CLASSIFICATION OF SOLUTIONS TO GENERAL TODA SYSTEMS WITH SINGULAR SOURCES

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Abstract. We classify all the solutions to the elliptic Toda system associated to a general simple Lie algebra with singular sources at the origin and with finite energy. The solution space is shown to be parametrized by a subgroup of the corresponding complex Lie group. We also show the quantization result for the finite integrals. This work generalizes the previous works in [LWY12] and [Nie16] for Toda systems of types A and B, C. However, a more Lie-theoretic method is needed here for the general case, and the method relies heavily on the structure theories of the local solutions and of the W-invariants for the Toda system. This work will have applications to nonabelian Chern-Simons-Higgs gauge theory and to the mean field equations of Toda type.

1. Introduction

In this paper, we consider the following Toda systems on the plane. Let $\mathfrak{g}$ be a complex simple Lie algebra of rank n, and let (a_{ij}) be its Cartan matrix (see [Hel78, Kna02, FH91] and the Appendix for basic Lie theory). The Toda system associated to $\mathfrak{g}$ with singular sources at the origin and with finite energy is the following system of semilinear elliptic PDEs

$$
\begin{cases} \Delta u_i + 4\sum_{j=1}^{n} a_{ij} e^{u_j} = 4\pi\gamma_i\delta_0 \quad \text{on } \mathbb{R}^2, \quad \gamma_i > -1, \\ \int_{\mathbb{R}^2} e^{u_i}\, dx < \infty, \qquad\qquad 1 \le i \le n, \end{cases} \tag{1.1}
$$

where δ_0 is the Dirac delta function at the origin. Here the solutions u_i are required to be real and well-defined on the whole $\mathbb{R}^2$ minus the origin.

When the Lie algebra $\mathfrak{g} = A_1 = \mathfrak{sl}_2$ whose Cartan matrix is (2), the Toda system becomes the Liouville equation

$$
\Delta u + 8e^u = 4\pi\gamma\delta_0 \quad \text{on } \mathbb{R}^2, \quad \gamma > -1, \quad \int_{\mathbb{R}^2} e^u\, dx < \infty. \tag{1.2}
$$

The Toda system (1.1) and the Liouville equation (1.2) arise in many physical and geometric problems. For example, in the Chern-Simons theory, the Liouville equation is related to the abelian gauge field theory, while the Toda system is related to nonabelian gauges (see [Yan01, Tar08]). On the geometric side, the Liouville equation is related to conformal metrics on S^2 with conical singularities whose Gaussian curvature is a constant. The Toda systems are related to holomorphic curves in projective spaces [Dol97] and the Plücker formulas [GH78], and the periodic Toda systems are related to harmonic maps [Gue97].

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From the analytic point of view, one would like to study the following mean field equation on a compact surface M with a Riemannian metric g

$$\Delta_g u + \rho\Big(\frac{he^u}{\int_M he^u} - \frac{1}{|M|}\Big) = 4\pi \sum_{j=1}^N \gamma_j \left(\delta_{p_j} - \frac{1}{|M|}\right) \quad \text{on } M, \tag{1.3}$$

where h is a positive smooth function on M and $|M|$ is the volume of M with respect to g. This equation again arises from both conformal change of metrics [KW74, Tro91] with prescribed Gaussian curvature and the Chern-Simons-Higgs theory on the compact surface M. There are intense interests and extensive literature on (1.3) concerning solvability, blow-up analysis and topological degrees [Lin14, Mal14, Tar10, CL03].

In general, we are interested in the following mean field equations of Toda type

$$\Delta_g u_i + \sum_{j=1}^n a_{ij}\rho_j \left(\frac{h_j e^{u_j}}{\int_M h_j e^{u_j}} - \frac{1}{|M|}\right) = 4\pi \sum_{j=1}^N \gamma_{ij} \left(\delta_{p_j} - \frac{1}{|M|}\right) \quad \text{on } M.$$

Such systems have been studied in [JW01, JLW06] for Lie algebras of type A. When carrying out the analysis of such systems, there often appears a sequence of bubbling solutions near blow-up points. For that purpose, the fundamental question is to completely classify all entire solutions of the Toda system with finite energy and with singular sources at the origin as in (1.1).

The classification problem for the solutions to the Toda systems has a long history. For the Liouville equation (1.2), Chen and Li [CL91] classified their solutions without the singular source, and Prajapat and Tarantello [PT01] completed the classification with the singular source. For general $A_n = \mathfrak{sl}_{n+1}$ Toda systems, Jost and Wang [JW02] classified the solutions without singular sources, and Ye and two of the authors [LWY12] completed the classification with singular sources. This later work also invented the method of characterizing the solutions by a complex ODE involving the W-invariants of the Toda system. The work [LWY12] has also established the corresponding quantization result for the integrals and the non-degeneracy result for the corresponding linearized systems. The case of G_2 Toda system was treated in [ALW15]. In [Nie16], one of us generalized the classification to Toda systems of types B and C by treating them as reductions of type A with symmetries and by applying the results from [Nie12].

In this paper, we complete the classification of solutions to Toda systems for all types of simple Lie algebras, and we also establish the quantization result for the corresponding integrals. We note that the remaining types of Toda systems can not be treated as reductions of type A, and a genuinely new method is needed for our purpose. We are able to achieve our goal by systematically applying and further developing the structure theories of local solutions to Toda systems similarly to [LS92, GL14] and of the W-invariants as in [Nie14]. We furthermore note that it is the finite energy condition and the strength of the singularities that combine to greatly restrict the form of the solutions. The current work will lay the foundation for future applications to the Chern-Simons-Higgs theory and to the mean filed equations.

Our approach of solving (1.1) will heavily use the complex coordinates and holomorphic functions. Let $x = (x_1, x_2)$ be the coordinates on $\mathbb{R}^2$, and we introduce the complex coordinates $z = x_1 + i\, x_2$ and $\bar z = x_1 - i\, x_2$. Therefore we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. For simplicity, we write $\partial_z = \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2})$, and similarly $\partial_{\bar z} = \frac{\partial}{\partial \bar z} = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2})$. The Laplace operator is then $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = 4\partial_z\partial_{\bar z}$. The coefficient 4 here is responsible for the slightly unconventional coefficient 4 on the left of (1.1), and this coefficient can be easily dealt with (see [Nie16, Remark 1.12]).

Furthermore, our results and proofs are more conveniently presented in a different set of dependent variables. Let $u_i = \sum_{j=1}^n a_{ij}U_j$. Then the U_i satisfy

$$\begin{cases} U_{i,z\bar z} + \exp\Big(\sum_{j=1}^n a_{ij}U_j\Big) = \pi\gamma^i\delta_0 \quad \text{on } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{\sum_{j=1}^n a_{ij}U_j}\, dx < \infty, \end{cases} \tag{1.4}$$

where $\gamma^i = \sum_{j=1}^n a^{ij}\gamma_j$ and (a^{ij}) is the inverse matrix of (a_{ij}). The first equation can also be written as

$$\Delta U_i + 4e^{u_i} = 4\pi\gamma^i\delta_0 \quad \text{on } \mathbb{R}^2. \tag{1.5}$$

(The γ^i were denoted by α_i in [LWY12, Nie16], but we will use α_i to denote the ith simple root of the Lie algebra $\mathfrak{g}$ in this paper.)

Throughout the paper, we use the Lie-theoretic setup detailed in the Appendix. Although for the Lie algebras of classical types A, B, C, D, the setup can be made fairly concrete, we have chosen to present our result in a Lie-theoretic and intrinsic way, which automatically covers the Lie algebras of exceptional types G_2, F_4, E_6, E_7 and E_8. Here is our main theorem.

Theorem 1.6. *Let G be a connected complex Lie group whose Lie algebra is $\mathfrak{g}$ with the Cartan matrix (a_{ij}). Let $G = KAN$ be the Iwasawa decomposition of G (see Eq.* (A.7)*) with K maximally compact, A abelian, and N nilpotent. Let N_Γ be the subgroup of N (see Definition 6.2) determined by the γ_i.*

Let $\Phi : \mathbb{C}\backslash\mathbb{R}_{\le 0} \to N \subset G$ be the unique solution of

$$\begin{cases} \Phi^{-1}\Phi_z = \sum_{i=1}^n z^{\gamma_i} e_{-\alpha_i} \quad \textit{on } \mathbb{C}\backslash\mathbb{R}_{\le 0}, \\ \lim_{z\to 0} \Phi(z) = Id, \end{cases} \tag{1.7}$$

where $Id \in G$ is the identity element, the limit exists because $\gamma_i > -1$, and the root vectors $e_{-\alpha_i} \in \mathfrak{g}_{-\alpha_i}$ are normalized as in (A.3) *and* (A.8).

Then all the solutions to (1.4) *are*

$$U_i = -\log\langle i|\Phi^* C^* \Lambda^2 C\Phi|i\rangle + 2\gamma^i \log|z|, \quad 1 \le i \le n, \tag{1.8}$$

where $C \in N_\Gamma$ and $\Lambda \in A$. Here for $g \in G$, $g^* = (g^\theta)^{-1}$ and θ is the Cartan involution of G, and $\langle i|\cdot|i\rangle$ is the highest matrix coefficient for the ith fundamental representation (see the Appendix).

Consequently, all the solutions to (1.1) *are*

$$u_i = -\sum_{j=1}^n a_{ij}\log\langle j|\Phi^* C^* \Lambda^2 C\Phi|j\rangle + 2\gamma_i \log|z|, \quad 1 \le i \le n, \tag{1.9}$$

and they satisfy the following quantization result for the integrals (see (7.22)*)*

$$\sum_{j=1}^{n} a_{ij} \int_{\mathbb{R}^2} e^{u_j}\, dx = \pi(2+\gamma_i-\kappa\gamma_i), \quad 1 \le i \le n, \tag{1.10}$$

where κ is the longest element in the Weyl group and if $-\kappa\alpha_i = \alpha_k$, then $-\kappa\gamma_i := \gamma_k$.

Sections 2 to 7 are devoted to the proof of this main theorem, and the approach can be summarized as follows. In Sections 2 and 3, we develop the the structure theories of the local solutions and of the W-invariants for the Toda system, and we relate them. In Sections 4 and 5, we use the finite energy condition and the strength of the singularities to greatly restrict the forms of the W-invariants and hence of the solutions. In Section 6, we take up the monodromy consideration for the solutions to be well-defined on the punctured plane. Finally in Section 7, we study the quantization result for the finite integrals by establishing the close relationship of our current work with that of Kostant [Kos79] on Toda ODE systems.

Consistently with the results in [LWY12, Nie16] for the Lie algebras of types A, B and C, the solution space to the Toda system is parametrized by the subgroup AN_Γ of a corresponding complex Lie group G. When all the γ_i are integers, $AN_\Gamma = AN$ (see Definition 6.2), and the solution space has the maximal dimension. Since N is a complex group, the real dimension of AN is the same as the real dimension of a real group corresponding to the real Lie algebra $\mathfrak{g}_0$ (A.3).

For an element g in a classical Lie group G, we have that $g^* = \bar{g}^t$ is the conjugate transpose. The abelian subgroup A can be chosen to consist of diagonal matrices with positive real entries, and the nilpotent subgroup N can be chosen to consist of unipotent lower-triangular matrices with complex entries. The fundamental representations are contained in the wedge products of the standard representations together with the spin representations for the B and D cases. In Section 8 of the paper, we will relate the above general theorem 1.6 to the previous results [LWY12, Nie16] in the A, B and C cases, and we will spell out more details for our general theorem in the D case.

The nondegeneracy of the linearized system in this general setting of the Toda system (1.1) for a simple Lie algebra will be pursued in a future work.

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2. Local solutions from holomorphic functions

In this section, we show that the solutions to the Toda system (1.4) for a general simple Lie algebra locally all come from holomorphic data, which are generalizations of the developing map in the Liouville case. Our approach follows [LS79], [LS92, §3.1, §4.1], and [GL14, Appendix]. Since we only consider real-valued solutions to the Toda systems, we only need the holomorphic data of [LS79] and the compact real form (unitary structure) comes into play. In [GL14], similar results were obtained for the periodic Toda systems and the loop groups.

Therefore this section generalizes the well-known fact that locally the solutions to the Liouville equation

$$U_{z\bar{z}} = -e^{2U} \quad \text{on an open set } D \subset \mathbb{C} \tag{2.1}$$

are

$$U(z) = \log \frac{|f'|}{1+|f|^2}, \tag{2.2}$$

where f is a holomorphic function on D whose derivative is nowhere vanishing.

For simplicity, we introduce the notation $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and also recall that $\mathfrak{g}_{-1} = \oplus_{i=1}^n \mathfrak{g}_{-\alpha_i}$ as in (A.14). We use the terminology that a domain is a connected open set in $\mathbb{C}$.

