THE TODA SYSTEM AND MULTIPLE-END SOLUTIONS OF AUTONOMOUS PLANAR ELLIPTIC PROBLEMS

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Abstract. We construct a new class of positive solutions for the classical elliptic problem

$$\Delta u - u + u^p = 0, \quad p > 2, \quad \text{in } \mathbb{R}^2.$$

We show that these solutions are of the form $u(x, z) \sim \sum_{j=1}^{k} w(x - f_j(z))$, where w is the unique even, positive, asymptotically vanishing solution of $w'' - w + w^p = 0$ in $\mathbb{R}$. Functions $f_j(z)$, representing the multiple ends of $u(x, z)$, solve the Toda system

$$c^2 f_j'' = e^{f_{j-1} - f_j} - e^{f_j - f_{j+1}} \quad \text{in } \mathbb{R}, \quad j = 1, \ldots, k,$$

are asymptotically linear, and satisfy

$$f_0 \equiv -\infty < f_1 \ll \cdots \ll f_k < f_{k+1} \equiv +\infty.$$

The solutions of the elliptic problem we construct have their counterparts in the theory of constant mean curvature surfaces. An analogy can also be made between their construction and the gluing of constant scalar curvature Fowler singular metrics in the sphere.

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Date: September 22, 2009.

1. Introduction

1.1. **The Dancer solution to the nonlinear Schrödinger equation.** This paper deals with the classical semilinear elliptic problem

$$\Delta u - u + u^p = 0, \quad u > 0, \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $p > 2$. Equation (1.1) arises for instance as the standing-wave problem for the standard nonlinear Schrödinger equation

$$i\psi_t = \Delta_y \psi + |\psi|^{p-1}\psi,$$

typically $p = 3$, corresponding to that of solutions of the form $\psi(y,t) = u(y)e^{-it}$. It also arises in nonlinear models in Turing's theory biological theory of pattern formation [45] such as the Gray-Scott or Gierer-Meinhardt systems, [19, 18]. The solutions of (1.1) which decay to zero at infinity are well understood. Problem (1.1) has a radially symmetric solution $w_N(y)$ which approaches 0 at infinity provided that

$$1 < p < \begin{cases} \frac{N+2}{N-2} & \text{if } N \geq 3, \\ +\infty & \text{if } N = 1, 2, \end{cases}$$

see [44, 4]. This solution is unique [25], and actually any positive solution to (1.1) which vanishes at infinity must be radially symmetric around some point [17].

Problem (1.1) and its variations have been broadly treated in the PDE literature in the last two decades. These variations are mostly of one of the two types: (1.1) is changed to a non-autonomous problem with a potential depending on the space variable; or (1.1) is considered in a bounded domain under suitable boundary conditions. Typically, in both versions a small parameter is introduced rendering (1.1) a singular perturbation problem. We refer the reader to the works [2, 3, 7, 11, 12, 13, 16, 20, 21, 27, 29, 30, 39, 37, 38] and references therein. Many constructions in the literature refer to "multi-bump solutions", built from a perturbation of the superposition of suitably scaled copies of the basic radial bump w_N. The location of their maxima is determined typically by a criterion related either with the potential or the geometry of the underlying domain.

Much less is known about solutions to this equation in entire space which do not vanish at infinity (while they are all known to be bounded, see [42]). For example, the solution w_N of (1.1) in $\mathbb{R}^N$ can be trivially extended to a solution in $\mathbb{R}^{N+1}$ which only depends on N variables. This solution vanishes asymptotically in all but one variable. For simplicity, we restrict our attention to the case $N = 2$, and consider positive solutions $u(x,z)$ to problem (1.1) which vanish as $|x| \to +\infty$, namely

$$\lim_{|x|\to+\infty} u(x,z) = 0 \quad \text{for all } z \in \mathbb{R}. \tag{1.2}$$

A natural example is given by the one-dimensional bump w_1, which we denote in the sequel just by w, namely the unique solution of the ODE

$$w'' - w + w^p = 0, \quad w > 0, \quad \text{in } \mathbb{R}, \tag{1.3}$$

$$w'(0) = 0, \quad w(x) \to 0 \quad \text{as } |x| \to +\infty, \tag{1.4}$$

corresponding in phase plane to a homoclinic orbit for the equilibrium 0. Using this function we can define a family of solutions u of equation (1.1) with the properties (1.2) setting $u(x,z) := w(x-a)$, $a \in \mathbb{R}$. By analogy with the above terminology, we may call these solutions "single bump-lines". A natural question is whether a solution that satisfies (1.2) and which is in addition even in x must equal $w(x)$. The solution w of (1.1) was found to be isolated in a uniform topology which avoids oscillations at infinity by Busca and Felmer in [6]. On the other hand, a second class of solutions which are even both in z and x was discovered by Dancer in [9] via local bifurcation arguments. They form a one-parameter family of solutions which are periodic in the z variable and originate from $w(x)$. Let us briefly review their construction: we consider problem (1.1) with T-periodic conditions in z,

$$u(x, z+T) = u(x,z) \quad \text{for all } (x,z) \in \mathbb{R}^2, \tag{1.5}$$

and regard $T > 0$ as a bifurcation parameter. The linearized operator around the single bump line is

$$L(\phi) = \phi_{zz} + \phi_{xx} + (pw^{p-1} - 1)\phi.$$

It is well known that the eigenvalue problem

$$\phi_{xx} + (pw^{p-1} - 1)\phi = \lambda\phi, \tag{1.6}$$

has a unique positive eigenvalue λ_1, with $Z(x)$ being a positive eigenfunction. We observe that as long as $0 < T < \frac{2\pi}{\sqrt{\lambda_1}} = T_1$ the operator L has a one dimensional kernel of bounded, periodic and even functions spanned by w'. When $T = T_1$ another bounded, periodic and even element in this kernel, given by

$$Z(x)\cos(\sqrt{\lambda_1} z),$$

appears. Crandall-Rabinowitz bifurcation theorem can then be adapted to yield existence of a continuum of solutions bifurcating at $T = T_1$. They are periodic in z with period $T_\varepsilon = \frac{2\pi}{\sqrt{\lambda_1}} + \mathcal{O}(\varepsilon)$, even and bounded. In addition they are uniformly close to $w(x)$ and for $\varepsilon \ll 1$ their asymptotic form is

$$u(x,z;\varepsilon) = w(x) + \varepsilon Z(x)\cos(\sqrt{\lambda_1} z) + \mathcal{O}(\varepsilon^2) e^{-|x|}.$$

Of course once we have found the family of solutions $u(\cdot;\varepsilon)$, for all sufficiently small ε we have trivially that for all φ:

$$u(x, z+\varphi;\varepsilon) = w(x) + \varepsilon Z(x)\cos(\sqrt{\lambda_1}z+\varphi) + \mathcal{O}(\varepsilon^2)e^{-|x|}, \tag{1.7}$$

is also a solution. Introducing now parameters:

$$\delta = \varepsilon\cos\varphi, \quad \tau = -\varepsilon\sin\varphi,$$

we define:

$$w_{\delta,\tau}(x,z) := u(x, z+\varphi;\varepsilon). \tag{1.8}$$

We refer to the functions $w_{\delta,\tau}$ in what follows as the *Dancer solution*. We observe that the Dancer solution is in reality a two parameter family of solutions to (1.1). The interpretation of the original parameters $\varepsilon, \varphi \in \mathbb{R}$ is that they represent, respectively, the amplitude and the phase shift of the oscillations superposed over the homoclinic profile $w(x)$. Finally we observe that for sufficiently small δ, τ we have:

$$w_{\delta,\tau}(x,z) = w(x) + \delta Z(x)\cos(\sqrt{\lambda_1}z) + \tau Z(x)\sin(\sqrt{\lambda_1}z) + \mathcal{O}(|\delta|^2+|\tau|^2)e^{-|x|}. \tag{1.9}$$

1.2. **Multiple bump lines. The statement of the main result.** The purpose of this paper is to construct a new type of solutions of (1.1) in $\mathbb{R}^2$ that have multiple ends in the form of multiple bump-lines, and that satisfy in addition (1.2).

At this point it will be convenient to define a multi bump line solution of (1.1) in a more precise way:

Definition 1.1. *We say that u, a solution of (1.1), is a* multiple bump line with $2k$ ends *if there exist $2k$ oriented half lines $\{\mathbf{a}_j\cdot\mathbf{x}+b_j=0\}$, $j=1,\dots,2k$ (for some choice of $\mathbf{a}_j\in\mathbb{R}^2$, $|\mathbf{a}_j|=1$ and $b_j\in\mathbb{R}$) such that along these half lines and away from a compact set K containing the origin, the solution is asymptotic to $w_{\delta_j,\tau_j}(\mathbf{a}_j\cdot\mathbf{x}+b_j)$ for certain numbers δ_j,τ_j, $j=1,\dots,2k$, that is there exist positive constants C, c such that:*

$$\|u(\mathbf{x}) - \sum_j^{2k} w_{\delta_j,\tau_j}(\mathbf{a}_j\cdot\mathbf{x}+b_j)\|_{L^\infty(\mathbb{R}^2\setminus K)} \le Ce^{-c|\mathbf{x}|}. \tag{1.10}$$

What we actually look for is a solution is a multiple bump line solution of of (1.1) whose asymptotic behavior is determined by k curves

$$\gamma_j = \{(x,z)\mid x = f_j(z)\}, \quad j=1,\dots,k, \quad f_1(z)\ll f_2(z)\ll\cdots\ll f_k(z),$$

which asymptotically resemble straight lines. The functions f_j defining the curves γ_j are not arbitrary and turn out to be related to a second order system of differential equations, the *Toda system*, given by

$$c_p^2 f_j'' = e^{f_{j-1}-f_j} - e^{f_j-f_{j+1}} \quad \text{in } \mathbb{R}, \quad j=1,\dots,k, \tag{1.11}$$

with the conventions $f_0=-\infty$, $f_{k+1}=+\infty$, where c_p is an explicit positive constant that will be specified later (see (4.44)). The Toda system has a special scaling property which we will explain now. We observe that if $\mathbf{f}=(f_1,\dots,f_k)$ is a solution of this system, then function $\mathbf{f}_\alpha$ defined by

$$\mathbf{f}_\alpha = (f_{\alpha,1},\dots,f_{\alpha,k}), \quad f_{\alpha,j}(z) := f_j(\alpha z) + 2(j-\frac{k+1}{2})\log\frac{1}{\alpha}, \tag{1.12}$$

is also a solution. As we will see later the functions f_j are asymptotically linear, namely we have (globally) for α small

$$f_{\alpha,1}(z) \ll f_{\alpha,2}(z) \ll \cdots \ll f_{\alpha,k}(z), \qquad f'_{\alpha,j}(\pm\infty) = a_{\pm,j}\alpha,$$

and

(1.13)

$$f_{\alpha,j}(z) \;=\; a_{\pm,j}\alpha\, z + b_{\pm,j} + 2(j - \frac{k+1}{2}) \log\frac{1}{\alpha} + \mathcal{O}((\cosh z)^{-\vartheta\alpha}), \quad \text{as } z \to \pm\infty,$$

for certain scalars $a_{\pm,j}, b_{\pm,j}$ and $\vartheta > 0$. These are standard facts about the Toda system that can be found for instance in [23, 36]. The Toda system is a classical model describing scattering of k particles distributed on a straight line, which interact only with their closest neighbors with forces given by a potential depending on the exponentials of their mutual distances. Here the z variable is interpreted as time. This and other properties of the Toda system will be discussed in detail in section 2.

Before stating our result it will be convenient to agree that χ^+ (resp. χ^-) is a smooth cutoff function defined on $\mathbb{R}$ which is identically equal to 1 for $z > 1$ (resp. for $z < -1$) and identically equal to 0 for $z < -1$ (resp. for $z > 1$) and additionally $\chi^- + \chi^+ \equiv 1$. With these cutoff functions at hand, we define the 4 dimensional space

$$D := \operatorname{Span}\left\{z \longmapsto \chi^\pm(z),\, z \longmapsto z\,\chi^\pm(z)\right\}, \tag{1.14}$$

and, for all $\mu \in (0,1)$ and all $\theta \in \mathbb{R}$, we define the space $\mathcal{C}^{2,\mu}_\theta(\mathbb{R})$ of $\mathcal{C}^{2,\mu}$ functions h which satisfy

$$\|h\|_{\mathcal{C}^{2,\mu}_\theta(\mathbb{R})} := \|(\cosh z)^\theta\, h\|_{\mathcal{C}^{2,\mu}(\mathbb{R})} < \infty\,.$$

Given the notion of multiple bump lines in Definition 1.1 our main result is:

Theorem 1.1. *Assume that $N = 2$ and $p > 2$. Given $k \geq 2$, for any sufficiently small number $\alpha > 0$, there exists a $4k$ parameter family of multiple bump line solutions of equation (1.1) with $2k$ ends. Their asymptotic profiles are determined by k curves*

$$\gamma_{\alpha,j} = \{x = f_{\alpha,j}(z) + h_{\alpha,j}(\alpha z)\}.$$

Here $\mathbf{f}_\alpha$ is the scaling (1.12) of $\mathbf{f}$, which in turn is a solution to the Toda system (1.11) and in particular formula (1.13) holds, that is functions $f_{\alpha,j}$ are asymptotically linear. Functions $h_{\alpha,j} \in \mathcal{C}^{2,\mu}_\theta(\mathbb{R}) \oplus D$ representing small perturbations satisfy

$$\|h_{\alpha,j}\|_{\mathcal{C}^{2,\mu}_\theta(\mathbb{R})\oplus D} \leq C\,\alpha^\kappa$$

with some constants $\theta, \kappa > 0$.

As we will see the proof of the above theorem starts with building an approximate solution to (1.1). Each bump line of this solution (represented by one curve $\gamma_{\alpha,j}$) consists of three parts: two Dancer ends and a middle "connector" which is a curved piece of the homoclinic inserted between the wiggling Dancer pieces. Each of the $2k$ Dancer ends depends on 2 free parameters. Each curve $\gamma_{\alpha,j}$ depends on 2 initial conditions for the Toda system. Thus in all there are $4k$ Dancer parameters and $2k$ initial conditions for the Toda system. This gives $6k$ parameters of which $2k$ Dancer parameters must be adjusted at the end. As a consequence we obtain $4k$ parameter family of solutions.

There is another, more geometric way, to see this. Let us observe that each end of the bump line can be translated in any of the two directions, can be rotated and in addition depends of the Dancer parameter denoted by ε in (1.7). Taking derivatives of the solution with respect to these parameters leads to 4 elements in the kernel of the nonlinear Schrödinger operator linearized around the Dancer solution. These functions are the analogues of the geometric Jacobi fields discussed in section 1.3. Thus we have $8k$ "geometric" elements of the kernel corresponding to $2k$ ends of the multiple bump line. This seems at first sight to be inconsistent with the previous count which gave us only $6k$ parameters, however we recall that above both ends of any given bump line were associated with a single curve $\gamma_{\alpha,j}$ and consequently the first way of counting "misses" $2k$ parameters. Now, accepting that we have $8k$ geometric parameters, it is known [24] that this should imply the existence of $\frac{1}{2}\times 8k = 4k$ solutions to the original problem. Our result is in agreement with this intuition.

Remark 1.1. *We observe that, by choosing suitable solution to the Toda system in the case $k = 2$, it is possible to show the existence a a multiple bump line such that*

(1.15)
$$\lim_{x\to\pm\infty} u_\alpha(x,z) = 0, \quad \text{for all } z\in\mathbb{R}, \qquad \lim_{z\to\pm\infty} u_\alpha(x,z) = 0, \quad \text{for all } x\in\mathbb{R}.$$

By the well known result of Gidas, Ni and Nirenberg [17] a positive solution of equation (1.1) that satisfies the limit conditions (1.15) uniformly *must be radially symmetric around the origin. Theorem 1.1 shows that uniformity cannot be relaxed in this classical result.*

1.3. **Geometric counterpart of the Dancer solution.** One of the striking features of the existence result in Theorem 1.1, which is a purely PDE result, is that its counterparts can be found in geometric framework. To illustrate this, we will concentrate on what is perhaps the most appealing one: the analogy between the theory of complete constant mean curvature surfaces in Euclidean 3-space and the theory of entire solutions of (1.1). For simplicity we will restrict ourselves to constant mean curvature surfaces in $\mathbb{R}^3$ which have embedded coplanar ends. In the following we will draw parallels between these geometric objects and some solutions of (1.1).

Embedded constant mean curvature surfaces of revolution were found by Delaunay in the mid 19th century [10]. They constitute a smooth one-parameter family of singly periodic surfaces D_t, for $t\in(0,1]$, which interpolate between the cylinder $D_1 = S^1(1)\times\mathbb{R}$ and the singular surface $D_0 := \lim_{t\to 0} D_t$, which is the union of infinitely many spheres of radius $1/2$ centered at each of the points $(0,0,n)$, $n\in\mathbb{Z}$. The Delaunay surface D_t can be parametrized by

$$X_t(x,z) = (\varphi(z)\,\cos x, \varphi(z)\,\sin x, \psi(z)) \in D_t \subset \mathbb{R}^3,$$

for $(x,z)\in\mathbb{R}\times\mathbb{R}/2\pi\mathbb{Z}$. Here the function φ is the smooth solution of

$$(\varphi')^2 + \left(\frac{\varphi^2+t}{2}\right)^2 = \varphi^2,$$

and the function ψ is defined by

$$\psi' = \frac{\varphi^2+t}{2}.$$

As already mentioned, when $t = 1$, the Delaunay surface is nothing but a right circular cylinder $D_1 = S^1(1) \times \mathbb{R}$, with the unit circle as the cross section. This cylinder is clearly invariant under the continuous group of vertical translations, in the same way that the single bump-line solution of (1.1) is invariant under a one parameter group of translations. It is then natural to agree on the correspondence between

$$\boxed{\begin{array}{c}\text{The cylinder}\\ D_1 = S^1 \times \mathbb{R}\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{The single bump-line}\\ (x,z) \longmapsto w(x)\end{array}}.$$

Let us denote by w_2 the unique radially symmetric, decaying solution of (1.1). Inspection of the other end of the Delaunay family, namely when the parameter t tends to 0, suggests the correspondence between

$$\boxed{\begin{array}{c}\text{The sphere}\\ S^1(1/2)\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{The radially symmetric solution}\\ (x,z) \longmapsto w_2(\sqrt{x^2+z^2})\end{array}}.$$

It is tempting to extend this correspondence for the whole range of the Delaunay parameter by associating the "intermediate" Delaunay surfaces with the Dancer solutions. To do this, first of all, we need to find a curve in the function space that would represent these solutions. However, since we do not have any explicit formula for the Dancer solution it is not immediately obvious how this curve should be defined. A natural possibility is to built a one parameter family solution of (1.1) by using the variational structure of the problem as follows: let $S_T = \mathbb{R} \times (0, T)$ and consider a least energy (mountain pass) solution in $H^1(S_T)$ for the energy

$$\frac{1}{2}\int_{S_T} |\nabla u|^2 + \frac{1}{2}\int_{S_T} u^2 - \frac{1}{p+1}\int_{S_T} u_+^{p+1},$$

for $T > 0$. We denote the least energy solution by u_T. Let us summarize what has been proven about it as T varies between $T = 0$ and $T = \infty$ in [5]. In general the curve $T \mapsto u_T$ is analytic except for possibly finitely many T (see also [1] for related results). After translating and reflecting with respect to line $z = T/2$, it can be shown that for all $T > 0$, $u_T > 0$ must be even in x and with respect to the line $z = T/2$, it has a maximum located at $(0, T/2)$ and it is non-increasing in x, z away from it. Moreover when $T < T_1$ the least energy solution is precisely the homoclinic while for $T > T_1$ it must depend on 2 variables in a non-trivial way, and as long as $T - T_1$ is small it is the bifurcating solution described above. For T sufficiently large the least energy solution is unique and as $T \to \infty$ it converges uniformly over compacts to w_2.

To give further credit to this correspondence, let us recall that the Jacobi operator about the cylinder D_1 corresponds to the linearized mean curvature operator when nearby surfaces are considered as normal graphs over D_1. In the above parameterization, the Jacobi operator reads $J_1 = \left(\partial_x^2 + \partial_z^2 + 1\right)$. In this geometric context, it plays the role of the linear operator L which is the linearization of (1.1) about the single bump-line solution w. Hence we have the correspondence

$$\boxed{\begin{array}{c}\text{The Jacobi operator}\\ J_1 = \left(\partial_x^2 + \partial_z^2 + 1\right)\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{The linearized operator}\\ L = \partial_x^2 + \partial_z^2 - 1 + p\, w^{p-1}\end{array}}.$$

Notice that the emergence of the family of Delaunay surfaces due to the loss of stability of a cylinder when its height varies is the analogue to the emergence of the Dancer solutions through a bifurcation from the homoclinic branch at $T = T_1$.

In our construction bounded elements of the kernel of the linearized operator L play a crucial role. As we will see they correspond to the natural invariances of the problem: two translations and the derivative of the solution with respect to the Dancer parameter $\varepsilon = T - T_1$ taken at $\varepsilon = 0$. Viewed this way they turn out to have the same geometric interpretation as the bounded elements in the kernel of the Jacobi operator J_1, which again correspond to translations (3 this time) and the derivative with respect to the Delaunay parameter. Considering, more generally, the elements of the kernel with at most polynomial growth we have in the case of the homoclinic additionally one more function that corresponds to the rotational invariance of the operator and in the case of D_1 two more functions which represent the rotations of the surface about the two coordinate axes that are orthogonal to the axis of the cylinder. Counting gives the 4 dimensional kernel of geometric eigenfunctions for the homoclinic and the 6 dimensional kernel in the case of D_1, but the difference comes from the number degrees of freedom in $\mathbb{R}^2$ versus $\mathbb{R}^3$. This geometric eigenfunctions are commonly called the geometric Jacob fields.

