On Helmholtz equation and Dancer's type entire solutions for nonlinear elliptic equations

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Abstract

Starting from a bound state (positive or sign-changing) solution to

$$-\Delta\omega_m = |\omega_m|^{p-1}\omega_m - \omega_m \ \text{ in } \mathbb{R}^n,\ \omega_m \in H^2(\mathbb{R}^n)$$

and solutions to the Helmholtz equation

$$\Delta u_0 + \lambda u_0 = 0 \ \text{ in } \mathbb{R}^n,\ \lambda > 0,$$

we build new Dancer's type entire solutions to the nonlinear scalar equation

$$-\Delta u = |u|^{p-1}u - u \ \text{ in } \mathbb{R}^{m+n}.$$

1 Introduction

The purpose of this note is to construct new entire (positive or sign-changing) solutions for the elliptic equation

$$-\Delta u = |u|^{p-1}\, u - u \quad \text{in } \ \mathbb{R}^{m+n}, \tag{1}$$

where the exponent $p > 1$ and satisfies some conditions. A solution to (1) corresponds to a standing wave to the nonlinear Schrödinger equation

$$-iu_t = \Delta u + |u|^{p-1}u, \quad \text{in } \mathbb{R}^{m+n}. \tag{2}$$

It also serves as models in different areas of applied mathematics such as pattern formation in mathematical biology.

Equation (1) has been studied extensively and there is a vast literature on this subject. Let us first describe the classical example of Dancer's solutions. To describe these solutions, we recall that when $n = 0$, equation (1) reduces to

$$-\Delta u = |u|^{p-1}\, u - u \text{ in } \ \mathbb{R}^m, \tag{3}$$

which admits a unique positive radially symmetric ground state solution w_m decaying exponentially fast to zero at infinity, provided that p is subcritical, i.e. $1 < p < \frac{m+2}{m-2}$. The linearized equation of (3) around w_m is

$$L_{w_m}\eta = -\Delta\eta + \left(1 - p w_m^{p-1}\right)\eta,$$

acting on $H^2(\mathbb{R}^m)$. The essential spectrum of L_{w_m} is $[1, +\infty)$. It is known that this operator has a unique negative eigenvalue $-\lambda_1$. We choose a corresponding positive eigenfunction $Z_1(x)$. Dancer [8] first constructed new positive solutions with only partial decaying using bifurcation theory. He proved that for $n = 1$, there exists a family of solutions to (1) with the following behavior:

$$u(x,y) = w_m(x) + \epsilon Z_1(x)\cos(\sqrt{\lambda_1}y) + o(\epsilon e^{-\frac{1}{2}|x|}),\ |\epsilon| << 1,\ (x,y) \in \mathbb{R}^{m+1}. \tag{4}$$

Moreover such solutions are periodic in y. We call this type of solutions as Dancer's type.

The existence of Dancer's solution generates lots of interest in constructing other type of solutions. Variational and gluing methods have been successfully applied in the construction of solutions. There is also a deep connection between the solutions of (1) and the constant mean curvature surfaces (CMC). We refer to [2, 8, 14, 15, 17] and the references therein for more discussions.

In this paper we extend Dancer's type solutions to general dimensions $n \geq 3$. Let ω_m be a bound state solution (not necessary positive) of equation (3) with $\omega_m(x) \to 0$ as $|x| \to +\infty$. Existence results of this type solutions for subcritical exponent p could be found in [3, 4]. There are also plenty of sign-changing radial solutions. In fact, for each integer $k \geq 0$ there are sign-changing radial solutions with k zeroes. There are also infinitely many sign-changing solutions without any symmetry. See [5, 2, 7]. Slightly abusing the notation, we use (x, y) to denote the vectors in $\mathbb{R}^{m+n}$, where x represents the first m coordinates. Denote the positive eigenvalues of $-L_{\omega_m}$ as $\lambda_1, ..., \lambda_k$, with the corresponding eigenfunctions $Z_1, ..., Z_k$. Let $Z_{k+1}, ..., Z_l$ be the eigenfunctions of the zero eigenvalue. We could also assume that $Z_1, ..., Z_l$ consists an orthonormal basis for the nonnegative eigenspace in $L^2(\mathbb{R}^m)$. (Note that we don't assume the non-degeneracy of w.)

