\Gamma_{12}} \quad (14)$$

where

$$\Gamma_{12} = 1 - \frac{\alpha_0^2}{\alpha_1\alpha_2}. \quad (15)$$

In particular,

$$\alpha_0\gamma_1\gamma_1\rho_1\rho_2 = \frac{1-\Gamma_{12}}{16\alpha_1\Gamma_{12}^2}. \quad (16)$$

THOMAS-FERMI PROFILE OF THE GROUND STATE

When $\alpha_1\alpha_2 - \alpha_0^2 \geq 0$, that is $\Gamma_{12} \geq 0$, and ε is small, both components are in the Thomas-Fermi regime. We study the properties of the solutions of (6)-(7). The following non-dimensional parameters are introduced

$$\Gamma_1 = 1 - \frac{\alpha_0}{\alpha_1} \quad (17)$$

$$\Gamma_2 = 1 - \frac{\alpha_0}{\alpha_2}. \quad (18)$$

We refer to [3] for detailed computations. To begin, assume that both components are circular with radii R_1 and R_2 and with $R_1 < R_2$. Thus in the Thomas-Fermi (TF) regime, the density profiles for $r < R_1$ are

$$|\eta_1|^2 = \frac{1}{2\alpha_1\Gamma_{12}}\left(\mu_1 - \frac{\alpha_0}{\alpha_2}\mu_2 - r^2\Gamma_2\right) \quad (19)$$

$$|\eta_2|^2 = \frac{1}{2\alpha_2\Gamma_{12}}\left(\mu_2 - \frac{\alpha_0}{\alpha_1}\mu_1 - r^2\Gamma_1\right) \quad (20)$$

and for $R_1 < r < R_2$ are

$$|\eta_2|^2 = \frac{\mu_2 - r^2}{2\alpha_2} \quad (21)$$

with $|\eta_1|^2 = 0$. The chemical potentials μ_1 and μ_2, and the radii, R_1 and R_2, are to be found. In addition we have the normalisation condition

$$\int |\eta_k|^2 d\boldsymbol{r} = N_k, \quad (22)$$

where, for generality, $N_1 \neq N_2$. We denote $\tilde{\alpha}_k = N_k\alpha_k$ and $\tilde{\alpha}_0 = \sqrt{N_1N_2}\alpha_0$ and get

$$R_1 = \left(\frac{4\tilde{\alpha}_1\Gamma_{12}}{\pi\Gamma_2}\right)^{1/4}, \quad (23)$$

$$R_2 = \left(\frac{4(\tilde{\alpha}_2 + \tilde{\alpha}_1(1-\Gamma_1))}{\pi}\right)^{1/4}, \quad (24)$$

$$\mu_1 = \left(\frac{4\tilde{\alpha}_1\Gamma_{12}\Gamma_2}{\pi}\right)^{1/2} \quad (25)$$

$$+(1-\Gamma_2)\left(\frac{4}{\pi}\left[\tilde{\alpha}_2 + \tilde{\alpha}_1(1-\Gamma_1)\right]\right)^{1/2}, \quad (26)$$

$$\mu_2 = \left(\frac{4}{\pi}\left[\tilde{\alpha}_2 + \tilde{\alpha}_1(1-\Gamma_1)\right]\right)^{1/2}. \quad (27)$$

We find from (19) and (14) that

$$\rho_1 = \eta_1^2(0) = \frac{\Gamma_2 R_1^2}{2\alpha_1\Gamma_{12}} = \sqrt{\frac{\Gamma_2 N_1}{\pi\alpha_1\Gamma_{12}}} \quad (28)$$

and

$$\gamma_1 = \sqrt{\frac{\pi}{16N_1\alpha_1\Gamma_2\Gamma_{12}}}$$

while

$$\rho_2 = \eta_2^2(0) = \frac{1}{\alpha_2}\left((\frac{1}{\pi}(N_2\alpha_2 + N_1\alpha_0))^{1/2} - \alpha_0(\frac{N_1\Gamma_2}{\pi\alpha_1\Gamma_{12}})^{1/2}\right) \quad (29)$$

and γ_2 follows from (14).