With these analogies in mind, we can now translate our main result into the constant mean curvature surface framework. The result of Theorem 1.1 corresponds to the connected sum of finitely many copies of the cylinder $S^1(1) \times \mathbb{R}$ which have a common plane of symmetry. The connected sum construction is performed by inserting small catenoidal necks between two consecutive cylinders and this can be done in such a way that the ends of the resulting surface are coplanar. Such a result, in the context of constant mean curvature surfaces, follows at once from [33]. It is observed that, once the connected sum is performed the ends of the cylinder have to be slightly bent and moreover, the ends cannot be kept asymptotic to the ends of right cylinders but have to be asymptotic to Delaunay ends, in agreement with the result of Theorem 1.1. In fact in [33] a $6k$ (not $4k$ as in the present paper) parameter family of constant mean curvature surfaces whose ends are asymptoticly Delaunay is constructed. The difference is due to the fact that the number of the geometric Jacobi fields for the Delaunay end is 6, while for the Dancer solution the analogous number is 4, as we have pointed out.

There is yet another difference between the two cases which indeed is much more substantial. The Toda system which governs the location of the multiple bump lines does not have a counterpart in the connected sum construction of the constant mean curvature surfaces. This difference is due to the strong interactions in the elliptic equations.

Another (older) construction of complete noncompact constant mean curvature surfaces was performed by N. Kapouleas [22] (see also [32]) starting with finitely many halves of Delaunay surfaces with parameter t close to 0 which are connected to a central sphere. The corresponding solutions of (1.1) have recently been constructed by A. Malchiodi in [28].

It is well known that the story of complete constant mean curvature surfaces in $\mathbb{R}^3$ parallels that of complete locally conformally flat metrics with constant, positive scalar curvature. Therefore, it is not surprising that there should be a correspondence between these objects in conformal geometry and solutions of (1.1). For example, Delaunay surfaces and Dancer solutions should now be replaced by Fowler solutions which correspond to constant scalar curvature metrics on the cylinder $\mathbb{R} \times S^{n-1}$ which are conformal to the product metric $dz^2 + g_{S^{n-1}}$, when $n \geq 3$.

These are given by

$$v^{\frac{4}{n-2}}(dz^2 + g_{S^{n-1}}),$$

where $z \longmapsto v(z)$ is a smooth positive solution of

$$(v')^2 - v^2 + \tfrac{n-2}{n}\, v^{\frac{2n}{n-2}} = -\tfrac{2}{n}\,\tau^2.$$

When $\tau = 1$ and $v \equiv 1$ the solution is a straight cylinder while as τ tends to 0 the metrics converge on compacts to the round metric on the unit sphere. The connected sum construction for such Fowler type metrics was carried out by R. Mazzeo, D. Pollack and K. Uhlenberk [35] (where it is called the dipole construction). N. Kapouleas' construction mentioned above is due to R. Schoen [43] (see also R. Mazzeo and F. Pacard [32]).

2. The Toda system and its linearization

2.1. The Toda system. In the sequel we will consider vector valued smooth functions $\mathbf{g}\colon \mathbb{R} \mapsto \mathbb{R}^k$. To measure the size of such functions we will use weighted Hölder spaces $\mathcal{C}^{\ell,\mu}_\theta(\mathbb{R};\mathbb{R}^k)$ with the norm:

$$\|\mathbf{g}\|_{\mathcal{C}^{\ell,\mu}_\theta(\mathbb{R};\mathbb{R}^k)} = \|\mathbf{g}(\cdot)(\cosh z)^\theta\|_{\mathcal{C}^{\ell,\mu}(\mathbb{R};\mathbb{R}^k)}.$$

In this paper the Toda system (1.11) plays a crucial role and thus we will begin with outlining the basic theory of this system and its linearization, see [23, 36] for details. It is convenient to consider our problem in a slightly more general framework then that of the system (1.11). Thus for given functions $q_j(z), p_j(z)$, $j = 1,\dots,k$ we define the Hamiltonian

$$H = \sum_{j=1}^{k} \frac{p_j^2}{2} + V, \quad V = \sum_{j=1}^{k-1} e^{(q_j - q_{j+1})}.$$

We consider the following Toda system

$$\begin{aligned} &\frac{dq_j}{dz} = p_j,\\ &\frac{dp_j}{dz} = -\frac{\partial H}{\partial q_j},\\ &q_j(0) = q_{0j}, \quad p_j(0) = p_{0j}, \qquad j = 1,\dots,k. \end{aligned} \tag{2.1}$$

Observe that that the center of mass moves with constant velocity and the momentum remains constant since if

$$\sum_{j=1}^{k} q_{0j} = \bar q, \quad \sum_{j=1}^{k} p_{0j} = \bar p, \tag{2.2}$$

then from $\sum_{j=1}^k q_j''(z) = 0$ it follows:

$$\sum_{j=1}^{k} q_{0j}(z) = \bar p z + \bar q.$$

We will now give a more precise description of these solutions and in particular their asymptotic behavior as $z \to \pm\infty$. To this end we will often make use of classical results of Kostant [23] and in particular we will use the explicit formula for the solutions of (2.1) (see formula (7.7.10) in [23]).

We will first introduce some notation. Given numbers $w_1, \dots, w_k \in \mathbb{R}$ such that

$$\sum_{j=1}^{k} w_j = 0, \quad \text{and } w_j > w_{j+1}, \quad j = 1, \dots, k, \tag{2.3}$$

we define the matrix

$$\mathbf{w}_0 = \operatorname{diag}(w_1, \dots, w_k).$$

Next, given numbers $g_1, \dots, g_k \in \mathbb{R}$ such that

$$\prod_{j=1}^{k} g_j = 1, \quad \text{and } g_j > 0, \quad j = 1, \dots, k, \tag{2.4}$$

we define the matrix

$$\mathbf{g}_0 = \operatorname{diag}(g_1, \dots, g_k).$$

The matrices $\mathbf{w}_0$ and $\mathbf{g}_0$ can be parameterized by introducing the following two sets of parameters

$$c_j = w_j - w_{j+1}, \quad d_j = \log g_{j+1} - \log g_j, \quad j = 1, \dots, k. \tag{2.5}$$

Furthermore, we define functions $\Phi_j(\mathbf{g}_0, \mathbf{w}_0; z)$, $z \in \mathbb{R}$, $j = 0, \dots, k$, by

$$\begin{aligned} &\Phi_0 = \Phi_k \equiv 1 \\ &\Phi_j(\mathbf{g}_0, \mathbf{w}_0; z) = \\ &(-1)^{j(k-j)} \sum_{1 \le i_i < \dots < i_j \le k} r_{i_1 \dots i_j}(\mathbf{w}_0) g_{i_1} \dots g_{i_j} \exp[-z(w_{i_1} + \dots + w_{i_j})], \end{aligned} \tag{2.6}$$

where $r_{i_1 \dots i_j}(\mathbf{w}_0)$ are rational functions of the entries of the matrix $\mathbf{w}_0$. It is proven in [23] that all solutions of (2.1) are of the form

$$q_j(z) = \log \Phi_{j-1}(\mathbf{g}_0, \mathbf{w}_0; z) - \log \Phi_j(\mathbf{g}_0, \mathbf{w}_0; z), \quad j = 1, \dots, k. \tag{2.7}$$

Namely, given initial conditions in (2.1) there exist matrices $\mathbf{w}_0$ and $\mathbf{g}_0$ satisfying (2.3)–(2.4) and the solution is given in the form (2.7). According to Theorem 7.7.2 of [23], it holds

$${q_j}'(+\infty) = w_{k+1-j}, \quad {q_j}'(-\infty) = w_j, \quad j = 1, ..., k. \tag{2.8}$$

We introduce variables

$$u_j = q_j - q_{j+1}. \tag{2.9}$$

In terms of $\mathbf{u} = (u_1, \dots, u_{k-1})$ system (2.1) becomes

$$\begin{aligned} &\mathbf{u}'' - M e^{\mathbf{u}} = 0, \\ &u_j(0) = q_{0j} - q_{0j+1}, \quad u_j'(0) = p_{0j} - p_{0j+1}, \quad j = 1, \dots, k-1, \end{aligned} \tag{2.10}$$

where

$$M = \begin{pmatrix} 2 & -1 & 0 \cdots & 0 \\ -1 & 2 & -1 \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & 2 & -1 \\ 0 & \cdots & -1 & 2 \end{pmatrix}, \quad e^{-\mathbf{u}} = \begin{pmatrix} e^{u_1} \\ \vdots \\ e^{u_{k-1}} \end{pmatrix}.$$

As a consequence of (2.6) all solutions to (2.10) are given by

$$\begin{aligned} u_j(z) &= q_j(z) - q_{j+1}(z) \\ &= -2 \log \Phi_j(\mathbf{g}_0, \mathbf{w}_0; z) + \log \Phi_{j-1}(\mathbf{g}_0, \mathbf{w}_0; z) + \log \Phi_{j+1}(\mathbf{g}_0, \mathbf{w}_0; z). \end{aligned} \tag{2.11}$$

Conversely, given a solution $\mathbf{u}$ of (2.10) and $\bar{p}, \bar{q} \in \mathbb{R}$, the functions

$$q_j = \frac{1}{k}\left(\sum_{i=0}^{j-1} iu_i - \sum_{i=0}^{k-j} iu_{k-i}\right) + \bar{p}z + \bar{q}, \tag{2.12}$$

for $j = 1, \ldots, k$ (we agree that $u_0 = u_k \equiv 0$), are solutions of (2.1) satisfying (2.2).

We will need the following result which has been proven in [14]:

Lemma 2.1. *Let $\mathbf{w}_0$ be such that*

$$\min_{j=1,\ldots,k-1}(w_j - w_{j+1}) = \vartheta > 0. \tag{2.13}$$

Then there holds

$$u_j(z) = \begin{cases} -c_{k-j}z - d_{k-j} + \tau_j^+(\mathbf{c}) + \mathcal{O}(e^{-\vartheta|z|}), \text{ as } z \to +\infty, & j = 1, \ldots, k-1, \\ c_j z + d_j + \tau_j^-(\mathbf{c}) + \mathcal{O}(e^{-\vartheta|z|}), \text{ as } z \to -\infty, & j = 1, \ldots, k-1, \end{cases} \tag{2.14}$$

where $\tau_j^{\pm}(\mathbf{c})$ are smooth functions of the vector $\mathbf{c} = (c_1, \ldots, c_{k-1})$.

To find a family of solutions of the Toda system (1.11) starting from a solution of (2.1) we calculate functions q_j using (2.12) and set

$$f_j(z) = q_j(z) + (j - \frac{k+1}{2})\log\frac{1}{c_p}. \tag{2.15}$$

Observe that as a consequence of Lemma 2.1 we get that there exist w_j, g_j, $j = 1, \ldots, k$ such that (2.3) and (2.4) holds, that

$$\min_{j=1,\ldots,k}(w_j - w_{j+1}) = \vartheta > 0,$$

and functions f_j satisfy

$$\begin{aligned} \|f_j''\|_{\mathcal{C}^{0,\mu}_\vartheta(\mathbb{R};\mathbb{R}^k)} &:= \|f_j''(\cosh z)^\vartheta\|_{\mathcal{C}^{0,\mu}(\mathbb{R};\mathbb{R}^k)} \leq C, \\ f_j(z) &= a_{\pm,j}z + b_{\pm,j} + \mathcal{O}\big((\cosh z)^{-\vartheta}\big), \quad z \to \pm\infty. \end{aligned} \tag{2.16}$$

We also have, taking ϑ smaller if necessary:

$$\min_j |a_{\pm,j} - a_{\pm,j-1}| \geq \vartheta. \tag{2.17}$$

2.2. **The linearized Toda system.** Given a solution to the Toda system (1.11) we will consider its linearization:

$$c_p\mathbf{h}'' + \mathbf{N}\mathbf{h} = \mathbf{p}, \quad \mathbf{h} = (h_1, \ldots, h_k), \quad \mathbf{N} = (\mathbf{N}_1, \ldots, \mathbf{N}_k)^T, \tag{2.18}$$

where

$$\mathbf{N}_j = -e^{f_{j-1}-f_j}\mathsf{e}_{j-1} + [e^{f_{j-1}-f_j} + e^{f_j - f_{j+1}}]\mathsf{e}_j - e^{f_{j-1}-f_j}\mathsf{e}_{j+1}, \tag{2.19}$$

and e_j are the vectors of the canonical basis in $\mathbb{R}^k$. Thanks to the results of Lemma 2.1 and in particular estimates (2.16), (2.17) the rows of the matrix $\mathbf{N}$ decay exponentially as $|z| \to \infty$. Also we observe that the fundamental set of the system (2.18) is given by the following $2k$ functions:

$$\begin{aligned} \mathbf{v}_j^\sharp &= \partial_{c_j}\mathbf{f}, \quad j = 1, \ldots, k-1, \quad \mathbf{v}_k^\sharp = \partial_{\bar{p}}\mathbf{f}, \\ \mathbf{v}_j^\flat &= \partial_{d_j}\mathbf{f}, \quad j = 1, \ldots, k-1, \quad \mathbf{v}_k^\flat = \partial_{\bar{q}}\mathbf{f}, \end{aligned}$$

where c_j, d_j are the parameters given in the statement of Lemma 2.1 and $\bar p, \bar q$ are the parameters in (2.2). The kernel of the system (2.18) is given by

$$\mathcal{K} = \text{span}\,\{\mathbf{v}_j^\sharp, \mathbf{v}_j^\flat\}.$$

Notice that functions $\mathbf{v}_j^\sharp$ are linearly growing, while $\mathbf{v}_j^\flat$ are bounded as $|z| \to \infty$. In fact from Lemma 2.1 it follows:

$$\begin{aligned} \mathbf{v}_j^\sharp(z) &= \mathbf{a}^\sharp_{\pm,j} z + \mathbf{b}^\sharp_{\pm,j} + \mathcal{O}\big((\cosh z)^{-\vartheta}\big), \\ \mathbf{v}_j^\flat(z) &= \mathbf{b}^\flat_{\pm,j} + \mathcal{O}\big((\cosh z)^{-\vartheta}\big). \end{aligned} \tag{2.20}$$

Let χ^+, χ^- be smooth cut off function such that $\chi^+(z) = 1$, $z > 1$, $\chi^+(z) = 0$, $z < 0$, $\chi^-(z) = \chi^+(-z)$ and finally $\chi^+ + \chi^- \equiv 1$. We will define a $4k$ dimensional *deficiency space* by

$$\mathcal{D} = \text{span}\,\{\chi^\pm \mathbf{v}_j^\sharp, \chi^\pm \mathbf{v}_j^\flat\}.$$

Let us observe that the kernel $\mathcal{K}$ of the linearized Toda system is a $2k$ subspace of $\mathcal{D}$. Therefore, we can certainly decompose

$$\mathcal{D} = \mathcal{K} \oplus \mathcal{E}\,, \tag{2.21}$$

where $\mathcal{E}$ is a complement of $\mathcal{K}$ in $\mathcal{D}$. With this decomposition at hand, we have the following result which follows from standard arguments in ordinary differential equations.

Lemma 2.2. *Assume that $\theta > 0$. Then the mapping*

$$\begin{array}{rccc} T: & \mathcal{C}^{2,\mu}_\theta(\mathbb{R};\mathbb{R}^k) \oplus \mathcal{E} & \longrightarrow & \mathcal{C}^{0,\mu}_\theta(\mathbb{R};\mathbb{R}^k), \\ & \mathbf{v} & \longmapsto & c_p\,\mathbf{v}'' + \mathbf{N}\,\mathbf{v}, \end{array}$$

is an isomorphism.

Proof. For convenience of the reader we reproduce here the proof, which can be found in [15]. Standard arguments in ordinary differential equations imply that there exists a unique solution of (2.18) which satisfies $\mathbf{v}(0) = \mathbf{v}'(0) = 0$. We will denote $\mathbf{v} = S_0(\mathbf{p})$.

We now prove that $\mathbf{v} \in \mathcal{C}^{2,\mu}_\theta(\mathbb{R};\mathbb{R}^k) \oplus \mathcal{D}$. To this end, we observe that one can also find a (unique) solution $\bar{\mathbf{v}}$ of (2.18) which satisfies

$$|\bar{\mathbf{v}}(z)| \le C\, e^{\theta z}\, \|\mathbf{p}\|_{\mathcal{C}^{0,\mu}_\theta(\mathbb{R};\mathbb{R}^k)},$$

in $(-\infty, 0]$. Indeed, using the variation of parameters formula it is easy to show the existence of a unique solution decaying to 0 at $-\infty$ at some exponential rate. Integrating the equation twice over $(-\infty, z]$ shows that in fact $\bar{\mathbf{v}} \in \mathcal{C}^{2,\mu}_\theta((-\infty,0];\mathbb{R}^k)$. Then $\mathbf{v} - \bar{\mathbf{v}}$ is a linear combination of the functions $\mathbf{v}_j^\sharp$ and $\mathbf{v}_j^\flat$. This proves that, in $(-\infty, 0]$, the vector valued function $\mathbf{v}$ can be decomposed into the sum of a linear combination of elements in $\mathcal{D}$ and a vector valued function which is bounded by a constant times $e^{\theta z}$. A similar decomposition can be derived on $[0, +\infty)$. Once this decomposition is proven, the estimates for the Hölder norm of $\mathbf{v}$ follow at once.

In other words, $S_0 : \mathcal{C}^{0,\mu}_\theta(\mathbb{R};\mathbb{R}^k) \longrightarrow \mathcal{C}^{2,\mu}_\theta(\mathbb{R};\mathbb{R}^k) \oplus \mathcal{D}$ is a right inverse for T. The decomposition $\mathcal{D} = \mathcal{K} \oplus \mathcal{E}$ induces the decomposition $S_0(\mathbf{p}) = \bar S_0(\mathbf{p}) + e(\mathbf{p}) + k(\mathbf{p})$, where $\bar S_0(\mathbf{p}) \in \mathcal{C}^{2,\mu}_\theta(\mathbb{R};\mathbb{R}^k)$, $e(\mathbf{p}) \in \mathcal{E}$ and $k(\mathbf{p}) \in \mathcal{K}$. The operator $S := S_0 - k$ is also a right inverse of T and maps onto $\mathcal{C}^{2,\mu}_\theta(\mathbb{R};\mathbb{R}^k) \oplus \mathcal{E}$ as desired. This completes the proof of the Lemma. □

2.3. **Another important ODE.** We will finish this section with a discussion of a simple problem which, however not directly related to the Toda system considered above, plays as important role in the sequel. The problem we have in mind is the following equation:

$$e'' + \kappa^2 e = g, \quad \|g(\cos z)^\theta\|_{\mathcal{C}^{0,\mu}(\mathbb{R})} < \infty. \tag{2.22}$$

We are interested in solutions to this problem which decay exponentially at both $\pm\infty$. It is clear that if we define

$$e(z) = -\frac{1}{\kappa}\cos(\kappa z)\int_{-\infty}^{z} g(\zeta)\sin(\kappa\zeta)\,d\zeta + \frac{1}{\kappa}\sin(\kappa z)\int_{-\infty}^{z} g(\zeta)\cos(\kappa\zeta)\,d\zeta, \tag{2.23}$$

then this function is the unique solution that decays exponentially at $-\infty$. If we assume that in addition

$$\int_{-\infty}^{\infty} g(\zeta)\sin(\kappa\zeta)\,d\zeta = 0, \quad \int_{-\infty}^{\infty} g(\zeta)\cos(\kappa\zeta)\,d\zeta, \tag{2.24}$$

then we have

$$\|e(\cosh z)^\theta\|_{\mathcal{C}^{2,\mu}(\mathbb{R})} < \infty, \tag{2.25}$$

as required. The necessity of imposing the extra condition (2.24) has its direct consequence for our problem of constructing the multiple bump lines for (1.1). As we will see it is precisely because of (2.24) that we can fix arbitrarily the amplitudes and phase shifts of only $2k$ ends (say all lower ends if we chose so) of the bump lines and we need to adjust suitably the amplitudes and the phase shifts of the remaining $2k$ ends (say upper ends) and thus we have only $2k$ (and not $4k$ as one might expect) free parameters corresponding to the amplitudes and the phase shifts.

3. The approximate solution

3.1. **Local coordinates near model bump lines.** We will fix from now on a solution to the Toda system $\mathbf{f}$ with the properties described in the previous section. We will also choose $\mathbf{v} \in \mathcal{E}$. We will assume that

$$\|\mathbf{v}\|_{\mathcal{E}} \le \alpha^{\kappa_1}, \tag{3.1}$$

where $\kappa_1 > 0$ is a small number to be chosen later on. With these two functions at hand we define for each $j = 1, \ldots, k$ the model for a bump line to be the curve:

$$\bar\gamma_{\alpha,j} = \{\mathbf{x} = (x, z) \in \mathbb{R}^2 \mid x = f_{\alpha,j}(z) + v_j(\alpha z)\},$$

where $\mathbf{f}_\alpha = (f_{\alpha,1}, \ldots, f_{\alpha,k})$ is the rescaled solution to the Toda system, see (1.12).

We will introduce local coordinates associated with each $\bar\gamma_{\alpha,j}$. For convenience we will denote $\bar{\mathbf{f}}_\alpha(z) = \mathbf{f}_\alpha(z) + \mathbf{v}(\alpha z)$. We will fix the orientation of $\bar\gamma_{\alpha,j}$ in such a way that the pair of vectors $(T_{\alpha,j}, N_{\alpha,j})$, where the unit tangent $T_{\alpha,j} = \frac{1}{\sqrt{1+\alpha(\bar f'_{\alpha,j})^2}}(\alpha\bar f'_{\alpha,j}, 1)$ and the unit normal $N_{\alpha,j} = \frac{1}{\sqrt{1+(\alpha\bar f'_{\alpha,j})^2}}(1, -\alpha\bar f'_{\alpha,j})$ are negatively oriented (and the functions $\bar f'_{\alpha,j}$ are evaluated at αz). Let $\mathbf{z}_j$ be the arc length parameter on $\bar\gamma_{\alpha,j}$ i.e.

$$\mathbf{z}_j = \int_0^z \sqrt{1 + \alpha^2(\bar f'_{\alpha,j})^2(\alpha\zeta)}\,d\zeta, \tag{3.2}$$

and let $q_{\alpha,j} = q_{\alpha,j}(\mathbf{z}_j)$ be the corresponding arc length parametrization.