Our main result can be stated roughly as follows: Starting from solutions to the Helmholtz equation

$$\Delta u_j + \lambda_j u_j = 0 \text{ in } \mathbb{R}^n, j = 1, ..., k, \tag{5}$$

we find a family of solutions to (1) with the following asymptotic behavior:

$$u(x, y) = \omega_m(x) + \epsilon \sum_{j=1}^{k} Z_j(x) u_j(y) + o(\epsilon e^{-\frac{1}{2}|x|}), \quad (x, y) \in \mathbb{R}^{m+n}. \tag{6}$$

As a consequence, for $n \geq 3$, there are abundance of solutions near $\omega_m(x)$. Unlike the classical Dancer's solution, these solutions are not periodic in the y variable. As a matter of fact, they converge to $\omega_m(x)$ as $|y| \to \infty$. The existence of these type of solutions makes it more difficult to classify entire solutions near the ground state profile. Nevertheless we believe that all solutions near ω_m should be described by (6). We refer to [6, 12] and the references therein for partial progress on this issue.

Our idea of the proof is in the spirit similar to that of [13], where existence of small amplitude solutions to the Ginzburg-Landau equation in dimension 3 and 4 has been proved. In [11], using dual variational method, the existence of a sequence of solutions u_k of

$$\Delta u + u + |u|^{p-1} u = 0 \quad \text{in } \mathbb{R}^n,$$

with $\|u_k\|_{L^p} \to +\infty$, has been proved under certain condition on p and n. A similar functional-analytical frame was used.

To describe our main results, we need to introduce some notations. Let $\lambda > 0$ be a fixed positive number. Consider the so-called Helmholtz equation

$$\Delta u + \lambda u = 0 \text{ in } \mathbb{R}^n. \tag{7}$$

We are interested in solutions to (7) with the following decaying property

$$|u(y)| \leq C(1 + |y|)^{-\frac{n-1}{2}}. \tag{8}$$

There are plenty many solutions to (7) satisfying (8). We start with radial solutions. Let s be a parameter and $n \geq 2$. Consider the equation

$$\varphi'' + \frac{n-1}{r}\varphi' + \left(1 - \frac{s^2}{r^2}\right)\varphi = 0. \tag{9}$$

For $n = 2$, it has a regular solution $J_{2,s}$, which is the Bessel functions of the first kind. For general $n \geq 2$, (9) has a regular solution

$$J_{n,s}(r) = r^{1-\frac{n}{2}} J_{2,\sqrt{\left(\frac{n}{2}-1\right)^2 + s^2}}(r).$$

It is known that

$$J_{n,s}(r) \leq C(1 + r)^{-\frac{n-1}{2}}.$$

Hence $J_{n,0}(\sqrt{\lambda} r)$ is a solution to (7)-(8).

Given finitely many points $y_j \in \mathbb{R}^n$, $j = 1, ..., q$, functions of the form

$$\sum_{j=1}^{q} J_{n,0}\left(\sqrt{\lambda}(y - y_j)\right)$$

are also solutions of (7) satisfying (8) . In [10], solutions of (7) satisfying (8) with zero level sets arbitrarily close to any compact surfaces are constructed.

Our main result states that from these solutions of the Helmholtz equation we could construct solutions of (1) .

Theorem 1. *Let $n \geq 6$ and $p \geq \frac{n+2}{n-2}$. For any ε with $|\varepsilon|$ small enough, there is a solution u_ε to the equation*

$$\Delta u + |u|^{p-1} u - u = 0, \text{ in } \mathbb{R}^{m+n},$$

such that

$$u_\varepsilon = \omega_n(x) + \varepsilon \sum_{j=1}^{k} Z_j(x) u_j(y) + o(\varepsilon)$$

where u_j are solutions of (7)-(8), with λ_j being the negative eigenvalues of L_{ω_m}.

As a corollary of the proof of this theorem, in the case that ω_m is the positive radially symmetric solution w_m, we have the following result.

Corollary 2. *Let $n \geq 5$ and $p \geq \frac{n+3}{n-1}$. Let w_m be the unique positive solutions of (1). For any ε with $|\varepsilon|$ small enough, there is a positive solution u_ε to* (1) *such that*

$$u_\varepsilon = w_m(x) + \varepsilon Z_1(x) u_1(y) + o(\varepsilon).$$

Observe that in this corollary, we allow $n = 5$. This is partly due to the fact that w_m is nondegenerate in certain sense.

Remark 3. *We don't know the decay rates of these solutions to ω_m as y goes to infinity.*

When we are considering the existence of solutions radially symmetric in the y variable, the requirement that $p \geq \frac{n+3}{n-1}$ could be slightly relaxed. This is the content of our next theorem.