Theorem 2.3. *Let $\{U_i\}$ be a set of solutions to the Toda system* (1.4)*. Then there exists a domain $D \subset \mathbb{C}\backslash\mathbb{R}_{\leq 0}$ containing 1 and a holomorphic map*

$$\eta : D \to \mathfrak{g}_{-1};\ \eta(z) = \sum_{i=1}^n f_i(z) e_{-\alpha_i}, \tag{2.4}$$

where the f_i are holomorphic and nowhere zero on D, such that

$$U_i = -\log\langle i|L^* L|i\rangle + \sum_{j=1}^n a^{ij} \log|f_j|^2, \tag{2.5}$$

where $L : D \to N \subset G$ satisfies

$$L^{-1}L_z = \eta, \quad L(1) = Id. \tag{2.6}$$

Here $L^* = (L^\theta)^{-1}$, and $\langle i| \cdot |i\rangle$ is the highest matrix coefficient between the ith fundamental representation (see the Appendix).

On the other hand, a holomorphic map $\eta = \sum_{i=1}^n f_i e_{-\alpha_i} : D \to \mathfrak{g}_{-1}$ as in (2.4) *on a simply connected domain D where the f_i are nowhere zero gives rise to a set of solutions $\{U_i\}$ to the Toda system on D.*

Proof. Using the notation and the setup in the Appendix, Eq. (1.4) has the following zero-curvature equation on $\mathbb{C}^*$

$$[\partial_z + A, \partial_{\bar{z}} + A^\theta] = 0, \quad \text{that is,} \tag{2.7}$$

$$-A_{\bar{z}} + (A^\theta)_z + [A, A^\theta] = 0, \quad \text{where}$$

$$A = -\sum_{i=1}^n \tfrac{1}{2} U_{i,z} h_{\alpha_i} + \sum_{i=1}^n \exp\Big(\tfrac{1}{2}\sum_{j=1}^n a_{ij}U_j\Big) e_{-\alpha_i}, \tag{2.8}$$

$$A^\theta = \sum_{i=1}^n \tfrac{1}{2} U_{i,\bar{z}} h_{\alpha_i} - \sum_{i=1}^n \exp\Big(\tfrac{1}{2}\sum_{j=1}^n a_{ij}U_j\Big) e_{\alpha_i}. \tag{2.9}$$

The zero-curvature equation can also be written as the Maurer-Cartan equation

$$d\omega + \frac{1}{2}[\omega, \omega] = 0$$

for the following Lie algebra valued differential form

$$\omega = A\,dz + A^\theta d\bar{z} \in \Omega^1(\mathbb{C}^*, \mathfrak{g}).$$

With $dz = dx_1 + i\,dx_2$ and $d\bar{z} = dx_1 - i\,dx_2$, it is also

$$\omega = (A + A^\theta)dx_1 + i(A - A^\theta)dx_2.$$

Since the Cartan involution θ on $\mathfrak{g}$ is conjugate linear, we see that ω takes value in the fixed subalgebra $\mathfrak{g}^\theta = \mathfrak{k}$ (see the Appendix Subsection A.3). Therefore by [Sha97, Theorems 6.1 and 7.14], there exists a map on the simply connected domain

$$F : \mathbb{C}\backslash\mathbb{R}_{\leq 0} \to K \subset G$$

to the compact subgroup $K = G^\theta$ such that

$$\begin{cases} F^{-1}dF = \omega \\ F(1) = Id. \end{cases} \tag{2.10}$$

Therefore,

$$F^{-1}F_z = A, \quad F^{-1}F_{\bar{z}} = A^\theta. \tag{2.11}$$

The F has the following Gauss decomposition (A.2) in a domain $1 \in D \subset \mathbb{C}\backslash\mathbb{R}_{\leq 0}$

$$F = LM\exp(H) \tag{2.12}$$

where $L : D \to N = N_-$ takes value in the negative nilpotent subgroup, and $M : D \to N_+$ takes value in the positive nilpotent subgroup. Furthermore $H = \sum_{i=1}^n b_i h_{\alpha_i} : D \to \mathfrak{h}$ takes value in the Cartan subalgebra, and $\exp : \mathfrak{h} \to \mathcal{H}$ is the exponential map to the Cartan subgroup. From $F(1) = Id$ in (2.10), we see clearly that $L(1) = Id$.

Now we show that L is holomorphic on D. By the second equation in (2.11), we have

$$\exp(-H)M^{-1}(L^{-1}L_{\bar{z}})M\exp(H) + \exp(-H)M^{-1}M_{\bar{z}}\exp(H) + H_{\bar{z}} = A^\theta.$$

In view of (2.9), the components in $\mathfrak{n}_- = \oplus_{\alpha\in\Delta^+}\mathfrak{g}_{-\alpha}$ (see (A.1)) of the above equation give

$$L^{-1}L_{\bar{z}} = 0, \tag{2.13}$$

and the components in $\mathfrak{h}$ give

$$b_{i,\bar{z}} = \tfrac{1}{2}U_{i,\bar{z}}, \quad 1 \leq i \leq n.$$

Thus we see that $b_{i,z\bar{z}} = \frac{1}{2}U_{i,z\bar{z}}$. Taking the conjugate, we also have $\bar{b}_{i,z\bar{z}} = \frac{1}{2}U_{i,z\bar{z}}$ since U_i is real. Therefore,

$$(b_i + \bar{b}_i)_{z\bar{z}} = U_{i,z\bar{z}}.$$

Hence we have, for $1 \leq i \leq n$,

$$b_i + \bar{b}_i = U_i - p_i \tag{2.14}$$

for some real-valued harmonic function p_i on D.

By the first equation in (2.11), we have

$$\exp(-H)M^{-1}(L^{-1}L_z)M\exp(H) + \exp(-H)M^{-1}M_z\exp(H) + H_z = A.$$

Since $A \in \mathfrak{g}_{-1} \oplus \mathfrak{h}$ by (2.8), we see that $L^{-1}L_z \in \mathfrak{g}_{-1}$. We denote it by η and write it out in terms of the basis

$$L^{-1}L_z = \eta = \sum_{i=1}^n f_i(z)e_{-\alpha_i}. \tag{2.15}$$

Then the f_i are holomorphic by (2.13). Furthermore, by (2.8) the component of A in $\mathfrak{g}_{-1}$ is $\sum_{i=1}^n \exp\left(\frac{1}{2}\sum_{j=1}^n a_{ij}U_j\right)e_{-\alpha_i}$ where all the coordinates are nowhere zero, so the f_i are nowhere zero on D. Thus we have shown (2.6) and (2.4).

Now following the physicists, we denote by $|i\rangle$ a highest weight vector in the ith fundamental representation of G, and $\langle i|$ a lowest weight vector in its dual right representation such that $\langle i|Id|i\rangle = 1$ (see the Appendix Subsection A.8).

From (2.12), we have

$$L = F\exp(-H)M^{-1}.$$

Therefore using the $*$ operation from the Appendix Subsection A.4, we have

$$\begin{aligned}\langle i|L^*L|i\rangle &= \langle i|(M^{-1})^*\exp(-\bar{H})F^*F\exp(-H)M^{-1}|i\rangle \\ &= \Big\langle i\Big|\exp\Big(-\sum_{j=1}^{m}(b_j+\bar{b}_j)h_{\alpha_j}\Big)\Big|i\Big\rangle = e^{-(b_i+\bar{b}_i)},\end{aligned} \tag{2.16}$$

where we have used the following facts. First, by $F \in K$ we have $F^*F = Id$. Secondly, since $M^{-1} \in N_+$ and $|i\rangle$ is a highest weight vector, we have $M^{-1}|i\rangle = |i\rangle$. Similarly, $(M^{-1})^* \in N_-$ and $\langle i|$ is a lowest weight vector, so $\langle i|(M^{-1})^* = \langle i|$. Finally, we have $h_{\alpha_j}|i\rangle = \delta_{ij}|i\rangle$ (see Eq. (A.16)). Eq. (2.16) actually shows that $\langle i|L^*L|i\rangle$ is real, and this also follows from the Appendix Subsection A.9. Therefore by (2.14),

$$U_i = -\log\langle i|L^*L|i\rangle + p_i. \tag{2.17}$$

Now we show that for the above U_i to satisfy (1.4) with L in (2.15), we must have

$$p_i = \sum_{j=1}^{n} a^{ij}\log|f_j(z)|^2.$$

This follows from [LS92, §4.1.2] using the Jacobi identity from [LS92, §1.6.4], which is more rigorously proved in [Nie15]. The identity says that for a general element $g \in G^s$, the simply connected Lie group with Lie algebra $\mathfrak{g}$ (see the Appendix Subsection A.8), we have

$$\langle i|g|i\rangle\langle i|e_{\alpha_i}ge_{-\alpha_i}|i\rangle - \langle i|ge_{-\alpha_i}|i\rangle\langle i|e_{\alpha_i}g|i\rangle = \prod_{j\neq i}\langle j|g|j\rangle^{-a_{ij}}. \tag{2.18}$$

From (2.17) and that p_i is harmonic, we have

$$U_{i,z\bar{z}} = -\frac{\langle i|L^*L|i\rangle\langle i|L^*L|i\rangle_{z\bar{z}} - \langle i|L^*L|i\rangle_z\langle i|L^*L|i\rangle_{\bar{z}}}{\langle i|L^*L|i\rangle^2}. \tag{2.19}$$

Now by (2.13) and (2.15), we have

$$\langle i|L^*L|i\rangle_z = \langle i|L^*L\eta|i\rangle = f_i(z)\langle i|L^*Le_{-\alpha_i}|i\rangle,$$

where we have used that for the ith fundamental representation we have $e_{-\alpha_j}|i\rangle = 0$ for $j \neq i$ (see Eq. (A.16)). Taking the $*$ operation and noting (A.6), (2.15) also gives

$$(L^*)_{\bar{z}}(L^*)^{-1} = \eta^* = \sum_{i=1}^{n}\overline{f_i(z)}e_{\alpha_i}. \tag{2.20}$$

Therefore, similarly we have

$$\langle i|L^*L|i\rangle_{\bar{z}} = \langle i|\eta^*L^*L|i\rangle = \overline{f_i(z)}\langle i|e_{\alpha_i}L^*L|i\rangle.$$

Furthermore, we have

$$\langle i|L^*L|i\rangle_{z\bar{z}} = |f_i|^2\langle i|e_{\alpha_i}L^*Le_{-\alpha_i}|i\rangle.$$

Now applying the Jacobi identity (2.18) to (2.19) with $g = L^*L$ gives

$$U_{i,z\bar{z}} = -|f_i|^2 \prod_{j=1}^{n} \langle j|L^*L|j\rangle^{-a_{ij}}$$

by $a_{ii} = 2$. By (2.17), this is

$$U_{i,z\bar{z}} = -|f_i|^2 \exp\Big(\sum_{j=1}^{n} a_{ij}U_j - \sum_{j=1}^{n} a_{ij}p_j\Big)$$
$$= -\exp\Big(\log|f_i|^2 - \sum_{j=1}^{n} a_{ij}p_j\Big)\exp\Big(\sum_{j=1}^{n} a_{ij}U_j\Big).$$

Therefore for the U_i to satisfy (1.4), we need $\log|f_i|^2 - \sum_{j=1}^{n} a_{ij}p_j = 0$ which gives

$$p_i = \sum_{j=1}^{n} a^{ij}\log|f_j|^2.$$

This proves the formula (2.5).

Now given a holomorphic map $\eta = \sum_{i=1}^{n} f_i e_{-\alpha_i} : D \to \mathfrak{g}_{-1}$ as in (2.4) on a simply connected domain D where the f_i are nowhere zero, we can construct a $L : D \to G$, which solves $L^{-1}L_z = \eta$ as in (2.6) but may not necessarily satisfy the condition $L(1) = Id$. Construct the U_i as in (2.5), and they are checked to satisfy the Toda system (1.4) in the same way as above, where the important point is again the Jacobi identity (2.18). $\square$

3. W-Invariants of the Toda systems

In this section, we first present the algebraic theories of W-invariants of Toda systems as developed in [FF96, Nie14]. Then we present a result relating the W-invariants with the local solutions from the last section.

By definition, a W-invariant (also called a characteristic integral) for the Toda system (1.4) is a polynomial in the $\partial_z^k U_i$ for $k \geq 1$ and $1 \leq i \leq n$ whose derivative with respect to $\bar{z}$ is zero if the U_i are solutions.

For example, for the Liouville equation (2.1),

$$W = U_{zz} - U_z^2 \tag{3.1}$$

is a W-invariant since $W_{\bar{z}} = 0$ for a solution U. Furthermore, plugging in the local solution (2.2), we have that

$$W = \frac{1}{2}\Big(\frac{f'''}{f'} - \frac{3}{2}\Big(\frac{f''}{f'}\Big)^2\Big), \tag{3.2}$$

that is, the W-invariant of the local solution becomes one half of the Schwarzian derivative of the function f. We aim to generalize such results to general Toda systems in this section.

For a general Toda system associated to a simple Lie algebra of rank n, there are n basic W-invariants (see [FF96]) so that the other W-invariants are differential polynomials in these. One of us in [Nie14] has given a concrete construction and a direct proof of the basic W-invariants W_j for $1 \leq j \leq n$. They are obtained by conjugating one side of the following zero-curvature equation of the Toda system to its Drinfeld-Sokolov gauge [DS84], which is in turn related to a Kostant slice of the corresponding Lie algebra [Kos63].