As it turns out the true asymptotic behavior of the bump line is not exactly linear but it has an extra exponentially small correction. This correction is an unknown to be determined, and in fact this is one of the most important steps in this paper which involves the linearized Toda system discussed in the previous section. To describe this perturbation we let $\mathbf{h} = (h_1, \dots, h_k)$ to be a fixed function such that

$$\|\mathbf{h}\|_{\mathcal{C}^{2,\mu}_\theta(\mathbb{R},\mathbb{R}^k)} \le \alpha^{\kappa_2}, \tag{3.3}$$

with some small parameter κ_2. In the sequel we will use the function $\mathbf{h}$ of the stretched argument αz, namely we will write $\mathbf{h}(\alpha z)$. To measure the size of this function it is more suitable to use the weights of the form $(\cosh z)^{\theta\alpha}$ rather than $(\cosh z)^\theta$. Thus we will see norms like $\|\cdot\|_{\mathcal{C}^{\ell,\mu}_{\theta\alpha}(\mathbb{R};\mathbb{R}^k)}$. In general we have the following relations:

$$\|\mathbf{h}\|_{\mathcal{C}^{\ell,\mu}_{\theta\alpha}(\mathbb{R};\mathbb{R}^{\mathbf{k}})} \le \|\mathbf{h}\|_{\mathcal{C}^{\ell,\mu}_{\theta}(\mathbb{R};\mathbb{R}^{\mathbf{k}})}, \quad \|\mathbf{h}\|_{\mathcal{C}^{\ell,\mu}_{\theta}(\mathbb{R};\mathbb{R}^{\mathbf{k}})} \le \alpha^{-\ell-\mu}\|\mathbf{h}\|_{\mathcal{C}^{\ell,\mu}_{\theta\alpha}(\mathbb{R};\mathbb{R}^{\mathbf{k}})}. \tag{3.4}$$

These relations will be used for the function $\mathbf{h}$ as well as for several similar type functions appearing below without specially mentioning them. Thus for instance from (3.3) and (3.4) it follows:

$$\|\mathbf{h}\|_{\mathcal{C}^{2,\mu}_{\theta\alpha}(\mathbb{R};\mathbb{R}^k)} \le \varepsilon\alpha^{\kappa_2}.$$

A neighborhood of the curve $\bar\gamma_{\alpha,j}$ can be parametrized in the following way:

$$\mathbf{x} = X_{\alpha,j}(\mathtt{x}_j, \mathtt{z}_j) = q_{\alpha,j}(\mathtt{z}_j) + \big(\mathtt{x}_j + h_j(\alpha\mathtt{z}_j)\big)N_{\alpha,j}(\mathtt{z}_j). \tag{3.5}$$

Notice that $t_j = \mathtt{x}_j + h_j(\alpha\mathtt{z}_j)$ is simply the signed distance to $\bar\gamma_{\alpha,j}$. For this reason our local coordinates can be seen as shifted with respect to the Fermi coordinates of the curve $\bar\gamma_{\alpha,j}$.

The distance function is not a smooth function in the whole $\mathbb{R}^2$ however we observe that given $\mathbf{f}_\alpha, \mathbf{v}$ there exists a maximal subset of $\mathbb{R}^2$ in which t_j is a smooth function for $j = 1, \dots, k$. Using the asymptotic (linear) behavior of $\mathbf{f}_\alpha(z), \mathbf{v}(\alpha z)$ and estimate (3.1) it is not hard to prove that this set contains the set:

$$V_\varsigma = \{\mathbf{x} = (x, z) \mid |x| \le \frac{\varsigma}{\alpha}\sqrt{1+z^2}\},$$

with certain small constant ς. Indeed, the Fermi coordinates are defined as long as the map $(t_j, \mathtt{z}_j) \longmapsto \mathbf{x}$ is one-to-one. Using the fact that the curvature of each $\bar\gamma_{\alpha,j}$,

$$k_{\alpha,j}(\mathtt{z}_j) \sim \alpha^2(\cosh\mathtt{z}_j)^{-\vartheta\alpha},$$

and also the asymptotic behavior as $|\mathtt{z}_j| \to \infty$:

$$\bar\gamma_{\alpha,j}(\mathtt{z}_j) \sim (\mathcal{O}(\alpha)|\mathtt{z}_j| + \mathcal{O}(\log\frac{1}{\alpha}), \mathtt{z}_j(1+\mathcal{O}(\alpha^2))),$$

one can show that for each small ς_j and each sufficiently small α the Fermi coordinates are well defined around $\bar\gamma_{\alpha,j}(\mathtt{z}_j)$ as long as:

$$|t_j| \le \frac{\varsigma_j}{\alpha}\sqrt{1+\mathtt{z}_j^2}. \tag{3.6}$$

Noting that the distance between $\bar\gamma_{\alpha,j}$ and any other curve, say $\bar\gamma_{\alpha,i}$ behaves like

$$\operatorname{dist}\big(\bar\gamma_{\alpha,j}(\mathtt{z}_j), \bar\gamma_{\alpha,i}\big) \sim \mathcal{O}(\alpha)|\mathtt{z}_j| + \mathcal{O}(\log\frac{1}{\alpha}),$$

we conclude that the constant ς in the definition of the set V_ς can be taken as small as we wish and also, using (3.3), that it can be chosen in such a way that:

$$\mathbf{x} \in V_\varsigma \Longrightarrow |\mathtt{x}_j| = |t_j - h_j(\alpha\mathtt{z}_j)| \le \frac{\varsigma_j}{\alpha}\sqrt{1+\mathtt{z}_j^2}, \quad \mathbf{x} = X_{\alpha,j}(\mathtt{x}_j, \mathtt{z}_j). \tag{3.7}$$

To accomplish this it suffices to take V_ς to be the intersection of all the sets where (3.6) is satisfied.

In the sequel we will use convenient notation: for a given function $f\colon V_\varsigma \to \mathbb{R}$ we set:

$$X^*_{\alpha,j} f(\mathtt{x}_j, \mathtt{z}_j) = (f \circ X_{\alpha,j})(\mathtt{x}_j, \mathtt{z}_j). \tag{3.8}$$

We will also need a simple relation between the coordinates $\mathtt{x}_j$ and $\mathtt{x}_i$, which follows from the definition of the curves $\bar\gamma_{\alpha,j}$ together with elementary geometry. By definition of the coordinates (3.5) we get:

$$\begin{aligned} \mathtt{x}_i &= \big[q_{\alpha,j}(\mathtt{z}_j) - q_{\alpha,i}(\mathtt{z}_i)\big]\big(1 + \mathcal{O}(\alpha^2)\big) + \mathcal{O}(\alpha^{\kappa_2}) + \mathtt{x}_j\big(1 + \mathcal{O}(\alpha^2)\big), \\ \mathtt{z}_i &= \mathtt{z}_j\big(1 + \mathcal{O}(\alpha^2)\big) + \mathcal{O}(\alpha)(\mathtt{x}_j - \mathtt{x}_i) + \mathcal{O}(\alpha^{1+\kappa_2}). \end{aligned} \tag{3.9}$$

Since

$$q_{\alpha,i}(\mathtt{z}_j) - q_{\alpha,i}(\mathtt{z}_i) = \mathcal{O}(\alpha)(\mathtt{z}_j - \mathtt{z}_i),$$

in V_ς we have

$$\mathtt{x}_j - \mathtt{x}_i = 2(i-j)\log\frac{1}{\alpha} + \mathcal{O}(\alpha^2)\mathtt{x}_j + \mathcal{O}(\alpha)\mathtt{z}_j + \mathcal{O}(\alpha^{\kappa_2}), \tag{3.10}$$

$$\mathtt{z}_i - \mathtt{z}_j = \mathcal{O}(\alpha^2)\mathtt{z}_j + \mathcal{O}(\alpha\log\frac{1}{\alpha}) + \mathcal{O}(\alpha^3)\mathtt{x}_j. \tag{3.11}$$

as α tends to 0.

3.2. **Laplacian in the local coordinates.** It will be useful to have the expression of the Laplacian in the coordinates defined in (3.5). Let $k_{\alpha,j}$ be the curvature of the curve $\bar\gamma_{\alpha,j}$, which in its natural parametrization is given by:

$$k_{\alpha,j} = \frac{\alpha^2 \bar f''_{\alpha,j}(\alpha z)}{\big(1 + \alpha^2(\bar f'_{\alpha,j}(\alpha z))^2\big)^{\frac{3}{2}}}, \quad \mathtt{z}_j = \int_0^z \sqrt{1 + \alpha^2(\bar f'_{\alpha,j}(\alpha\zeta))^2}\, d\zeta. \tag{3.12}$$

We define the function A_j by

$$A_j := 1 - (\mathtt{x}_j + h_j)k_{\alpha,j}.$$

With this notation the following expression for the Laplacian is easy to derive:

$$\Delta = \frac{1}{A_j}\Big\{\partial_{\mathtt{x}_j}\Big(\frac{A_j^2 + \alpha^2(h_j')^2}{A_j}\partial_{\mathtt{x}_j}\Big) - \partial_{\mathtt{z}_j}\Big(\frac{\alpha h_j'}{A_j}\partial_{\mathtt{x}_j}\Big) - \partial_{\mathtt{x}_j}\Big(\frac{\alpha h_j'}{A_j}\partial_{\mathtt{z}_j}\Big) + \partial_{\mathtt{z}_j}\Big(\frac{1}{A_j}\partial_{\mathtt{z}_j}\Big)\Big\}.$$

This formula can be written in the form:

$$\Delta = \partial^2_{\mathtt{x}_j} + \partial^2_{\mathtt{z}_j} + a_{11,j}\partial^2_{\mathtt{x}_j} + a_{12,j}\partial_{\mathtt{x}_j\mathtt{z}_j} + a_{22,j}\partial^2_{\mathtt{z}_j} + b_{1,j}\partial_{\mathtt{x}_j} + b_{2,j}\partial^2_{\mathtt{z}_j}, \tag{3.13}$$

where:

$$\begin{aligned} a_{11,j} &= \frac{\alpha^2(h_j')^2}{A_j^2}, \quad a_{12,j} = -\frac{2\alpha h_j'}{A_j^2}, \quad a_{22,j} = \frac{1 - A_j^2}{A_j^2}, \\ b_{1,j} &= \frac{1}{A_j^3}(-k_{\alpha,j}A_j^2 - \alpha^2 h_j'' A_j + \alpha^2(h_j')^2 k_{\alpha,j} - \alpha(\mathtt{x}_j + h_j)h_j' k'_{\alpha,j}), \\ b_{2,j} &= \frac{1}{A_j^3}((h_j + \mathtt{x}_j)k'_{\alpha,j}). \end{aligned} \tag{3.14}$$

The reader should keep in mind that functions h_j, $k_{\alpha,j}$ are taken as functions of $\alpha \mathtt{z}_j$. Additionally we recall that

$$k_{\alpha,j} = \mathcal{O}_{\mathcal{C}^{2,\mu}_{\theta\alpha}(\mathbb{R})}(\alpha^2), \quad k'_{\alpha,j} = \mathcal{O}_{\mathcal{C}^{2,\mu}_{\theta\alpha}(\mathbb{R})}(\alpha^3),$$

and consequently we have in V_ς, taking into account (3.7):

(3.15)

$$\begin{aligned} a_{11,j} &= \mathcal{O}_{\mathcal{C}^{0,\mu}_{\theta\alpha}(\mathbb{R})}(\alpha^2), \quad a_{12,j} = \mathcal{O}_{\mathcal{C}^{0,\mu}_{\theta\alpha}(\mathbb{R})}(\alpha), \quad a_{22,j} = \mathcal{O}_{\mathcal{C}^{0,\mu}_{\theta\alpha}(\mathbb{R})}\big(\alpha^2(1+|\mathtt{x}_j|)\big),\\ b_{1,j} &= \mathcal{O}_{\mathcal{C}^{0,\mu}_{\theta\alpha}(\mathbb{R})}\big(\alpha^2(1+|\mathtt{x}_j|)\big), \quad b_{2,j} = \mathcal{O}_{\mathcal{C}^{0,\mu}_{\theta\alpha}(\mathbb{R})}\big(\alpha^3(1+|\mathtt{x}_j|)\big). \end{aligned}$$

3.3. **Asymptotic formulas for the homoclinic and the Dancer solution.** In this section we will list some well known or standard properties of the functions we will use in the sequel. We will use them without special making any special reference since there are rather ubiquitous. First we recall that for the homoclinic solution defined in (1.3)–(1.4) we have:

$$w(x) = e^{-|x|} + \mathcal{O}(e^{-2|x|}), \quad \text{as } |x| \to \infty.$$

Second, let us recall that the linearized operator

$$L = \partial_x^2 - 1 + p\, w^{p-1}, \tag{3.16}$$

has a unique principal eigenvalue $\lambda_1 > 0$ with corresponding eigenfunction $Z(x) > 0$. In fact we have

$$\lambda_1 = \frac{1}{4}(p-1)(p+3), \qquad Z = \frac{w^{(p+1)/2}}{\sqrt{\int_{\mathbb{R}} w^{p+1}}},$$

and in paricular

$$Z(x) = e^{-\frac{p+1}{2}|x|} + \mathcal{O}(e^{-(p+1)|x|}), \quad \text{as } |x| \to \infty.$$

It is also known that $\lambda_2 = 0$ and the corresponding eigenfunction is w' while the rest of the spectrum is strictly negative.

Finally, using the results of [9] and the standard facts about the bifurctatin solutions, with the aid of barriers, we find that the Dancer solution $w_{\delta,\tau}$ has an expansion of the form

$$w_{\delta,\tau}(x,z) = w(x) + \delta Z(x)\cos(\sqrt{\lambda_1}z) + \tau Z(x)\sin(\sqrt{\lambda_1}z) + \mathcal{O}\big((|\delta|^2 + |\tau|^2)e^{-|x|}\big),$$

for all small δ, τ.

3.4. **Definition of the approximate solution.** Before giving a precise definition of the approximate solution let us explain the ingredients from which it is built. Considering just one of the bump lines we require that its lower and upper ends be asymptotic to two (possibly distinct) Dancer solutions. These two functions are "glued" together by some cutoff function. Let us observe that this way the amplitudes and the phase shifts of the ends do not change along the end of the bump line but instead are fixed. This is possible because the ends, whose shape is determined through the Toda system, are asymptotically linear. However, in the middle the bump line is curved and there the amplitude and the phase shift must be allowed to vary. This is quite analogous to bending of a corrugated, plastic pipe which "wrinkles" are stretched on the outside but piled up on the inside. To achieve this extra degree of freedom a function, whose local form is given by $e_j(\alpha \mathtt{z}_j)Z(\mathtt{x}_j)$ is added to our approximation. Comparing with the asymptotic formula for the

Dancer solution we see that this form of the extra correction to the approximate solution is natural.

Let us be more precise now. We will consider vector functions $\mathbf{e} \in \mathcal{C}^{2,\mu}_\theta(\mathbb{R}, \mathbb{R}^k)$ with the property:

$$\|\mathbf{e}\|_{\mathcal{C}^{2,\mu}_\theta(\mathbb{R},\mathbb{R}^k)} \leq \alpha^{2+\kappa_3} \tag{3.17}$$

where ε, κ_3 are small numbers to be chosen later on. In addition we will use $4k$ real parameters $\boldsymbol{\delta}_\pm = (\delta_{\pm,1}, \dots, \delta_{\pm,k})$ and $\boldsymbol{\tau}_\pm = (\tau_{\pm,1}, \dots, \tau_{\pm,k})$, such that with some small κ_4:

$$\|\boldsymbol{\delta}_\pm\| + \|\boldsymbol{\tau}_\pm\| \leq \alpha^{1+\kappa_4}. \tag{3.18}$$

Denoting by w the homoclinic solution, by $w_{\delta,\tau}$ the Dancer solution to (1.1) and by Z the principal eigenvector of the operator L defined in (3.16) we define (using the notation (3.8)) functions:

$$\begin{aligned} X^*_{\alpha,j} w_{\pm,j}(\mathtt{x}_j, \mathtt{z}_j) &= w_{\delta_{\pm,j},\tau_{\pm,j}}(\mathtt{x}_j, \mathtt{z}_j), \\ X^*_{\alpha,j} w_{0,j}(\mathtt{x}_j, \mathtt{z}_j) &= w(\mathtt{x}_j), \\ X^*_{\alpha,j} Z_j(\mathtt{x}_j, \mathtt{z}_j) &= Z(\mathtt{x}_j). \end{aligned} \tag{3.19}$$

Now, let $\varXi_\pm \geq 0$, $\varXi_0 \geq 0$ be cutoff functions such that

$$\begin{aligned} &\varXi_+(t) + \varXi_0(t) + \varXi_-(t) = 1, \quad \forall t \in \mathbb{R}, \\ &\operatorname{supp} \varXi_+ = (1, \infty), \quad \operatorname{supp} \varXi_0 = (-2, 2), \quad \operatorname{supp} \varXi_- = (-\infty, -1), \end{aligned}$$

and let

$$X^*_{\alpha,j} \varXi_{\pm,j}(\mathtt{x}_j, \mathtt{z}_j) := \varXi_\pm(\alpha \mathtt{z}_j), \quad X^*_{\alpha,j} \varXi_{0,j}(\mathtt{x}_j, \mathtt{z}_j) := \varXi_0(\alpha \mathtt{z}_j).$$

We will introduce the following convenient notation:

$$\mathtt{w}_j = \varXi_{+,j} w_{+,j} + \varXi_{0,j} w_{0,j} + \varXi_{-,j} w_{-,j}. \tag{3.20}$$

Given these notations we will define the approximate solution of (1.1) in V_ς by:

$$\bar{w}(\mathbf{x}) = \sum_{j=1}^k \mathtt{w}_j + e_j(\alpha \mathtt{z}_j) Z_j. \tag{3.21}$$

Notice that $\bar{w}$ depends on the parameters $\mathbf{f}_\alpha, \mathbf{v}, \mathbf{h}, \mathbf{e}, \boldsymbol{\delta}_\pm, \boldsymbol{\tau}_\pm$. We will not emphasize this dependence unless necessary. Taking now a smooth cutoff function η_ς supported in V_ς and such that $\eta_\varsigma \equiv 1$ in $V_{\frac{\varsigma}{2}}$ we define the global approximate solution of (1.1) by:

$$\mathtt{w} := \eta_\varsigma \big(\sum_{j=1}^k \mathtt{w}_j + e_j(\alpha \mathtt{z}_j) Z_j \big) = \eta_\varsigma \bar{w}. \tag{3.22}$$

4. Proof of Theorem 1.1

4.1. **Reduction to the nonlinear projected problem.** For the proof of the theorem it is convenient to modify (1.1) slightly. As customary we will consider initially

$$\Delta u - u + u_+^p = 0, \tag{4.1}$$

where u_+ is the positive part of u. The modification of the nonlinearity has no effect on the preceding considerations. Also, once the existence of a solution to

(4.1) is established, as an immediate consequence of the maximum principle we will obtain the existence for (1.1) as well.

Let ρ be a cutoff function such that

$$\rho(s) = \begin{cases} 1, & |s| \le \frac{3}{4}, \\ 0, & |s| > \frac{7}{8}. \end{cases} \tag{4.2}$$

We define:

$$X^*_{\alpha,j}\rho_j = \rho\big(\frac{\mathtt{x}_j}{\log\frac{1}{\alpha}}\big). \tag{4.3}$$

Finally, we define the function $w'_{0,j}$ by:

$$X^*_{\alpha,j} w'_{0,j} = w'(\mathtt{x}_j),$$

where w is the homoclinic solution of (1.1).

We look for a solution to (4.1) in the form $u = \mathtt{w} + \varphi$ where φ is a function to be determined. Denoting by $S(u)$ the nonlinear Schrödinger operator in (4.1) we expand:

$$S(\mathtt{w}+\varphi) = \mathbb{L}(\varphi) + S(\mathtt{w}) + N(\varphi),$$

where $S(\mathtt{w})$ is defined in (4.15) and

$$\mathbb{L}(\varphi) = \Delta\varphi - \varphi + p\mathtt{w}^{p-1}\varphi,$$
$$N(\varphi) = (\mathtt{w}+\varphi)^p_+ - \mathtt{w}^p - p\mathtt{w}^{p-1}\varphi.$$

This way our problem can be written in the form:

$$\mathbb{L}(\varphi) + S(\mathtt{w}) + N(\varphi) = 0,$$

and in principle it should be possible to reduce to a fixed point problem for the nonlinear function

$$\varphi + \mathbb{L}^{-1}\big(S(\mathtt{w}) + N(\varphi)\big) = 0,$$

provided that the operator $\mathbb{L}^{-1}$ is, in a suitable sense, bounded. But this is of course what we do not expect in general since in some sense $\mathbb{L}$ is a small perturbation, at least near a fixed bump line, of the operator

$$L = (\partial_x^2 + \partial_z^2 - 1 + p\,w^{p-1}),$$

which has bounded kernel spanned by the functions $w'(x)$, and $Z(x)\cos(\sqrt{\lambda_1}z)$, $Z(x)\sin(\sqrt{\lambda_1}z)$.

To deal with this (indeed fundamental) difficulty we will reduce the problem to the following *projected nonlinear problem*:

$$\mathbb{L}(\varphi) + S(\mathtt{w}) + N(\varphi) + \sum_{j=1}^k \mathtt{c}_j w'_{0,j}\rho_j + \sum_{j=1}^k \mathtt{d}_j Z_j \rho_j = 0. \tag{4.4}$$

In the following sections we will describe:

(1) how to solve (4.4) for the unknowns φ and $\mathtt{c} = (\mathtt{c}_1,\dots,\mathtt{c}_k)$, $\mathtt{d} = (\mathtt{d}_1,\dots,\mathtt{d}_k)$ with given fixed parameters $\mathbf{v},\mathbf{h},\mathbf{e},\boldsymbol{\delta}_\pm,\boldsymbol{\tau}_\pm$, and
(2) we will show how to adjust these parameters to achieve $\mathtt{c}\equiv 0$, $\mathtt{d}\equiv 0$.