Equations (23)-(24) are valid provided $\Gamma_{12}/\Gamma_2 > 0$ (to ensure that R_1 and μ_1 are real) and $\tilde{\alpha}_1\Gamma_1 < \tilde{\alpha}_2\Gamma_2$ (to ensure that $R_2 > R_1$). If instead the initial assumption on the size of the radii was taken to be $R_1 > R_2$, then the appropriate expressions would also be given by Eq.'s (23)-(24), however with the indices 1 and 2 alternated. In this case, the conditions would be $\tilde{\alpha}_1\Gamma_1 > \tilde{\alpha}_2\Gamma_2$ and $\Gamma_{12}/\Gamma_1 > 0$.

Returning now to $R_2 > R_1$, one must ensure that $|\eta_2|^2 > 0$ for all $r < R_1$. Suppose that there is a point at the origin, where $|\eta_2|^2 = 0$. Then, from Eq. (20),

$$\alpha_0 = \bar{\alpha}_0 = \frac{\alpha_1\mu_2}{\mu_1} = \frac{N_1\alpha_1}{2(N_1+N_2)} + \frac{1}{2}\sqrt{\frac{\alpha_1^2N_1^2}{(N_1+N_2)^2} + \frac{4N_2\alpha_1\alpha_2}{N_1+N_2}} \quad (30)$$

The existence of some Γ_{12} at which the density in

component-2 hits zero at the origin is the indication of a spatial separation of the components. Notice that this critical value for Γ_{12} is independent of Ω. In the spatial separation regime, component-1 is circular while component-2 is annular, provided $R_2 > R_1$. It is not possible for an annulus to develop in component-1; this can be seen by writing down the TF density expressions for an annular component-1 and a circular component-2 in which the chemical potentials become multi-valued. Thus under the assumption $R_2 > R_1$, an annulus can only develop in component-2. Similarly, an annulus can only develop in component-1 if $R_1 > R_2$. The condition to have two disks is thus $\alpha_0 < \bar{\alpha}_0$.

VORTEX INTERACTION

Let us call $\rho_{TF,1}$, $\rho_{TF,2}$ the Thomas-Fermi limits of $|\eta_1|^2$ and $|\eta_2|^2$ given by (19)-(20)-(21) and (23), (24), (25), (26), (27) in the case of two disks. Then

$$\rho_{TF,1} = \frac{\Gamma_2}{2\alpha_1\Gamma_{12}}(R_1^2 - r^2) \tag{31}$$

$$\rho_{TF,2} = \frac{\Gamma_1}{2\alpha_2\Gamma_{12}}(R_1^2 - r^2) + \frac{1}{2\alpha_2}(R_2^2 - R_1^2) \text{ if } r < R_1 \tag{32}$$

$$\text{and } \frac{1}{2\alpha_2}(R_2^2 - r^2) \text{ if } r > R_1. \tag{33}$$

We want to estimate the various terms in the energy $F_{\alpha,\Omega}$ as in [8] and we are going to show that, if p_i are the vortices for component-1 and q_j are the vortices for component-2, then they minimize the point energy

$$\begin{aligned} &- \pi\varepsilon^2 \sum_{i,j} \rho_1 \log|p_i - p_j| - \pi\varepsilon^2 \sum_{i,j} \rho_2 \log|q_i - q_j| \\ &\quad + \pi\left(-\varepsilon^2|\log\varepsilon|\frac{\Gamma_2}{2\alpha_1\Gamma_{12}} + \varepsilon\Omega\rho_1\right)\sum_i |p_i|^2 \\ &\quad + \pi\left(-\varepsilon^2|\log\varepsilon|\frac{\Gamma_1}{2\alpha_2\Gamma_{12}} + \varepsilon\Omega\rho_2\right)\sum_i |q_i|^2 \\ &+ \pi\frac{1-\Gamma_{12}}{16\Gamma_{12}^2}\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right)\varepsilon^4|\log\varepsilon|\sum_{i\neq j}\frac{1}{|p_i - q_j|^2}. \end{aligned} \tag{34}$$

Estimate of the kinetic energy term

Let us call p_i the vortices in component-1, and q_j in component-2. Then the kinetic energy term $(1/2)\int \eta_1^2\varepsilon^2|\nabla f_1|^2$ provides a leading order term due to the kinetic energy of the phase (which behaves locally like $1/r$ outside a disk of radius ε around each vortex), which is

$$\pi\varepsilon^2 \sum_i \rho_{TF,1}(p_i)|\log\varepsilon| - \pi\varepsilon^2 \sum_{i,j} \rho_{TF,1}(p_i)\log|p_i - p_j| \tag{35}$$

with a similar term for component-2, where p_i is replaced by q_j.