The Toda system (1.4) on $\mathbb{C}^*$ is also expressed by the following zero-curvature equation

$$\Big[-\partial_z + \epsilon + \sum_{i=1}^n U_{i,z} h_{\alpha_i}, \partial_{\bar z} + \sum_{i=1}^n e^{u_i} e_{\alpha_i}\Big] = 0,$$

where $\epsilon = \sum_{i=1}^n e_{-\alpha_i} \in \mathfrak{g}_{-1}$, and $u_i = \sum_{j=1}^n a_{ij} U_j$. (This zero-curvature equation is related to but different from the one in (2.7), and the current one is not invariant under the Cartan involution.) Let $\mathfrak{s}$ be a Kostant slice of $\mathfrak{g}$, that is, a homogeneous subspace $\mathfrak{s}$ with respect to the principal grading (A.13) such that $\mathfrak{g} = [\epsilon, \mathfrak{g}] \oplus \mathfrak{s}$. Then it is known [Kos63] that $\dim \mathfrak{s} = n$. Let $\{s_j\}_{j=1}^n$ be a homogeneous basis of $\mathfrak{s}$ ordered with nondecreasing gradings by (A.13).

Theorem 3.3 ([Nie14])**.** *There exists a unique $M_0 \in N_+$, an element in the positive nilpotent subgroup whose coefficients depend on the derivatives $\partial_z^k U_i$ for $1 \le i \le n$ and $k \ge 1$, such that*

$$M_0\Big(-\partial_z + \epsilon + \sum_{i=1}^n U_{i,z} h_{\alpha_i}\Big) M_0^{-1} = -\partial_z + \epsilon + \sum_{j=1}^n W_j s_j. \tag{3.4}$$

Then the W_j for $1 \le j \le n$ are the basic W-invariants of the Toda system.

The uniqueness of M_0 is easily proved by induction on the grading (A.13) and the fact from [Kos59] that

$$\ker \mathrm{ad}_\epsilon \cap \mathfrak{b}_+ = 0. \tag{3.5}$$

The right hand side of (3.4) is said to be in the Drinfeld-Sokolov gauge [DS84].

Here is the main result in this section which relates the W-invariants with the holomorphic data η in (2.4) for local solutions. It generalizes the relation (3.2) to a general Lie algebra.

Theorem 3.6. *For the local solutions* (2.5) *on a simply connected domain D, there exists a $P_1 \in B_+$, an element in the Borel subgroup whose coefficients depend on the derivatives $\partial_z^k f_i$ for $1 \le i \le n$ and $k \ge 0$, such that*

$$P_1\Big(-\partial_z + \sum_{i=1}^n f_i(z) e_{-\alpha_i}\Big) P_1^{-1} = -\partial_z + \epsilon + \sum_{j=1}^n W_j s_j, \tag{3.7}$$

where the W_j are the W-invariants computed by (3.4) *for the local solutions* (2.5)*.*

Furthermore P_1 is unique up to the finite center of G and is holomorphic on D.

Proof. First we show that for the local solutions (2.5), the W-invariants are also computed by

$$M_1\Big(-\partial_z + \epsilon + \sum_{i=1}^n F_i h_{\alpha_i}\Big) M_1^{-1} = -\partial_z + \epsilon + \sum_{j=1}^n W_j s_j, \tag{3.8}$$

where $M_1 \in N_+$, and

$$F_i = \sum_{j=1}^n a^{ij} \partial_z \log f_j = \sum_{j=1}^n a^{ij} \frac{f_j'}{f_j} \tag{3.9}$$

is the part of $U_{i,z}$ in (3.4) using just the summand $p_i = \sum_{j=1}^n a^{ij} \log |f_j|^2$ in (2.5). We note that the log's are well-defined since D is simply connected.

If we do not impose that the solutions U_i in (2.5) are real-valued, then instead of (2.20) we can solve for $T : D \to G$ such that

$$\begin{cases} T_{\bar z}T^{-1} = \sum_{i=1}^n g_i(\bar z)e_{\alpha_i}, \\ T(1) = Id, \end{cases} \tag{3.10}$$

for n anti-holomorphic functions $g_1(\bar z), \dots, g_n(\bar z)$. Then in analogy to the local solutions (2.5), all the

$$U_i^T = -\log\langle i|T(\bar z)L(z)|i\rangle + \sum_{j=1}^n a^{ij}\log(f_j(z)g_j(\bar z)), \quad 1 \le i \le n,$$

are solutions to the Toda system (1.4) on D. Clearly the U_i in (2.5) corresponds to $T = L^*$, that is, $U_i^{L^*} = U_i$ in (2.5).

Now the W-invariants W_j^T of the solutions U_i^T satisfy $\partial_{\bar z}W_j^T = 0$. Since the W_j^T are polynomials in the $\partial_z^k U_i^T$ for $k \ge 1$, we see that when computing them, we can in place of the U_i^T use

$$\tilde U_i^T = -\log\langle i|T(\bar z)L(z)|i\rangle + \sum_{j=1}^n a^{ij}\log f_j(z),$$

that is, we can discard the anti-holomorphic part $\sum_{j=1}^n a^{ij}\log g_j(\bar z)$.

Now since $\partial_{\bar z}W_j^T = 0$, we see that the W_j^T are independent of the $T(\bar z)$ in (3.10). Therefore the $W_j^{L^*} = W_j$ in (3.4) is that same as the W_j^{Id} using the simplest $T(\bar z) = Id$. Then

$$\tilde U_i^{Id} = \sum_{j=1}^n a^{ij}\log f_j(z)$$

since $\langle i|Id \cdot L(z) = \langle i|L(z) = \langle i|$ by $L(z) \in N_-$ and hence $\langle i|Id \cdot L(z)|i\rangle = 1$. Replacing the U_i in (3.4) by the above $\tilde U_i^{Id}$ proves (3.8) since $\tilde U_{i,z}^{Id} = F_i$ in (3.9).

Now we show that on the simply connected D there is a conjugation to transform $-\partial_z + \sum_{i=1}^n f_i e_{-\alpha_i}$ to $-\partial_z + \epsilon + \sum_{i=1}^n F_i h_{\alpha_i}$. We choose

$$Q_1 = \exp\Big(\sum_{k=1}^n \big(\sum_{j=1}^n a^{kj}\log f_j\big)h_{\alpha_k}\Big) \in \mathcal{H}. \tag{3.11}$$

Since $[h_{\alpha_k}, e_{-\alpha_i}] = -a_{ik}e_{-\alpha_i}$ by (A.10), we have

$$\mathrm{Ad}_{Q_1} e_{-\alpha_i} = \exp\Big(-\sum_{k=1}^n \big(\sum_{j=1}^n a^{kj}\log f_j\big)a_{ik}\Big)e_{-\alpha_i} = \exp\big(-\log f_i\big)e_{-\alpha_i} = \frac{1}{f_i}e_{-\alpha_i}.$$

It is also clear that

$$-Q_1\partial_z Q_1^{-1} = \partial_z Q_1 Q_1^{-1} = \sum_{i=1}^n \big(\sum_{j=1}^n a^{ij}\partial_z\log f_j\big)h_{\alpha_i} = \sum_{i=1}^n F_i h_{\alpha_i}.$$

Therefore we have

$$\begin{aligned} Q_1\Big(-\partial_z + \sum_{i=1}^n f_i e_{-\alpha_i}\Big)Q_1^{-1} &= -\partial_z - Q_1\partial_z Q_1^{-1} + \sum_{i=1}^n f_i\frac{1}{f_i}e_{-\alpha_i} \\ &= -\partial_z + \sum_{i=1}^n F_i h_{\alpha_i} + \epsilon. \end{aligned} \tag{3.12}$$

Then Eq. (3.7) is proved with $P_1 = M_1 Q_1 \in B_+$.

Again P_1 is unique up to the finite center of G by (3.5). Therefore P_1 is holomorphic since both the f_i and the W_j in (3.7) are holomorphic. $\square$

Remark 3.13. Consider a curve $L : D \to N_-$ on a simply connected domain such that $L^{-1}L_z = \eta = \sum_{i=1}^n f_i(z)e_{-\alpha_i}$. Such a curve is called an integral curve of the standard differential system on $N_- \subset G/B_+$ [Nie15]. The above proof shows that there exists a P_0 in B_+ unique up to the center of G such that

$$P_0(-\partial_z + \eta)P_0^{-1} = -\partial_z + \sum_{j=1}^n I_j s_j.$$

It can be shown by the method of moving frames [MB08] that the I_j are the differential invariants for the curve $L(z)$ under the natural action of G on $N_- \subset G/B_+$.

These are the natural generalizations of the Schwarzian derivative for the group $SL_2\mathbb{C}$ to other simple Lie groups. See also [DZ13].

4. Use the finite energy condition

In this section, we adapt the analytical estimates from [BM91, LWY12] using the finite energy condition and the strength of the singularities to determine the simple forms of the W-invariants, which will be shown to further restrict the solutions in the next section.

For a differential monomial in the U_i, we call by its *degree* the sum of the orders of differentiation multiplied by the algebraic degrees of the corresponding factors. For example the above $W = U_{zz} - U_z^2$ in (3.1) for the Liouville equation has a homogeneous degree 2. It is known from [FF96] and also clear from (3.4) that the W-invariants W_j involve the $\partial_z^k U_i$ for $k \geq 1$ and that the homogeneous degree of W_j is the same as the degree d_j of the corresponding primitive adjoint-invariant function of the Lie algebra $\mathfrak{g}$ [Kos59]. We call such degrees the degrees of the simple Lie algebra and we have listed them in the Appendix Subsection A.11.

Proposition 4.1. *The W-invariants for the Toda system* (1.4) *are*

$$W_j = \frac{w_j}{z^{d_j}}, \quad z \in \mathbb{C}^*, \ 1 \leq j \leq n, \tag{4.2}$$

where the d_j are the degrees of the Lie algebra $\mathfrak{g}$ and the w_j are constants.

Proof. This proof is an adaption of the proof in [LWY12] of the corresponding assertion in their Eq. (5.9).

Following [LWY12, Eq. (5.10)], introduce

$$V_i = U_i - 2\gamma^i \log|z|, \quad 1 \leq i \leq n. \tag{4.3}$$

Then system (1.4) becomes

$$\begin{cases} \Delta V_i = -4|z|^{2\gamma_i} \exp\Big(\sum_{j=1}^n a_{ij}V_j\Big), \\ \int_{\mathbb{R}^2} |z|^{2\gamma_i} \exp\Big(\sum_{j=1}^n a_{ij}V_j\Big)\, dx < \infty. \end{cases}$$

As $\gamma_i > -1$, applying Brezis-Merle's argument in [BM91], we have that $V_i \in C^{0,\alpha}$ on $\mathbb{C}$ for some $\alpha \in (0,1)$ and that they are upper bounded over $\mathbb{C}$. Therefore we can express V_i by the integral representation formula, and we have

$$\begin{aligned} &\partial_z^k V_i(z) = O(1+|z|^{2+2\gamma_i-k}) \quad \text{near } 0, \\ &\partial_z^k V_i(z) = O(|z|^{-k}) \quad \text{near } \infty, \forall\, k \geq 1. \end{aligned} \tag{4.4}$$

Therefore from (4.3), we have

$$\begin{aligned} &\partial_z^k U_i(z) = O(|z|^{-k}) \quad \text{near } 0, \forall\, k \geq 1, \\ &\partial_z^k U_i(z) = O(|z|^{-k}) \quad \text{near } \infty, \forall\, k \geq 1. \end{aligned} \tag{4.5}$$

By $W_{j,\bar z} = 0$ and that W_j has degree d_j, we see from the above estimates that $z^{d_j} W_j$ is holomorphic and bounded on $\mathbb{C}^*$. Therefore $z^{d_j} W_j = w_j$ is a constant by the Liouville theorem, and so (4.2) holds. □

Theorem 4.6. *The W-invariants for the Toda systems* (1.4) *are also computed by*

$$P_2\Big(-\partial_z + \sum_{i=1}^n z^{\gamma_i} e_{-\alpha_i}\Big) P_2^{-1} = -\partial_z + \epsilon + \sum_{j=1}^n W_j s_j, \tag{4.7}$$

where $P_2 : \mathbb{C}\backslash\mathbb{R}_{\leq 0} \to B_+$ is holomorphic and unique up to the finite center of G.

Proof. First we show that the W-invariants for the Toda system (1.4) are also computed by

$$M_2\Big(-\partial_z + \epsilon + \sum_{i=1}^n \frac{\gamma^i}{z} h_{\alpha_i}\Big) M_2^{-1} = -\partial_z + \epsilon + \sum_{j=1}^n W_j s_j, \tag{4.8}$$

where $M_2 \in N_+$.

The W_j are polynomials in the $\partial_z^k U_i$ for $k \geq 1$, and $W_j = \frac{w_j}{z^{d_j}}$ by (4.2). Since $\gamma_i > -1$, by (4.4) and (4.5) for the orders at 0, we see that all the terms involving $\partial_z^k V_i$ will not appear in the final form of W_j since their orders of pole are not high enough. Therefore the W_j can also be computed using just the

$$2\gamma^i \log|z| = \gamma^i \log z + \gamma^i \log \bar z$$

component of U_i in (4.3). Then the term $U_{i,z}$ in (3.4) can be replaced by $\frac{\gamma^i}{z}$, and we get (4.8).