This clearly will yield a solution to (4.1) (and (1.1)) as described in Theorem 1.1.

4.2. **The decomposition procedure.** In this section we will explain how to decompose the projected nonlinear problem into $k+1$ coupled equations. The advantage of this procedure is that we can deal separately with k problems, each of which is associated with a single bump line, and an extra $(k+1)$th problem that accounts for a cumulative, far field behavior of the bump lines.

To begin with we need to introduce cutoff functions χ, χ_j, $j=1,\dots,k$ as folows:

$$\chi(s)=\begin{cases}1, & |s|\le \frac{7}{8},\\ 0, & |s|>\frac{15}{16}.\end{cases} \tag{4.5}$$

We define:

$$X^*_{\alpha,j}\chi_j=\chi\big(\frac{\mathtt{x}_j}{\log\frac{1}{\alpha}}\big). \tag{4.6}$$

Comparing this with the definition of the cutoff functions ρ, ρ_j in (4.2)–(4.3) we see that

$$\chi_j\rho_j=\rho_j,\quad \chi_j\chi_i=0,\quad j\neq i. \tag{4.7}$$

This last statement follows from the fact that the distance between any two model bump lines is at least like $2\log\frac{1}{\alpha}+\mathcal{O}(1)$ and the definition of χ_j.

We look for a solution of (4.4) in the form

$$\varphi=\sum_{j=1}^k\phi_j\rho_j+\psi. \tag{4.8}$$

It is straightforward to check that this function is the solution if we require that functions ϕ_j, $j=1,\dots,k$ and ψ satisfy the following system of equations:

$$\chi_j\mathbb{L}(\phi_j)+\mathtt{c}_jw'_{0,j}\chi_j+\mathtt{d}_jZ_j\chi_j=\chi_j(S(\mathtt{w})+N)-\chi_j(\mathbb{L}-\Delta+1)\psi, \tag{4.9}$$

$$\begin{aligned}(\Delta-1)\psi=&\big(1-\sum_{i=1}^k\rho_i\big)(S(\mathtt{w})+N)-\sum_{i=1}^k\big[\mathbb{L}(\phi_i\rho_i)-\rho_i\mathbb{L}(\phi_i)\big]\\ &-\big(1-\sum_{i=1}^k\rho_i\big)(\mathbb{L}-\Delta+1)\psi,\end{aligned} \tag{4.10}$$

where $N=N(\sum_{j=1}^k\phi_j\rho_j+\psi)$. Indeed, multiplying (4.9) by ρ_j, using (4.7) and adding all the equations we get (4.4). This is a coupled system however the coupling terms are of higher order (in α). Additionally the linear operator on the right hand side of (4.9) expressed in the local coordinates is a small perturbation of the basic linearized opertator L already seen above. We will take advantage of these facts in what follows.

We further recast (4.9)–(4.10). Clearly ϕ_j is a solution of (4.9) if

$$\big[\partial^2_{\mathtt{x}_j}+\partial^2_{\mathtt{z}_j}+g'_p\big(w_{0,j}\big)\big]X^*_{\alpha,j}\phi_j=X^*_{\alpha,j}k_j-X^*_{\alpha,j}(\mathtt{c}_jw'_j\chi_j)-X^*_{\alpha,j}(\mathtt{d}_jZ_j\chi_j), \tag{4.11}$$

where

$$\begin{aligned}X^*_{\alpha,j}k_j=&X^*_{\alpha,j}\big[\chi_j\big(S(\mathtt{w})+N\big)\big]-X^*_{\alpha,j}\big(\chi_j(\mathbb{L}-\Delta+1)\psi\big)\\ &-X^*_{\alpha,j}\big(\chi_j\mathbb{L}(\phi_j)\big)+(X^*_{\alpha,j}\chi_j)\big[\partial^2_{\mathtt{x}_j}+\partial^2_{\mathtt{z}_j}+g'_p\big(w_{0,j}\big)\big]X^*_{\alpha,j}\phi_j.\end{aligned} \tag{4.12}$$

Again this is evident because of (4.7). We observe that (4.11) can be seen as an equation in $(\mathtt{x}_j,\mathtt{z}_j)\in\mathbb{R}^2$. In particular functions $X^*_{\alpha,j}\phi_j$, as solutions of (4.11) are defined for all $(\mathtt{x}_j,\mathtt{z}_j)\in\mathbb{R}^2$, although in reality these variables correspond to the local coordinates of $\bar{\gamma}_{\alpha,j}$ in a subset of $\mathbb{R}^2$ only. It is important to remember that this subset contains supp χ_j.

Let us now consider equation (4.10). Denoting $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_k)$ and the right hand side of (4.10) by $Q = Q(\boldsymbol{\phi}, \psi)$ we can write:

$$(\Delta - 1)\psi = Q(\boldsymbol{\phi}, \psi). \tag{4.13}$$

This way (4.9)–(4.10) is reduced to the system of equations given by (4.11) and (4.13). This is a nonlinear system for the unknowns ϕ_j, $j = 1, \ldots, k$ and ψ with functions $\mathtt{c}_j$ and $\mathtt{d}_j$ to be determined as well. Because (4.11) carries all long range interactions between the bump lines we will refer to it and its modifications as the *interaction system*. Equation (4.13) will be called the *background equation*.

4.3. **The error of the initial approximation.** Let us analyze the right hand sides of the (4.11), (4.13). We introduce the following weighted Hölder norms:

$$\|\phi\|_{\mathcal{C}^{\ell,\mu}_{\sigma,a}(\mathbb{R}^2)} = \sup_{\mathbf{x}\in\mathbb{R}^2} \Big((\cosh x)^\sigma (\cosh z)^a \|\phi\|_{\mathcal{C}^{\ell,\mu}(B_1(\mathbf{x}))} \Big). \tag{4.14}$$

The error of the global approximation $\mathtt{w}$ is defined by:

$$S(\mathtt{w}) = \Delta \mathtt{w} - \mathtt{w} + \mathtt{w}^p. \tag{4.15}$$

This function depends in particular on the parameters $\mathbf{v}, \mathbf{h}, \mathbf{e}, \boldsymbol{\delta}_\pm, \boldsymbol{\tau}_\pm$, and although this dependence is usually not emphasized sometimes it will be necessary to denote:

$$S(\mathtt{w}) = S(\mathtt{w}; \mathbf{v}, \mathbf{h}, \mathbf{e}, \boldsymbol{\delta}_\pm, \boldsymbol{\tau}_\pm).$$

We always assume that these parameters satisfy estimates (3.1), (3.3), (3.17) and (3.18) with some fixed $\kappa_i > 0$, $i = 1, \ldots, 4$. In particular we notice that the most involved is the dependence of the error on $\mathbf{h}$ through the local variables $(\mathtt{x}_j, \mathtt{z}_j)$. We will go back to this issue in more detail later.

We state the main result of this section.

Proposition 4.1. *The function $S(\mathtt{w}; \mathbf{v}, \mathbf{h}, \mathbf{e}, \boldsymbol{\delta}_\pm, \boldsymbol{\tau}_\pm)$ is a continuous function of its parameters and for each sufficiently small α the following estimate holds:*

$$\|X^*_{\alpha,j}(\chi_j S(\mathtt{w}))\|_{\mathcal{C}^{0,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)} \leq C\alpha^2, \tag{4.16}$$

where $0 < \sigma < \min\{p-2, 1\}$, $\theta \in (0, \vartheta)$ and ϑ is the constant defined in (2.13). Moreover $S(\mathtt{w})$ is a Lipschitz function of its parameters $\mathbf{h}, \mathbf{e}$, and denoting $S^{(\ell)} = S(\mathtt{w}; \mathbf{v}, \mathbf{h}^{(\ell)}, \mathbf{e}^{(\ell)}, \boldsymbol{\delta}_\pm, \boldsymbol{\tau}_\pm)$, $\ell = 1, 2$, we have:

$$\begin{aligned} &\|X^{(1)*}_{\alpha,j}(\chi^{(1)}_j S^{(1)}) - X^{(2)*}_{\alpha,j}(\chi^{(2)}_j S^{(2)})\|_{\mathcal{C}^{0,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)} \\ &\leq C\big(\alpha^2 \|\mathbf{h}^{(1)} - \mathbf{h}^{(2)}\|_{\mathcal{C}^{2,\mu}_{\theta\alpha}(\mathbb{R}^k;\mathbb{R})} + \|\mathbf{e}^{(1)} - \mathbf{e}^{(2)}\|_{\mathcal{C}^{2,\mu}_{\theta\alpha}(\mathbb{R}^k;\mathbb{R})}\big). \end{aligned} \tag{4.17}$$

Observe that we regard functions $X^*_{\alpha,j}(\chi_j S(\mathtt{w}))$ as defined on the whole plane $\mathbb{R}^2$. This is correct since these functions are supported in the region where the local coordinates are well defined.

The proof of this lemma is fairly technical but at the same standard (see [14, 15] for similar results) and is postponed to section 5. We should make a comment regarding the Lipschitz property (4.17). We observe that expressing the error $S^{(\ell)}$ in local variables $(\mathtt{x}_j, \mathtt{z}_j)$ we have to use relations (3.9) to express variables $(\mathtt{x}_i, \mathtt{z}_i)$ in terms of $(\mathtt{x}_j, \mathtt{z}_j)$. These relations involve the components of the function $\mathbf{h}^{(\ell)}$ as higher order terms. Using the Implicit Function Theorem one can prove that in fact local coordinates with respect to different bump lines are $\mathcal{C}^{2,\mu}$ functions of the local coordinates of one fixed line.

So far we have estimated the error near the bump lines. Another proposition is needed to estimate the norm in the complement of the sets $\operatorname{supp}\rho_j$. Recall that we have $S(\mathtt{w}) \equiv 0$ in $\mathbb{R}^2 \setminus V_\varsigma$. We will denote

$$V_\varsigma^o = V_\varsigma \setminus \bigcup_{j=1}^k \operatorname{supp}\rho_j.$$

Proposition 4.2. *Under the hypothesis of the previous Proposition we have for each $j = 1, \ldots, k$:*

$$\|S(\mathtt{w})(\cosh \mathtt{z}_j)^{\theta\alpha}\|_{\mathcal{C}^{0,\mu}(V_\varsigma^o)} \leq C\alpha^{2+\frac{3}{4}\sigma}. \tag{4.18}$$

Similarly to (4.17) we have

(4.19)

$$\|S^{(1)} - S^{(2)}\|_{\mathcal{C}^{0,\mu}(V_\varsigma^o)} \leq C\alpha^{\frac{3}{4}\sigma}[\alpha^2\|\mathbf{h}^{(1)} - \mathbf{h}^{(2)}\|_{\mathcal{C}^{2,\mu}_{\theta\alpha}(\mathbb{R}^k;\mathbb{R})} + \|\mathbf{e}^{(1)} - \mathbf{e}^{(2)}\|_{\mathcal{C}^{2,\mu}_{\theta\alpha}(\mathbb{R}^k;\mathbb{R})}].$$

We prove this result in section 5. Here we comment only that in Proposition 4.2 we consider the error as a function of the variable $\mathbf{x} \in \mathbb{R}^2$ and the weight function depends in particular on z, since $\mathtt{z}_j = \mathtt{z}_j(z)$ by its definition as the arc length parameter of $\bar{\gamma}_{\alpha,j}$.

4.4. **Existence of the background function.** In order to solve the system (4.11)–(4.13) we will use the Banach fixed point theorem. A convenient way to implement it is to solve first (4.13) with given $\boldsymbol{\phi}$. To accomplish this we need to make some assumptions regarding the initial size of the functions ϕ_j. We will assume from now on that functions ϕ_j are such that with σ and θ as in the hypothesis of Proposition 4.1 we have

$$\|X^*_{\alpha,j}\phi_j\|_{\mathcal{C}^{2,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)} < \infty, \quad j = 1, \ldots, k. \tag{4.20}$$

We assume above that $X^*_{\alpha,j}\phi_j$ is a function defined in the whole plane and the weight functions are taken with respect to the variables $(\mathtt{x}_j, \mathtt{z}_j)$. We have the following Lemma:

Lemma 4.1. *Assuming that (4.20) holds there exists a unique solution of (4.13) such that for all $j = 1, \ldots, k$ we have:*

$$\|\psi(\cosh \mathtt{z}_j)^{\theta\alpha}\|_{\mathcal{C}^{2,\mu}(\mathbb{R}^2)} \leq C\alpha^{\frac{3}{4}\sigma}\big(\alpha^2 + \sum_{j=1}^k \|X^*_{\alpha,j}\phi_j\|_{\mathcal{C}^{2,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)}\big). \tag{4.21}$$

In addition ψ is a continuous function of the parameters $\mathbf{v}, \mathbf{h}, \mathbf{e}, \boldsymbol{\delta}_\pm, \boldsymbol{\tau}_\pm$ and a Lipschitz function of $\boldsymbol{\phi}$ and also of the parameters $\mathbf{h}, \mathbf{e}$ and the following estimates hold:

(4.22)

$$\|(\psi(\boldsymbol{\phi}^{(1)}) - \psi(\boldsymbol{\phi}^{(2)}))(\cosh \mathtt{z}_j)^{\theta\alpha}\|_{\mathcal{C}^{2,\mu}(\mathbb{R}^2)} \leq C\alpha^{\frac{3}{4}\sigma}\sum_{j=1}^k \big\|X^*_{\alpha,j}\big(\phi_j^{(1)} - \phi_j^{(2)}\big)\big\|_{\mathcal{C}^{2,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)},$$

and

$$\begin{aligned}&\|(\psi(\mathbf{h}^{(1)}, \mathbf{e}^{(1)}) - \psi(\mathbf{h}^{(1)}, \mathbf{e}^{(1)}))(\cosh \mathtt{z}_j)^{\theta\alpha}\|_{\mathcal{C}^{2,\mu}(\mathbb{R}^2)}\\ &\quad\leq C\alpha^{\frac{3}{4}\sigma}\big(\alpha^2\|\mathbf{h}_1 - \mathbf{h}_2\|_{\mathcal{C}^{2,\mu}_{\theta\alpha}(\mathbb{R}^k;\mathbb{R})} + \|\mathbf{e}_1 - \mathbf{e}_2\|_{\mathcal{C}^{2,\mu}_{\theta\alpha}(\mathbb{R}^k;\mathbb{R})}\big).\end{aligned} \tag{4.23}$$

The proof of this Lemma is postponed to section 6.

4.5. **Invertibility of the basic linearized operator.** We will develop now the main functional analytic tool needed to solve the system of equations (4.11). Let us recall the definition of the basic linearized operator L in (3.16):

$$L = (\partial_x^2 + \partial_z^2 - 1 + p\, w^{p-1}).$$

We will consider the problem of existence of the unique solution of

$$L(\phi) = h, \quad \text{in } \mathbb{R}^2, \tag{4.24}$$

which additionally satisfies:

$$\int_{\mathbb{R}} w'(x)\phi(x,z)\, dx = 0 = \int_{\mathbb{R}} Z(x)\phi(x,z)\, dx. \tag{4.25}$$

We will assume below that

$$\int_{\mathbb{R}} w'(x)h(x,z)\, dx = 0 = \int_{\mathbb{R}} Z(x)h(x,z)\, dx, \tag{4.26}$$

and

$$\|(\cosh x)^\sigma (\cosh z)^a h\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)} < \infty. \tag{4.27}$$

Proposition 4.3. *There exists an $a_0 > 0$ such that given h satsifying (4.26)–(4.27) with $\sigma \in (0,1)$, $a \in [0, a_0)$, there exists a unique bounded solution $\phi = T(h)$ to problem (4.24) which defines a bounded linear operator of h in the sense that*

$$\|(\cosh x)^\sigma (\cosh z)^a \phi\|_{\mathcal{C}^{2,\mu}(\mathbb{R}^2)} \le C \|(\cosh x)^\sigma (\cosh z)^a h\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)},$$

and ϕ satisfies additionally the orthogonality conditions (4.25).

The proof of this Proposition is postponed to section 7.

We will adopt the above theory to deal with the system of nonlinear and nonlocal equations (4.11).

4.6. **Existence of solutions to the interaction system.** Given what we said above we will describe the procedure that will give the solution of (4.11). By what we said in previous section we are reduced to considering the following fixed point problem

$$X^*_{\alpha,j}\phi_j = T\big(X^*_{\alpha,j}k_j - X^*_{\alpha,j}(\mathtt{c}_j w'_{0,j}\chi_j) - X^*_{\alpha,j}(\mathtt{d}_j Z_j \chi_j)\big), \tag{4.28}$$

where $\mathtt{c}_j$ and $\mathtt{d}_j$ must chosen in such a way that the orthogonality conditions in (4.26) are satisfied. These conditions read in this case:

$$\begin{aligned} \mathtt{c}_j \int_{\mathbb{R}} X^*_{\alpha,j}((w'_{0,j})^2\chi_j)\, d\mathtt{x}_j &= \int_{\mathbb{R}} X^*_{\alpha,j}(k_j w'_{0,j}) d\mathtt{x}_j, \\ \mathtt{d}_j \int_{\mathbb{R}} X^*_{\alpha,j}(Z^2\chi_j)\, d\mathtt{x}_j &= \int_{\mathbb{R}} X^*_{\alpha,j}(k_j Z)\, d\mathtt{x}_j. \end{aligned} \tag{4.29}$$

Let us make a comment about the structure of the system (4.28). Of course it can be written, alternatively as a system oh PDEs:

$$\big[\partial^2_{\mathtt{x}_j} + \partial^2_{\mathtt{z}_j} + g'_p\big(w(\mathtt{x}_j)\big)\big] X^*_{\alpha,j}\phi_j = X^*_{\alpha,j}k_j - X^*_{\alpha,j}(\mathtt{c}_j w'_{0,j}\chi_j) - X^*_{\alpha,j}(\mathtt{d}_j Z\chi_j). \tag{4.30}$$

This system is coupled only through the background function ψ (hidden in $X^*_{\alpha,j}k_j$) considered in each equation restricted to the set $\operatorname{supp}\chi_j$. As given in Lemma 4.1 this function is a function of $\mathbf{x} = (x,z) \in \mathbb{R}^2$. Since we can express these variables in terms of the local coordinates in $\operatorname{supp}\chi_j \subset V_\varsigma$ we are justified in writing something like $X^*_{\alpha,j}(\chi_j\psi)$. Similar observation applies to other functions appearing on the right hand side of (4.30). The key point is that functions $X^*_{\alpha,j}k_j$ are supported in V_ς where the local coordinates of all curves $\bar\gamma_{\alpha,j}$ are well defined.

We will examine the size of the functions $X^*_{\alpha,j}k_j$ in the weighted Hölder norms.

Lemma 4.2. *We assume that*

$$\|X^*_{\alpha,j}\phi_j\|_{\mathcal{C}^{2,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)} \le \alpha^{\frac{3}{4}\sigma}. \tag{4.31}$$

With the notations of Proposition 4.1 the following estimate holds for $j = 1, \dots, k$:

$$\|X^*_{\alpha,j}k_j\|_{\mathcal{C}^{0,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)} \le C\alpha^2 + C\alpha^{\frac{3}{8}\sigma}\sum_{i=1}^k \|X^*_{\alpha,i}\phi_i\|_{\mathcal{C}^{2,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)}. \tag{4.32}$$

Moreover, functions $X^*_{\alpha,j}k_j$ are Lipschitz as functions of $\boldsymbol{\phi}$ and we have

(4.33)

$$\|X^*_{\alpha,j}k_j(\boldsymbol{\phi}^{(1)}) - X^*_{\alpha,j}k_j(\boldsymbol{\phi}^{(2)})\|_{\mathcal{C}^{0,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)} \le C\alpha^{\frac{3}{8}\sigma}\sum_{j=1}^k \|X^*_{\alpha,j}\phi^{(1)}_j - X^*_{\alpha,j}\phi^{(2)}_j\|_{\mathcal{C}^{2,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)},$$

We prove this Lemma in section 8.

We will now turn our attention to functions $\mathtt{c}_j$, $\mathtt{d}_j$ given by (4.29). It is easy to see that we have in fact:

$$\|\mathtt{c}_j\|_{\mathcal{C}^{0,\mu}_{\theta\alpha}(\mathbb{R})} + \|\mathtt{d}_j\|_{\mathcal{C}^{0,\mu}_{\theta\alpha}(\mathbb{R})} \le C\|X^*_{\alpha,j}k_j\|_{\mathcal{C}^{0,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)}, \tag{4.34}$$

and consequently,

$$\begin{aligned}\|X^*_{\alpha,j}(\mathtt{c}_j w'_{0,j}\chi_j)\|_{\mathcal{C}^{0,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)} + \|X^*_{\alpha,j}(\mathtt{d}_j Z\chi_j)\|_{\mathcal{C}^{0,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)} &\le C\|X^*_{\alpha,j}k_j\|_{\mathcal{C}^{0,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)}\\ &\le C\alpha^2 + C\alpha^{\frac{3}{8}\sigma}\sum_{i=1}^k \|X^*_{\alpha,i}\phi_i\|_{\mathcal{C}^{2,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)},\end{aligned}$$

by (4.32). The Lipschitz property of the functions $\mathtt{c}_j$, $\mathtt{d}_j$ in terms of the unknowns ϕ_j is also clear. Using these facts, the results of Lemma 4.2 and (4.28) we can apply Banach contraction mapping theorem to conclude:

Proposition 4.4. *The interaction system (4.28)–(4.29) has a unique solution $\boldsymbol{\phi} = (\phi_1, \dots, \phi_k)$ such that*

$$\sum_{j=1}^k \|X^*_{\alpha,j}\phi_j\|_{\mathcal{C}^{2,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)} \le C\alpha^2. \tag{4.35}$$

The proof of this proposition is rather straightforward. We need to set up the fixed point scheme for the operator defined in (4.28) in the space of functions $\boldsymbol{\phi}\colon (\mathbb{R}^2)^k \to \mathbb{R}^k$ with the weighted norm defined, component by component, as in the statement of the proposition. We do this in the set of functions satisfying in addition (4.31). Observe that while $X^*_{\alpha,j}k_j$ depends on the component functions of $\boldsymbol{\phi}$ the coupling between the equation is only through the operator ψ, which is nonlocal but easy to handle thanks to Lemma 4.1. We leave the details of the proof to the reader.