Rotation term

We call $X_1(r)$ the primitive of $r\rho_{TF,1}(r)$ which vanishes at R_1 and $X_2(r)$ the primitive of $r\rho_{TF,2}(r)$ which vanishes at R_2. Then for $r < R_1$

$$X_1(r) = \frac{\Gamma_2}{8\alpha_1\Gamma_{12}}(R_1^2 - r^2)^2, \tag{36}$$

$$X_2(r) = \frac{\Gamma_1}{8\alpha_2\Gamma_{12}}(R_1^2 - r^2)^2 + \frac{1}{8\alpha_2}(R_2^2 - R_1^2)(R_2^2 + R_1^2 - 2r^2). \tag{37}$$

Thus the rotation term $-\varepsilon\Omega\int \eta_1^2 \times r(if_1, \nabla f_1)$ is well approximated by $-\varepsilon\Omega\int \nabla X_1 \times (if_1, \nabla f_1)$. An integration by parts around each vortex yields

$$-2\pi\varepsilon\Omega\sum_i X_1(p_i) \tag{38}$$

with a similar contribution for component-2.

First vortices

The leading order approximation of the kinetic and rotation energy yields (assuming vortices at points p_i for component-1 and q_j for component-2):

$$\begin{aligned} &\pi\varepsilon^2 \sum_i \rho_{TF,1}(p_i)|\log\varepsilon| + \pi\varepsilon^2 \sum_j \rho_{TF,2}(q_j)|\log\varepsilon| \\ &\qquad - 2\pi\varepsilon\Omega\sum_i X_1(p_i) - 2\pi\varepsilon\Omega\sum_j X_2(q_j). \end{aligned} \tag{39}$$

The vortex first appears at the point where $X_1/\rho_{TF,1}$ or $X_2/\rho_{TF,2}$ reaches its maximum. We find that

$$\frac{X_1(r)}{\rho_{TF,1}(r)} = \frac{1}{4}(R_1^2 - r^2) \tag{40}$$

$$\frac{X_2(r)}{\rho_{TF,2}(r)} = \frac{1}{4}(R_1^2 - r^2) + \frac{1}{4}\frac{(R_2^2 - R_1^2)(R_2^2 - r^2)}{\frac{\Gamma_1}{\Gamma_{12}}(R_1^2 - r^2) + (R_2^2 - R_1^2)} \tag{41}$$

This implies that above a critical value Ω^c, vortices become energetically favorable and Ω^c is given by

$$\Omega^c = \frac{1}{2}\varepsilon|\log\varepsilon| \min_{i,r} \frac{\rho_{TF,i}}{X_i}. \tag{42}$$

For a harmonic potential, and in the case of a disk, the minimum of $\frac{\rho_{TF,i}}{X_i}$ occurs at the origin since

$$\frac{\rho_{TF,2}}{X_2} = \frac{\rho_{TF,1}}{X_1} - D(r) \tag{43}$$

where

$$D(r) = \frac{4(R_2^2 - R_1^2)(R_2^2 - r^2)}{(R_1^2 - r^2)} \times \frac{1}{\left[\frac{\Gamma_1}{\Gamma_{12}}(R_1^2 - r^2)^2 + (R_2^2 - R_1^2)(R_1^2 + R_2^2 - 2r^2)\right]}, \tag{44}$$

and given the signs of the parameters, we see that $D(0) > 0$ so that $\rho_{TF,1}/X_1 > \rho_{TF,2}/X_2$ always.

The first vortex is thus preferred in component-2 (i.e. the component with larger support) and occurs at the origin with the critical velocity given by

$$\Omega^c = \frac{\sqrt{\pi}}{\tilde{\alpha}_2\Gamma_{12}}\varepsilon|\log\varepsilon|\Big[\Gamma_{12}\sqrt{\tilde{\alpha}_2 + \tilde{\alpha}_1(1-\Gamma_1)} - (1-\Gamma_1)\sqrt{\tilde{\alpha}_1\Gamma_2\Gamma_{12}}\Big]. \tag{45}$$

Note that this expression gives $\Omega^c = 0$ when $\alpha_0 = \bar{\alpha}_0$ (provided $\alpha_1 \neq \alpha_2$ otherwise Ω^c reduces to a non-zero constant).