Next we use again the conjugation in (3.11) in our situation, and let

$$Q_2 = \exp\Big(\sum_{k=1}^n \log z^{\gamma^k} h_{\alpha_k}\Big).$$

Then by (3.12) or by direct computation, we have

$$Q_2\Big(-\partial_z + \sum_{i=1}^n z^{\gamma_i} e_{-\alpha_i}\Big) Q_2^{-1} = -\partial_z + \epsilon + \sum_{i=1}^n \frac{\gamma^i}{z} h_{\alpha_i}.$$

Then (4.7) is proved with $P_2 = M_2 Q_2 \in B_+$.

Similarly to Theorem 3.6, P_2 is unique up to the finite center of G and is holomorphic. □

5. The holomorphic functions in the local solutions

The W-invariants play essential roles in our approach to classify the solutions. The work [LWY12] classified the solutions to the Toda systems of type A by relating them to an ODE whose coefficients are the W-invariants. In this section, we will use the W-invariants to largely restrict the holomorphic functions $f_i(z)$ (2.4) in the local solutions (2.5) to be $f_i(z) = z^{\gamma_i}$ as long as we allow some constant group element.

Theorem 5.1. *The local solutions U_i (2.5) on a simply connected domain $D \subset \mathbb{C}\backslash\mathbb{R}_{\leq 0}$ that are solutions to (1.4) have the following form*

$$U_i = -\log\langle i|\Phi^* g^* g\Phi|i\rangle + 2\gamma^i \log|z|, \quad 1 \leq i \leq n, \tag{5.2}$$

where Φ satisfies (1.7) and $g \in G$ is a constant group element.

Proof. We introduce the notation

$$\zeta = \sum_{i=1}^{n} z^{\gamma_i} e_{-\alpha_i}, \tag{5.3}$$

and then (1.7) becomes $\Phi^{-1}\Phi_z = \zeta$.

Consider $s = \exp(\pi i E_0)$ where E_0 is defined in (A.12) such that $\alpha_i(E_0) = 1$ for $1 \leq i \leq n$. Then $\mathrm{Ad}_s\, e_{-\alpha_i} = -e_{-\alpha_i}$ and also $\mathrm{Ad}_{s^{-1}}\, e_{-\alpha_i} = -e_{-\alpha_i}$. Actually s^2 is in the center of the group G. We also note that such an element s plays an important role in Kostant's approach to the Toda lattices [Kos79, Eq (3.5.5)], where it is called m.

By Theorem 3.6 and Theorem 4.6, we see that for $P = s^{-1}P_2^{-1}P_1 s : D \to B_+$ we have

$$\begin{aligned} P^{-1}(\partial_z + \zeta)P &= s^{-1}P_1^{-1}P_2 s(\partial_z + \zeta)s^{-1}P_2^{-1}P_1 s \\ &= s^{-1}P_1^{-1}P_2(\partial_z - \zeta)P_2^{-1}P_1 s \\ &= -s^{-1}P_1^{-1}\Big(-\partial_z + \epsilon + \sum_{j=1}^{n} W_j s_j\Big)P_1 s \\ &= s^{-1}(\partial_z - \eta)s = \partial_z + \eta. \end{aligned} \tag{5.4}$$

Therefore

$$P^{-1}P_z + P^{-1}\zeta P = \eta.$$

Now consider the product $\Phi P : D \to G$ from the simply connected domain D to the group G. We have

$$(\Phi P)^{-1}(\Phi P)_z = P^{-1}\Phi^{-1}\Phi_z P + P^{-1}P_z = P^{-1}\zeta P + P^{-1}P_z = \eta.$$

Comparison with (2.6) gives, on the basis that both L and ΦP are holomorphic on D, that

$$L = g\Phi P, \tag{5.5}$$

where $g \in G$ is a constant element and actually $g = ((\Phi P)(1))^{-1}$ by the requirement that $L(1) = Id$ in (2.6).

Now let $P = QM$ be its decomposition with $Q \in \mathcal{H}$ and $M \in N_+$, and write $Q = \exp\big(\sum_{j=1}^{n} q_j h_{\alpha_j}\big)$. Then considering the components in $\mathfrak{g}_{-1}$ from (5.4), we have

$$\eta = Q^{-1}\zeta Q.$$

By $[-h_{\alpha_j}, e_{-\alpha_i}] = a_{ij} e_{-\alpha_i}$ from (A.10) and from the definitions (2.4), (5.3), we see that

$$f_i = z^{\gamma_i} e^{\sum_{j=1}^n a_{ij} q_j}.$$

It follows that

$$\sum_{j=1}^n a^{ij} \log f_j = \log z^{\gamma^i} + q_i, \quad \text{and}$$

$$\sum_{j=1}^n a^{ij} \log |f_j|^2 = 2\gamma^i \log |z| + q_i + \bar{q}_i, \quad 1 \le i \le n. \tag{5.6}$$

Since $|i\rangle$ is a highest weight vector of the ith fundamental representation, we see that

$$P|i\rangle = QM|i\rangle = Q|i\rangle = e^{q_i}|i\rangle,$$

by $M \in N_+$ and $h_{\alpha_j}|i\rangle = \delta_{ij}|i\rangle$. Taking the $*$ operation (and using (A.17)), we have $\langle i|P^* = e^{\bar{q}_i}\langle i|$.

Therefore by (5.5) and (5.6), the U_i from (2.5) becomes

$$\begin{aligned} U_i &= -\log\langle i|L^*L|i\rangle + \sum_{j=1}^n a^{ij} \log |f_j|^2 \\ &= -\log\langle i|P^*\Phi^* g^* g \Phi P|i\rangle + \sum_{j=1}^n a^{ij} \log |f_j|^2 \\ &= -\log\langle i|\Phi^* g^* g \Phi|i\rangle - (q_i + \bar{q}_i) + \sum_{j=1}^n a^{ij} \log |f_j|^2 \\ &= -\log\langle i|\Phi^* g^* g \Phi|i\rangle + 2\gamma^i \log |z|. \end{aligned}$$

□

6. The final monodromy consideration

In the local solution (5.2), write the Iwasawa decomposition (A.7)

$$g = F\Lambda C$$

with $F \in K$, $\Lambda \in A$ and $C \in N$. Then (5.2) becomes (1.8) by $F^*F = Id$ and $\Lambda^* = \Lambda$.

The solutions U_i in (1.8) are well-defined on $\mathbb{C}\backslash\mathbb{R}_{\le 0}$ after the branch cut for the functions z^{γ_i}. Clearly the U_i satisfy the Toda system (1.4) by the other direction in Theorem 2.3 and have the right strength of singularities at the origin. Therefore we have shown that all local solutions are at least defined on $\mathbb{C}\backslash\mathbb{R}_{\le 0}$ and are of the form (1.8).

We now want to show that for the U_i in (1.8) to be well-defined on $\mathbb{C}^*$, C needs to belong to a suitable subgroup N_Γ of N. This subgroup was introduced in [Nie16] already in the classification result for Toda systems of types B and C.

With the $E_j \in \mathfrak{h}_0$ as in (A.11), consider the following element in the Cartan subgroup

$$t_\Gamma := \exp\Big(2\pi i \sum_{j=1}^n \gamma_j E_j\Big) = \exp\Big(2\pi i \sum_{j=1}^n \gamma^j h_{\alpha_j}\Big) \in \mathcal{H}. \tag{6.1}$$

Clearly $t_\Gamma^{-1} = t_\Gamma^*$.

Definition 6.2. The subgroup $N_\Gamma \subset N$ is the centralizer of t_Γ in N, that is,

$$N_\Gamma = \{C \in N \mid Ct_\Gamma = t_\Gamma C\}.$$

Remark 6.3. Here is a more concrete description of N_Γ. For a positive root $\alpha \in \Delta^+$, write $\alpha = \sum_{i=1}^n m_i \alpha_i$ in terms of the simple roots $\{\alpha_i\}_{i=1}^n$. Define the number $\alpha(\Gamma) = \sum_{i=1}^n m_i \gamma_i$ where we replace α_i by γ_i. Also define the subset Δ_Γ of Δ^+ as $\Delta_\Gamma = \{\alpha \in \Delta^+ \mid \alpha(\Gamma) \in \mathbb{Z}\}$, and the Lie subalgebra $\mathfrak{n}_\Gamma$ of $\mathfrak{n}$ as $\mathfrak{n}_\Gamma = \bigoplus_{\alpha \in \Delta_\Gamma} \mathfrak{g}_{-\alpha}$.

Then N_Γ is the subgroup of N corresponding to $\mathfrak{n}_\Gamma$. The reason is that clearly $\alpha(\Gamma) = \alpha(\sum_{j=1}^n \gamma_j E_j)$ by (A.11), and so $\mathrm{Ad}_{t_\Gamma} e_\alpha = \exp(2\pi i \alpha(\Gamma)) e_\alpha$. Hence we see that

$$\mathfrak{n}_\Gamma = \mathfrak{n}^{\mathrm{Ad}_{t_\Gamma}} \quad \text{and} \quad N_\Gamma = N^{\mathrm{Ad}_{t_\Gamma}} \tag{6.4}$$

are the fixed point sets of the adjoint actions by t_Γ.

Theorem 6.5. *The U_i in* (1.8) *are well-defined on $\mathbb{C}^*$ if and only if*

$$C \in N_\Gamma.$$

Proof. For the U_i in (1.8) to be well-defined on $\mathbb{C}^*$, we need that the U_i are invariant under the change of $z \mapsto e^{-2\pi i} z$, that is, when one travels once (clockwise) around the origin.

By the definition (6.1), we have

$$\sum_{j=1}^n (ze^{-2\pi i})^{\gamma_j} e_{-\alpha_j} = \sum_{j=1}^n z^{\gamma_j} e^{-2\pi \gamma_j i} e_{-\alpha_j} = \mathrm{Ad}_{t_\Gamma} \Big(\sum_{j=1}^n z^{\gamma_j} e_{-\alpha_j} \Big).$$

Hence the corresponding solution to

$$\begin{cases} \tilde{\Phi}^{-1} \tilde{\Phi}_z = -\displaystyle\sum_{j=1}^n (ze^{-2\pi i})^{\gamma_j} e_{-\alpha_j} \quad \text{on } \mathbb{C} \backslash \mathbb{R}_{\leq 0}, \\ \tilde{\Phi}(0) = Id, \end{cases}$$

is

$$\tilde{\Phi}(z) = \mathrm{Ad}_{t_\Gamma} \Phi(z) = t_\Gamma \Phi(z) t_\Gamma^{-1}.$$

Therefore the corresponding solution (1.8) to the Toda system is

$$\begin{aligned} \tilde{U}_i &= -\log \langle i | \tilde{\Phi}^* C^* \Lambda^2 C \tilde{\Phi} | i \rangle + 2\gamma^i \log |z| \\ &= -\log \langle i | \Phi^* t_\Gamma^* C^* \Lambda^2 C t_\Gamma \Phi | i \rangle + 2\gamma^i \log |z|, \end{aligned}$$

where we have used that $t_\Gamma^* = t_\Gamma^{-1}$, $t_\Gamma^{-1} | i \rangle = e^{-2\pi i \gamma^i} | i \rangle$, and $\langle i | t_\Gamma = e^{2\pi i \gamma^i} \langle i |$ from (6.1). Note that $\Lambda t_\Gamma = t_\Gamma \Lambda$ since both belong to the Cartan subgroup $\mathcal{H}$. From $\gamma_i > -1$, it can be shown that the above $\tilde{U}_i$ is equal to the U_i in (1.8) iff

$$Ct_\Gamma = t_\Gamma C,$$

that is, $C \in N_\Gamma$. $\square$

7. Relation with Kostant's work and quantization

In this section, we obtain an explicit expression for the Φ in (1.7) inspired by [Kos79] and this allows us to obtain the quantization result for the integrals of our solutions. This section thus establishes a very concrete relationship of our work with that of Kostant for Toda ODEs (see Proposition 7.14).

Following [LWY12], we denote

$$\mu_i = \gamma_i + 1 > 0, \quad 1 \le i \le n. \tag{7.1}$$

Inspired by [Kos79], we introduce the following notation, using (A.11) and (5.3),

$$\begin{aligned} w_0 &= \sum_{i=1}^n \mu_i E_i \in \mathfrak{h}, \\ \xi &= z\zeta = \sum_{i=1}^n z^{\mu_i} e_{-\alpha_i} \in \mathfrak{g}_{-1}. \end{aligned} \tag{7.2}$$

(It is easy to see from (6.1) that $t_\Gamma^{-1}\exp(2\pi i w_0)$ belongs to the center of G.) For the concrete expression for Φ in (1.7), we introduce the following setup after [Kos79].

Let $\mathcal{S}$ be the set of all finite sequences

$$s = (i_1, \dots, i_k), \quad k \ge 0,\ 1 \le i_j \le n. \tag{7.3}$$

We write $|s|$ for the length k of the element $s \in \mathcal{S}$, and we also write

$$\varphi(s) = \sum_{j=1}^{|s|} \alpha_{i_j}, \quad \varphi(s, w_0) = \varphi(s)(w_0) = \langle \varphi(s), w_0 \rangle = \sum_{j=1}^{|s|} \mu_{i_j}, \tag{7.4}$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing between $\mathfrak{h}_0'$ and $\mathfrak{h}_0$ (see (A.3)). Note that $\varphi(s)$ is equal to the constant function 0 on $\mathfrak{h}$ if $|s| = 0$.