In the sequel we will need one more property of the solution of the interaction system. We observe that $X^*_{\alpha,j}\phi_j$ is a function of the parameters $\mathbf{v}, \mathbf{h}, \mathbf{e}, \boldsymbol{\delta}_\pm, \boldsymbol{\tau}_\pm$. As for the nature of the dependence of $X^*_{\alpha,j}\phi_j$ on these parameters we have:

Lemma 4.3. *The solution of the system (4.28)–(4.29) is a continuous function of the parameters* $\mathbf{v}, \mathbf{h}, \mathbf{e}, \boldsymbol{\delta}_\pm, \boldsymbol{\tau}_\pm$ *and a Lipschitz function of* $\mathbf{h}$, $\mathbf{e}$. *Moreover we have:*

(4.36)
$$\|X^{(1)*}_{\alpha,j}\phi_j(\mathbf{h}^{(1)}, \mathbf{e}^{(1)}) - X^{(2)*}_{\alpha,j}\phi_j(\mathbf{h}^{(2)}, \mathbf{e}^{(2)})\|_{\mathcal{C}^{2,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)} \le C\alpha^2\|\mathbf{h}^{(1)} - \mathbf{h}^{(2)}\|_{\mathcal{C}^{2,\mu}_{\theta\alpha}(\mathbb{R}^k;\mathbb{R})} + C\|\mathbf{e}^{(1)} - \mathbf{e}^{(2)}\|_{\mathcal{C}^{2,\mu}_{\theta\alpha}(\mathbb{R}^k;\mathbb{R})}.$$

To prove Lemma 4.3 we observe that the operator defined in (4.28) is a uniform contraction in the set of functions satisfying (4.31) as long as (3.1), (3.3), (3.17) and (3.18) are satisfied. In addition for each fixed $\boldsymbol{\phi}$ the right hand side of (4.28) is a continuous function of the parameters $\mathbf{v}, \mathbf{h}, \mathbf{e}, \boldsymbol{\delta}_\pm, \boldsymbol{\tau}_\pm$ and Lipschitz function of $\mathbf{h}, \mathbf{e}$. This follows from Proposition 4.1, Lemma 4.1. From the Banach contraction mapping theorem we conclude (4.36).

We will finish this section with the discussion of the rate of decay of the solution φ to (4.4) which is given by (4.8), namely

$$\varphi = \sum_{j=1}^{k} \phi_j \rho_j + \psi,$$

in terms of the original variables $\mathbf{x} = (x, z)$ rather than the local variables. We observe that whenever

$$\|X^*_{\alpha,j}(\rho_j\phi_j)\|_{\mathcal{C}^{2,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)} \le C\alpha^2,$$

then

$$\|\rho_j\phi_j\|_{\mathcal{C}^{2,\mu}_{\sigma_*,\theta_*\alpha}(\mathbb{R}^2)} \le C\alpha^{2-\sigma_*(k+1)} \tag{4.37}$$

since, because of (3.5), we have:

$$x = \mathtt{x}_j(1 + \mathcal{O}(\alpha^2)) + \mathtt{z}_j\mathcal{O}(\alpha) + 2(j - \frac{k+1}{2})\log\frac{1}{\alpha}. \tag{4.38}$$

Of course in (4.37) we must take $\sigma_* < \sigma$ and $\theta_* < \theta$. Estimate of a similar type can be shown for the background function ψ as well by a slight modification of the proof of Lemma 4.1 (see also Remark 6.1 in section 6). Thus taking σ_* sufficiently small we get:

$$\|\varphi\|_{\mathcal{C}^{2,\mu}_{\sigma_*,\theta_*\alpha}(\mathbb{R}^2)} \le C\alpha,$$

which is the estimate we claimed in the statement of Theorem 1.1 (see (1.10) in Definition 1.1) .

4.7. **Derivation of the reduced equations.** In order to finish the proof of the Theorem 1.1 we need to adjust the (so far undetermined) parameters $\mathbf{v}, \mathbf{h}, \mathbf{e}, \boldsymbol{\delta}_\pm, \boldsymbol{\tau}_\pm$ in such a way that $\mathtt{c}_j = 0$, $\mathtt{d}_j = 0$. In other words, by (4.29), we need:

$$\int_{\mathbb{R}} X^*_{\alpha,j}(k_j w'_{0,j})\, d\mathtt{x}_j = 0, \tag{4.39}$$

$$\int_{\mathbb{R}} X^*_{\alpha,j}(k_j Z)\, d\mathtt{x}_j = 0. \tag{4.40}$$

We will refer to (4.39) as the reduced system. We will first show that it is equivalent to a nonlinear and nonlocal system of second order in variables $\mathbf{h} = (h_1, \ldots, h_k)$ and $\mathbf{e} = (e_1, \ldots, e_k)$. This is a system of $2k$ equations with $2k$ unknowns. The first k equation which determine $\mathbf{h}$ have the form, to main order, of the linearized Toda system discussed already in section 2. In particular a solution which decays exponentially exists only if we can chose suitably the unknown function $\mathbf{v} = (v_1, \dots, v_k)$. On the other hand the system for $\mathbf{e}$ consists of decoupled (to main order) linear equations of the form considered in section 2.3. As we have indicated each of the k equations requires 2 extra solvability conditions if we seek solutions in the exponentially decaying class. These requirements lead to $2k$ constraints on $4k$ parameters $\boldsymbol{\delta}_\pm, \boldsymbol{\tau}_\pm$. Considering (4.39) we have the following

Proposition 4.5. *Equations (4.39) are equivalent to the following system of equations:*

$$c_p(\mathbf{h}+\mathbf{v})'' + \mathbf{N}(\mathbf{h}+\mathbf{v}) = \mathbf{P}, \quad \mathbf{h} = (h_1, \dots, h_k), \quad \mathbf{v} = (v_1, \dots, v_k), \tag{4.41}$$

and $\mathbf{N} = (\mathbf{N}_1, \dots, \mathbf{N}_k)^T$, *where*

$$\mathbf{N}_j = -e^{\,f_{j-1}-f_j}\mathtt{e}_{j-1} + [e^{\,f_{j-1}-f_j} + e^{\,f_j - f_{j+1}}]\mathtt{e}_j - e^{\,f_j-f_{j+1}}\mathtt{e}_{j+1}, \tag{4.42}$$

and $\mathtt{e}_j$ *are the vectors of the canonical basis in* $\mathbb{R}^k$. *The function* $\mathbf{P}$ *satisfies:*

$$\|\mathbf{P}\|_{\mathcal{C}^{0,\mu}_\theta(\mathbb{R};\mathbb{R}^k)} \le C\alpha^{\nu_1}, \tag{4.43}$$

where we choose

$$\nu_1 = \min\{1-\mu, 2\kappa_1 - \mu, 2\kappa_2 - \mu, 1+\kappa_4 - \mu, \kappa_2 + \kappa_4 - \mu, \frac{3}{4}\sigma - \mu\},$$

provided that (3.1), (3.3), (3.17) and (3.18) are satisfied. The constant c_p *is defined by*

$$c_p = \frac{\int_{\mathbb{R}} (w')^2\,dx}{-p\int_{\mathbb{R}} w^{p-1}w'e^{\,x}\,dx} > 0. \tag{4.44}$$

In addition $\mathbf{P}$ *is a continuos function of* $\mathbf{v}, \mathbf{h}, \mathbf{e}, \boldsymbol{\delta}_\pm, \boldsymbol{\tau}_\pm$ *and a Lipschitz function of* $\mathbf{h}, \mathbf{e}$ *and we have:*

$$\begin{aligned}&\|\mathbf{P}(\mathbf{h}^{(1)}, \mathbf{e}^{(1)}; \cdot) - \mathbf{P}(\mathbf{h}^{(2)}, \mathbf{e}^{(2)}; \cdot)\|_{\mathcal{C}^{0,\mu}_\theta(\mathbb{R};\mathbb{R}^k)} \\ &\qquad \le C\alpha^{\nu_1-\mu}(\|\mathbf{h}^{(1)} - \mathbf{h}^{(2)}\|_{\mathcal{C}^{0,\mu}_\theta(\mathbb{R};\mathbb{R}^k)} + \|\mathbf{e}^{(1)} - \mathbf{e}^{(2)}\|_{\mathcal{C}^{0,\mu}_\theta(\mathbb{R};\mathbb{R}^k)}).\end{aligned} \tag{4.45}$$

Proof. It is not hard to show that the main order terms in the projection of the function $X^*_{\alpha,j}k_j$ onto $w'_{0,j}$ come from the projection of $X^*_{\alpha,j}(\chi_j S(\mathtt{w}))$. Accepting this fact for now (in section 9 we will provide some more details to justify this claim) we will focus on computing the asymptotic form of this term. In order to make the calculations more accessible we will assume that $k=2$. This way we are able to emphasize the important points without obscuring them with complicated notations. We will compute first the projection of $X^*_{\alpha,1}(\chi_1 S(\mathtt{w}))$ onto $w'_{0,1}$. Expressing Δ in local coordinates, using the notation (3.13)–(3.14), and neglecting the higher order terms (in α) we get

$$\int_{\mathbb{R}} X^*_{\alpha,1}(\chi_1 S(\mathtt{w})w'_{0,1})\,d\mathtt{x}_1 \sim \int_R b_{1,1}(\partial_{\mathtt{x}_1} w_{0,1})^2\,d\mathtt{x}_1 + p\int_{\mathbb{R}} w^{p-1}_{0,1} w_{0,2}\partial_{\mathtt{x}_1} w_{0,1}\,d\mathtt{x}_1. \tag{4.46}$$

In section 9 we will show that the difference between the left and the right member in (4.46) is negligible. Now to compute the integrals we use (3.14) to get:

$$\begin{aligned}
&\int_R b_{1,1}(\partial_{\mathtt{x}_1} w_{0,1})^2\,d\mathtt{x}_1 \\
&= \int_{\mathbb{R}} (\partial_{\mathtt{x}_1} w_{0,1})^2[-k_{\alpha,1}A_1^{-1} - \alpha^2 h_1'' A_1^{-2} + \alpha^2(h_1')^2 k_{\alpha,1} - \alpha(\mathtt{x}_1 + h_1)h_1' k_{\alpha,1}']\,d\mathtt{x}_1 \\
&= -\alpha^2(f_1'' + h_1'')\int_{\mathbb{R}}(w')^2\,dx + \mathcal{O}_{\mathcal{C}^{0,\mu}_\theta(\mathbb{R})}(\alpha^{3-\mu})(\|h_1\|^2_{\mathcal{C}^{2,\mu}_\theta(\mathbb{R})} + \|f_1\|^2_{\mathcal{C}^{3,\mu}_\theta(\mathbb{R})})
\end{aligned} \tag{4.47}$$

where we use (3.12) to replace $k_{\alpha,1}$ by f_1''. Notice that the exponential weights we take are like $(\cosh z)^\theta$. In other words in estimating $\mathbf{P}$ we take $\mathcal{C}^{0,\mu}_\theta(\mathbb{R})$ norm instead of $\mathcal{C}^{0,\mu}_{\theta\alpha}(\mathbb{R})$, which is the norm we in which we have actually measured the errors. This entails loss of a power of α hence the remainder is a factor of $\alpha^{3-\mu}$. This small detail, which we have already mentioned in (3.4) will be present in all subsequent calculations. Finally we remind the reader that above all functions of the arc length $\mathtt{z}_1$ are taken of the scaled argument $\alpha\mathtt{z}_1$.

To compute the second term in (4.46) we will use a refinement of (3.9) which reads:

$$\begin{aligned}
\mathtt{x}_2 &= \big[q_{\alpha,1}(\mathtt{z}_1) + h_1(\alpha\mathtt{z}_1) - q_{\alpha,2}(\mathtt{z}_1) - h_2(\alpha\mathtt{z}_1)\big]\big(1 + \mathcal{O}(\alpha^{2-\mu})\big) \\
&\quad + \mathcal{O}(\alpha^{2-\mu})\mathtt{z}_1 + \mathcal{O}(\alpha^{2-\mu}\log\frac{1}{\alpha}) + \mathtt{x}_1\big(1 + \mathcal{O}(\alpha^{2-\mu})\big).
\end{aligned} \tag{4.48}$$

Using this we can write:

$$\begin{aligned}
\int_{\mathbb{R}} w_{0,1}^{p-1} w_{0,2}\partial_{\mathtt{x}_1} w_{0,1}\,d\mathtt{x}_1 &= \int_{\mathbb{R}} w^{p-1}(\mathtt{x}_1)w'(\mathtt{x}_1)w(\mathtt{x}_1 + \tilde q_{\alpha,1}(\mathtt{z}_1) - \tilde q_{\alpha,2}(\mathtt{z}_1))\,d\mathtt{x}_1 \\
&\quad + \int_{\mathbb{R}} w^{p-1}(\mathtt{x}_1)w'(\mathtt{x}_1)[w(\mathtt{x}_2) - w(\mathtt{x}_1 + \tilde q_{\alpha,1}(\mathtt{z}_1) - \tilde q_{\alpha,2}(\mathtt{z}_1))]\,d\mathtt{x}_1,
\end{aligned} \tag{4.49}$$

where $\tilde q_{\alpha,j}(\mathtt{z}_1) = q_{\alpha,j}(\mathtt{z}_1) + h_j(\alpha\mathtt{z}_1)$. To evaluate the first integral above we observe that its leading order behavior comes from integration over the set where $|\mathtt{x}_1| \leq \frac{3}{2}\log\frac{1}{2}$, which means $\mathtt{x}_2 < -\frac{1}{2}\log\frac{1}{\alpha} + \mathcal{O}(1)$. Using the asymptotic formula

$$w(x) = e^{-|x|} + \mathcal{O}((\cosh x)^{-2}),$$

and denoting:

$$c_1 = \big(p\int_{\mathbb{R}} w^{p-1}(x)w'(x)e^{x}\,dx\big),$$

we get:

$$\begin{aligned}
&\int_{\mathbb{R}} w^{p-1}(\mathtt{x}_1)w'(\mathtt{x}_1)w(\mathtt{x}_1 + \tilde q_{\alpha,1}(\mathtt{z}_1) - \tilde q_{\alpha,2}(\mathtt{z}_1) - h_2(\alpha\mathtt{z}_1))\,d\mathtt{x}_1 \\
&= c_1\exp\big(\tilde q_{\alpha,1}(\mathtt{z}_1) - \tilde q_{\alpha,2}(\mathtt{z}_1)\big)(1 + \mathcal{O}_{\mathcal{C}^{2,\mu}_\theta(\mathbb{R})}(\alpha^{\frac{3}{2}-\mu})) \\
&= c_1\alpha^2 e^{f_1(\alpha\mathtt{z}_1) - f_2(\alpha\mathtt{z}_1)} \\
&\quad + c_1\alpha^2 e^{f_1(\alpha\mathtt{z}_1) - f_2(\alpha\mathtt{z}_1)}(h_1(\alpha\mathtt{z}_1) + v_1(\alpha\mathtt{z}_1) - h_2(\alpha\mathtt{z}_1) - v_2(\alpha\mathtt{z}_1)) \\
&\quad + \mathcal{O}_{\mathcal{C}^{2,\mu}_\theta(\mathbb{R})}(\alpha^{2+\nu_1}).
\end{aligned} \tag{4.50}$$

The second term in (4.49) is estimated in a similar way. Notice that since $w'(x) < 0$, $x > 0$ therefore the factor $c_1 = p\int_{\mathbb{R}} w^{p-1}(x)w'(x)e^{x}\,dx < 0$.

In combining (4.47), (4.49) and (4.50) we use the fact that $\mathbf{f}$ is a solution of the Toda system (1.11). In this manner we get:

$$c_p(h_1+v_1)'' + e^{f_1-f_2}(h_1+v_1-h_2-v_2) = \mathcal{O}_{\mathcal{C}^{0,\mu}_\theta(\mathbb{R})}(\alpha^{\nu_1}).$$

Analogous calculations can be done of course for the projection onto $w'_{0,2}$. This gives the assertion of the Proposition, (4.43), except for the detailed calculations which we will discuss in section 9.

Finally, for the continuity and the Lipschitz property (4.45) of $\mathbf{P}$ we observe that the former follows from the corresponding statements for $S(\mathtt{w})$, ψ and $\boldsymbol{\phi}$, see Proposition 4.1, Lemma 4.1, Lemma 4.3 respectively. The details are somewhat tedious but at the same time standard. □

We will now turn our attention to the second projected equation (4.40). We have:

Proposition 4.6. *Formula (4.40) is equivalent to the following system of equations:*

$$\mathbf{e}'' + \frac{\lambda_1}{\alpha^2}\mathbf{e} = \mathbf{Q}, \tag{4.51}$$

where

$$\|\mathbf{Q}\|_{\mathcal{C}^{0,\mu}_\theta(\mathbb{R};\mathbb{R}^k)} \leq C\alpha^{\nu_1}. \tag{4.52}$$

In addition statement (4.45) holds, with obvious modifications, for $\mathbf{Q}$ *in place of* $\mathbf{P}$.

Proof. We will again present simply the main point in the proof and postpone some details to section 9. We consider the leading order term in (4.40)

(4.53)
$$\int_{\mathbb{R}} X^*_{\alpha,j}(\chi_j S(\mathtt{w}))X^*_{\alpha,j}Z_j\,d\mathtt{x}_j \sim \int_{\mathbb{R}}[\partial^2_{\mathtt{x}_j} + \partial^2_{\mathtt{z}_j} + g'_p(w(\mathtt{x}_j))](e_j(\alpha\mathtt{z}_j)Z(\mathtt{x}_j))\chi_j Z(\mathtt{x}_j)\,d\mathtt{x}_j.$$

Terms that we have neglected above are of smaller order, and in fact they satisfy an estimate similar to (4.52) but with an extra factor α^2. We have in particular interaction terms similar to the ones considered in (4.49) but with $Z(\mathtt{x}_j)Z(\mathtt{x}_i)$ in place of the products $w_{0,j}w_{0,i}$. Because we have $Z(x) \sim e^{-a_p|x|}$, as $|x| \to \infty$ with $a_p \geq \frac{3}{2}$ we can neglect them in this case.

To calculate the right hand side of (4.53) we use the fact that Z is the principal eigenfunction of $\partial^2_{\mathtt{x}_j} + g'_p(w(\mathtt{x}_j))$. This gives immediately

$$\int_{\mathbb{R}} X^*_{\alpha,j}(\chi_j S(\mathtt{w}))X^*_{\alpha,j}Z_j\,d\mathtt{x}_j \sim \big(\alpha^2 e''_j(\alpha\mathtt{z}_j) + \lambda_1 e_j(\alpha\mathtt{z}_j)\big)\Big(\int_{\mathbb{R}} \chi Z^2\,dx\Big).$$

Formula (4.51) follows after dividing by α^2. The proof of the Lipschitz property is left to the reader. □

Now we recall that from our considerations in section 2.3 it follows that problem (4.51) is solvable in exponentially decaying class if in addition to (4.40) the following conditions hold:

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} X^*_{\alpha,j}(k_j Z) Z(\mathtt{x}_j)\cos(\sqrt{\lambda_1}\mathtt{z}_j)\,d\mathtt{x}_j d\mathtt{z}_j = 0,\\ \int_{\mathbb{R}}\int_{\mathbb{R}} X^*_{\alpha,j}(k_j Z) Z(\mathtt{x}_j)\sin(\sqrt{\lambda_1}\mathtt{z}_j)\,d\mathtt{x}_j d\mathtt{z}_j = 0.\end{aligned} \tag{4.54}$$

We will now show that (4.54) leads to conditions on $\boldsymbol{\delta}_\pm$, and $\boldsymbol{\tau}_\pm$. Let us denote the first integral above by Υ_j. We have (see section 9 for details):

$$\Upsilon_j = \int_{\mathbb{R}}\int_{\mathbb{R}} X^*_{\alpha,j}(\chi_j S(\mathtt{w}_j)) Z(\mathtt{x}_j)\cos(\sqrt{\lambda_1}\mathtt{z}_j)\,d\mathtt{x}_j d\mathtt{z}_j + \mathcal{O}(\alpha^{1+\nu_1}),$$

where $\mathtt{w}_j$ is defined in (3.20). With the notation (3.13)–(3.14) we get

$$X^*_{\alpha,j}(S(\mathtt{w}_j)) \sim [\partial^2_{\mathtt{x}_j} + \partial_{\mathtt{z}_j}]\mathtt{w}_j + g_p(\mathtt{w}_j), \tag{4.55}$$

where the neglected terms give at the end contributions of order $\mathcal{O}(\alpha^{1+\nu_1})$ to Υ_j.

