ASYMPTOTIC BEHAVIOR OF A FOURTH ORDER MEAN FIELD EQUATION WITH DIRICHLET BOUNDARY CONDITION

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ABSTRACT. We consider asymptotic behavior of the following fourth order equation

$$\Delta^2 u = \rho \frac{e^u}{\int_\Omega e^u \, dx} \quad \text{in } \Omega, \; u = \partial_\nu u = 0 \text{ on } \partial\Omega$$

where Ω is a smooth oriented bounded domain in $\mathbb{R}^4$. Assuming that $0 < \rho \leq C$, we completely characterize the asymptotic behavior of the unbounded solutions.

1. INTRODUCTION

In this paper, we study the asymptotic behavior of unbounded solutions for the following fourth order mean field equation under Dirichlet boundary condition

$$\begin{cases} \Delta^2 u = \rho \frac{e^u}{\int_\Omega e^u \, dx} \text{ in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\rho > 0$ and $\Omega \subset \mathbb{R}^4$ is a smooth oriented bounded domain. In dimension two, the analogous problem

$$\begin{cases} -\Delta u = \rho \frac{e^u}{\int_\Sigma e^u \, dx} \text{ in } \Sigma, \\ u = 0 \quad \text{on } \partial\Sigma \end{cases} \tag{1.2}$$

where Σ is a smooth bounded domain in $\mathbb{R}^2$, has been extensively studied by many authors. Let (u_k, ρ_k) be a unbounded sequence of solutions to (1.2) with $\rho_k \leq C, \max_{x \in \Sigma} u_k(x) \to +\infty$. Then it has been proved that

(P1) (**no boundary bubbles**) u_k is uniformly bounded near a neighborhood of $\partial\Sigma$ (Nagasaki-Suzuki [33], Ma-Wei [29]);

(P2) (**bubbles are simple**) $\rho_k \to 8m\pi$ for some $m \geq 1$ and $u_k(x) \to 8\pi \sum_{j=1}^m G(\cdot, x_j)$ in $C^2_{loc}(\Sigma \backslash \{x_1, ..., x_m\})$ (Brézis-Merle [5], Li-Shafrir [24], Nagasaki-Suzuki [33], Ma-Wei [29]), where G is the Green function of $-\Delta$ with Dirichlet boundary condition. Furthermore, it holds that

$$\nabla_x R(x_j, x_j) + \sum_{i \neq j} \nabla_x G(x_i, x_j) = 0, j = 1, ..., m \tag{1.3}$$

where $R(x,y) = G(x,y) - \frac{1}{2\pi} \log \frac{1}{|x-y|}$ is the regular part of $G(x,y)$.

Date: May 30th 2007.
1991 *Mathematics Subject Classification.* Primary 35B40, 35B45; Secondary 35J40.
Key words and phrases. Asymptotic Behavior, Biharmonic Equations.

On the other hand, giving m points satisfying (1.3), Baraket and Pacard [6] constructed multiple bubbling solutions to (1.2) when the bubble points satisfy non-degeneracy condition. Del Pino, Kowalczyk and Musso [15] constructed multiple bubbling solutions to (1.2) when the bubble points are *topologically nontrivial.* Furthermore, Chen and Lin [12, 13] obtained the sharp estimates for the bubbling rate and the Leray-Schauder degree of all solutions to (1.2) for all $\rho \notin 8\pi\mathbb{N}$. A related question connected to physics consists in adding Dirac masses to the nonlinear parts: we refer to Bartolucci-Chen-Lin-Tarantello [3] and to Tarantello [36] for results and asymptotics in this context.

In [38], the second author considered the following fourth order equation under Navier boundary condition

$$\begin{cases} \Delta^2 u = \rho \frac{e^u}{\int_\Omega e^u\, dx} \text{ in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^4$ is a smooth and bounded domain. Assuming that Ω is convex, the corresponding property (P1) and (P2) are established in [38]. Later, Lin and Wei [26] considered the attainment of least energy solution and removed the convexity assumption of [38]. Therefore, property (P1) and (P2) are established for (1.4). Sharp estimates for the bubbles and the computation of topological degree are contained in [27] and [28].