Theorem 4. *Let $n \geq 4$ and $p > \frac{n+1}{n-1}$. For any ε with $|\varepsilon|$ small enough, there is a positive solution u_ε to* (1) *which is radially symmetric in the y variable and*

$$u_\varepsilon(x, y) = w_m(x) + \varepsilon J_{n,0}\left(\sqrt{\lambda_1}|y|\right) + o(\varepsilon).$$

When p is an integer, using the same method, we could obtain similar result for $n = 3$.

Theorem 5. *Let $n = 3$ and $p > 1$ be an integer. For any ε with $|\varepsilon|$ small enough, there is a positive solution u_ε to* (1) , *radially symmetric in the y, such that*

$$u_\varepsilon(x, y) = w_m(x) + \varepsilon J_{n,0}\left(\sqrt{\lambda_1}|y|\right) + o(\varepsilon).$$

Remark 6. *An open question is the case of $n = 2$. We don't know whether or not there are similar solutions. We believe that our method of proof for these theorems could potentially be used in other settings.*

We will use contraction mapping principle to prove these results. The conditions on p and n are used to ensure the contraction mapping properties. In Section 2, we prove Theorem 1 and sketch the proof of Corollary 2. In section 3, we prove Theorem 4 and Theorem 5.

Throughout the paper, we use C to denote a general constant which may vary from step to step.

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2 Proof of Theorem 1 and Corollary 2

We first prove Theorem 1. For each ε with $|\varepsilon|$ small enough, we will construct a solution u_ε to (1) in the form:

$$u_\varepsilon(x,y) = \omega_m(x) + \sum_{i=1}^{l} Z_i(x) f_i(y) + \Phi(x,y),$$

where we require that Φ is orthogonal to Z_i in the following sense:

$$\int_{\mathbb{R}^m} \Phi(x,y) Z_i(x)\, dx = 0, \text{ for any } y, i.$$

We obtain that the equations satisfied by f_i and Φ are

$$-\Delta\Phi + \left(1 - p|\omega_m|^{p-1}\right)\Phi + \left[-\Delta f_i - \lambda_i f_i\right] Z_i(x) = N(f,\Phi).$$

Here

$$N(f,\Phi) = |u_\varepsilon|^{p-1} u_\varepsilon - |\omega_m|^{p-1}\omega_m - p|\omega_m|^{p-1}\left(\sum_{i=1}^{l} Z_i(x) f_i(y) + \Phi\right).$$

Introduce the constant $\bar{p} := \min\{p, 2\}$. We have the following estimate for N :

$$|N(f,\Phi)| \leq C\left(\sum_{i=1}^{l} |f_i|^{\bar{p}} + |\Phi|^{\bar{p}},\right) \tag{10}$$

provided that $|f_i|$ and $|\Phi|$ are small.

Let $f_i(y) = \varepsilon u_i(|y|) + h_i(y)$, where for $i = 1, ..., k$, $\lambda_i > 0$ and u_i satisfies

$$\Delta u_i + \lambda_i u_i = 0 \text{ in } \mathbb{R}^n,\ |u_i| \leq C(1+|y|)^{-\frac{n-1}{2}} \tag{11}$$

For $i = k+1, ..., l$, $\lambda_i = 0$ and $u_i \equiv 0$.

Then we need to solve

$$-\Delta\Phi + \left(1 - p|\omega_m|^{p-1}\right)\Phi + \left[-\Delta h_i - \lambda_i h\right] Z_i(x) = N(f,\Phi). \tag{12}$$

For the sake of convenience, we introduce the notation

$$L\Phi := -\Delta\Phi + \left(1 - p|\omega_m|^{p-1}\right)\Phi.$$

To get a solution for (12), it will be sufficient to deal with the system

$$\begin{cases} -\Delta h_i - \lambda_i h_i = \int_{\mathbb{R}^m} \left[Z_i(x) N(f,\Phi)\right] dx, i = 1, ..., l, \\ L\Phi = N(f,\Phi) - \sum_{i=1}^{l} Z_i(x) \int_{\mathbb{R}^m} \left[Z_i(x) N(f,\Phi)\right] dx. \end{cases} \tag{13}$$

Throughout the paper, we use q' to denote the conjugate exponent of q. That is,

$$q' = \frac{q}{q-1}.$$

We need the following important generalized Sobolev type inequality (Theorem 2.3 in [16]).