It is not hard to see that, after neglecting lower order terms, (c.f. considerations in section 5, (5.1) and also section 9)

$$\begin{aligned}[\partial^2_{\mathtt{x}_j} + \partial^2_{\mathtt{z}_j}]\mathtt{w}_j + g_p(\mathtt{w}_j) &\sim \alpha^2[\Xi''_{+,j} w_{+,j} + \Xi''_{0,j} w_{0,j} + \Xi''_{-,j} w_{-,j}]\\ &\quad + 2\alpha[\Xi'_{+,j}\partial_{\mathtt{z}_j} w_{+,j} + \Xi'_{0,j}\partial_{\mathtt{z}_j} w_{0,j} + \Xi'_{-,j}\partial_{\mathtt{z}_j} w_{-,j}]\\ &= \alpha^2[\Xi''_{+,j}(w_{+,j} - w_{0,j}) + \Xi''_{-,j}(w_{-,j} - w_{0,j})]\\ &\quad + 2\alpha[\Xi'_{+,j}\partial_{\mathtt{z}_j} w_{+,j} + \Xi'_{-,j}\partial_{\mathtt{z}_j} w_{-,j}].\end{aligned}$$

We note that by (3.3) we have:

$$\begin{aligned}\partial_{\mathtt{z}_j} w_{\pm,j} &\sim \sqrt{\lambda_1} Z[-\delta_{\pm,j}\sin(\sqrt{\lambda_1}\mathtt{z}_j) + \tau_{\pm,j}\cos(\sqrt{\lambda_1}\mathtt{z}_j)],\\ w_{\pm,j} - w_{0,j} &\sim Z[\delta_{\pm,j}\cos(\sqrt{\lambda_1}\mathtt{z}_j) + \tau_{\pm,j}\sin(\sqrt{\lambda_1}\mathtt{z}_j)],\end{aligned}$$

where the neglected parts are of order $\mathcal{O}_{\mathcal{C}^\infty(\mathbb{R}^2)}\big(|\delta_{\pm,j}|^2 + |\tau_{\pm,j}|^2)(\cosh \mathtt{x}_j)^{-1}\big)$ and consequently their contribution is relatively smaller. Denoting

$$\Theta_{\pm,j} = [\delta_{\pm,j}\cos(\sqrt{\lambda_1}\mathtt{z}_j) + \tau_{\pm,j}\sin(\sqrt{\lambda_1}\mathtt{z}_j)], \quad \zeta_0 = \int_{\mathbb{R}} \chi Z^2, \tag{4.56}$$

we calculate:

$$\begin{aligned}\Upsilon_j &\sim \zeta_0 \int_{\mathbb{R}} [\alpha^2 \Xi''_{+,j}\Theta_{+,j} + 2\alpha \Xi'_{+,j}\Theta'_{+,j}]\cos(\sqrt{\lambda_1}\mathtt{z}_j)\,d\mathtt{z}_j\\ &\quad + \zeta_0 \int_{\mathbb{R}} [\alpha^2 \Xi''_{-,j}\Theta_{-,j} + 2\alpha \Xi'_{-,j}\Theta'_{-,j}]\cos(\sqrt{\lambda_1}\mathtt{z}_j)\,d\mathtt{z}_j\\ &= \sqrt{\lambda_1}\zeta_0(\tau_{+,j} - \tau_{-,j}).\end{aligned}$$

Similar calculations can be done for the second integral in (4.54). Denoting it by Λ_j we can summarize our considerations as follows:

Lemma 4.4. *With the notation introduced above it holds:*

$$\begin{aligned}\Upsilon_j &= \sqrt{\lambda_1}\zeta_0(\tau_{+,j} - \tau_{-,j}) + \mathcal{O}(\alpha^{1+\nu_1}),\\ \Lambda_j &= \sqrt{\lambda_1}\zeta_0(\delta_{+,j} - \delta_{-,j}) + \mathcal{O}(\alpha^{1+\nu_1}).\end{aligned} \tag{4.57}$$

For future references we will denote:

$$\bar{\Upsilon}_j = \Upsilon_j - \sqrt{\lambda_1}\zeta_0(\tau_{+,j} - \tau_{-,j}),$$
$$\bar{\Lambda}_j = \Lambda_j - \sqrt{\lambda_1}\zeta_0(\delta_{+,j} - \delta_{-,j}),$$

and $\boldsymbol{\Upsilon} = (\bar{\Upsilon}_1, \dots, \bar{\Upsilon}_k)$ and $\boldsymbol{\Lambda} = (\bar{\Lambda}_1, \dots, \bar{\Lambda}_k)$.

4.8. **Solution of the reduced system.** We will now complete the proof of Theorem 1.1. To this end we have to solve the following system of equations (see Proposition 4.5, Proposition 4.6 and Lemma 4.4):

$$c_p(\mathbf{h} + \mathbf{v})'' + \mathbf{N}(\mathbf{h} + \mathbf{v}) = \mathbf{P}(\mathbf{v}, \mathbf{h}, \mathbf{e}, \boldsymbol{\delta}_\pm, \boldsymbol{\tau}_\pm), \tag{4.58}$$

$$\mathbf{e}'' + \frac{\lambda_1}{\alpha^2}\mathbf{e} = \mathbf{Q}(\mathbf{v}, \mathbf{h}, \mathbf{e}, \boldsymbol{\delta}_\pm, \boldsymbol{\tau}_\pm) \tag{4.59}$$

$$\begin{cases} \sqrt{\lambda_1}\zeta_0(\boldsymbol{\tau}_+ - \boldsymbol{\tau}_-) = \boldsymbol{\Upsilon}(\mathbf{v}, \mathbf{h}, \mathbf{e}, \boldsymbol{\delta}_\pm, \boldsymbol{\tau}_\pm), \\ \sqrt{\lambda_1}\zeta_0(\boldsymbol{\delta}_+ - \boldsymbol{\delta}_-) = \boldsymbol{\Lambda}(\mathbf{v}, \mathbf{h}, \mathbf{e}, \boldsymbol{\delta}_\pm, \boldsymbol{\tau}_\pm). \end{cases} \tag{4.60}$$

Proposition 4.7. *System (4.58)–(4.60) has a $2k$ parameter family of solutions in the sense that for each choice of k components of the vector $(\boldsymbol{\delta}_-, \boldsymbol{\delta}_+) \in \mathbb{R}^{2k}$, and k components of the vector $(\boldsymbol{\tau}_-, \boldsymbol{\tau}_+) \in \mathbb{R}^{2k}$ this system has a solution for the remaining $2k$ components of $(\boldsymbol{\delta}_-, \boldsymbol{\delta}_+)$, $(\boldsymbol{\tau}_-, \boldsymbol{\tau}_+)$ and the functions $\mathbf{v}, \mathbf{h}, \mathbf{e}$.*

Proof. First we choose κ_i, $\mu \in (0, 1)$, and $0 < \sigma < \min\{p - 2, 1\}$ in such a way that

$$\min\{1 - \mu, 2\kappa_1 - \mu, 2\kappa_2 - \mu, 1 + \kappa_4 - \mu, \kappa_2 + \kappa_4 - \mu, \frac{3}{4}\sigma - \mu\} = \nu_1 > \max\{\kappa_i\}.$$

Second we fix k components of $(\boldsymbol{\delta}_-, \boldsymbol{\delta}_+) \in \mathbb{R}^{2k}$. For simplicity we assume that the fixed components correspond to the lower ends of the bump lines, however it is easy to see that any combination of k ends will do. We will denote them by $\boldsymbol{\delta}_-$. Similarly we fix $\boldsymbol{\tau}_-$. We assume that the fixed vectors satisfy

$$\|\boldsymbol{\delta}_\pm| + \|\boldsymbol{\tau}_\pm\| \le \frac{1}{2}\alpha^{1+\kappa_4}. \tag{4.61}$$

(c.f. (3.18)). Now we will solve the system by a fixed point argument following the three steps below.

Step 1. We fix $\tilde{\mathbf{v}}, \tilde{\mathbf{h}}, \tilde{\mathbf{e}}, \bar{\boldsymbol{\delta}}, \bar{\boldsymbol{\tau}}$ satisfying, respectively, (3.1), (3.3), (3.17) and (4.61). We set $\tilde{\boldsymbol{\delta}}_+ = \bar{\boldsymbol{\delta}} + \boldsymbol{\delta}_-$ and $\tilde{\boldsymbol{\tau}}_+ = \bar{\boldsymbol{\tau}} + \boldsymbol{\tau}_-$ and use these functions and parameters, together with $\boldsymbol{\delta}_-, \boldsymbol{\tau}_-$ to calculate the right hand sides of the equations (4.58)–(4.60) above. We observe that these functions satisfy the assertions of Proposition 4.5, Proposition 4.6 and Lemma 4.4. In particular they are Lipschitz as functions of $\tilde{\mathbf{h}}$ and $\tilde{\mathbf{e}}$ and continuous as functions of $\tilde{\mathbf{v}}$ and $\tilde{\boldsymbol{\delta}}_\pm, \tilde{\boldsymbol{\tau}}_\pm$.

Step 2.

Next, we use the Banach contraction mapping theorem to solve (4.58)–(4.60) for $\mathbf{h}$ and $\mathbf{e}$. We observe that as a result we get the following system:

$$c_p(\mathbf{h} + \mathbf{v})'' + \mathbf{N}(\mathbf{h} + \mathbf{v}) = \mathbf{P}(\tilde{\mathbf{v}}, \mathbf{h}, \mathbf{e}, \boldsymbol{\delta}_-, \boldsymbol{\tau}_-, \boldsymbol{\delta}_- + \bar{\boldsymbol{\delta}}, \boldsymbol{\tau}_- + \bar{\boldsymbol{\tau}}),$$

$$\begin{cases} \sqrt{\lambda_1}\zeta_0\boldsymbol{\tau} = \boldsymbol{\Upsilon}(\tilde{\mathbf{v}}, \mathbf{h}, \mathbf{e}, \boldsymbol{\delta}_-, \boldsymbol{\tau}_-, \boldsymbol{\delta}_- + \bar{\boldsymbol{\delta}}, \boldsymbol{\tau}_- + \bar{\boldsymbol{\tau}}), \\ \sqrt{\lambda_1}\zeta_0\boldsymbol{\delta} = \boldsymbol{\Lambda}(\tilde{\mathbf{v}}, \mathbf{h}, \mathbf{e}, \boldsymbol{\delta}_-, \boldsymbol{\tau}_-, \boldsymbol{\delta}_- + \bar{\boldsymbol{\delta}}, \boldsymbol{\tau}_- + \bar{\boldsymbol{\tau}}), \end{cases}$$

Using the theory developed in section 2 we find in addition that

$$\begin{aligned}\|\mathbf{h}\|_{\mathcal{C}^{2,\mu}_\theta(\mathbb{R};\mathbb{R}^k)} &\le C\|\mathbf{P}\|_{\mathcal{C}^{0,\mu}_\theta(\mathbb{R};\mathbb{R}^k)} \le C\alpha^{\nu_1},\\ \|\mathbf{e}\|_{\mathcal{C}^{2,\mu}_\theta(\mathbb{R};\mathbb{R}^k)} &\le C\alpha^2\|\mathbf{Q}\|_{\mathcal{C}^{0,\mu}_\theta(\mathbb{R};\mathbb{R}^k)} \le C\alpha^{2+\nu_1},\end{aligned} \tag{4.62}$$

and that $\mathbf{v}$, $\boldsymbol{\delta}$, $\boldsymbol{\tau}$ satisfy

$$\begin{aligned}\|\mathbf{v}\|_{\mathcal{E}} &\le C\alpha^{\nu_1},\\ \|\boldsymbol{\delta}\| + \|\boldsymbol{\tau}\| &\le C\alpha^{1+\nu_1}.\end{aligned} \tag{4.63}$$

Step 3. We notice that the map

$$\begin{aligned}(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\tau})\colon \mathcal{E}\times\mathbb{R}^k\times\mathbb{R}^k &\longrightarrow \mathcal{E}\times\mathbb{R}^k\times\mathbb{R}^k,\\ (\tilde{\mathbf{v}},\bar{\boldsymbol{\delta}},\bar{\boldsymbol{\tau}}) &\longmapsto (\mathbf{v},\boldsymbol{\delta},\boldsymbol{\tau}),\end{aligned}$$

is continuous and, because of the choice of ν_1 and (4.63), we can use Browder's theorem to find a fixed point of this map. In summary we obtain a solution to (4.58)–(4.60) as claimed. □

We recall that in the statement of Theorem (1.1) we assert the existence of $4k$ parameter family of solutions. So far we have only demonstrated a 2k parameter family of solutions of the system (4.58)–(4.60) however the missing $2k$ parameters are easy to find. Indeed at the beginning of our considerations we have chosen a solution of the Toda system (1.11) represented by $\mathbf{f}$. Of course this solution depends on $2k$ real parameters representing its initial conditions. These, together with the choice of $2k$ Dancer parameters give the $4k$ parameter family of solutions.

5. Proof of Propositions 4.1 and 4.2

5.1. **Evaluation of the error in the case of two bump lines.** In order to make the argument more transparent we will consider the special case of two bump lines i.e. $k = 2$ Recall that we have $g_p(t) = -t + t_+^p$, $p > 2$. Let us consider the error restricted to the set:

$$U_1 := \{\mathtt{x}_1 + \mathtt{x}_2 \le 0\} \cap V_{\frac{\varsigma}{2}}.$$

In this set it s convenient to write (with the notation (3.20)):

$$\begin{aligned}S(\mathtt{w}) = {}& \underbrace{\Delta(\mathtt{w}_1+\mathtt{w}_2) + g_p(\mathtt{w}_1) + g_p(\mathtt{w}_2)}_{E_1} + \underbrace{[\Delta + g_p'(\mathtt{w}_1+\mathtt{w}_2)](e_1Z_1+e_2Z_2)}_{E_2}\\ &+ \underbrace{g_p(\mathtt{w}_1+\mathtt{w}_2+e_1Z_1+e_2Z_2) - g_p(\mathtt{w}_1) - g_p(\mathtt{w}_2) - g_p'(\mathtt{w}_1+t w_2)(e_1Z_1+e_2Z_2)}_{E_3}.\end{aligned}$$

To estimate the first term we notice that using Taylor's expansion we get

$$\begin{aligned}\mathtt{w}_j^p = {}& \Xi_{+,j}w_{+,j}^p + \Xi_{0,j}w_{0,j}^p + \Xi_{-,j}w_{-,j}^p\\ &+ \big(w_{0,j} + \Xi_{+,j}(w_{+,j}-w_{0,j}) + \Xi_{-,j}(w_{-,j}-w_{0,j})\big)^p\\ &- \Xi_{+,j}\big(w_{0,j} + (w_{+,j}-w_{0,j})\big)^p - \Xi_{0,j}w_{0,j}^p - \Xi_{-,j}\big(w_{0,j}+(w_{-,j}-w_{0,j})\big)^p\\ = {}& \Xi_{+,j}w_{+,j}^p + \Xi_{0,j}w_{0,j}^p + \Xi_{-,j}w_{-,j}^p\\ &+ \mathcal{O}_{\mathcal{C}^{0,\mu}(U_1)}\big((|\delta_{\pm,j}|^2 + |\tau_{\pm,j}|^2)(\cosh\mathtt{x}_j)^{-2}(\cosh\mathtt{z}_j)^{-\vartheta\alpha}\big),\end{aligned}$$

since the 0th and the 1st order term in $(w_{\pm,j} - w_{0,j})$ in the two middle lines cancel out, and the the quality $\mathtt{w}_j^p = w_{\pm,j}^p$ holds whenever $\Xi_{\pm,j} = 1$. Using this and denoting by P_j the differential operator $(\Delta - \partial^2_{\mathtt{x}_j} - \partial^2_{\mathtt{z}_j})$ we can write:

$$
\begin{aligned}
E_1 = &\sum_{j=1}^{2}\big[P_j(\Xi_{+,j}w_{+,j}) + P_j(\Xi_{0,j}w_{0,j}) + P_j(\Xi_{-,j}w_{-,j})\\
&+ 2\sum_{j=1}^{2}\big[\partial_{\mathtt{z}_j}\Xi_{+,j}\partial_{\mathtt{z}_j}w_{+,j} + \partial_{\mathtt{z}_j}\Xi_{0,j}\partial_{\mathtt{z}_j}w_{0,j} + \partial_{\mathtt{z}_j}\Xi_{-,j}\partial_{\mathtt{z}_j}w_{-,j}\big]\\
&+ \sum_{j=1}^{2}\big[\partial^2_{\mathtt{z}_j}\Xi_{+,j}w_{+,j} + \partial^2_{\mathtt{z}_j}\Xi_{0,j}w_{0,j} + \partial^2_{\mathtt{z}_j}\Xi_{-,j}w_{-,j}\big]\\
&+ \mathcal{O}_{\mathcal{C}^\infty(\mathbb{R})}\big((|\delta_{\pm,,j}|^2 + |\tau_{\pm,j}|^2)(\cosh\mathtt{x}_j)^{-2}(\cosh\mathtt{z}_j)^{-\vartheta\alpha}\big).
\end{aligned}
\tag{5.1}
$$

We observe that the term involving P_j above is, because of (3.15), of order α^2 and in addition it decays in $\mathtt{x}_1$ and $\mathtt{z}_j$ exponentially, like $(\cosh\mathtt{x}_j)^{-\sigma}(\cosh\mathtt{z}_j)^{-\theta\alpha}$, for any $\sigma < 1$. In making this claim we use of the asymptotic form of the Dancer solution and also estimates (3.3), (3.18). Similar estimate can be proven for the two following terms since for example we have

$$
\partial_{\mathtt{z}_j}\Xi_{+,j}(\alpha\mathtt{z}_j) = \alpha\Xi'_{+,j}(\alpha\mathtt{z}_j), \quad \partial^2_{\mathtt{z}_j}\Xi_{+,j}(\alpha\mathtt{z}_j) = \alpha\Xi''_{+,j}(\alpha\mathtt{z}_j)
$$

and $\delta_{\pm,j}, \tau_{\pm,j} \sim \alpha^{1+\kappa_4}$, while on the other hand $\partial_{\mathtt{z}_j}w_{0,j} = 0$. Thus we get:

$$
\|E_1\|_{\mathcal{C}^{0,\mu}_{\sigma,\theta\alpha}(U_1)} \le C\alpha^2. \tag{5.2}
$$

The second term denoted by E_2 above satisfies an estimate of the same type by (3.17). We observe also that for any $\sigma < 1$:

$$
\|E_1\|_{\mathcal{C}^{0,\mu}_{\theta\alpha}(U_1\cap V^o_\varsigma)} \le C\alpha^{2+\frac{3}{4}\sigma}, \tag{5.3}
$$

since in $U_1 \cap V^o_\varsigma$ we have $|\mathtt{x}_1| \ge \frac{3}{4}\log\frac{1}{\alpha}$. It is important that in (5.3) we take the exponential weight in the norm only in the $\mathtt{z}_1$ direction. Again, same estimate is true for E_2.

Finally, we estimate the term denoted by E_3. It is not hard to see that the leading order in E_3 comes from the first three terms in its definition and thus we have:

$$
\begin{aligned}
E_3 &\sim g_p(\mathtt{w}_1 + \mathtt{w}_2) - g_p(\mathtt{w}_1) - g_p(\mathtt{w}_2)\\
&= p\mathtt{w}_1^{p-1}\mathtt{w}_2 - \mathtt{w}_2^p + \frac{p(p-1)}{2}(\zeta\mathtt{w}_1 + (1-\zeta)\mathtt{w}_2)^{p-2}\mathtt{w}_2^2\\
&\sim p\mathtt{w}_1^{p-1}\mathtt{w}_2,
\end{aligned}
$$

with some $\zeta \in (0,1)$. The last relation is easily justified, since in U_1 we have $\mathtt{w}_1 \gg \mathtt{w}_2$. We need to consider the product $\mathtt{w}_1^{p-1}\mathtt{w}_2$. We use (3.9) to express $\mathtt{x}_2$ in terms of $\mathtt{x}_1$ to get, as $\mathtt{z}_1 \to \pm\infty$:

$$
\mathtt{x}_2 = \mathtt{x}_1 + (a_{\pm,1} - a_{\pm,2})\alpha\mathtt{z}_1 - 2\log\frac{1}{2} + (\mathtt{x}_1 + \mathtt{z}_1)\mathcal{O}(\alpha^2) + \mathcal{O}(1),
$$

where the coefficients $a_{\pm,j}$ satisfy (2.17). From this we find:

$$
|\mathtt{w}_1\mathtt{w}_2| \le C\alpha^2(\cosh\mathtt{x}_1)^{c\alpha^2}(\cosh\mathtt{z}_1)^{-\vartheta\alpha + c\alpha^2}, \tag{5.4}
$$

with some $c > 0$. In all we have then, with $0 < \sigma < p - 2$:

$$|\mathtt{w}_1^{p-1}\mathtt{w}_2| \le C\alpha^2(\cosh \mathtt{x}_1)^{-\sigma}(\cosh \mathtt{z}_1)^{-\theta\alpha}, \tag{5.5}$$

hence, with $\theta < \vartheta$,

$$\|E_3\|_{\mathcal{C}^{0,\mu}_{\sigma,\theta\alpha}(U_1)} \le C\alpha^2. \tag{5.6}$$

From this (4.16) it follows in the set $U_1 \cap \mathrm{supp}\chi_1$. Exactly same argument can be carried out in the set

$$U_2 := \{\mathtt{x}_1 + \mathtt{x}_2 > 0\} \cap V_{\frac{\varsigma}{2}}.$$

It is also easy to see from the above considerations that $S(\mathtt{w})$ is continuous as a function of its parameters.

To conclude (4.18) restricted to the set $U_1 \cap V_\varsigma^o$ we observe that as $\mathtt{x}_1 > \frac{3}{4}\log\frac{1}{\alpha}$ in $U_1 \cap V_\varsigma^o$ from (5.5) we get:

$$\|\mathtt{w}_1^{p-1}\mathtt{w}_2\|_{\mathcal{C}^{0,\mu}_{\theta\alpha}(U_1\cap V_\varsigma^o)} \le C\alpha^{2+\frac{3}{4}\sigma}.$$

Finally in the complement of $U_1 \cup U_2$ in V_ς we have for instance the following terms to estimate for each $j = 1, 2$:

$$\partial^2_{\mathtt{x}_j}(X^*_{\alpha,j}\eta_\varsigma)(X^*_{\alpha,j}\mathtt{w}_j) + 2\partial_{\mathtt{x}_j}(X^*_{\alpha,j}\eta_\varsigma)\partial_{\mathtt{x}_j}(X^*_{\alpha,j}\mathtt{w}_j) \le Ce^{-|\mathtt{x}_j|} = Ce^{-\sigma|\mathtt{x}_j|}e^{-(1-\sigma)|\mathtt{x}_j|}.$$

In the support of $\partial^2_{\mathtt{x}_j}(X^*_{\alpha,j}\eta_\varsigma)$, $\partial_{\mathtt{x}_j}(X^*_{\alpha,j}\eta_\varsigma)$ we have

$$|\mathtt{x}_j| \ge \frac{\varsigma}{2\alpha}\sqrt{1 + |\mathtt{z}_j|^2},$$

hence we can estimate, with some constants C_1, C_2 depending on σ and ς:

$$e^{-(1-\sigma)|\mathtt{x}_j|} \le e^{-\frac{C_1}{\alpha}}e^{-\frac{C_2}{\alpha}|\mathtt{z}_j|} \le C\alpha^2 e^{-\theta\alpha|\mathtt{z}_j|},$$

provided that α is taken sufficiently small. It follows from this:

$$|\partial^2_{\mathtt{x}_j}(X^*_{\alpha,j}\eta_\varsigma)(X^*_{\alpha,j}\mathtt{w}_j)| + |2\partial_{\mathtt{x}_j}(X^*_{\alpha,j}\eta_\varsigma)\partial_{\mathtt{x}_j}(X^*_{\alpha,j}\mathtt{w}_j)| \le C\alpha^2 e^{-\sigma|\mathtt{x}_j|}e^{-\theta\alpha|\mathtt{z}_j|}. \tag{5.7}$$

We obtain (4.18) noting that in V_ς^o we have

$$|\mathtt{x}_j| \ge \frac{3}{4}\log\frac{1}{\alpha}. \tag{5.8}$$

(Notice that the estimate (4.18) does not carry any weight in $\mathtt{x}_j$).