The purpose of this paper is to establish the corresponding property (P1) and (P2) for equation (1.1): indeed, equation (1.1) is more natural than (1.4) from the viewpoint of the Adams inequality (see (1.12) below). Our main result can be stated as follows.

Theorem 1.1. *Assume that Ω is a bounded smooth domain in $\mathbb{R}^4$. Let (u_k, ρ_k) be a sequence of solutions to (1.1) such that*

$$0 < \rho_k \le C, \max_{x\in\Omega} u_k(x) \to +\infty. \tag{1.5}$$

Then

(a) $\rho_k \to 64\pi^2 m$ for some positive integer m.

(b) u_k has $m-$point blow up, i.e., there exists a set $S = \{x_1, ..., x_m\} \subset \Omega$ such that $\{u_k\}$ have a limit $u_0(x)$ for $x \in \overline{\Omega}\backslash S$, where the limit function $u_0(x)$ has the form

$$u_0(x) = 64\pi^2 \sum_{i=1}^{m} G(x, x_j) \tag{1.6}$$

where $G(x, y)$ denotes the Green's function of Δ^2 under the Dirichlet condition, that is

$$\Delta^2 G(x, y) = \delta(x - y) \text{ in } \Omega,\ G(x, y) = \partial_\nu G(x, y) = 0 \text{ on } \partial\Omega. \tag{1.7}$$

Furthermore, blow up points $x_j \in \Omega$ $(1 \le j \le m)$ satisfy the following relation

$$\nabla_x R(x_j, x_j) + \sum_{l\neq j} \nabla_x G(x_j, x_l) = 0 \ (1 \le j \le m) \tag{1.8}$$

where

$$R(x, y) = G(x, y) + \frac{\log|x - y|}{8\pi^2}. \tag{1.9}$$

The main difficulty (and main difference) between (1.1) and (1.4) is that for fourth order equations, Maximum Principle works for Navier boundary conditions but doesn't work for Dirichlet boundary conditions. More precisely, Green's function for the Navier boundary condition

$$\Delta^2 G(x,y) = \delta(x-y) \text{ in } \Omega,\ G(x,y) = \Delta G(x,y) = 0 \text{ on } \partial\Omega \tag{1.10}$$

is positive but the Green's function for Dirichlet boundary condition may become negative (see [14] and [19]). This poses a major difficulty in using the method of moving planes (as in [26]) to exclude the boundary bubbles. We overcome this by using the Pohozaev identity and by proving strong pointwise estimates for blowing-up solutions to (1.1).

As an application of Theorem 1.1, we consider the following minimization problem

$$J_\rho(u) = \frac{1}{2}\int_\Omega |\Delta u|^2\, dx - \rho \log \int_\Omega e^u\, dx, \tag{1.11}$$

where Ω is a bounded and smooth domain of $\mathbb{R}^4$ and $u \in H_0^2(\Omega)$. Here, $H_0^2(\Omega)$ denotes the completion of $C_c^\infty(\Omega)$ for the norm $u \mapsto \|\Delta u\|_2$. Adam's version of the Moser-Trudinger inequality [1] asserts that there exists $C(\Omega) > 0$ such that

$$\int_\Omega e^{32\pi^2 u^2}\, dx \le C(\Omega) \tag{1.12}$$

for all $u \in H_0^2(\Omega)$ such that $\|\Delta u\|_2 = 1$. It follows from (1.12) that J_ρ is bounded from below if and only if $\rho \le 64\pi^2$ (for the proof, see the appendix of [26]). Furthermore, if $\rho < 64\pi^2$, the minimizer of J_ρ actually exists, that is, there exists a $u_\rho \in H_0^2(\Omega)$ such that

$$J_\rho(u_\rho) := \inf_{u \in H_0^2(\Omega)} J_\rho(u) := c_\rho. \tag{1.13}$$

For $J_{64\pi^2}$, it is an interesting question to ask whether the minimum $c_{64\pi^2}$ can be attained or not. The Euler-Lagrange equation of J_ρ is just (1.1). For the corresponding problem in two dimension, given Σ a smooth two-dimensional domain, we consider