For $0 \le j \le |s| - 1$, let $s_j \in \mathcal{S}$ be the sequence obtained from s by "cutting off" the *first* j terms (different from [Kos79])

$$s_j = (i_{j+1}, \dots, i_{|s|}), \tag{7.5}$$

and define

$$p(s, w_0) = \prod_{j=0}^{|s|-1} \langle \varphi(s_j), w_0 \rangle. \tag{7.6}$$

Note then when $|s| = 0$, we have $p(s, w_0) = 1$. Clearly when $|s| \ge 1$, we have

$$p(s, w_0) = \varphi(s, w_0) p(s_1, w_0). \tag{7.7}$$

Let $U(\mathfrak{n}) = U(\mathfrak{n}_-)$ be the enveloping algebra of $\mathfrak{n}$. For convenience write $e_{-i} = e_{-\alpha_i}$ for $i = 1, \dots, n$. For $s \in \mathcal{S}$ as in (7.3), put

$$e_{-s} = e_{-i_k} \cdots e_{-i_2} e_{-i_1}. \tag{7.8}$$

We note that the ξ in (7.2) is

$$\xi = \sum_{i=1}^n z^{\varphi((i), w_0)} e_{-(i)}, \tag{7.9}$$

where the s are the simplest (i) for $1 \le i \le n$.

Proposition 7.10. *In the space $\hat{D}(N)$ (see the Appendix Subsection A.8), we have*

$$\Phi = \sum_{s\in\mathcal{S}} \frac{z^{\varphi(s,w_0)}e_{-s}}{p(s,w_0)}. \tag{7.11}$$

Proof. We denote the right hand side of (7.11) by Υ, and we show that it satisfies (1.7) and hence it is equal to Φ. Clearly $\Upsilon(0) = 1$ corresponding to the empty s. We also have

$$\Upsilon_z = \sum_{s\in\mathcal{S}} \frac{\varphi(s,w_0)z^{\varphi(s,w_0)-1}e_{-s}}{p(s,w_0)} = \frac{1}{z}\sum_{|s|\geq 1} \frac{z^{\varphi(s,w_0)}e_{-s}}{p(s_1,w_0)} \tag{7.12}$$

by (7.7).

Let $s \in \mathcal{S}$ and let $\{t^1, \dots, t^n\}$ be the set of all $t \in \mathcal{S}$ such that $t_1 = s$ according to (7.5). Then by (7.9) and (7.4), we have

$$\frac{z^{\varphi(s,w_0)}e_{-s}}{p(s,w_0)}\xi = \sum_{i=1}^{n} \frac{z^{\varphi(t^i,w_0)}e_{-t^i}}{p(s,w_0)} = \sum_{i=1}^{n} \frac{z^{\varphi(t^i,w_0)}e_{-t^i}}{p((t^i)_1,w_0)}.$$

Therefore we have

$$\Upsilon\xi = \sum_{|s|\geq 1} \frac{z^{\varphi(s,w_0)}e_{-s}}{p(s_1,w_0)}. \tag{7.13}$$

Combining (7.12) and (7.13), we see $\Upsilon_z = \frac{1}{z}\Upsilon\xi = \Upsilon\zeta$. Therefore $\Upsilon^{-1}\Upsilon_z = \zeta$, and (7.11) is proved. □

For its own interest, we have the following algebraic characterization of our Φ in (1.7), where it was originally defined by an ODE.

Proposition 7.14. *The unique map $\Psi : \mathbb{C} \setminus \mathbb{R}_{<0} \to N = N_-$ such that*

$$\begin{cases} \Psi^{-1}w_0\Psi = w := w_0 - \xi, \\ \Psi(0) = Id \end{cases} \tag{7.15}$$

is $\Psi = \Phi$ from (1.7).

Proof. Our Ψ is modeled after [Kos79], and its existence and uniqueness (up to the center of G) follow from Lemma 3.5.2 there. Note that when $z = 0$, $\xi(0) = 0$ and so $\Psi(0)$ can be chosen to be Id. This makes the Ψ unique.

Now we show that $\Psi = \Phi$ by verifying that it satisfies the ODE in (1.7). We note that by (7.2) and (A.11), we have

$$w_z = -\sum_{i=1}^{n} \mu_i z^{\gamma_i} e_{-\alpha_i} = [w_0, \zeta].$$

With $\psi := \Psi^{-1}\Psi_z \in \mathfrak{n} = \mathfrak{n}_-$, we differentiate (7.15) to get $[w, \psi] = w_z$, which is then

$$[w_0 - z\zeta, \psi] = [w_0, \zeta]. \tag{7.16}$$

We now show that this equation implies that $\psi = \zeta$. We write $\psi = \sum_{i\leq -1} \psi_i$ according to the grading (A.13), and we also note that $\zeta \in \mathfrak{g}_{-1}$. Furthermore by (7.2) and (7.1), we see that

$$\ker \operatorname{ad}_{w_0} \cap \mathfrak{n}_- = 0.$$

The components in $\mathfrak{g}_{-1}$ of (7.16) give that $[w_0, \psi_{-1}] = [w_0, \zeta]$ and so $\psi_{-1} = \zeta$. The components in $\mathfrak{g}_{-2}$ of (7.16) give that $[w_0, \psi_{-2}] - [z\zeta, \psi_{-1}] = 0$, and so by $\psi_{-1} = \zeta$ we have $\psi_{-2} = 0$. A quick induction then shows that all $\psi_i = 0$ for $i \leq -3$. Therefore $\psi = \zeta$, and $\Psi = \Phi$ by (1.7). $\square$

Remark 7.17. A proof following [Kos79, Prop. 5.8.2] can also be constructed to show directly that $\Psi = \Upsilon$ for the Υ and Ψ in Propositions 7.10 and 7.14.

Let λ be a dominant weight, and let V^λ be the corresponding irreducible representation. Let κ be the longest Weyl group element which maps positive roots to negative roots. Then $\kappa\lambda$ is the lowest weight of V^λ (see the Appendix Subsection A.10). Throughout the paper we denote the action of κ on $\mathfrak{h}_0'$ without parentheses. In the following calculations, we use the Hermitian metric $\{\cdot, \cdot\}$ on V^λ which is invariant under the compact subgroup K^s of G^s (see the Appendix Subsection A.9). Choose vectors $v^\lambda \in V_\lambda$ and $v^{\kappa\lambda} \in V_{\kappa\lambda}$ in the one-dimensional highest and lowest weight spaces such that

$$\{v^\lambda, v^\lambda\} = 1, \quad \{v^{\kappa\lambda}, v^{\kappa\lambda}\} = 1.$$

(One can choose $v^{\kappa\lambda}$ to be $s_0(\kappa)v^\lambda$ where $s_0(\kappa) \in G^s$ induces the longest Weyl element κ. See [Kos79, Eq. (5.2.10)].)

With the notation (7.4) and (7.8), the vector $e_{-s}v^\lambda$ is a weight vector with weight $-\varphi(s) + \lambda$. Since different weight spaces are orthogonal, by Proposition 7.10 we have

$$\{\Phi v^\lambda, v^{\kappa\lambda}\} = \Big(\sum_{s \in \mathcal{S}^\lambda} \frac{c_{s,\lambda}}{p(s, w_0)} \Big) z^{\langle \lambda - \kappa\lambda, w_0 \rangle}, \tag{7.18}$$

where $\mathcal{S}^\lambda = \{s \in \mathcal{S} \,|\, \varphi(s) = \lambda - \kappa\lambda\}$, and $c_{s,\lambda} = \{e_{-s}v^\lambda, v^{\kappa\lambda}\}$. It is known that coefficient in the parentheses is nonzero by [Kos79, Prop. 5.9.1] (and can be made positive real with suitable choices).

The following theorem is the main result of this section. It provides the asymptotic behavior of our solutions and proves the quantization result for our integrals.

Theorem 7.19. *The solutions U_i in* (1.8) *satisfy that*

$$U_i = 2(\gamma^i - \langle \omega_i - \kappa\omega_i, w_0 \rangle) \log |z| + O(1), \quad \textit{as } z \to \infty, \tag{7.20}$$

where ω_i is the ith fundamental weight. Consequently, the solutions u_i in (1.9) *satisfy that*

$$u_i = -2(2 - \kappa\gamma_i) \log |z| + O(1), \quad \textit{as } z \to \infty, \tag{7.21}$$

where if $-\kappa\alpha_i = \alpha_k$ then $-\kappa\gamma_i = \gamma_k$.

The finite integrals in (1.1) *are quantized as*

$$\sum_{j=1}^n a_{ij} \int_{\mathbb{R}^2} e^{u_j} \, dx = \pi(\mu_i - \kappa\mu_i), \tag{7.22}$$

where $-\kappa\mu_i = \langle -\kappa\alpha_i, w_0 \rangle$. Therefore we have

$$\sum_{j=1}^n a_{ij} \int_{\mathbb{R}^2} e^{u_j} \, dx = 2\pi(1 + \gamma_i), \quad 1 \leq i \leq n, \tag{7.23}$$

except the following cases:

(i) $\mathfrak{g} = A_n$, *we recover the quantization in* [LWY12]

$$\sum_{j=1}^{n} a_{ij} \int_{\mathbb{R}^2} e^{u_j}\, dx = \pi(2 + \gamma_i + \gamma_{n+1-i}), \quad 1 \le i \le n. \tag{7.24}$$

(ii) $\mathfrak{g} = D_{2n+1}$ *and* $i = 2n$ *or* $2n+1$, *we have*

$$\begin{aligned} -\int_{\mathbb{R}^2} e^{u_{2n-1}}\, dx + 2\int_{\mathbb{R}^2} e^{u_{2n}}\, dx &= \pi(2 + \gamma_{2n} + \gamma_{2n+1}), \\ -\int_{\mathbb{R}^2} e^{u_{2n-1}}\, dx + 2\int_{\mathbb{R}^2} e^{u_{2n+1}}\, dx &= \pi(2 + \gamma_{2n} + \gamma_{2n+1}). \end{aligned} \tag{7.25}$$

(iii) $\mathfrak{g} = E_6$ *and using the labeling in the Appendix Subsection A.5, we have*

$$\begin{aligned} \sum_{j=1}^{6} a_{ij} \int_{\mathbb{R}^2} e^{u_j}\, dx &= \pi(2 + \gamma_1 + \gamma_6), \quad i = 1, 6, \\ \sum_{j=1}^{6} a_{ij} \int_{\mathbb{R}^2} e^{u_j}\, dx &= \pi(2 + \gamma_3 + \gamma_5), \quad i = 3, 5, \end{aligned} \tag{7.26}$$

Proof. To show the asymptotic behavior (7.20), we prove

$$e^{-U_i} = |z|^{-2\gamma^i} |z|^{2\langle \omega_i - \kappa\omega_i, w_0\rangle}(c_i + o(1)), \quad \text{as } z \to \infty. \tag{7.27}$$

By (1.8) and (A.18), we have

$$e^{-U_i} = |z|^{-2\gamma^i} \langle i|\Phi^* C^* \Lambda^2 C\Phi|i\rangle = |z|^{-2\gamma^i} \{\Lambda C\Phi v^{\omega_i}, \Lambda C\Phi v^{\omega_i}\}.$$

We know that

$$\{\Lambda C\Phi v^{\omega_i}, \Lambda C\Phi v^{\omega_i}\} = \sum_v \left|\{\Lambda C\Phi v^{\omega_i}, v\}\right|^2,$$

where the sum ranges over an orthonormal basis of weight vectors of the ith fundamental representation V_i. By Proposition 7.10 and that $\mu_i > 0$ from (7.1), we see that the highest power of z appears in the norm squared of $\{\Lambda C\Phi v^{\omega_i}, v^{\kappa\omega_i}\}$, that is, the coordinate of $\Lambda C\Phi v^{\omega_i}$ in the lowest weight vector $v^{\kappa\omega_i}$. Furthermore, by (A.17) we have

$$\{\Lambda C\Phi v^{\omega_i}, v^{\kappa\omega_i}\} = \{\Phi v^{\omega_i}, C^*\Lambda v^{\kappa\omega_i}\} = \Lambda^{\kappa\omega_i}(\{\Phi v^{\omega_i}, v^{\kappa\omega_i}\} + \text{lower order terms}),$$

where $\Lambda^{\kappa\omega_i} = \exp(\langle \kappa\omega_i, H\rangle)$ if $\Lambda = \exp(H)$ for $H \in \mathfrak{h}$. Using (7.18), we see that (7.27) holds with

$$c_i = (\Lambda^{\kappa\omega_i})^2 \Big|\sum_{s \in \mathcal{S}^{\omega_i}} \frac{c_{s,\omega_i}}{p(s, w_0)}\Big|^2.$$

By $u_i = \sum_{j=1}^n a_{ij} U_j$, $\gamma_i = \sum_{j=1}^n a_{ij}\gamma^j$, $\alpha_i = \sum_{j=1}^n a_{ij}\omega_j$, and $\langle \alpha_i, w_0\rangle = \mu_i$, and assuming that $-\kappa\alpha_i = \alpha_k$ for some $1 \le k \le n$, we see

$$\begin{aligned} u_i = \sum_{j=1}^{n} a_{ij} U_j &= 2(\gamma_i - \langle \alpha_i - \kappa\alpha_i, w_0\rangle)\log|z| + O(1) \\ &= 2(\gamma_i - \mu_i - \mu_k)\log|z| + O(1) = -2(2 + \gamma_k)\log|z| + O(1) \\ &= -2(2 - \kappa\gamma_i)\log|z| + O(1), \end{aligned}$$

which is (7.21).