To show the Lipschitz property (4.17) we observe that the dependence on the function $\mathbf{h}$ appears in the expression for the operator P_j above and also in the nonlinearity because of the formula (3.9), through terms of order α^{κ_2}. In particular the leading order term for $S(\cdot, \mathbf{h}^{(1)}) - S(\cdot, \mathbf{h}^{(2)})$ comes from estimating an expression similar to (5.4). This gives the factor α^2 in the first line in the estimate (4.17). As for the the Lipschitz dependence on $\mathbf{e}$ we observe that the leading behavior of $S(\cdot, \mathbf{e}^{(1)}) - S(\cdot, \mathbf{e}^{(2)})$ comes form the linear term (in $\mathbf{e}$) denoted above by E_2. Thus the second part of the estimate (4.17) follows. We omit somewhat tedious details. Again using (5.8) we conclude (4.19).

5.2. **The error in the general case.** In the general case $S(\mathtt{w})$, i.e. when $k > 2$ and $p > 2$ we consider the following subsets of $\mathbb{R}^2$:

$$U_j := \{\mathtt{x}_j + \mathtt{x}_{j-1} \geq 0\} \cap \{\mathtt{x}_j + \mathtt{x}_{j+1} \leq 0\} \cap V_{\frac{\varsigma}{2}},$$
$$U_1 := \{\mathtt{x}_1 \leq 0\} \cap \{\mathtt{x}_1 + \mathtt{x}_2 \leq 0\} \cap V_{\frac{\varsigma}{2}},$$
$$U_k := \{\mathtt{x}_k + \mathtt{x}_{k-1} \geq 0\} \cap \{\mathtt{x}_k \geq 0\} \cap V_{\frac{\varsigma}{2}}.$$

Since, by (3.22), $\mathtt{w} = \bar{w}$ in $V_{\frac{\varsigma}{2}}$ we can write

$$S(\mathtt{w}) = \sum_{j=1}^{k} \chi_{U_j} S(\bar{w}) + S\big((1-\eta_\varsigma)\bar{w}\big), \tag{5.9}$$

where χ_{U_j} denotes the characteristic function of the set U_j.

We fix a j and consider the error restricted to the set U_j. Setting for convenience $g_p(t) = -t + t_+^p$ and using the notation (3.20) we have in U_j:

$$\begin{aligned} S(\bar{w}) = {} & \underbrace{\Delta \mathtt{w}_j + g_p(\mathtt{w}_j)}_{E_{1,j}} + \underbrace{\Delta\big(e_j(\alpha \mathtt{z}_j)Z_j\big) + g_p'(\mathtt{w}_j)\big(e_j(\alpha \mathtt{z}_j)Z_j\big)}_{E_{2,j}} \\ & + \sum_{i\neq j} \underbrace{\Delta \mathtt{w}_i + g_p(\mathtt{w}_i)}_{E_{1,i}} + \sum_{i \neq j} \underbrace{\Delta\big(e_i(\alpha \mathtt{z}_i)Z_j\big) + g_p'(\mathtt{w}_i)\big(e_i(\alpha \mathtt{z}_j)Z_i\big)}_{E_{2,i}} \\ & + \underbrace{g_p\Big(\sum_{i=1}^{k} \mathtt{w}_i + e_i(\alpha \mathtt{z}_i)Z_j\Big) - \sum_{i=1}^{k} g_p(\mathtt{w}_i) - \sum_{i=1}^{k} g_p'(\mathtt{w}_i)\big(e_i(\alpha \mathtt{z}_j)Z_i\big)}_{E_3}. \end{aligned} \tag{5.10}$$

All the components above can be estimated using the same argument as in the case of two lines noting that the error due to the interactions between the bump lines is the biggest when the closest neighbors are considered. Another observation is that in the Taylor expansion of the nonlinear function $g_p(\mathtt{w})$, $p > 2$ around w_j all components with powers higher than 2 give rise to terms that are negligible. We leave the details to the reader.

6. The background equation: Proof of Lemma 4.1

Let us consider first the following problem:

$$(\Delta - 1)\psi = h, \quad \text{in } \mathbb{R}^2, \tag{6.11}$$

where $h \in \mathcal{C}^{0,\mu}(\mathbb{R}^2)$ is such that

$$\|h(\cosh \mathtt{z}_j)^{\theta\alpha}\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)} < \infty, \tag{6.12}$$

for $j = 1, \dots, k$ (here $\mathtt{z}_j = \mathtt{z}_j(z)$ via (3.2)). Since by assumption $h \in \mathcal{C}^{0,\mu}(\mathbb{R}^2)$ as well, by maximum principle and elliptic regularity theory we get the existence of a unique solution ψ such that

$$\|\psi\|_{\mathcal{C}^{2,\mu}(\mathbb{R}^2)} \leq C\|h\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)}.$$

We will now prove:

$$\|\psi(\cosh \mathtt{z}_j)^{\theta\alpha}\|_{\mathcal{C}^{2,\mu}(\mathbb{R}^2)} \leq C\|h(\cosh \mathtt{z}_j)^{\theta\alpha}\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)}. \tag{6.13}$$

As we will see (4.21)) will follow from this. Using (3.2) we see that functions of the form:

$$\psi_{\theta\alpha,\nu} = (\cosh \mathtt{z}_j)^{-\theta\alpha} + \nu\big[\cosh(\frac{x}{2}) + \cosh(\frac{z}{2})\big],$$

with $\nu \geq 0$ and α sufficiently small are positive supersolutions for $\Delta - 1$ in $\mathbb{R}^2$. In fact:

$$(\Delta - 1)\psi_{\theta\alpha,\nu} \leq -\frac{1}{4}\psi_{\theta\alpha,\nu}.$$

Considering now the function

$$\omega_{\theta\alpha,\nu,M} = M\|h(\cosh \mathtt{z}_j)^{\theta\alpha}\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)}\psi_{\theta\alpha,\nu} - \psi,$$

where M large is to be chosen, we get:

$$\begin{aligned}(\Delta - 1)\omega_{\theta\alpha,\nu,M} &\leq -\frac{M}{4}\|h(\cosh \mathtt{z}_j)^{\theta\alpha}\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)}\psi_{\theta\alpha,\nu} + h\\ &\leq -\frac{M}{4}\|h(\cosh \mathtt{z}_j)^{\theta\alpha}\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)}\psi_{\theta\alpha,\nu}\\ &\quad + \|h(\cosh \mathtt{z}_j)^{\theta\alpha}\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)}(\cosh \mathtt{z}_j)^{-\theta\alpha}\\ &\leq 0.\end{aligned}$$

By letting $\nu \to 0$ we get the upper bound:

$$\psi(\cosh \mathtt{z}_j)^{\theta\alpha} \leq C\|h(\cosh \mathtt{z}_j)^{\theta\alpha}\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)}.$$

The lower bound and the rest of the proof of (6.13) follow by a straightforward argument and are left to the reader.

Next we need to examine the size of the function Q and also its dependence on $\boldsymbol{\phi}$ and $\mathbf{h}$, $\mathbf{e}$ and other parameters. We will now assume $\boldsymbol{\phi}$ to be given and of finite $\mathcal{C}^{2,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)$ norm. We will show that

$$\|Q(\cosh \mathtt{z}_j)^{\theta\alpha}\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)} \leq C\alpha^{2+\frac{3}{4}\sigma} + C\alpha^{\frac{3}{4}\sigma}\sum_{j=1}^{k}\|X^*_{\alpha,j}\phi_j\|_{\mathcal{C}^{0,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)}.$$

from which, using (6.13) the required estimate will follow. In the remainder of the proof we will use the fact that in the $\operatorname{supp} Q$ we have

$$|\mathtt{x}_j| \geq \frac{3}{4}\log\frac{1}{\alpha},$$

(c.f. (5.8)) to estimate terms whose norm (including the exponential weight in $\mathtt{x}_j$) is bounded (see for example the proof of estimate (4.18)). We observe that the first term on the right hand side above comes from $(1 - \sum_{i=1}^{k}\rho_i)S(\mathtt{w})$ and has already been estimated in (4.18). To estimate the remaining terms involved in Q we observe that they depend on the functions $\boldsymbol{\phi}$ and ψ, see (4.10). For example, using the fact that the derivatives of the functions ρ_j are supported in the set where

$$\frac{3}{4}\log\frac{1}{\alpha} \leq |\mathtt{x}_j| \leq \log\frac{1}{\alpha},$$

we get for all $j = 1, \ldots, k$

$$\|(\mathbb{L}\phi_j\rho_j) - \rho_j\mathbb{L}(\phi_j))(\cosh \mathtt{z}_j)^{\theta\alpha}\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)} \leq C\alpha^{\frac{3}{4}\sigma}\|X^*_{\alpha,j}\phi_j\|_{\mathcal{C}^{2,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)}.$$

Finally we will use (5.8) and the fact that $\mathbb{L}-\Delta+1 = p\mathtt{w}_+^{p-1}$ with $p>2$ to get:

$$\big\|\big[\big(1-\sum_{i=1}^{k}\rho_i\big)(\mathbb{L}-\Delta+1)\psi\big](\cosh \mathtt{z}_j)^{\theta\alpha}\big\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)} \leq C\alpha^{\frac{3}{4}\sigma}\|\psi(\cosh z)^{\theta\alpha}\|_{\mathcal{C}^{2,\mu}(\mathbb{R}^2)}.$$

Summarizing, we have found:

$$\begin{aligned}\|Q(\boldsymbol{\phi},\psi)(\cosh \mathtt{z}_j)^{\theta\alpha}\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)} &\leq C\alpha^{\frac{3}{2}\sigma}\|\psi(\cosh \mathtt{z}_j)^{\theta\alpha}\|_{\mathcal{C}^{2,\mu}(\mathbb{R}^2)}\\ &\quad + C\alpha^{\frac{3}{4}\sigma}\big[\alpha^2+\sum_{j=1}^{k}\|X^*_{\alpha,j}\phi_j\|_{\mathcal{C}^{2,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)}\big].\end{aligned} \tag{6.14}$$

Now assuming that $\boldsymbol{\phi}$ is given, using (6.11)–(6.13) and a standard fixed point argument we find a $\psi=\psi(\boldsymbol{\phi})$ that satisfies (4.13). Moreover we have:

$$\|\psi(\boldsymbol{\phi})(\cosh \mathtt{z}_j)^{\theta\alpha}\|_{\mathcal{C}^{2,\mu}(\mathbb{R}^2)} \leq C\alpha^{\frac{3}{4}\sigma}\big[\alpha^2+\sum_{j=1}^{k}\|X^*_{\alpha,j}\phi_j\|_{\mathcal{C}^{2,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)}\big]. \tag{6.15}$$

Since the function $Q(\boldsymbol{\phi},\psi)$ is a uniform contraction (as a function of ψ) and it is continuous (as function of its parameters, assuming of course that $\boldsymbol{\phi}$ is continuous), we conclude that ψ is a continuous function of $\mathbf{v},\mathbf{h},\mathbf{e},\boldsymbol{\delta}_\pm,\boldsymbol{\tau}_\pm$. It is also easy to see that $\psi(\boldsymbol{\phi})$ is Lipschitz as a function of $\boldsymbol{\phi}$ and in fact we have:

(6.16)

$$\|[\psi(\boldsymbol{\phi}^{(1)})-\psi(\boldsymbol{\phi}^{(2)})](\cosh \mathtt{z}_j)^{\theta\alpha}\|_{\mathcal{C}^{2,\mu}(\mathbb{R}^2)} \leq C\alpha^{\frac{3}{4}\sigma}\sum_{j=1}^{k}\big\|X^*_{\alpha,j}\big(\phi_j^{(1)}-\phi_j^{(2)}\big)\big\|_{\mathcal{C}^{2,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)}.$$

Final estimate in Lemma 4.1, namely (4.23) follows from (4.19).

Remark 6.1. *We observe that a slight modification of the proof of (6.13) gives*

$$\|\psi(\cosh z)^{a}(\cos z)^{\sigma}\|_{\mathcal{C}^{2,\mu}(\mathbb{R}^2)} \leq C\|h\|_{\mathcal{C}^{0,\mu}_{\sigma,a}(\mathbb{R}^2)}. \tag{6.17}$$

In the case at hand we have, with $\sigma_*<\sigma$, $\theta_*<\theta$

$$\|Q\|_{\mathcal{C}^{0,\mu}_{\sigma_*,\theta_*\alpha}(\mathbb{R}^2)} \leq C\alpha^{\frac{3}{4}\sigma-\sigma_*(k+1)}\big[\alpha^2+\sum_{j=1}^{k}\|X^*_{\alpha,j}\phi_j\|_{\mathcal{C}^{2,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)}\big],$$

because of (4.38). Therefore when $\boldsymbol{\phi}$ is the true solution of (4.4) we get:

$$\|Q\|_{\mathcal{C}^{0,\mu}_{\sigma_*,\theta_*\alpha}(\mathbb{R}^2)} \leq C\alpha^{2-\sigma_*(k+1)},$$

which is the same type of estimate as (4.37).

7. A priori estimates and invertibility of the basic linear operator

7.1. **Non-degeneracy of the homoclinic.** In this section we will consider first the following linearized operator

$$L_0(\phi)=\phi_{xx}+g_p'(w)\phi, \quad g_p'(w)=pw^{p-1}-1.$$

We recall some well known facts about L_0. First notice that $L_0(w')=0$ i.e. has one dimensional kernel. Second we observe that

$$\lambda_1=\frac{1}{4}(p-1)(p+3), \qquad Z=\frac{w^{(p+1)/2}}{\sqrt{\int_{\mathbb{R}} w^{p+1}}},$$

correspond, respectively, to principal eigenvalue and eigenfunction of L_0. Except for $\lambda_1 > 0$ and $\lambda_2 = 0$ the rest of the spectrum of L_0 is negative. This means in particular that there exists a positive constant γ_0 such that

$$\langle L_0(\phi), \phi\rangle \geq \gamma_0 \|\phi\|^2_{L^2(\mathbb{R})}, \tag{7.1}$$

whenever

$$\langle \phi, w'\rangle = 0 = \langle \phi, Z\rangle.$$

From (7.1) it also follows that there exists a $\gamma > 0$ such that:

$$\langle L_0(\phi), \phi\rangle \geq \gamma(\|\phi_x\|^2_{L^2(\mathbb{R})} + \|\phi\|^2_{L^2(\mathbb{R})}), \tag{7.2}$$

As another consequence of these facts we observe that problem

$$L_0(\phi) - \xi^2\phi = h, \tag{7.3}$$

is uniquely solvable whenever $\xi \neq \pm\sqrt{\lambda_1}, 0$ for $h \in L^2(\mathbb{R})$. Actually, rather standard argument, using comparison principle and the fact that L_0 is of the form

$$L_0(\phi) = \phi_{xx} - \phi + q(x)\phi, \quad |q(x)| \leq Ce^{-c|x|},$$

can be used to show that whenever h is for instance a compactly supported function then the solution of (7.3) is an exponentially decaying function.

Let us consider now the basic linearized operator

$$L(\phi) = L_0(\phi) + \phi_{zz},$$

defined in the whole plane $(x, z) \in \mathbb{R}^2$. Using (7.1) we get that

$$\langle L(\phi), \phi\rangle \geq \gamma_0 \|\phi\|^2_{L^2(\mathbb{R}^2)}, \tag{7.4}$$

whenever

$$\int_{\mathbb{R}} \phi(x, z)w'(x)\, dx = 0 = \int_{\mathbb{R}} \phi(x, z)Z(x)\, dx, \quad \text{for all } z.$$

Equation $L(\phi) = 0$, has 3 obvious bounded solutions

$$w'(x), \quad Z(x)\cos(\sqrt{\lambda_1}z), \quad Z(x)\sin(\sqrt{\lambda_1}z).$$

Our first result shows that converse is also true.

Lemma 7.1. *Let ϕ be a bounded solution of the problem*

$$L(\phi) = 0 \quad \text{in } \mathbb{R}^2. \tag{7.5}$$

Then $\phi(x, z)$ is a linear combination of the functions $w'(x)$, $Z(x)\cos(\sqrt{\lambda_1}z)$, and $Z(x)\sin(\sqrt{\lambda_1}z)$.

Proof. Let assume that ϕ is a bounded function that satisfies

$$\phi_{zz} + \phi_{xx} + (pw^{p-1} - 1)\phi = 0. \tag{7.6}$$

Let us consider the Fourier transform of $\phi(x, z)$ in the z variable, $\hat\phi(x, \xi)$ which is by definition the distribution defined as

$$\langle \hat\phi(x, \cdot), \mu\rangle_{\mathbb{R}} = \langle \phi(x, \cdot), \hat\mu\rangle_{\mathbb{R}} = \int_{\mathbb{R}} \phi(x, \xi)\hat\mu(\xi)d\xi,$$

where $\mu(\cdot)$ is any smooth rapidly decreasing function. Let us consider a smooth rapidly decreasing function ψ of the two variables (x, ξ). Then from equation (7.6) we find

$$\int_{\mathbb{R}} \langle \hat\phi(x, \cdot), \psi_{xx} - \xi^2\psi + (pw^{p-1} - 1)\psi\rangle_{\mathbb{R}} dx = 0.$$

Let $\varphi(x)$ and $\mu(\xi)$ be smooth and compactly supported functions such that

$$\{\sqrt{\lambda_1}, -\sqrt{\lambda_1}, 0\} \cap \text{supp}\,(\mu) = \emptyset.$$

Then we can solve the equation

$$\psi_{xx} - \xi^2\psi + (pw^{p-1} - 1)\psi = \mu(\xi)\varphi(x), \quad x \in \mathbb{R},$$

uniquely for a smooth, rapidly decreasing function $\psi(x, \xi)$ such that $\psi(x, \xi) = 0$ whenever $\xi \notin \text{supp}\,(\mu)$. We conclude that

$$\int_{\mathbb{R}} \langle \hat{\phi}(x, \cdot), \mu \rangle_{\mathbb{R}} \varphi(x)\, dx = 0,$$

so that for all $x \in \mathbb{R}$, $\langle \hat{\phi}(x, \cdot), \mu \rangle_{\mathbb{R}} = 0$, whenever $\{\sqrt{\lambda_1}, -\sqrt{\lambda_1}, 0\} \cap \text{supp}\,(\mu) = \emptyset$, in other words

$$\text{supp}\,(\hat{\phi}(x, \cdot)) \subset \{\sqrt{\lambda_1}, -\sqrt{\lambda_1}, 0\}.$$

By distribution theory we find that $\hat{\phi}(x, \cdot)$ is a linear combination (with coefficients depending on x) of derivatives up to a finite order of Dirac masses supported in $\{\sqrt{\lambda_1}, -\sqrt{\lambda_1}, 0\}$. Taking inverse Fourier transform, we get that

$$\phi(x, z) = p_0(z, x) + p_1(z, x)\cos(\sqrt{\lambda_1} z) + p_2(z, x)\sin(\sqrt{\lambda_1} z),$$

where p_j are polynomials in z with coefficients depending on x. Since ϕ is bounded these polynomials are of zero order, i.e. $p_j(z, x) \equiv p_j(x)$, and the bounded functions p_j must satisfy the equations

$$L_0(p_0) = 0, \quad L_0(p_1) - \lambda_1 p_1 = 0, \quad L_0(p_2) - \lambda_1 p_2 = 0,$$

and the desired result follows. □

7.2. *A priori* **estimates for the basic linearized operator.** The linear theory used in this paper is based on *a priori* estimates for the solutions of the following problem

$$L(\phi) = h, \quad \text{in } \mathbb{R}^2. \tag{7.7}$$

The results of Lemma 7.1 imply that such estimates without imposing extra conditions on ϕ may not exist. The form of the bounded solutions of $L(\phi) = 0$ and (7.4) suggest the following orthogonality conditions:

$$\int_{\mathbb{R}} \phi(x, z) w'(x)\, dx = 0 = \int_{\mathbb{R}} \phi(x, z) Z(x)\, dx, \quad \text{for all } z \in \mathbb{R}. \tag{7.8}$$

With these restrictions imposed we have the following result concerning *a priori* estimates for this problem.