Lemma 7. *Suppose the exponent q satisfies*

$$\frac{2}{n+1} \leq \frac{1}{q'} - \frac{1}{q} \leq \frac{2}{n}. \tag{14}$$

Then

$$\|\Phi\|_{L^q(R^n)} \leq C\|\Delta\Phi + \Phi\|_{L^{q'}(R^n)}. \tag{15}$$

Note that if $u \in L^q$, then $u^{q-1} \in L^{q'}$. Inequality (14) is equivalent to

$$\frac{n-2}{n} \leq \frac{2}{q} \leq \frac{n-1}{n+1}.$$

That is,

$$\frac{2(n+1)}{n-1} \leq q \leq \frac{2n}{n-2}.$$

Recall that $\bar{p} = \min\{p, 2\}$. Hence under the assumption that $n \geq 6$ and $p \geq \frac{n+2}{n-2}$, we have

$$\bar{p} \geq \frac{2n}{n-2} - 1 = \frac{n+2}{n-2}.$$

To solve (13) ,we first consider the solvability of the equation

$$L\Phi = \xi \tag{16}$$

for given function ξ. For this purpose, we introduce the functional space to work with.

Definition 8. *The space E_α consists of functions ξ defined on $\mathbb{R}^{m+n}$ satisfying*

$$\|\xi\|_{*,\alpha} := \|\xi\|_{L^\alpha(\mathbb{R}^{m+n})} + \|\xi\|_{L^\infty(\mathbb{R}^{m+n})} < +\infty.$$

The space $\bar{E}_\alpha$ consists of l-tuple of functions $\eta = (\eta_1, ..., \eta_l)$ with η_i defined on $\mathbb{R}^n$, satisfying

$$\|\eta\|_{**,\alpha} := \Sigma_{i=1}^l \|\eta_i\|_{L^\alpha(\mathbb{R}^n)} + \Sigma_{i=1}^l \|\eta_i\|_{L^\infty(\mathbb{R}^n)} < +\infty.$$

We will choose q to be $\frac{2n}{n-2}$. Then

$$q' = \frac{q}{q-1} = \frac{2n}{n+2}.$$

Lemma 9. *Suppose $\|\xi\|_{*,q'} < +\infty$ and*

$$\int_{\mathbb{R}^m} \xi(x, y) Z_i(x) \, dx = 0, \textit{ for any } y, i.$$

Then the equation (16) *has a solution Φ satisfying*

$$\|\Phi\|_{*,q} \leq C \|\xi\|_{*,q'}$$

and

$$\int_{\mathbb{R}^m} \Phi(x, y) Z_i(x) \, dx = 0, \textit{ for any } y, i.$$

Proof. Note that $q = \frac{2n}{n-2} > 2$ and hence $q' < 2$. We then have

$$\|\xi\|_{L^2} \leq C \|\xi\|_{*,q'}.$$

Consider the operator L acting on the space of functions in $H^2(\mathbb{R}^{m+n})$ even in x variable and which additionally orthogonal to $Z_i(x)$ for each y, i. Then 0 is not in the spectrum of L and hence we have an even solution Φ for the equation $L\Phi = \xi$, with $\|\Phi\|_{L^2} \leq C \|\xi\|_{L^2} \leq C \|\xi\|_{*,q'}$ and

$$\int_{\mathbb{R}^m} \Phi(x, y) Z_i(x) \, dx = 0, \text{ for any } y, i.$$

On the other hand, since we impose the orthogonality condition on Φ, we also have the L^∞ bounds for Φ, that is,

$$\|\Phi\|_{L^\infty} \leq C \|\xi\|_{L^\infty} \leq C \|\xi\|_{*,q'}.$$

Therefore, using the fact that $q > 2$, we find that $\|\Phi\|_{L^q} \leq C \|\Phi\|_{*,2} \leq C \|\xi\|_{*,q'}$. This finishes the proof. □

With all these estimates at hand, we could use the contraction mapping principle to prove Theorem 1.

Proof of Theorem 1. For each $h = (h_1, ..., h_l)$ with $\|h\|_{**,q} \leq M_1 \varepsilon^{\bar{p}}$, where M_1 is a large constant, we consider the equation

$$L\Phi = N(f, \Phi) - \sum_{i=1}^{l} Z_i(x) \int_{\mathbb{R}^m} [Z_i(x) N(f, \Phi)] \, dx.$$

Note that the right hand side of this equation is automatically orthogonal to $Z(x)$ for each y. By Lemma 9, we could write it as

$$\Phi = L^{-1} \left[N(f, \Phi) - \sum_{i=1}^{l} Z_i(x) \int_{\mathbb{R}^m} Z_i(x) N(f, \Phi) \, dx \right] := \bar{N}(h, \Phi).$$