$$E_\rho(u) = \frac{1}{2}\int_\Sigma |\nabla u|^2\, dx - \rho \log\left(\int_\Sigma e^u dx\right), u \in H_0^1(\Sigma)$$

where $H_0^1(\Sigma)$ denotes the completion of $C_c^\infty(\Sigma)$ for the norm $u \mapsto \|\nabla u\|_2$. Again, by the Moser-Trudinger inequality, E_ρ is bounded from below if and only if $\rho \le 8\pi$, and moreover, the minimum of E_ρ is always attained if $\rho < 8\pi$. However, it has been noted that minimizers do not always exist for $E_{8\pi}$. Actually, it depends on the geometry of Σ in a very subtle way. For example, the minimum of $E_{8\pi}$ is not attained if Σ is a ball in $\mathbb{R}^2$, but, it is attained if Σ is a long and thin domain, see [11]. So, it is rather surprising to have the following claim.

Theorem 1.2. *Let Ω be a bounded C^4 domain in $\mathbb{R}^4$, and u_ρ denote a minimizer of J_ρ for $\rho < 64\pi^2$. Assume that*

$$R_1(Q_0, Q_0) + 16\pi^2 \Delta_x R(Q_0, Q_0) > 0 \tag{1.14}$$

for $Q_0 \in \Omega$ such that $R(Q_0, Q_0) = \max_{P \in \Omega} R(P, P)$, where $R_1(x, P)$ is defined by

$$\begin{cases} \Delta^2 R_1(x,P) = 0 \text{ in } \Omega, \\ R_1(x,P) = \frac{4}{|x-P|^2}, \partial_\nu R_1(x,P) = \partial_\nu(\frac{4}{|x-P|^2}), \text{ on } \partial\Omega. \end{cases} \tag{1.15}$$

Then u_ρ is uniformly bounded in C^4 as $\rho \uparrow 64\pi^2$. Consequently, the minimum of $J_{64\pi^2}$ can be attained. As an example, when Ω is a ball in $\mathbb{R}^4$, $J_{64\pi^2}$ is attained.

Semilinear equations involving exponential nonlinearity and fourth order elliptic operator appear naturally in conformal geometry and in particular in prescribing $Q-$curvature on 4-dimensional Riemannian manifold M (see e.g. Chang-Yang [9])

$$P_g w + 2Q_g = 2\tilde{Q}_{g_w} e^{4w} \tag{1.16}$$

where P_g is the so-called Paneitz operator:

$$P_g = (\Delta_g)^2 + \delta\left(\frac{2}{3} R_g I - 2\mathrm{Ric}_g\right) d,$$

$g_w = e^{2w} g$, Q_g is $Q-$ curvature under the metric g, and $\tilde{Q}_{g_w}$ is the Q-curvature under the new metric g_w. Integrating (1.16) over M, we obtain

$$k_g := \int_M Q_g \, dv_g = \int_M (\tilde{Q}_{g_w}) e^{4w} \, dv_g = \int_M \tilde{Q}_{g_w} \, dv_{g_w}$$

and k_g is conformally invariant (here dv_g denote the Riemannian element of volume). Thus, we can write (1.16) as

$$P_g w + 2Q_g = 2k_g \frac{\tilde{Q}_{g_w} e^{4w}}{\int_M \tilde{Q}_{g_w} e^{4w} \, dv_g} \tag{1.17}$$

In the special case, where the manifold is the Euclidean space, $P_g = \Delta^2$, and (1.17) becomes

$$\Delta^2 w = 2k_g \frac{h(x) e^{4w}}{\int_\Omega h(x) e^{4w} \, dx} \tag{1.18}$$

With $u = 2w, \rho = 4k_g, h \equiv 1$, we arrive at equation (1.1). There is now an extensive litterature about this problem. For instance, we refer to Adimurthi-Robert-Struwe [2], Baraket-Dammak-Ouni-Pacard [7], Druet [16], Druet-Robert [17], Hebey-Robert [20], Hebey-Robert-Wen [21], Malchiodi [30], Malchiodi-Struwe [31], Robert [34], Robert-Struwe [35] and the references therein.