Lemma 7.2. *Assuming that ϕ is a bounded solution of* (7.7) *satisfying* (7.8) *we have*

$$\|\phi\|_{L^\infty(\mathbb{R}^2)} \le C\|h\|_{L^\infty(\mathbb{R}^2)}.$$

Proof. We will argue by contradiction. Assuming the opposite means that there are sequences ϕ_n, h_n such that

$$\|\phi_n\|_\infty = 1, \quad \|h_n\|_\infty \to 0,$$

and

$$L(\phi_n) = h_n, \quad \text{in } \mathbb{R}^2, \tag{7.9}$$

$$\int_{\mathbb{R}} \phi_n(x,z) w_x(x)\, dx = 0 = \int_{\mathbb{R}} \phi_n(x,z) Z(x)\, dx, \quad \text{for all } z \in \mathbb{R}. \tag{7.10}$$

Let us assume that $(x_n, z_n) \in \mathbb{R}^2$ is such that

$$|\phi_n(x_n, z_n)| \to 1.$$

We claim that the sequence x_n is bounded. Indeed, if not, using the fact that $L\phi = \Delta\phi - \phi + O(e^{-c|x|})\phi$ and employing elliptic estimates we find that the sequence of functions

$$\tilde{\phi}_n(x,z) = \phi_n(x_n + x, z_n + z),$$

converges, up to a subsequence, locally uniformly to a solution $\tilde{\phi}$ of the equation

$$\Delta\tilde{\phi} - \tilde{\phi} = 0, \quad \text{in } \mathbb{R}^2,$$

whose absolute value attains its maximum at $(0,0)$, This implies $\tilde{\phi} \equiv 0$, so that x_n is indeed bounded. Let now

$$\tilde{\phi}_n(x,z) = \phi_n(x, z_n + z).$$

Then $\tilde{\phi}_n$ converges uniformly over compacts to a bounded, nontrivial solution $\tilde{\phi}$ of

$$L(\tilde{\phi}) = 0 \quad \text{in } \mathbb{R}^2,$$

$$\int_{\mathbb{R}} \tilde{\phi}(x,z) w_x(x)\, dx = 0 = \int_{\mathbb{R}} \tilde{\phi}(x,z) Z(x)\, dx, \quad \text{for all } z \in \mathbb{R}.$$

Lemma 7.1 then implies $\tilde{\phi} \equiv 0$, a contradiction and the proof is concluded. □

Using Lemma 7.2 we can also find a priori estimates with norms involving exponential weights. When the weights involve only the x variable we have the following a priori estimates.

Lemma 7.3. *Assuming that $\|(\cosh x)^\sigma h\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)} < +\infty$, $\sigma \in [0,1)$, then a bounded solution ϕ of (7.7)–(7.8) satisfies*

$$\|(\cosh x)^\sigma \phi\|_{\mathcal{C}^{2,\mu}(\mathbb{R}^2)} \le C\|(\cosh x)^\sigma h\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)}. \tag{7.11}$$

Proof. We already know that

$$\|\phi\|_{L^\infty(\mathbb{R}^2)} \le C\|(\cosh x)^\sigma h\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)}.$$

We set $\tilde{\phi} = \phi\|(\cosh x)^\sigma h\|^{-1}_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)}$. Then we have

$$L(\tilde{\phi}) = \tilde{h}, \quad \text{where } \|(\cosh x)^\sigma \tilde{h}\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)} = 1,$$

and also $\|\tilde{\phi}\|_{L^\infty(\mathbb{R}^2)} \le C$. Let us fix a number $R_0 > 0$ such that for $x > R_0$ we have

$$pw^{p-1}(x) < \frac{1-\sigma^2}{2},$$

which is always possible since $w(x) = O(e^{-c|x|})$. For an arbitrary number $\rho > 0$ let us set

$$\bar{\phi}(x,z) = \rho[\cosh(z/2) + e^{\sigma x}] + Me^{-\sigma x},$$

where M is to be chosen. Then we find that,

$$L(\bar{\phi}) \le -\frac{M(1-\sigma^2)}{4} e^{-\sigma x}, \quad \text{for } x > R_0.$$

Thus

$$L(\bar\phi) \le \tilde h, \quad \text{for } x > R_0,$$

if

$$\frac{M(1-\sigma^2)}{4} \ge \|(\cosh x)^\sigma \tilde h\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)} = 1.$$

If we also also assume

$$Me^{-\sigma R_0} \ge \|\tilde\phi\|_\infty,$$

we conclude from maximum principle that $\tilde\phi \le \bar\phi$. Letting $\rho \to 0$ we get (since M can be fixed independent on ρ),

$$\tilde\phi \le Me^{-\sigma x}, \quad \text{for } x > 0,$$

hence

$$\phi \le M\|(\cosh x)^\sigma h\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)} e^{-\sigma x}, \quad \text{for } x > 0.$$

In a similar way we obtain the lower bound

$$\phi \ge -M\|(\cosh x)^\sigma h\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)}, \quad \text{for } x > 0.$$

Finally, the same argument for $x < 0$ yields

$$\|(\cosh x)^\sigma \phi\|_{L^\infty(\mathbb{R}^2)} \le C\|(\cosh x)^\sigma h\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)}.$$

The required estimate now follows from local elliptic estimates and the proof is concluded. □

When we also take into account the exponential decay in the z variable we have the following a priori estimates.

Lemma 7.4. *There exists $a_0 > 0$ such that assuming $\|h(\cosh x)^\sigma (\cosh z)^a\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)} < +\infty$, $\sigma \in (0,1)$, $a \in [0,1)$, for any bounded solution ϕ to problem (7.7)-(7.8) we have*

$$\|(\cosh x)^\sigma (\cosh z)^a \phi\|_{\mathcal{C}^{2,\mu}(\mathbb{R}^2)} \le C_\sigma \|(\cosh x)^\sigma (\cosh z)^a h\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)}.$$

Proof. We already know that

$$\|(\cosh x)^\sigma \phi\|_{\mathcal{C}^{2,\mu}(\mathbb{R}^2)} \le C\|(\cosh x)^\sigma (\cosh z)^a h\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)}.$$

Then we may write

$$\psi(z) = \int_{\mathbb{R}} \phi^2(x,z)\,dx,$$

and differentiate twice weakly to get

$$\psi''(z) = 2\int_{\mathbb{R}} \phi_z^2\,dx + 2\int_{\mathbb{R}} \phi_{zz}\phi\,dx.$$

We have

$$\int_{\mathbb{R}} \phi_{zz}\phi\,dx = \int_{\mathbb{R}} \phi_x^2\,dx + \int_{\mathbb{R}} (1-pw^{p-1})\phi^2\,dx + \int_{\mathbb{R}} h\phi. \tag{7.12}$$

Because of the orthogonality conditions (7.8) we also have by (7.2) that,

$$\int_{\mathbb{R}} \phi_x^2\,dx + \int_{\mathbb{R}} (1-pw^{p-1})\phi^2\,dx \ge \gamma \int_{\mathbb{R}} (\phi_x^2 + \phi^2)\,dx, \quad \gamma > 0.$$

Hence we find that for a certain constant $C > 0$

$$\psi''(z) \ge \frac{\gamma}{4}\psi(z) - C\int_{\mathbb{R}} h^2(x,z)\,dx,$$

so that

$$-\psi''(z) + \frac{\gamma}{4}\psi(z) \le \frac{C}{\sigma} e^{-2a|z|} \|(\cosh x)^\sigma (\cosh z)^a h\|^2_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)}.$$

Since we also know that ψ is bounded by:

$$|\psi(z)| \le \frac{C}{\sigma} \|(\cosh x)^\sigma (\cosh z)^a h\|^2_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)},$$

we can use a barrier of the form $\psi^+(z) = M\|h\|^2_{\sigma,a} e^{-2az} + \rho e^{2az}$, with M sufficiently large and $\rho > 0$ arbitrary, to get the bound $0 \le \psi \le \psi^+$ for $z \ge 0$ and any $a < \frac{\sqrt{\gamma}}{4} \equiv a_0$. A similar argument can be used for $z < 0$. Letting $\rho \to 0$ we get then

$$\int_{\mathbb{R}} \phi^2(x,z)\, dx \le C_\sigma e^{-2a|z|} \|(\cosh x)^\sigma (\cosh z)^a h\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)}, \quad a < a_0.$$

Elliptic estimates yield that for R_0 fixed and large

$$|\phi(x,z)| \le C_\sigma e^{-a|z|} \|(\cosh x)^\sigma (\cosh z)^a h\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)} \quad \text{for } |x| < R_0.$$

The corresponding estimate in the complementary region can be found by barriers. For instance in the quadrant $\{x > R_0, z > 0\}$ we may consider a barrier of the form

$$\bar\phi(x,z) = M \|(\cosh x)^\sigma (\cosh z)^a h\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)} e^{-(\sigma x + az)} + \rho e^{\frac{x}{2} + \frac{z}{2}},$$

with $\rho > 0$ arbitrarily small. Fixing M depending on R_0 we find the desired estimate for $|(\cosh x)^\sigma (\cosh z)^a \phi|$ in this quadrant by letting $\rho \to 0$. The argument in the remaining quadrants is similar. The corresponding bound for the $\mathcal{C}^{2,\mu}(\mathbb{R}^2)$ weighted norm is then deduced from local elliptic estimates. This concludes the proof. □

7.3. **The existence result for the basic linearized operator: Proof of Proposition 4.3.**

Proof. We will argue by approximations. Let us replace h in (4.24) by the function $h(x,z)\chi_{(-R,R)}(z)$ extended $2R$-periodically to the whole plane. With this right hand side we can give to the problem (4.24) a weak formulation in the closed subspace $H^1_R \subset H^1(\mathbb{R}^2)$ of functions that are $2R$-periodic in z and which also satisfy the orthogonality conditions in (4.24). To be more precise we say that ϕ_R is a weak solution of this problem if for

$$\langle L(\phi_R), \eta\rangle := \int_{-\infty}^{\infty}\int_{-R}^{R} \nabla\psi \cdot \nabla\eta\, dxdz + \int_{-\infty}^{\infty}\int_{-R}^{R} (1 - pw^{p-1})\psi\eta\, dxdz,$$

we have

$$\langle L(\phi_R), \eta\rangle = \int_{-\infty}^{\infty}\int_{-R}^{R} h\eta\, dxdz$$

for all tests functions $\eta \in H^1(\mathbb{R}^2)$ which are $2R$ periodic and which satisfy

$$\int_{\mathbb{R}} w'(x)\eta(x,z)\, dx = 0 = \int_{\mathbb{R}} Z(x)\eta(x,z)\, dx, \quad \text{for all } z \in (-R,R).$$

Because of the orthogonality conditions the bilinear form $\mathfrak{a}(\psi,\eta) = \langle L(\psi), \eta\rangle$ is actually positive definite in H^1_R and consequently there exists a unique $\phi_R \in H^1_R$ which satisfies

$$\mathfrak{a}(\phi_R, \eta) = \int_{-\infty}^{\infty}\int_{-R}^{R} h\eta\, dxdz, \quad \text{for all } \eta \in H^1_R.$$

Given that ϕ_R satisfies the orthogonality conditions we check that also for any smooth, compactly supported in $(-R,R)$ function $\tilde\eta(z)$ we have

$$\mathfrak{a}(\phi_R, w'(x)\tilde\eta(z)) = 0 = \int_{-\infty}^{\infty}\int_{-R}^{R} hw'(x)\tilde\eta(z)\,dxdz,$$
$$\mathfrak{a}(\phi_R, Z(x)\tilde\eta(z)) = 0 = \int_{-\infty}^{\infty}\int_{-R}^{R} hZ(x)\tilde\eta(z)\,dxdz.$$

This proves that ϕ_R is the unique weak solution of $L(\phi_R) = h$ in the space of $H^1(\mathbb{R}^2)$ functions which are $2R$ periodic in z. Letting $R \to +\infty$ and using the uniform a priori estimates valid for the approximations completes the proof of the Proposition. □

8. Estimates for the interaction system

We begin by proving Lemma 4.2.

Proof. We will use the definition of $X^*_{\alpha,j}k_j$ in (4.12) to estimate term by term. First we observe

$$\|X^*_{\alpha,j}(\chi_j S(\mathtt{w}))\|_{\mathcal{C}^{0,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)} \le C\alpha^2,$$

by (4.16). Next we will consider the nonlinear (in ϕ_j and ψ) term. Since by assumption

$$\|X^*_{\alpha,j}\phi_j\|_{\mathcal{C}^{2,\mu}_{\sigma,\theta\alpha}} \le \alpha^{\frac{3}{4}\sigma},$$

therefore by (4.21) we have

$$\|X^*_{\alpha,j}(\chi_j\psi)(\cosh \mathtt{z}_j)^{\theta\alpha}\|_{\mathcal{C}^{2,\mu}(\mathbb{R}^2)} \le C\alpha^{\frac{3}{4}\sigma}\big(\alpha^2 + \sum_{i=1}^k \|X^*_{\alpha,j}\phi_j\|_{\mathcal{C}^{2,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)}\big) \le C\alpha^{\frac{3}{2}\sigma}. \tag{8.1}$$

We will now estimate the nonlinear term, for which we get:

$$\chi_j N = N(\sum_{i=1}^k \rho_i\phi_i + \psi) = \chi_j N(\rho_j\phi_j + \psi),$$

using (4.7). Let us observe that N is a "quadratic" function of its argument. Indeed, with $p > 2$ we have for any $t, s \in \mathbb{R}$, $t \ge 0$:

$$|(s+t)_+^p - t^p - pt^{p-1}s| \le C\max\{t^{p-2}, |s|^{p-2}\}|s|^2,$$

Then it follows:

$$|X^*_{\alpha,j}(\chi_j N)| \le C(|X^*_{\alpha,j}\phi_j|^2 + |X^*_{\alpha,j}(\chi\psi)|^2)$$

We have in supp $X^*_{\alpha,j}(\chi_j)$:

$$|\mathtt{x}_j| \le \frac{15}{16}\log\frac{1}{\alpha}, \tag{8.2}$$

hence, by (8.1)

$$\|(\cosh \mathtt{x}_j)^\sigma(\cosh \mathtt{z}_j)^{\theta\alpha}X^*_{\alpha,j}(\chi_j\psi)\|^2_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)} \le C\alpha^{-\frac{15}{8}\sigma}\|(\cosh \mathtt{z}_j)^{\theta\alpha}X^*_{\alpha,j}(\chi_j\psi)\|^2_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)}$$
$$\le C\alpha^{-\frac{3}{8}\sigma}\big(\alpha^2 + \sum_{i=1}^k \|X^*_{\alpha,i}\phi_i\|_{\mathcal{C}^{2,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)}\big)^2.$$

Using this we find:

(8.3)

$$\|X^*_{\alpha,j}(\chi_j N)\|_{\mathcal{C}^{0,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)} \le C\big[\alpha^{4-\frac{3}{4}\sigma} + \|X^*_{\alpha,j}\phi_j\|^2_{\mathcal{C}^{2,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)} + \alpha^{\frac{3}{8}\sigma}\big(\sum_{i=1}^k \|X^*_{\alpha,i}\phi_i\|_{\mathcal{C}^{2,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)}\big)\big]$$

The next term we need to estimate is

$$X^*_{\alpha,j}\big(\chi_j(\mathbb{L}-\Delta+1)\psi\big) = X^*_{\alpha,j}(p\chi_j \mathtt{w}_+^{p-1}\psi).$$

Using the fact that $X^*_{\alpha,j}(\chi_j\mathtt{w}_+^{p-1})$ is an exponentially decaying function (in $\mathtt{x}_j$)

$$\begin{aligned}\|X^*_{\alpha,j}(\chi_j\mathtt{w}_+^{p-1}\psi)\|_{\mathcal{C}^{0,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)} &\le C\|(\cosh \mathtt{z}_j)^{\theta\alpha}X^*_{\alpha,j}(\chi_j\psi)\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)}\\ &\le \alpha^{\frac{3}{4}\sigma}\big(\alpha^2 + \sum_{i=1}^k \|X^*_{\alpha,i}\phi_i\|_{\mathcal{C}^{2,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)}\big).\end{aligned} \tag{8.4}$$

To estimate the last term we observe that using (3.15) we get :

$$\|X^*_{\alpha,j}[\chi_j(\Delta - \partial^2_{\mathtt{x}_j} - \partial^2_{\mathtt{z}_j})\phi_j]\| \le C\alpha\|X^*_{\alpha,j}\phi_j\|_{\mathcal{C}^{2,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)},$$

and also

$$\big\|X^*_{\alpha,j}\big[\chi_j\big(g_p'(\mathtt{w}) - g_p'(w_{0,j})\big)\big]\phi_j\big\| \le C\alpha\|X^*_{\alpha,j}\phi_j\|_{\mathcal{C}^{2,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)},$$

making use of (3.3), (3.17), (3.18). The proof of the Lipschitz property (4.33) is standard and is omitted. □

9. The reduced problem: error of the projections

In this sections we will fill in some details in the computations in section 4.7. We will begin with (4.39). We have computed the leading order of

$$\int_{\mathbb{R}} X^*_{\alpha,j}(\chi_j S(\mathtt{w})w'_{0,j})\,d\mathtt{x}_j,$$

which in particular gives rise to the Toda system, see (4.46)–(4.50). In particular we have neglected terms denoted by $P_j(\Xi_{\pm,j}w_{\pm,j}), P_j(\Xi_{0,j}w_{0,j})$ in (5.1). Among these lower order terms we will concentrate on one, representative term, namely, using the notation (3.13)–(3.14) and (4.56),

$$\int_{\mathbb{R}} a_{12,j}\Xi_{\pm,j}\chi_j(\partial^2_{\mathtt{x}_j,\mathtt{z}_j}w_{\pm,j})w_{0,j}\,d\mathtt{x}_j \sim -\alpha\sqrt{\lambda_1}h_j'\Theta'_{\pm,j}\Xi_{\pm,j}\int_{\mathbb{R}} w'Z'\,dx.$$

Now we observe that

$$\|\alpha\sqrt{\lambda_1}h_j'\Theta'_{\pm,j}\Xi_{\pm,j}\|_{\mathcal{C}^{0,\mu}_\theta(\mathbb{R})} \le C\alpha^{2+\kappa_2+\kappa_4-\mu} \le C\alpha^{2+\nu_1},$$

as long as (3.3) and (3.18) hold. Another important term comes from

$$X^*_{\alpha,j}\big(\chi_j(\mathbb{L}-\Delta+1)\psi\big) \sim X^*_{\alpha,j}\big(p\chi_j(\mathtt{w}_j)_+^{p-1}\psi\big). \tag{9.5}$$

Using Lemma 4.1 we get

$$\|\int_{\mathbb{R}} X^*_{\alpha,j}\big(\chi_j(\mathtt{w}_j)_+^{p-1}\psi\big)w'_{0,j}\,d\mathtt{x}_j\|_{\mathcal{C}^{0,\mu}_\theta(\mathbb{R})} \le C\alpha^{2+\frac{3}{4}\sigma-\mu} \le \alpha^{2+\nu_1}.$$

Other calculations can be done a similar way.

To see a representative term (slightly different than the ones we have seen above) in (4.40) we will recall the definition of k_j (4.12) and in particular consider this component of k_j that depends on the unknown function ϕ_j linearly, namely:

$$-X^*_{\alpha,j}\big(\chi_j \mathbb{L}(\phi_j)\big) + (X^*_{\alpha,j}\chi_j)\big[\partial^2_{\mathtt{x}_j} + \partial^2_{\mathtt{z}_j} + g'_p(w_{0,j})\big] X^*_{\alpha,j}\phi_j.$$

Although perhaps not immediately obvious but rather straightforward is the following relation

$$\int_R \big[-X^*_{\alpha,j}\big(\chi_j \mathbb{L}(\phi_j)\big) + (X^*_{\alpha,j}\chi_j)\big[\partial^2_{\mathtt{x}_j} + \partial^2_{\mathtt{z}_j} + g'_p(w_{0,j})\big] X^*_{\alpha,j}\phi_j\big] Z(\mathtt{x}_j)\, dx_j$$
$$\sim \int_{\mathbb{R}} X^*_{\alpha,j}\big(\chi_j (g'_p(w_{0,j}) - g'_p(\mathtt{w}_j))\phi_j\big) Z(\mathtt{x}_j)\, d\mathtt{x}_j.$$

Then we get

$$\Big\| \int_{\mathbb{R}} X^*_{\alpha,j}\big(\chi_j (g'_p(w_{0,j}) - g'_p(\mathtt{w}_j))\phi_j\big) Z(\mathtt{x}_j)\, d\mathtt{x}_j \Big\|_{\mathcal{C}^{0,\mu}_{\theta}(\mathbb{R})} \le C\alpha^{3+\kappa_4-\mu} \le C\alpha^{2+\nu_1}.$$

Let us now consider some of the terms we have neglected while considering Υ_j. One of them is

$$\Big| \int_{\mathbb{R}} \int_R X^*_{\alpha,j}\big(\chi_j (g'_p(w_{0,j}) - g'_p(\mathtt{w}_j))\phi_j\big) Z(\mathtt{x}_j) \cos(\sqrt{\lambda_1}\mathtt{z}_j)\, d\mathtt{x}_j d\mathtt{z}_j \Big|$$
$$\le C\alpha^{1+\kappa_4} \|\phi_j\|_{\mathcal{C}^{2,\mu}_{\sigma,\theta\alpha}(\mathbb{R}^2)} \int_{\mathbb{R}} (\cosh z)^{-\theta\alpha}\, dz$$
$$\le C\alpha^{2+\kappa_4}.$$

Another, similar in type term, is (c.f. (9.5)):

$$\Big| \int_{\mathbb{R}} \int_{\mathbb{R}} X^*_{\alpha,j}\big(\chi_j (\mathtt{w}_j)_+^{p-1}\psi\big) Z(\mathtt{x}_j) \cos(\sqrt{\lambda_1}\mathtt{z}_j)\, d\mathtt{x}_j d\mathtt{z}_j \Big|$$
$$\le C\|\psi (\cosh \mathtt{z}_j)^{\theta\alpha}\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)} \int_{\mathbb{R}} (\cosh z)^{-\theta\alpha}\, dz$$
$$\le C\alpha^{1+\frac{3}{4}\sigma}.$$

These terms are bounded by $\alpha^{1+\nu_1}$ for sufficiently small α. The rest of the calculations follow the same scheme and are omitted.

Acknowledgments: This work has been partly supported by Chilean research grants Fondecyt 1070389, 1050311, FONDAP, Nucleus Millennium grant P04-069-F, an Ecos-Conicyt contract, a grant under the MathAmSud program and an Earmarked Grant from RGC of Hong Kong. The fourth author thanks Professor Fanhua Lin for a nice conversation on the classification of solutions to (1.1), which motivated this research.

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