We have plotted $\Omega^c - \Gamma_{12}$ curves for two cases in Fig. 1, the first with distinct intracomponent coupling strengths and the second with equal intracomponent coupling strengths (where Eq. (45) reduces to Eq. (46)) and compared then to the numerical results of [3] (these parameter sets correspond to sets 'ES1' and 'ES3' respectively from [3]).

For the next computations, we can assume $N_1\alpha_1 = N_2\alpha_2$ so that $R_1 = R_2$ and we have a lattice of peaks and vortices close to the origin. Since $\min_{i,r}\rho_{TF,i}/X_i = 4/R_1^2$, we have

$$\Omega^c = \varepsilon|\log\varepsilon|\sqrt{\frac{\pi\Gamma_1}{\alpha_1\Gamma_{12}}}. \tag{46}$$

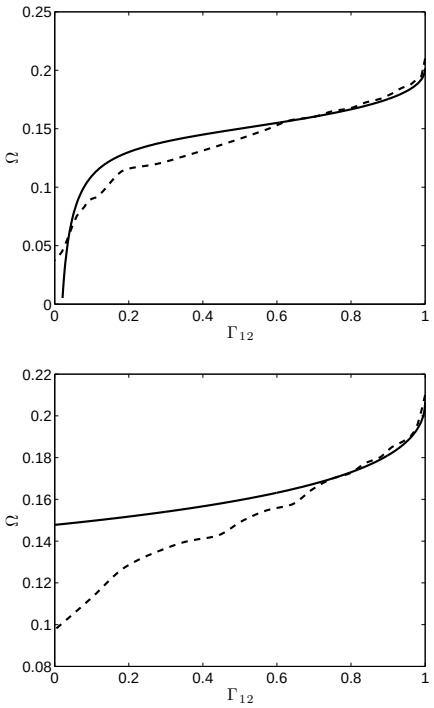

FIG. 1: The critical velocity for creation of the first vortex plotted analytically from Eq. 45 [solid line] and numerically (dotted line) as a function of Γ_{12} for two parameter sets: (a) $\varepsilon = 0.0352$, $\alpha_1 = 0.97$, $\alpha_2 = 1.03$ and (b) $\varepsilon = 0.0358$, $\alpha_1 = \alpha_2 = 1$.

Energy expansion

We assume that the vortices appear close to the origin. Then (35) and (38) can be expanded around the origin, using (31)-(33) and (36)-(37), which yields

$$-\pi\varepsilon^2\sum_{i,j}\rho_1\log|p_i - p_j| - \pi\varepsilon^2\sum_{i,j}\rho_2\log|q_i - q_j| + \pi\varepsilon\left(-\varepsilon|\log\varepsilon|\frac{\Gamma_2}{2\alpha_1\Gamma_{12}} + \Omega\rho_1\right)\sum_i|p_i|^2 + \pi\varepsilon\left(-\varepsilon|\log\varepsilon|\frac{\Gamma_1}{2\alpha_2\Gamma_{12}} + \Omega\rho_2\right)\sum_i|q_i|^2 \tag{47}$$

Interaction energy

We find from (9) that the interaction energy is

$$\alpha_0\rho_1\rho_2\int(1-|v_1|^2)(1-|v_2|^2).$$

Near a vortex-peak, this reduces to

$$\alpha_0\rho_1\rho_2\gamma_1\gamma_2\int\frac{1}{r_{(1,0)}^2}\frac{1}{r_{(0,1)}^2}$$

where we take the notations of [4]: $r_{(1,0)}$ is the local distance to the vortex in component-1, and $r_{(0,1)}$ is the distance to the next peak in component-1, or equivalently

to the vortex in component-2. From (16), we find that the coefficient in front of the integral is equal to $\pi\frac{1-\Gamma_{12}}{16\alpha_1\Gamma_{12}^2}$. The computations in [4] allow to estimate the integral term and we find for the interaction term

$$\pi\frac{1-\Gamma_{12}}{16\Gamma_{12}^2}\frac{1}{\alpha_1}\varepsilon^4|\log\varepsilon|\frac{1}{|p_i-q_j|^2}.$$

This is for a vortex in v_1. Of course, if the vortex is in v_2, it would be different by a factor $1/\alpha_2$.