Observe that the function $|u_i|^{\bar{p}}$ is in $L^{q'}$:

$$\int_{\mathbb{R}^n} |u_i|^{\bar{p}q'} \, dy \leq C \int_0^{+\infty} (1+r)^{-\frac{n-1}{2} \frac{n+2}{n-2} \frac{2n}{n+2}} r^{n-1} dr \leq C.$$

Now suppose $\|\Phi\|_{*,q} \leq M_2 \varepsilon^{\bar{p}}$, where M_2 is a large constant. Then using the fact that $\bar{p} \geq q-1$, we deduce

$$\begin{aligned} \left\| |\Phi|^{\bar{p}} \right\|_{L^{q'}(\mathbb{R}^{m+n})} &= \left(\int |\Phi|^{\bar{p}q'} \right)^{\frac{1}{q'}} \leq \left(|\Phi|^{(\bar{p}-(q-1))q'} \int |\Phi|^{(q-1)q'} \right)^{\frac{1}{q'}} \\ &\leq M_2 \varepsilon^{\bar{p}^2}. \end{aligned}$$

Also we observe that

$$\begin{aligned} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} Z_i(x) |\eta(x,y)| \, dx \right)^{q'} dy &\leq \int_{\mathbb{R}^n} \left[\left(\int_{\mathbb{R}^m} [Z(x)]^q \, dx \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^m} |\eta(x,y)|^{q'} \, dx \right)^{\frac{1}{q'}} \right]^{q'} dy \\ &\leq C \|\eta\|_{L^{q'}}^{q'}. \end{aligned}$$

This together with (10) implies that

$$\|N(f, \Phi)\|_{L^{q'}} \leq C\varepsilon^{\bar{p}} + (M_1 + M_2) \varepsilon^{\bar{p}^2}.$$

On the other hand, it follows from direct computation that

$$\|N(f, \Phi)\|_{L^\infty} \leq C\varepsilon^{\bar{p}} + (M_1 + M_2) \varepsilon^{\bar{p}^2}.$$

We could then check that $\bar{N}$ maps the balls of radius $M_2 \varepsilon^{\bar{p}}$ of E_q into itself for M_2 large enough and ε sufficiently small.

Next we show that $\bar{N}$ is a contraction mapping. To see this, for two function Φ_1, Φ_2, we compute

$$\begin{aligned} \|N(h, \Phi_1) - N(h, \Phi_2)\|_{L^{q'}} &\leq C\varepsilon^{\bar{p}-1} \|\Phi_1 - \Phi_2\|_{L^{q'}}, \\ \|N(h, \Phi_1) - N(h, \Phi_2)\|_{L^\infty} &\leq C\varepsilon^{\bar{p}-1} \|\Phi_1 - \Phi_2\|_{L^\infty}. \end{aligned}$$

This in turn will imply that $\bar{N}$ is a contraction map. We then conclude that it has a unique fixed point in the ball of radius $M_2 \varepsilon^{\bar{p}}$, denote it by Φ_h.

To solve the system (13), it remains to solve the system of equations

$$-\Delta h_j - \lambda_j h_j = \int_{\mathbb{R}^m} \left[Z_j(x) N(f, \Phi_h) \right] dx, \; j = 1, ..., l.$$

We write it as

$$h_j = \mathfrak{D}_j \left(\int_{\mathbb{R}^m} Z_j(x) N(f, \Phi_h) \, dx \right) := D_j(h), \; j = 1, ..., l.$$

Here the operator $\mathfrak{D}_j = \lim_{\varepsilon\to 0}\left(-\Delta - \lambda_j + i\varepsilon\right)^{-1}$. Then we are finally lead to solve the fixed point problem

$$h = D(h) := (D_1(h), ..., D_l(h)). \tag{17}$$

Making use of the estimate of Lemma 7, we could show that $D(h)$ maps the ball of radius $M_1\varepsilon^{\bar{p}}$ of $\bar{E}_q$ into itself for M_1 large and ε small and is a contraction mapping. We therefore get a fixed point h for this mapping, which yields a solution for our original problem. □

In the rest of this section, we sketch the proof of Corollary 2. In this case, we seek for a solution in the form

$$u_\varepsilon(x, y) = w_m(x) + Z_1(x) f(y) + \Phi(x, y).$$

Note that Z_1 is radially symmetric in x. Here Φ is required to be radially symmetric in x and orthogonal to Z_1 for each y. We write the function f in the form $f(y) = \varepsilon u_1(y) + h(y)$. Then similar as before, we prove the existence of solution by contraction mapping principle in suitable Banach spaces. Here we could work with functions h in the space F_α, and Φ in the space $\bar{F}_\alpha$, defined similarly as before.