Our paper is organized as follows. In Section 2, we present two useful lemmas. Theorem 1.1 is proved in Section 3 and Theorem 1.2 is proved in Section 4.

Notation: Throughout this paper, the constant C will denote various constants which are independent of ρ: the value of C might change from one line to the other, and even in the same line. The equality $B = O(A)$ means that there exists $C > 0$ such that $|B| \le CA$. All the convergence results are stated up to the extraction of a subsequence.

2. Some preliminaries

We state two results in this section. The first one concerns the properties of the Green's function (1.7). The second one is Pohozaev's identity. Recall that $G(x,y)$ is defined by (1.7). As we remarked earlier, in general, $G(x,y)$ is not positive. We collect properties of G in the following lemma.

Lemma 2.1. *There exists $C > 0$ such that for all $x, y \in \Omega$, $x \neq y$, we have that*

$$|G(x,y)| \le C \log\left(2 + \frac{1}{|x-y|}\right) \tag{2.1}$$

$$|\nabla^i G(x,y)| \le C|x-y|^{-i},\ i \ge 1 \tag{2.2}$$

Proof. These estimates are originally due to Krasovskiĭ [22]. We also refer to Dall'Acqua-Sweers [14] and Grunau-Robert [18]. □

Next we state a Pohozaev identity for equation (1.1).

Lemma 2.2. *Let $u \in C^4(\overline{\Omega})$ be a solution of $\Delta^2 u = f(u)$ in Ω. Then we have for any $y \in \mathbb{R}^4$,*

$$4\int_\Omega F(u)\,dx = \int_{\partial\Omega} \langle x-y,\nu\rangle F(u)\,d\sigma + \frac{1}{2}\int_{\partial\Omega} v^2\langle x-y,\nu\rangle d\sigma + 2\int_{\partial\Omega}\frac{\partial u}{\partial\nu} v\,d\sigma$$
$$+\int_{\partial\Omega}\left(\frac{\partial v}{\partial\nu}\langle x-y,Du\rangle + \frac{\partial u}{\partial\nu}\langle x-y,Dv\rangle - \langle Dv,Du\rangle\langle x-y,\nu\rangle\right)d\sigma$$

where $F(u) = \int_0^u f(s)\,ds$, $-\Delta u = v$ and $\nu(x)$ is the normal outward derivative of x on $\partial\Omega$.

Proof. More general version of this formula can be seen, for example in [32]. In our case, integrating the identity on Ω

$$\begin{aligned}&\operatorname{div}((x-y,\nabla v)\nabla u + (x-y,\nabla u)\nabla v - (\nabla u,\nabla v)(x-y))\\ &= (x-y,\nabla v)\Delta u + (x-y,\nabla u)\Delta v - 2(\nabla u,\nabla v)\end{aligned}$$

for $u, v \in C^2(\bar{\Omega})$, $\nabla = \nabla_x$, and noting that

$$\operatorname{div}((x-y)F(u)) = f(u)(x-y,\nabla u) + 4F(u)$$

and

$$\operatorname{div}\left(\frac{1}{2}v^2(x-y) + 2v\nabla u\right) = v(\nabla v, x-y) + 2(\nabla u,\nabla v)$$

if $v = -\Delta u$, we get the desired formula. □

3. Proof of Theorem 1.1

Let u_k be a family of solutions to problem (1.1) such that there exists $\Lambda > 0$ such that

$$0 < \rho_k \le \Lambda. \tag{3.1}$$

In this section, we study the asymptotic behavior of unbounded solutions and prove Theorem 1.1. Let

$$\alpha_k := \log\left(\frac{\int_\Omega e^{u_k}\,dx}{\rho_k}\right) \text{ and } \hat{u}_k := u_k - \alpha_k.$$

Theorem 1.1 is proved by a series of claims. We first claim that

Claim 1: There exists $C \in \mathbb{R}$ such that $\alpha_k \ge C$ for all $k \in \mathbb{N}$.