Integrating (1.5) and using (7.20), we have

$$\begin{aligned} 4\int_{\mathbb{R}^2} e^{u_i}\,dx &= 4\pi\gamma^i - \lim_{R\to\infty}\int_{\partial B_R}\frac{\partial U_i}{\partial \nu}ds \\ &= 4\pi\langle \omega_i - \kappa\omega_i, w_0\rangle. \end{aligned}$$

The linear combinations of the above using the Cartan matrix clearly give (7.22), which implies the more concrete version (1.10) in Theorem 1.6.

Except three cases, $-\kappa = Id$ and so (7.23) follows immediately. The three exceptions are for the Lie algebras of type A_n, D_{2n+1} and E_6, where $-\kappa$ represents the symmetry of the Dynkin diagram of the Lie algebra. In the A_n case,

$$-\kappa\alpha_i = \alpha_{n+1-i}, \quad 1 \le i \le n,$$

that is, $-\kappa$ is the reflection of the Dynkin diagram about its center. Therefore we have (7.24), and this is the quantization result in [LWY12, Theorem 1.3], up to the factor of 4.

In the D_{2n+1} case, we have

$$-\kappa\alpha_{2n} = \alpha_{2n+1},\ -\kappa\alpha_{2n+1} = \alpha_{2n},\ -\kappa\alpha_i = \alpha_i,\ 1 \le i \le 2n-1,$$

that is, $-\kappa$ permutes the last two roots at the node and preserves the others. This proves (7.25).

In the E_6 case, using the labeling in the Appendix Subsection A.5 we have

$$\begin{aligned} -\kappa\alpha_1 = \alpha_6,\ -\kappa\alpha_6 = \alpha_1, \\ -\kappa\alpha_3 = \alpha_5,\ -\kappa\alpha_5 = \alpha_3, \\ -\kappa\alpha_2 = \alpha_2,\ -\kappa\alpha_4 = \alpha_4. \end{aligned}$$

This then proves (7.26). □

8. Examples and relations with previous results

For classical groups, we discuss the Lie-theoretic setups in more concrete terms. To make our result more explicit especially to the analysts, we present the examples of A_2 Toda systems and relate them with the previous results in [LWY12, Nie16]. Then we will present the example of D_4 to illustrate our new result in this paper.

It is known [FH91] that the Lie groups $SL_{n+1}\mathbb{C}$ and $Sp_n\mathbb{C}$ are simply connected, but $\pi_1(SO_m\mathbb{C}) \cong \mathbb{Z}/2$ for $m \ge 3$ and its universal cover is $Spin_m\mathbb{C}$. However as discussed in the Appendix Subsection A.8, all the calculations can be done in the simpler $SO_m\mathbb{C}$.

Then for an element g in these classical groups, $g^* = \bar{g}^t$ is the conjugate transpose. The compact subgroups $K \subset G$ are characterized by $g^{-1} = g^*$, and they are

$$\begin{aligned} SU(n+1) \subset SL_{n+1}\mathbb{C},\ SO(2n+1) \subset SO_{2n+1}\mathbb{C}, \\ Sp(2n) \subset Sp_{2n}\mathbb{C},\ SO(2n) \subset SO_{2n}\mathbb{C}. \end{aligned}$$

Clearly for $SL_{n+1}\mathbb{C}$, the nilpotent subgroup N consists of unipotent lower-triangular matrices with complex entries, and the abelian subgroup A consists of diagonal matrices with positive entries that multiply to 1.

Choose the symmetric and skew-symmetric bilinear forms κ_m and Ω_{2n} for $SO_m\mathbb{C}$ and $Sp_{2n}\mathbb{C}$ to be

$$\kappa_m = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}_{m\times m}, \quad \Omega_{2n} = \begin{pmatrix} & \kappa_n \\ -\kappa_n & \end{pmatrix}. \tag{8.1}$$

Then the nilpotent subgroups $N \subset G$ in these cases consist of unipotent lower-triangular matrices with complex entries, which preserve the corresponding bilinear forms. The abelian subgroups $A \subset G$ consist of diagonal matrices with positive entries whose symmetric pairs with respect to the secondary diagonal multiply to 1.

The first fundamental representations of these classical groups are just their standard representations, that is, the standard actions of $SL_{n+1}\mathbb{C}$ on $\mathbb{C}^{n+1}$, $SO_m\mathbb{C}$ on $\mathbb{C}^m$, and $Sp_{2n}\mathbb{C}$ on $\mathbb{C}^{2n}$. The ith fundamental representations are the irreducible representations with the same highest weights in the ith wedge products of the standard representations, except that at the end there are 1 spin representation for B_n and two half-spin representations for D_n. Therefore the highest matrix coefficients $\langle i|\cdot|i\rangle$ are just the leading principal minors of rank i of the matrices in the standard representations, except for the spin representations where one needs to do more work.

Example 8.2 (A_2 Toda system). The group is $G = SL_3\mathbb{C}$, and the solution to

$$\begin{cases} \Phi^{-1}\Phi_z = \sum_{i=1}^{2} z^{\gamma_i} e_{-\alpha_i} = \begin{pmatrix} 0 & & \\ z^{\gamma_1} & 0 & \\ & z^{\gamma_2} & 0 \end{pmatrix} \\ \Phi(0) = Id \end{cases} \qquad \text{is}$$

$$\Phi(z) = \begin{pmatrix} 1 & & \\ \frac{z^{\mu_1}}{\mu_1} & 1 & \\ \frac{z^{\mu_1+\mu_2}}{\mu_2(\mu_1+\mu_2)} & \frac{z^{\mu_2}}{\mu_2} & 1 \end{pmatrix}.$$

General elements $\Lambda \in A$ and $C \in N$ are of the form

$$\Lambda = \begin{pmatrix} \lambda_0 & & \\ & \lambda_1 & \\ & & \lambda_2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & & \\ c_{10} & 1 & \\ c_{20} & c_{21} & 1 \end{pmatrix},$$

where $\lambda_i > 0$ and $\lambda_0\lambda_1\lambda_2 = 1$. Then

$$\mathrm{X} := \Lambda C \Phi = \begin{pmatrix} \lambda_0 & & \\ \lambda_1\left(\frac{z^{\mu_1}}{\mu_1} + c_{10}\right) & \lambda_1 & \\ \lambda_2\left(\frac{z^{\mu_1+\mu_2}}{\mu_2(\mu_1+\mu_2)} + c_{21}\frac{z^{\mu_1}}{\mu_1} + c_{20}\right) & \lambda_2\left(\frac{z^{\mu_2}}{\mu_2} + c_{21}\right) & \lambda_2 \end{pmatrix}.$$

Therefore by Theorem 1.6, we see that

$$\begin{aligned} e^{-U_1} &= |z|^{-2\gamma^1}\langle 1|\mathrm{X}^*\mathrm{X}|1\rangle = |z|^{-2\gamma^1}(\mathrm{X}^*\mathrm{X})_{1,1} \\ &= |z|^{-2\gamma^1}\left(\lambda_0^2 + \lambda_1^2\left|\frac{z^{\mu_1}}{\mu_1} + c_{10}\right|^2 + \lambda_2^2\left|\frac{z^{\mu_1+\mu_2}}{\mu_2(\mu_1+\mu_2)} + c_{21}\frac{z^{\mu_1}}{\mu_1} + c_{20}\right|^2\right). \end{aligned}$$

Noting that our γ^1 is their α_1, this clearly matches with the main Theorem 1.1 in [LWY12] after some suitable correspondence. In particular, Eq. (1.11) there matches our condition that $\lambda_0\lambda_1\lambda_2 = 1$ by the coefficient 4 in our equation (1.1) and that our power functions now have the coefficients for the μ's.

Furthermore, the condition that if $\gamma_{j+1}+\cdots+\gamma_i \notin \mathbb{Z}$ for some $j < i$, then $c_{ij} = 0$ exactly matches our current condition that $C \in N_\Gamma$.

For the Toda systems of types A (and C and B), the U_1 determines the other U_i's. So our current result is compatible with [LWY12].

Similarly for Toda systems of type C and B, the current results are related to those in [Nie16]. Now let us present the D_4 example in some detail, since this illustrates our new result.

Example 8.3 (D_4 Toda system). For simplicity, we present this example using concrete numbers. Let $\gamma_1 = -\frac{1}{2}, \gamma_2 = 1, \gamma_3 = 2, \gamma_4 = 3$, and then $\gamma^1 = 3, \gamma^2 = \frac{13}{2}, \gamma^3 = \frac{17}{4}, \gamma^4 = \frac{19}{4}$ by $\gamma^i = \sum_{j=1}^n a^{ij}\gamma_j$.

The solution to

$$\begin{cases} \Phi^{-1}\Phi_z = \sum_{i=1}^4 z^{\gamma_i} e_{-\alpha_i} = \begin{pmatrix} 0 & & & & & & & \\ \frac{1}{\sqrt{z}} & 0 & & & & & & \\ & z & 0 & & & & & \\ & & z^2 & 0 & & & & \\ & & z^3 & 0 & 0 & & & \\ & & & -z^3 & -z^2 & 0 & & \\ & & & & & -z & 0 & \\ & & & & & & -\frac{1}{\sqrt{z}} & 0 \end{pmatrix} \\ \Phi(0) = Id \end{cases} \quad \text{is}$$

$$\Phi(z) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\sqrt{z} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{5}z^{\frac{5}{2}} & \frac{1}{2}z^2 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{165}z^{\frac{11}{2}} & \frac{1}{15}z^5 & \frac{1}{3}z^3 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{156}z^{\frac{13}{2}} & \frac{1}{24}z^6 & \frac{1}{4}z^4 & 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{1026}z^{\frac{19}{2}} & -\frac{1}{108}z^9 & -\frac{1}{12}z^7 & -\frac{1}{4}z^4 & -\frac{1}{3}z^3 & 1 & 0 & 0 \\ \frac{1}{6210}z^{\frac{23}{2}} & \frac{1}{540}z^{11} & \frac{11}{540}z^9 & \frac{1}{12}z^6 & \frac{1}{10}z^5 & -\frac{1}{2}z^2 & 1 & 0 \\ -\frac{64}{312455}z^{12} & -\frac{768}{312455}z^{\frac{23}{2}} & -\frac{384}{13585}z^{\frac{19}{2}} & -\frac{8}{65}z^{\frac{13}{2}} & -\frac{8}{55}z^{\frac{11}{2}} & \frac{4}{5}z^{\frac{5}{2}} & -2\sqrt{z} & 1 \end{pmatrix}.$$

General elements $\Lambda \in A$ and $C \in N$ are of the form

$$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_4^{-1}, \lambda_3^{-1}, \lambda_2^{-1}, \lambda_1^{-1}), \quad \lambda_i > 0,$$

$$C = \begin{pmatrix} 1 & & & & & & & \\ c_{21} & 1 & & & & & & \\ c_{31} & c_{32} & 1 & & & & & \\ c_{41} & c_{42} & c_{43} & 1 & & & & \\ c_{51} & c_{52} & c_{53} & 0 & 1 & & & \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & 1 & & \\ c_{71} & c_{72} & c_{73} & c_{74} & c_{75} & c_{76} & 1 & \\ c_{81} & c_{82} & c_{83} & c_{84} & c_{85} & c_{86} & c_{87} & 1 \end{pmatrix}.$$

Solving $C^t\kappa_8 C = \kappa_8$ (see (8.1)), we see that the coordinates are the c's above the secondary diagonal, that is, the c_{ij} for $j < i \le 8 - j$, $1 \le j \le 3$, and the other c's are solved in these.

Furthermore, for the current γ's, $\alpha(\Gamma)$ in Remark 6.3 is not an integer if α contains α_1. Therefore the subgroup N_Γ in our case is obtained by letting $c_{i1} = 0$ for $2 \leq i \leq 7$, which are coordinates corresponding to the roots $\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$.

The solution space in this case has dimension $16 = 4+2\cdot 6$ and is parametrized by the positive numbers $\lambda_1, \ldots, \lambda_4$ and the complex numbers $c_{32}, c_{43}, c_{53}, c_{42}, c_{52}, c_{62}$, which are coordinates corresponding to the roots $\alpha_2, \alpha_3, \alpha_4, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4$.

Therefore with $\mathrm{X} = \Lambda C \Phi$, we see that

$$e^{-U_1} = |z|^{-2\gamma^1} (\mathrm{X}^*\mathrm{X})_{1,1},$$
$$e^{-U_2} = |z|^{-2\gamma^2} (\mathrm{X}^*\mathrm{X})_{[1,2],[1,2]},$$

where U_2 involves the leading principal 2×2 minor.