Definition 10. *The space F_α consists of functions ξ defined on $\mathbb{R}^{m+n}$ which is radially symmetric in x, satisfying*

$$\|\xi\|_{*,\alpha} := \|\xi\|_{L^\alpha(\mathbb{R}^{m+n})} + \|\xi\|_{L^\infty(\mathbb{R}^{m+n})} < +\infty.$$

The space $\bar{F}_\alpha$ consists of functions η defined on $\mathbb{R}^n$, satisfying

$$\|\eta\|_{**,\alpha} := \|\eta\|_{L^\alpha(\mathbb{R}^n)} + \|\eta\|_{L^\infty(\mathbb{R}^n)} < +\infty.$$

Then we choose the exponent α to be

$$\alpha = \frac{2(n+1)}{n-1}.$$

With this choice, under the assumption of Corollary 2, for $n \geq 5$, one has $\bar{p} = \min\{p, 2\} \geq \alpha - 1$. Then the rest of the proof follows from the arguments in that of Theorem **??**.

3 Proof of Theorem 4 and Theorem 5

We first prove Theorem 4. The idea is still using contraction mapping principle but we make the advantage of radial symmetry. For the sake of convenience, we drop the subscript of w_m and simply write it as w. We also write Z_1 as Z, λ_1 as λ. Denoting $r = |y|$. We are looking for a solution u_ε in the form

$$u_\varepsilon(x, y) = w(x) + Z(x) f(r) + \Phi.$$

Here Φ is radially symmetric in x and in y. Plug this into the equation

$$-\Delta u_\varepsilon + u_\varepsilon - |u_\varepsilon|^p = 0,$$

we get

$$L\Phi + \left[-f'' - \frac{n-1}{r} f' - \lambda f\right] Z(x) = N(f, \Phi).$$

Let $f(r) = \varepsilon J_{n,0}(r) + h(r)$. Then we need to solve

$$L\Phi + \left[-h'' - \frac{n-1}{r} h' - \lambda h\right] Z(x) = N(\varepsilon J_{n,0} + h, \Phi). \tag{18}$$

Here we require that

$$\int_{\mathbb{R}^m} \Phi(x, y) Z(x)\, dx = 0, \text{ for any } y.$$

We are lead to solve

$$\begin{cases} -h'' - \frac{n-1}{r} h' - \lambda h = \int_{\mathbb{R}^m} Z(x) N(\varepsilon J_{n,0} + h, \Phi)\, dx \\ L\Phi = N(\varepsilon J_{n,0} + h, \Phi) - Z(x)\int_{\mathbb{R}^m} Z(x) N(\varepsilon J_{n,0} + h, \Phi)\, dx. \end{cases}$$

We need to solve the equation

$$L\Phi = \xi \tag{19}$$

for given function ξ defined in $\mathbb{R}^{m+n}$.

Definition 11. *The space S_β consists of those functions ξ radially symmetric in the x and y variable, with*

$$\|\xi\| := \sup_{(x,y)} \left\| \xi(x,y)(1+|y|)^\beta e^{-\frac{1}{2}|x|} \right\|_{C^{0,\alpha}\left(B_{(x,y)}(1)\right)} < +\infty.$$

The space $\bar{S}_\beta$ consists of those radially symmetric functions η defined in $\mathbb{R}^n$, with

$$\|\eta\|_{\hat{,}\beta} := \sup \left\|(1+|y|)^\beta \eta(y)\right\|_{C^{0,\alpha}\left(B_y(1)\right)} < +\infty.$$

Lemma 12. *Suppose ξ is function radially symmetric in x, y with $\|\xi\| < +\infty$ and*

$$\int_{\mathbb{R}^m} \xi(x,y) Z(x)\, dx = 0, \text{ for any } y.$$

Then the equation (19) *has a solution* Φ, *radially symmetric in x, y, satisfying*

$$\|\Phi\| \le C \|\xi\|.$$

Proof. The proof of this type result is by now standard, we omit the details and refer to [9] for a proof. □