Proof. Note that $\hat{u}_k$ satisfies

$$\Delta^2 \hat{u}_k = e^{\hat{u}_k} \text{ in } \Omega,\ \hat{u}_k = -\alpha_k,\ \partial_\nu \hat{u}_k = 0 \text{ on } \partial\Omega \tag{3.2}$$

with $\int_\Omega e^{\hat{u}_k}\,dx < C$ for all k. So

$$\hat{u}_k(x) = \int_\Omega G(x,y)e^{\hat{u}_k(y)}dy - \alpha_k \tag{3.3}$$

and hence by (2.2),

$$\begin{aligned}\int_\Omega |\Delta \hat{u}_k(x)|\,dx &\leq \int_\Omega \left(\int_\Omega |\Delta_x G(x,y)| e^{\hat{u}_k(y)}\,dy\right)\,dx\\ &\leq C\int_\Omega \left(\int_\Omega \frac{1}{|x-y|^2} e^{\hat{u}_k(y)}\,dy\right)\,dx \leq C.\end{aligned}$$

Similarly, integrating (3.3), we get that there exists $C>0$ such that

$$\|\hat{u}_k+\alpha_k\|_{L^1(\Omega)}\leq C \tag{3.4}$$

for all $k\in\mathbb{N}$. It follows from Theorem 1.2 of [34] that there exists $S_1\subset\Omega$, where S_1 is at most finite, such that $\hat{u}_k\leq C(\omega)$ uniformly in ω for $\omega\subset\subset\Omega\backslash S_1$. Therefore, with (3.4), we get that (α_k) cannot go to $-\infty$ when $k\to+\infty$. This proves Claim 1. □

A consequence is the following proposition that concerns the case when u_k is bounded from above:

Lemma 3.1. *Let (u_k,ρ_k) be a sequence of solutions to (1.1) such that there exists $\Lambda>0$ such that $0<\rho_k\leq\Lambda$. Assume that there exists $C>0$ such that $u_k\leq C$ for all $k\in\mathbb{N}$. Then there exists $u\in C^4(\overline{\Omega})$ such that, up to a subsequence $\lim_{k\to+\infty}u_k=u$.*

Proof. It follows from the assumption of the lemma and Claim 1 that $\hat{u}_k\leq C_1$ on Ω. It then follows from (1.1) and (3.2) that (u_k) is bounded in $C^3(\overline{\Omega})$. The conclusion follows from elliptic theory. □

In the sequel, we assume that

$$\max_{x\in\Omega}u_k(x)\to+\infty. \tag{3.5}$$

Our second claim is an upper bound on the L^p-norm of $\nabla^i\hat{u}_k$:

Claim 2: For all $i=1,2,3, p\in(1,\frac{4}{i})$, there exists $C=C(i,p)$ such that $\|\nabla^i\hat{u}_k\|_{L^p(\Omega)}\leq C$.

Proof. By Green's representation formula (3.3) and (2.2), we have

$$\begin{aligned}|\nabla^i\hat{u}_k(x)| &\leq \int_\Omega |\nabla_x^i G(x,y)| e^{\hat{u}_k(y)}\,dy\\ &\leq C\int_\Omega \frac{1}{|x-y|^i}e^{\hat{u}_k}\,dy.\end{aligned}$$

Thus for any $\varphi\in C_c^\infty(\mathbb{R}^4)$, we have

$$\begin{aligned}&\int_\Omega |\nabla^i\hat{u}_k(x)|\varphi\,dx \leq \int_\Omega\left(\int_\Omega |\nabla_x^i G(x,y)|e^{\hat{u}_k(y)}\,dy\right)|\varphi(x)|\,dx\\ &\leq C\int_\Omega e^{\hat{u}_k}\left(\int_\Omega |x-y|^{-i}|\varphi(x)|\,dx\right)\,dy\\ &\leq C\int_\Omega e^{\hat{u}_k}\||x-y|^{-i}\|_{L^p(\Omega)}\|\varphi\|_{L^q(\Omega)}\,dy\\ &\leq C\|\varphi\|_{L^q(\Omega)}\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Here, we used that Ω is bounded. By duality, we derive that $\|\nabla^i\hat{u}_k\|_{L^p(\Omega)}\leq C$. □