Now to determine U_3 and U_4, we need the 3rd and the 4th fundamental representations of D_4, which are the two half-spin representations. (There is the triality, the symmetry for α_1, α_3 and α_4 in the case of D_4, but we disregard that since that symmetry is not valid for higher D_n.) Then U_3 and U_4 are expressed in terms of the highest matrix coefficient of $\mathrm{X}^*\mathrm{X}$ for the spin representations. These can be concretely computed, with the help of a computer algebra system such as Maple, by the Lie algebra spin representations and the exponential map. (See [FH91] for more details on the spin representations).

Appendix A. Background and setup from Lie theory

In this appendix, we provide the details for the Lie-theoretic background and setup needed in this paper. Our basic references are [Kna02, Hel78, FH91].

A.1. **Cartan subalgebra and root space decomposition.** Let $\mathfrak{g}$ be a complex simple Lie algebra. The Killing form $B(X,Y) = \mathrm{Tr}(\mathrm{ad}_X \, \mathrm{ad}_Y)$ is a symmetric nondegenerate bilinear form on $\mathfrak{g}$, where $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$; $Z \mapsto [X, Z]$ is the adjoint action of X. Let $\mathfrak{h}$ be a fixed Cartan subalgebra, whose dimension n is the rank of the Lie algebra.

Let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$ be the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$, where Δ denotes the set of roots. The roots are linear functions on the Cartan subalgebra $\mathfrak{h}$, and for $X_\alpha \in \mathfrak{g}_\alpha$ and $H \in \mathfrak{h}$, we have $[H, X_\alpha] = \alpha(H)X_\alpha$. It is known that $\dim_{\mathbb{C}} \mathfrak{g}_\alpha = 1$.

Let $\Delta = \Delta^+ \cup \Delta^-$ be a fixed decomposition of the set of roots into the sets of positive and negative roots, and let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be the set of positive simple roots.

We furthermore introduce the following standard subalgebras of $\mathfrak{g}$

$$\mathfrak{n} = \mathfrak{n}_- = \oplus_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha}, \quad \mathfrak{n}_+ = \oplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha, \quad \mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+. \tag{A.1}$$

A.2. **Gauss decomposition.** Let G be a connected complex Lie group whose Lie algebra is $\mathfrak{g}$. Let $\mathcal{H}$ be the Cartan subgroup of G corresponding to $\mathfrak{h}$. Denote the subgroups of G corresponding to $\mathfrak{n} = \mathfrak{n}_-$, $\mathfrak{n}_+$ and $\mathfrak{b}_+$ in (A.1) by $N = N_-, N_+$ and B_+. Here B_+ is called a Borel subgroup of G. Then by the Gauss decomposition (see [LS92, Eq. (1.5.6)] and [Kos79, Eq. (2.4.4)]), there exists an open and dense subset G_r of G, called the regular part, such that

$$G_r = N_- N_+ \mathcal{H}. \tag{A.2}$$

We note that $\mathcal{H}N_+ = N_+\mathcal{H}$ by $hn = (hnh^{-1})h$, where $h \in \mathcal{H}$ and $n \in N_+$. Clearly G_r contains the identity element of G.

A.3. **Split and compact real forms.** By [Kna02, Theorem 6.6 and Corollary 6.10], the complex Lie algebra $\mathfrak{g}$ has a split real form

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \bigoplus_{\alpha\in\Delta} \mathbb{R}e_\alpha, \tag{A.3}$$

where $\mathfrak{h}_0 = \{H \in \mathfrak{h} \,|\, \alpha(H) \in \mathbb{R}, \forall\alpha \in \Delta\}$, and the $e_\alpha \in \mathfrak{g}_\alpha$ form a Cartan-Weyl basis.

There exists a basis H_{α_i} for $1 \le i \le n$ of $\mathfrak{h}_0$ such that

$$B(H_{\alpha_i}, H) = \alpha_i(H), \quad \forall H \in \mathfrak{h}. \tag{A.4}$$

The positive definite Killing form B on $\mathfrak{h}_0$ also introduces an inner product on the real dual space $\mathfrak{h}_0' = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{h}_0, \mathbb{R})$ by $(\alpha_i, \alpha_j) = B(H_{\alpha_i}, H_{\alpha_j})$ for $1 \le i, j \le n$.

The split real form (A.3) defines a Cartan decomposition of the Lie algebra into vector subspaces

$$\begin{aligned}
&\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k},\\
&\mathfrak{k} = i\mathfrak{h}_0 + \sum_{\alpha\in\Delta^+} \mathbb{R}(e_\alpha - e_{-\alpha}) + \sum_{\alpha\in\Delta^+} \mathbb{R}i(e_\alpha + e_{-\alpha}),\\
&i\mathfrak{k} = \mathfrak{h}_0 + \sum_{\alpha\in\Delta^+} \mathbb{R}i(e_\alpha - e_{-\alpha}) + \sum_{\alpha\in\Delta^+} \mathbb{R}(e_\alpha + e_{-\alpha}).
\end{aligned} \tag{A.5}$$

Here $\mathfrak{k}$ is subalgebra and it is compact in the sense that the restriction of the Killing form $B|_{\mathfrak{k}\times\mathfrak{k}}$ is negative definite.

The Cartan decomposition (A.3) determines the Cartan involution θ of $\mathfrak{g}$ which is $+1$ on $\mathfrak{k}$ and -1 on $i\mathfrak{k}$. We often write X^θ for $\theta(X)$ with $X \in \mathfrak{g}$. Define $X^* = -X^\theta$, and we see that

$$H^*_{\alpha_i} = H_{\alpha_i}, \quad (iH_{\alpha_i})^* = -iH_{\alpha_i}, \quad e^*_\alpha = e_{-\alpha}, \text{ and } \quad (ie_\alpha)^* = -ie_{-\alpha}. \tag{A.6}$$

Therefore $*$ and θ are *conjugate linear* with respect to the split real form.

A.4. **Iwasawa decomposition.** From the Cartan decomposition (A.5), we also have the following Iwasawa decomposition [Kna02, Proposition 6.43] of the Lie algebra into Lie subalgebras

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}, \quad \text{where } \mathfrak{a} = \mathfrak{h}_0, \quad \mathfrak{n} = \mathfrak{n}_-.$$

Let G be a connected complex Lie group whose Lie algebra is $\mathfrak{g}$. Then we have the corresponding Iwasawa decomposition on the group level [Kna02, Theorem 6.46]

$$G = KAN, \tag{A.7}$$

where K, A and N are the subgroups in G corresponding to $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$. The subgroup K is compact, the subgroup A is abelian, and the subgroup N is nilpotent. The groups A and N are simply connected. (The usual version using $\mathfrak{n}_+ = \oplus_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$ implies our current version using $\mathfrak{n}_-$ after one application of the Cartan involution θ.)

The Cartan involution θ on $\mathfrak{g}$ also lifts to the group G [Kna02, Theorem 6.31], and we continue to denote it by the same notation. Then $K = G^\theta$ is the subgroup fixed by θ. For $g \in G$, define $g^* = (g^\theta)^{-1}$. Then $(gh)^* = h^*g^*$, and an element $F \in K$ if and only if $F^*F = Id$. Furthermore, from (A.6) we see that $(N_+)^* = N_-$.

For the purpose of doing representation theory, we let G^s be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Then it is known that G^s is the universal covering of a general G whose Lie algebra is $\mathfrak{g}$. Let $G^s = K^s A^s N^s$ be its Iwasawa decomposition, while $G = KAN$ is the Iwasawa decomposition of G. Then A and N are simply connected and hence isomorphic to A^s and N^s. Only K^s is a covering of K.

A.5. **Cartan matrices and classification of simple Lie algebras.** We further specify the normalization of the e_{α_i} and $e_{-\alpha_i}$ for $1 \le i \le n$ in (A.3) by requiring that

$$\alpha_i(h_{\alpha_i}) = 2, \quad \text{where } h_{\alpha_i} = [e_{\alpha_i}, e_{-\alpha_i}]. \tag{A.8}$$

By [Kna02, Eq. (2.93)], the relation of the h_{α_i} with the basis (A.4) is that

$$h_{\alpha_i} = \frac{2H_{\alpha_i}}{B(H_{\alpha_i}, H_{\alpha_i})} = \frac{2H_{\alpha_i}}{(\alpha_i, \alpha_i)}. \tag{A.9}$$

The benefit of the choice of the h_{α_i} is that the Lie subalgebra generated by $e_{\alpha_i}, h_{\alpha_i}$, and $e_{-\alpha_i}$ is isomorphic to a copy of $\mathfrak{sl}_2$ with the standard basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then the Cartan matrix (a_{ij}) of $\mathfrak{g}$ is

$$a_{ij} = \alpha_i(h_{\alpha_j}) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}, \quad 1 \le i, j \le n. \tag{A.10}$$

There are four infinite series of classical complex simple Lie algebras and five exceptional Lie algebras with the following Cartan matrices

$$A_n = \mathfrak{sl}_{n+1} : \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}, \qquad B_n = \mathfrak{so}_{2n+1} : \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -2 \\ & & & -1 & 2 \end{pmatrix},$$

$$C_n = \mathfrak{sp}_{2n} : \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -2 & 2 \end{pmatrix}, \qquad D_n = \mathfrak{so}_{2n} : \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & -1 \\ & & & -1 & 2 & \\ & & & -1 & & 2 \end{pmatrix},$$

$$G2 : \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \qquad F4 : \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -2 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix},$$

$$E6 : \begin{pmatrix} 2 & & -1 & & & \\ & 2 & & -1 & & \\ -1 & & 2 & -1 & & \\ & -1 & -1 & 2 & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}, \qquad E7 : \begin{pmatrix} 2 & & -1 & & & & \\ & 2 & & -1 & & & \\ -1 & & 2 & -1 & & & \\ & -1 & -1 & 2 & -1 & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{pmatrix},$$

$$E8 : \begin{pmatrix} 2 & & -1 & & & & & \\ & 2 & & -1 & & & & \\ -1 & & 2 & -1 & & & & \\ & -1 & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix}.$$

In the above, the labelling of the roots for the exceptional Lie algebras follows [Kna02, pp 180-1]. We require that $n \ge 2$ for B_n and C_n, and $n \ge 3$ for D_n.

Furthermore we have the following isomorphisms

$$B_2 \cong C_2, \quad D_3 \cong A_3.$$

A.6. **Principal grading.** There exist $E_j \in \mathfrak{h}_0$ such that

$$\alpha_i(E_j) = \delta_{ij}, \quad \text{for } 1 \le i, j \le n. \tag{A.11}$$

By (A.10), we have $E_j = \sum_{k=1}^n a^{kj} h_{\alpha_k}$, where (a^{kj}) is the inverse Cartan matrix. Define the so-called principal grading element

$$E_0 = \sum_{j=1}^n E_j \in \mathfrak{h}_0, \quad \text{such that } \alpha_i(E_0) = 1, \text{ for } 1 \le i \le n. \tag{A.12}$$

Define $\mathfrak{g}_j = \{x \in \mathfrak{g} \,|\, [E_0, x] = jx\}$. Then

$$\mathfrak{g} = \sum_j \mathfrak{g}_j \tag{A.13}$$

is the principal grading of $\mathfrak{g}$, and we have

$$\mathfrak{g}_{-1} = \oplus_{i=1}^n \mathfrak{g}_{-\alpha_i}. \tag{A.14}$$

A.7. **Representation spaces.** The weight lattice is $\Lambda_W = \{\beta \in \mathfrak{h}_0' \,|\, \beta(h_{\alpha_i}) \in \mathbb{Z}, \forall 1 \le i \le n\}$. A weight β is called dominant if $(\beta, \alpha_i) \ge 0$ for all $1 \le i \le n$. The weight lattice is a lattice generated by the fundamental weight ω_i for $1 \le i \le n$ such that

$$\omega_i(h_{\alpha_j}) = \frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}.$$

An irreducible representation ρ of $\mathfrak{g}$ on a finite-dimensional complex vector space V has the weight space decomposition $V = \oplus V_\beta$, where $\beta \in \Lambda_W$ and $\rho(H)(v) = \beta(H)v$ for $H \in \mathfrak{h}$ and $v \in V_\beta$. There exists a unique highest weight λ with a one-dimensional highest weight space V_λ such that $\rho(\mathfrak{n}_+)V_\lambda = 0$.

The Theorem of the Highest Weight [Kna02, Theorem 5.5] asserts that up to equivalence, the irreducible finite-dimensional complex representations of $\mathfrak{g}$ stand in one-one correspondence with the dominant weights which sends an irreducible representation to its highest weight. We denote the irreducible representation space corresponding to a dominant weight λ by V^λ.

There is a canonical pairing between the dual space $V^* = \mathrm{Hom}(V, \mathbb{C})$ and V denoted by $\langle w, v\rangle \in \mathbb{C}$ with $v \in V$ and $w \in V^*$. V^* has a *right representation* ρ^* of $\mathfrak{g}$ defined by

$$\langle w\rho^*(X), v\rangle = \langle w, \rho(X)v\rangle, \quad X \in \mathfrak{g}. \tag{A.15}$$

The representation corresponding to the ith fundamental weight ω_i is called the ith fundamental representation V_i of $\mathfrak{g}$. We choose a highest weight vector in V_i and following the physicists [LS92] we called it by $|i\rangle$. We choose a vector $\langle i|$ in the lowest weight space in V_i^* and require that $\langle i|Id|i\rangle = 1$ for the identity element $Id \in G$. For simplicity, we will omit the notation ρ for the representation.