Next we proceed to the analysis of the first equation of the system. Let η be a function which decays at certain rate at infinity. The homogeneous equation

$$h'' + \frac{n-1}{r} h' + \lambda h = 0$$

has two linearly independent solutions $J_n(\cdot)$ and $N_n(\cdot)$. They both decay like $r^{-\frac{n-1}{2}}$ at infinity. J_n is regular near 0 and N_n is singular near 0. Moreover, $N_n(r) = O\left(r^{2-n}\right)$ for r close to zero. Variation of parameter formula tells us that the function

$$N_n(r) \int_0^r J_n(s)\eta(s) s^{n-1} ds - J_n(r) \int_0^r N_n(s)\eta(s) s^{n-1} ds$$

is a solution of the nonhomogeneous equation

$$h'' + \frac{n-1}{r} h' + \lambda h = \eta. \tag{20}$$

Lemma 13. *Let $\eta \in \bar{S}_\beta$ with $\beta > \frac{n+1}{2}$. Then the equation* (20) *has a solution $h \in \bar{S}_{\frac{n-1}{2}}$ satisfies*

$$\|h\|_{\hat{,}\frac{n-1}{2}} \le C \|\eta\|_{\hat{,}\beta}.$$

Proof. Consider the following solution for (20) :

$$h(r) := N_n(r) \int_0^r J_n(s)\eta(s) s^{n-1} ds - J_n(r) \int_0^r N_n(s)\eta(s) s^{n-1} ds.$$

We have, for $r > 1$,

$$\begin{aligned} |h(r)| &\le C \|\eta\|_{\hat{,}\beta} r^{-\frac{n-1}{2}} \int_0^r s^{\frac{n-1}{2}-\beta} ds \\ &\le C \|\eta\|_{\hat{,}\beta} (1+r)^{-\frac{n-1}{2}}. \end{aligned}$$

This gives us the desired estimate. □

With these a priori estimate, one could prove Theorem 4 by contraction mapping principle, similar as before. We omit the details.

Now we proceed to prove Theorem 5. We consider the case $p = 2$. The other cases are similar, but notations will be heavier. The main point to prove this theorem is to prove a priori estimate for the solutions of the equation (20). Since $n = 3$, the fundamental solution J_3 and N_3 of the ODE

$$\varphi'' + \frac{2}{r}\varphi' + \lambda\varphi = 0$$

has the asymptotic behavior

$$J(r) := J_3(r) = r^{-1}\cos\left(\sqrt{\lambda}r - \zeta\right) + O\left(r^{-2}\right), \text{ as } r \to +\infty.$$
$$N(r) := N_3(r) = r^{-1}\sin\left(\sqrt{\lambda}r - \zeta\right) + O\left(r^{-2}\right), \text{ as } r \to +\infty.$$

where $\zeta \in \mathbb{R}$ is a constant depends on λ. Let ρ be a cutoff function such that

$$\rho(r) = \begin{cases} 1, r > 2, \\ 0, r < 1. \end{cases}$$

The key observation is the following

Lemma 14. *Let*

$$\eta(r) = \rho(r) r^{-2}\left[k_1 \sin^2\left(\sqrt{\lambda}r - \zeta\right) + k_2\cos^2\left(\sqrt{\lambda}r - \zeta\right) + k_3 \sin\left(\sqrt{\lambda}r - \zeta\right)\cos\left(\sqrt{\lambda}r - \xi\right)\right] + \bar{\eta}$$

with

$$\left\|\bar{\eta}(r)(1+r)^3\right\|_{C^{0,\alpha}} + \sum |k_i| < C.$$

Then the equation

$$h'' + \frac{1}{r}h' + \lambda h = \eta$$

has a solution $h \in C^{0,\alpha}[0, +\infty)$, *with*

$$h(r) = c_1 r^{-1}\sin\left(\sqrt{\lambda}r - \zeta\right) + \bar{h}(r), r > 1,$$

where

$$\left|\bar{h}(r)\right| + |c_1| \leq C(1+r)^{-2}.$$

The solution h *will be denoted by* $H(\eta)$.

Proof. Consider the solution

$$\begin{aligned} h(r) &= N(r)\int_0^r J(s)\eta(s)s^2 ds - J(r)\int_0^r N(s)\eta(s)s^2 ds \\ &= N(r)\int_0^r J(s)\rho(s)\sin^2\left(\sqrt{\lambda}s - \zeta\right)ds + J(r)\int_r^{+\infty} N(s)\rho(s)\sin^2\left(\sqrt{\lambda}s - \zeta\right)ds. \end{aligned}$$

We have

$$\int_0^r J(s)\rho(s)\sin^2\left(\sqrt{\lambda}s - \zeta\right)ds = \frac{1}{2}\int_0^r J(s)\rho(s)ds - \frac{1}{2}\int_0^r J(s)\rho(s)\cos\left(2\sqrt{\lambda}s - 2\xi\right)ds.$$