The interaction energy is thus

$$\pi\frac{1-\Gamma_{12}}{16\Gamma_{12}^2}\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}\right)\varepsilon^4|\log\varepsilon|\sum_{i\neq j}\frac{1}{|p_i-q_j|^2}. \quad (48)$$

Together with (47), this leads to (34).

NUMERICAL SIMULATION OF THE RENORMALISED ENERGY

We want to find the ground state of (34) when the radii of both components are equal. Under this condition we have $\rho_1=\rho_2$, $\Gamma_1=\Gamma_2$ and $\alpha_1=\alpha_2$. This allows us to perform a rescaling that leaves the renormalised energy only dependent on a single parameter. We write $\Omega=\omega\Omega^c$ for Ω^c defined in (46). Then

$$\rho_1\left(-\frac{|\log\varepsilon|}{R_1^2}+\frac{\Omega}{\varepsilon}\right)=\rho_1|\log\varepsilon|\left(\omega\sqrt{\frac{\pi\Gamma_2}{\alpha_1\Gamma_{12}}}-\frac{1}{R_1^2}\right) \quad (49)$$

which implies that we can rescale the p_i's and q_j's so that $p_i=\gamma\tilde{p}_i$ and $q_j=\gamma\tilde{q}_j$ with

$$\gamma^2=\frac{1}{2|\log\varepsilon|\left(\omega\sqrt{\frac{\pi\Gamma_2}{\alpha_1\Gamma_{12}}}-\frac{1}{R_1^2}\right)} \quad (50)$$

so that the new energy is

$$\frac{1}{2}\pi\varepsilon^2\rho_1\Bigg[-\sum_{i,j}\log|p_i-p_j|^2-\sum_{i,j}\log|q_i-q_j|^2 + \sum_i|p_i|^2+\sum_i|q_i|^2+c_\omega\sum_{i\neq j}\frac{1}{|p_i-q_j|^2}\Bigg] \quad (51)$$

with

$$c_\omega=\frac{\pi(1-\Gamma_{12})}{4\alpha_1\Gamma_{12}^2}(2\omega-1)\varepsilon^2|\log\varepsilon|^2. \quad (52)$$

We simulate this renormalised energy (51) using a conjugate gradient method varying the parameter c_ω and the number of vortex points. For a fixed number of lattice points, when increasing c_ω, the ground state lattice goes from triangular to square at a critical c_ω^{ts}. Note that when N gets large, c_ω^{ts} no longer depends on N. In Fig. 2 we plot the ground state for two values of c_ω which give a triangular ($c_\omega=0.05$) and a square lattice ($c_\omega=0.15$) when $N=?$. **to edit**

For each N, we can calculate the critical c_ω^{ts} and compare this with the simulations on the full GP equations, as performed in [3] (they provide the appropriate value of N). From (refcomega), we thus find the critical value of Ω for which the lattice goes from triangular to square:

$$\Omega^{ts}=\left(\frac{1}{2}+\frac{2c_\omega^{ts}\alpha_1\Gamma_{12}^2}{\pi(1-\Gamma_{12})\varepsilon^2|\log\varepsilon|^2}\right)\sqrt{\frac{\pi\Gamma_2}{g_1\Gamma_{12}}}|\log\varepsilon| \quad (53)$$

It turns out that when Ω gets close to 1, the condensate expands and one has to include in the TF profile a term $(1-\Omega^2)r^2$ instead of just r^2. The radii R_1 and R_2 vary like $(1-\Omega^2)^{1/4}$. This changes c_ω from [52] to

$$c_\omega=\frac{\pi(1-\Gamma_{12})}{4\alpha_1\Gamma_{12}^2}\left(\frac{2\omega-\sqrt{1-\Omega^2}}{\sqrt{1-\Omega^2}}\right)\varepsilon^2|\log\varepsilon|^2 \quad (54)$$

so that if we define

$$\beta_{ts}=\left(\frac{1}{2}+\frac{2c_\omega^{ts}g_1\Gamma_{12}^2}{\pi(1-\Gamma_{12})|\log\varepsilon|^2}\right)\sqrt{\frac{\pi\Gamma_2}{g_1\Gamma_{12}}}|\log\varepsilon| \quad (55)$$

then $\Omega^{ts}=\beta_{ts}\sqrt{1-(\Omega^{ts})^2}$, which yields

$$\Omega^{ts}=\frac{\beta_{ts}}{\sqrt{1+\beta_{ts}}}. \quad (56)$$

We plot the form of Ω^{ts} as a function of Γ_{12} in Fig. 3 where we have taken $\varepsilon=0.0358$ and $\alpha_1=\alpha_2=1$ (note that this parameter set corresponds to set 'ES3' in [3]). This provides good agreement and confirms that our point energy well describes the system.