Therefore we have (see [LS92, Eq. (1.4.19)])

$$X|i\rangle = 0, \ \forall X \in \mathfrak{n}_+; \quad h_{\alpha_j}|i\rangle = \delta_{ij}|i\rangle; \quad \text{and } e_{-\alpha_j}|i\rangle = 0, \ \forall j \ne i. \tag{A.16}$$

That is, in the ith fundamental representation, only $e_{-\alpha_i}$ may bring the highest weight vector down. Similarly we have $\langle i|Y = 0$ for $Y \in \mathfrak{n} = \mathfrak{n}_-$, and $\langle i|e_{\alpha_j} = 0$ if $j \ne i$.

A.8. **Lift the representation to the group.** Let G^s be a connected and simply connected Lie group whose Lie algebra is $\mathfrak{g}$. Then all the irreducible representations of $\mathfrak{g}$ lift to representations of G^s, and in particular the fundamental representations V_i do. For $g \in G^s$, the pairing $\langle i|g|i\rangle$ is called the highest matrix coefficient [LS92, p. 45] because it is the matrix coefficient for the highest weight vector.

It is clear that $g|i\rangle = |i\rangle$ for $g \in N_+$, $\langle i|g = \langle i|$ for $g \in N_-$, and $\exp(H)|i\rangle = e^{\omega_i(H)}|i\rangle$ for $H \in \mathfrak{h}$ from the corresponding Lie algebra facts in Subsection A.7.

A representation of $\mathfrak{g}$ also leads to a representation for the universal enveloping algebra $U(\mathfrak{g})$ [Kna02]. For $\mu, \nu \in U(\mathfrak{g})$ and $g \in G^s$, $\langle i|\nu g\mu|i\rangle$ denotes the pairing of $\langle i|\nu$ in V_i^* with $g(\mu|i\rangle)$ in V_i.

In our main Theorem 1.6, we can work with a general Lie group G whose Lie algebra is $\mathfrak{g}$ instead of only the simply connected G^s. The reason is that the simply connected compact subgroup K^s of G^s is used only in passing. Our results are expressed using N and A, and they are simply connected and the same for a general G and the simply connected G^s (see Subsection A.4).

In Section 7 on asymptotic behaviors and quantization, we also need the following setup from [Kos79, §5.7]. Let $\mathbb{C}[G^s]$ be the group algebra of G^s where G^s is regarded as an abstract group. Let $D(G^s) = \mathbb{C}[G^s] \otimes_{\mathbb{C}} U(\mathfrak{g})$. We endow $D(G^s)$ with the weak $*$ topology, and let $\hat{D}(G^s)$ be its completion.

Also let $D(N) = D(N_-)$ be the subalgebra of $D(G^s)$ generated by $N = N_-$ and $\mathfrak{n} = \mathfrak{n}_-$, and let $\hat{D}(N) = \hat{D}(N_-)$ be the closure of $D(N)$ in $\hat{D}(G^s)$. Our Proposition 7.10 is stated in this space $\hat{D}(N)$, where an element in N is expressed in $U(\mathfrak{n})$.

A.9. **The real structure using invariant Hermitian form.** There is a more concrete realization of the dual V_i^* in Subsection A.7. By averaging, there is a Hermitian metric $\{\cdot,\cdot\}$ on V_i (conjugate linear in the second position) invariant under the compact group K^s of a simply connected G^s. The important property of this Hermitian form is that [Kos79, Eq. (5.11)]

$$\{gu, v\} = \{u, g^* v\}, \quad g \in G^s,\ u, v \in V_i. \tag{A.17}$$

Therefore we have an isomorphism $\bar{V}_i \to V_i^*; v \mapsto \{\cdot, v\}$, where $\bar{V}$ is the vector space V with the conjugate scalar multiplication. Furthermore the right representation of G on V^* now corresponds to the right action $\bar{V} \times G \to \bar{V}; (v, g) \to g^* v$ by comparing (A.15) and (A.17). Choose $v^{\omega_i} \in V_i$ to be a highest weight vector for the ith fundamental representation, and we require that $\{v^{\omega_i}, v^{\omega_i}\} = 1$. Then the term in (1.8) is, with $g = \Lambda C \Phi$,

$$\langle i|g^* g|i\rangle = \{g^* g v^{\omega_i}, v^{\omega_i}\} = \{g v^{\omega_i}, g v^{\omega_i}\} > 0. \tag{A.18}$$

A.10. **Weyl group.** The Weyl group W of a Lie algebra $\mathfrak{g}$ is the finite group generated by the reflections in the simple roots on $\mathfrak{h}_0'$

$$r_i(\beta) = \beta - \frac{2(\beta, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i, \quad 1 \le i \le n.$$

In the Weyl group, there is a unique element $\kappa \in W$ that is the longest element in the sense that when one writes it as a product of the simple reflections it has the maximal length. (Actually the maximal length is the number of positive roots.) This κ on $\mathfrak{h}_0'$ maps all the positives roots to the negative roots and vice versa.

The κ is $-Id$ except in the following cases: A_n, D_{2n+1} and E_6, where $-\kappa$ represents the outer-automorphism of the corresponding Lie algebra represented by the symmetry of its Dynkin diagram. (See [Dav08, Remark 13.1.8].)

Therefore $-\kappa\alpha_i$ is still a simple root and it is hence α_k for some $1 \le k \le n$. In fact $-\kappa\alpha_i = \alpha_i$ except the three cases mentioned above.

For the irreducible representation V_λ with a highest weight λ, its lowest weight is $\kappa\lambda$.

A.11. **Degrees of primitive adjoint-invariant functions on $\mathfrak{g}$.** The degrees of the primitive homogeneous adjoint-invariant functions of the simple Lie algebras are listed as follows

Lie algebras	degrees
A_n	$2, 3, \cdots, n+1$
B_n	$2, 4, \cdots, 2n$
C_n	$2, 4, \cdots, 2n$
D_n	$2, 4, \cdots, 2n-2, n$
G_2	$2, 6$
F_4	$2, 6, 8, 12$
E_6	$2, 5, 6, 8, 9, 12$
E_7	$2, 6, 8, 10, 12, 14, 18$
E_8	$2, 8, 12, 14, 18, 20, 24, 30$

References

[ALW15] Weiwei Ao, Chang-Shou Lin, and Juncheng Wei, *On Toda system with Cartan matrix G_2*, Proc. Amer. Math. Soc. **143** (2015), no. 8, 3525–3536.

[BM91] H. Brezis and F. Merle, *Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions*, Comm. Partial Differential Equations **16** (1991), no. 8-9, 1223–1253.

[CL03] Chiun-Chuan Chen and Chang-Shou Lin, *Topological degree for a mean field equation on Riemann surfaces*, Comm. Pure Appl. Math. **56** (2003), no. 12, 1667–1727.

[CL91] Wen Xiong Chen and Congming Li, *Classification of solutions of some nonlinear elliptic equations*, Duke Math. J. **63** (1991), no. 3, 615–622.

[Dav08] Michael W. Davis, *The geometry and topology of Coxeter groups*, London Mathematical Society Monographs Series, vol. 32, Princeton University Press, Princeton, NJ, 2008.

[Dol97] Adam Doliwa, *Holomorphic curves and Toda systems*, Lett. Math. Phys. **39** (1997), no. 1, 21–32.

[DS84] V. G. Drinfel′d and V. V. Sokolov, *Lie algebras and equations of Korteweg-de Vries type*, Current problems in mathematics, Vol. 24, Itogi Nauki i Tekhniki, Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984, pp. 81–180.

[DZ13] Boris Doubrov and Igor Zelenko, *Geometry of curves in generalized flag varieties*, Transform. Groups **18** (2013), no. 2, 361–383.

[FF96] Boris Feigin and Edward Frenkel, *Integrals of motion and quantum groups*, Integrable systems and quantum groups (Montecatini Terme, 1993), Lecture Notes in Math., vol. 1620, Springer, Berlin, 1996, pp. 349–418.

[FH91] William Fulton and Joe Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991. A first course; Readings in Mathematics.

[GH78] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics.

[Gue97] Martin A. Guest, *Harmonic maps, loop groups, and integrable systems*, London Mathematical Society Student Texts, vol. 38, Cambridge University Press, Cambridge, 1997.

[GL14] Martin A. Guest and Chang-Shou Lin, *Nonlinear PDE aspects of the tt* equations of Cecotti and Vafa*, J. Reine Angew. Math. **689** (2014), 1–32.

[Hel78] Sigurdur Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics, vol. 80, Academic Press, Inc. New York-London, 1978.

[JLW06] Jürgen Jost, Changshou Lin, and Guofang Wang, *Analytic aspects of the Toda system. II. Bubbling behavior and existence of solutions*, Comm. Pure Appl. Math. **59** (2006), no. 4, 526–558.

[JW01] Jürgen Jost and Guofang Wang, *Analytic aspects of the Toda system. I. A Moser-Trudinger inequality*, Comm. Pure Appl. Math. **54** (2001), no. 11, 1289–1319.

[JW02] ———, *Classification of solutions of a Toda system in* $\mathbb{R}^2$, Int. Math. Res. Not. **6** (2002), 277–290.

[KW74] Jerry L. Kazdan and F. W. Warner, *Curvature functions for compact* 2*-manifolds*, Ann. of Math. (2) **99** (1974), 14–47.

[Kna02] Anthony W. Knapp, *Lie groups beyond an introduction*, 2nd ed., Progress in Mathematics, vol. 140, Birkhäuser Boston, Inc., Boston, MA, 2002.

[Kos59] Bertram Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*, Amer. J. Math. **81** (1959), 973–1032.

[Kos63] ———, *Lie group representations on polynomial rings*, Amer. J. Math. **85** (1963), 327–404.

[Kos79] ———, *The solution to a generalized Toda lattice and representation theory*, Adv. in Math. **34** (1979), no. 3, 195–338.

[Lez80] A. N. Leznov, *On complete integrability of a nonlinear system of partial differential equations in two-dimensional space*, Teoret. Mat. Fiz. **42** (1980), no. 3, 343–349.

[LS79] A. N. Leznov and M. V. Saveliev, *Representation of zero curvature for the system of nonlinear partial differential equations* $x_{\alpha,z\bar{z}} = \exp(kx)_\alpha$ *and its integrability*, Lett. Math. Phys. **3** (1979), no. 6, 489–494.

[LS92] ———, *Group-theoretical methods for integration of nonlinear dynamical systems*, Progress in Physics, vol. 15, Birkhäuser Verlag, Basel, 1992. Translated and revised from the Russian; Translated by D. A. Leuites.

[Lin14] Chang-Shou Lin, *Mean field equations, hyperelliptic curves and modular forms*, Proceedings of ICM Seoul **III** (2014), 331–344.

[LWY12] Chang-Shou Lin, Juncheng Wei, and Dong Ye, *Classification and nondegeneracy of* $SU(n+1)$ *Toda system with singular sources*, Invent. Math. **190** (2012), no. 1, 169–207.

[Mal14] Andrea Malchiodi, *Liouville equations from a variational point of view*, Proceedings of ICM Seoul **III** (2014), 345–361.

[MB08] G. Mari Beffa, *Geometric Hamiltonian structures on flat semisimple homogeneous manifolds*, Asian J. Math. **12** (2008), no. 1, 1–33.

[Nie12] Zhaohu Nie, *Solving Toda field theories and related algebraic and differential properties*, J. Geom. Phys. **62** (2012), no. 12, 2424–2442.

[Nie14] ———, *On characteristic integrals of Toda field theories*, J. Nonlinear Math. Phys. **21** (2014), no. 1, 120–131.

[Nie16] ———, *Classification of solutions to Toda systems of types C and B with singular sources*, Calc. Var. Partial Differential Equations **55** (2016), no. 3, 55:53, DOI 10.1007/s00526-016-0998-3.

[Nie15] ———, *Toda field theories and integral curves of standard differential systems* (2015), submitted, available at `arXiv:1510.04808`.

[PT01] J. Prajapat and G. Tarantello, *On a class of elliptic problems in* $\mathbb{R}^2$*: symmetry and uniqueness results*, Proc. Roy. Soc. Edinburgh Sect. A **131** (2001), no. 4, 967–985.

[Sha97] R. W. Sharpe, *Differential geometry*, Graduate Texts in Mathematics, vol. 166, Springer-Verlag, New York, 1997. Cartan's generalization of Klein's Erlangen program; With a foreword by S. S. Chern.

[Tar08] Gabriella Tarantello, *Selfdual gauge field vortices*, Progress in Nonlinear Differential Equations and their Applications, 72, Birkhäuser Boston, Inc., Boston, MA, 2008. An analytical approach.

[Tar10] ———, *Analytical, geometrical and topological aspects of a class of mean field equations on surfaces*, Discrete Contin. Dyn. Syst. **28** (2010), no. 3, 931–973.

[Tro91] Marc Troyanov, *Prescribing curvature on compact surfaces with conical singularities*, Trans. Amer. Math. Soc. **324** (1991), no. 2, 793–821.

[Yan01] Yisong Yang, *Solitons in field theory and nonlinear analysis*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2001.

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