Using the fact that $J(s) = s^{-1}\cos\left(\sqrt{\lambda}s - \zeta\right) + O\left(s^{-2}\right)$, we could estimate

$$\int_0^r J(s)\rho(s)ds = \int_0^{+\infty} J(s)\rho(s)ds + O\left(r^{-1}\right),$$
$$\int_0^r J(s)\rho(s)\cos\left(2\sqrt{\lambda}s - 2\xi\right)ds = \int_0^{+\infty} J(s)\rho(s)\cos\left(2\sqrt{\lambda}s - 2\zeta\right)ds + O\left(r^{-1}\right).$$

Similarly, for $r > 1$,

$$\left|\int_r^{+\infty} J(s)\rho(s)\bar{\eta}(s)s^2 ds\right| \leq \left|\int_r^{+\infty} s^{-1}s^{-3}s^2 ds\right| \leq Cr^{-1}.$$

Hence

$$h(r) = c_1 r^{-1}\sin\left(\sqrt{\lambda}r - \zeta\right) + \bar{h}(r), r > 1,$$

where

$$\left|\bar{h}(r)\right| + |c_1| \leq C(1+r)^{-2}.$$

This finishes the proof. □

Definition 15. *The space P_α consists of functions $\eta(r)$ of the form*

$$\eta(r) = c_1 \rho(r) r^{-1} \sin\left(\sqrt{\lambda} r - \zeta\right) + \bar{\eta}(r),$$

with

$$\|\eta\|_{\#,\alpha} = |c_1| + \sup_{k \geq 0} \left\|(1+r)^2 \bar{\eta}(r)\right\|_{C^{0,\alpha}([k,k++1])}.$$

With the previous results at hand, one could then use the contraction mapping principle to prove Theorem 5.

Proof of Theorem 3. The proof is similar as before. We sketch it.

We search for a solution of the form

$$w(x) + Z(x)(\varepsilon J(|y|) + h(r)) + \Phi(x,y),$$

where $\int_{\mathbb{R}^m} \Phi(x,y) Z(x)\, dx = 0$, for any y. We need to solve

$$\begin{cases} -h'' - \frac{1}{r}h' - \lambda h = \int_{\mathbb{R}^m} Z(x) N(\varepsilon J + h, \Phi)\, dx, \\ L\Phi = N(\varepsilon J + h, \Phi) - Z(x) \int_{\mathbb{R}^m} Z(x) N(\varepsilon J + h, \Phi)\, dx. \end{cases}$$

For each $h \in P_{0,\alpha}$ with $\|h\|_{\#,\alpha} \leq M\varepsilon^2$. We write

$$h(r) = c_1 \rho(r) \sin\left(\sqrt{\lambda} r - \zeta\right) + \bar{h}(r),$$

By Lemma 12, the second equation in this system could be solved and we obtain a solution $\Phi = \Phi_h$ with

$$\left\|\Phi_h(x,y)(1+r)^2\right\|_{C^{2,\alpha}} \leq C\varepsilon^2.$$

We insert this solution into the first equation of the system and proceed to solve

$$-h'' - \frac{1}{r}h' - \lambda h = \int_{\mathbb{R}^m} Z(x) N(\varepsilon J + h, \Phi_h)\, dx. \tag{21}$$

Note that

$$N(\varepsilon J + h, \Phi_h) = (\varepsilon J + h)^2 + \Phi_h^2 + 2(\varepsilon J + h)\Phi_h = 2\varepsilon J h + h^2 + \varepsilon^2 J^2 + O\left((1+r)^{-3}\right).$$

We write the equation (21) as

$$h = H\left(\int_{\mathbb{R}^m} Z(x) N(\varepsilon J + h, \Phi_h)\, dx\right) := \bar{H}(h).$$

Note that for the function $2\varepsilon J h + h^2 + \varepsilon^2 J^2$, the coefficient of the r^{-2} part has the form

$$k_1 \cos^2\left(\sqrt{\lambda} r - \xi\right) + k_2 \sin^2\left(\sqrt{\lambda} r - \xi\right) + k_3 \sin\left(\sqrt{\lambda} r - \xi\right) \cos\left(\sqrt{\lambda} r - \xi\right).$$

We would like to get a solution for this equation by contraction mapping principle. By Lemma 14, for each $h \in P_\alpha$ with $\|h\|_{\#,\alpha} \leq M\varepsilon^2$, we have the estimate

$$\left\|\bar{H}(h)\right\|_{\#,\alpha} \leq C\varepsilon^2 + CM\varepsilon^3.$$

Hence for ε small enough $\bar{H}$ maps the ball of radius $M\varepsilon^2$ into itself and one could also check it is a contraction map. We then get a fixed point for this map and thus complete the proof. □

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