We point out that [13] found a point energy with an interaction term $e^{-|p_i-q_j|^2}$, which is not so good numerically.

Acknowledgments

The authors acknowledge support from the French ministry Grant ANR-BLAN-0238, VoLQuan.

[1] Schweikhard, V et al 2004 *Phys. Rev. Lett.* **93**, 210403.
[2] Kasamatsu, K., Tsubota, M. and Ueda, M. 2005. *Int. J. Modern Phy. B* **19**, 11(1835-1904).
[3] Aftalion A, Mason P, A classification of the ground states and topological defects in a rotating two-component Bose-Einstein condensate, to be published in Phys. Rev. A (2011).
[4] M. Eto, K. Kasamatsu, M. Nitta, H. Takeuchi, M. Tsubota, Interaction of half-quantized vortices

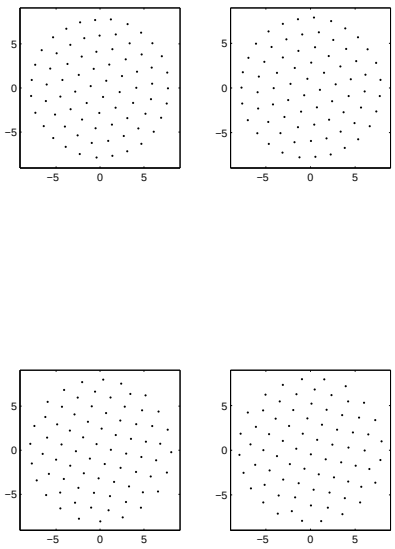

FIG. 2: The ground state of the renormalised energy (51) calculated for (a) $c_\omega = 0.05$ and (b) $c_\omega = 0.15$ with the number of points in each component equal to $N = 75$. Component-1 is shown in the left columns and component-2 in the right columns.

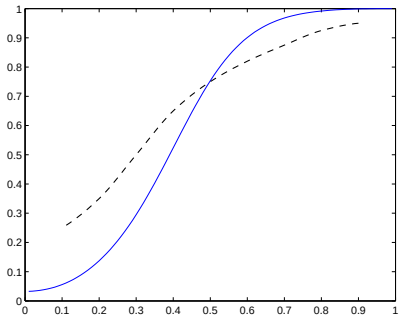

FIG. 3: The critical value of Ω^{ts} as a function of Γ_{12} calculated analytically (solid line) and numerically (dotted line) for the parameters $\varepsilon = 0.0358$, $\alpha_1 = \alpha_2 = 1$.

in two-component Bose-Einstein condensates, 2011, arxiv:1103.6144.
[5] Mueller, E. J. and Ho, T-L. 2002. *Phys. Rev. Lett.* **88**, 180403.
[6] Keçeli, M. and Oktel, Ö. 2006. *Phys. Rev. A* **73**, 023611.
[7] N.R.Cooper, Advances in Physics 57, 539 (2008).
[8] Aftalion A, Vortices in Bose Einstein condensates, Progress in Nonlinear Differential Equations and their Applications, Vol.67 Birkhäuser Boston, Inc., Boston, MA, (2006).
[9] Y.Castin and R.Dum, Eur. Phys. J. D, **7**, 399 (1999).
[10] A.L.Fetter, Rev. Mod. Phys. 81, 647 (2009).
[11] L. Lassoued, P. Mironescu, GinzburgLandau type energy with discontinuous constraint, J. Anal. Math. 77 (1999) 126.
[12] T.C. Lin, J. Wei, Half-Skyrmions and Spike-Vortex Solutions of Two-Component Nonlinear Schrödinger Systems, J. Math. Phys. 48, 053518 (2007).
[13] R.Barnett, G.Rafael, M.A.Porter, H.P.Buchler, New J. Phys. 10, 043030 (2008).