The third claim asserts that bubbles must have some distance from the boundary:

Claim 3: Let $(x_k)_{k\in\mathbb{N}} \in \Omega$ be such that $u_k(x_k) = \max_\Omega u_k$. Let $\mu_k := e^{-\frac{1}{4}\hat{u}_k(x_k)}$. Then $\lim_{k\to+\infty} \frac{d(x_k,\partial\Omega)}{\mu_k} = +\infty$.

Proof. Suppose otherwise, $d(x_k, \partial\Omega) = O(\mu_k)$. Let $\Omega_k := \frac{\Omega - x_k}{\mu_k}$. Then up to a rotation, we may assume that $\Omega_k \to (-\infty, t_0) \times \mathbb{R}^3$. Let $\tilde{u}_k(x) := \hat{u}_k(x_k + \mu_k x) + 4\log\mu_k$. Note that $\lim_{k\to+\infty} \mu_k = 0$ (otherwise $\hat{u}_k$ is bounded from above, and, as in the proof of Lemma 3.1, we get that (u_k) is bounded: a contradiction with (3.5)). Let $R > 0$ and $x \in B_R(0) \cap \Omega_k$, then we have by the representation formula (3.3) and (2.2)

$$\begin{aligned} |\nabla^i \tilde{u}_k(x)| &= |\mu_k^i \nabla^i \hat{u}_k(x_k + \mu_k x)| \\ &= \mu_k^i \left| \int_\Omega \nabla_x^i G(x_k + \mu_k x, y) e^{\hat{u}_k(y)}\, dy \right| \\ &\le C\mu_k^i \left(\int_{B_{2R\mu_k}(x_k)} \frac{1}{|x_k + \mu_k x - y|^i} e^{\hat{u}_k(y)}\, dy \right. \\ &\quad \left. + \int_{\Omega_k \backslash B_{2R\mu_k}(x_k)} \frac{1}{|x_k + \mu_k x - y|^i} e^{\hat{u}_k(y)}\, dy \right). \end{aligned}$$

On $\Omega_k \backslash B_{2R\mu_k}(x_k)$, $|x_k + \mu_k x - y| \ge |y - x_k| - \mu_k|x| \ge R\mu_k$, $e^{\hat{u}_k(y)} \le e^{\hat{u}_k(x_k)} = \mu_k^{-4}$. Hence

$$|\nabla^i \tilde{u}_k(x)| \le \mu_k^{i-4} \int_{B_{2R\mu_k}(x_k)} \frac{dy}{|x_k + \mu_k x - y|^i} + C\int_\Omega e^{\hat{u}_k}\, dy \le C(R).$$

In particular, this implies that $|\tilde{u}_k(x) - \tilde{u}_k(0)| \le C|x|$ for all $x \in B_R(0)$. Now let $x \in \partial\Omega_k$, we get $|\hat{u}_k(x_k) + \alpha_k| \le C$. This gives

$$4\log\frac{1}{\mu_k} + \alpha_k = O(1).$$

A contradiction with $\lim_{k\to+\infty} \mu_k = 0$ and Claim 1. Thus $\frac{d(x_k,\partial\Omega)}{\mu_k} \to +\infty$. □

Claim 4 concerns the first bubble:

Claim 4: We have that

$$\lim_{k\to+\infty} \hat{u}_k(x_k + \mu_k x) + 4\log\mu_k = -4\log\left(1 + \frac{|x|^2}{8\sqrt{6}}\right)$$

in $C^4_{loc}(\mathbb{R}^4)$.

Proof. By Claim 3, we have $\Omega_k \to \mathbb{R}^4$. Since $\tilde{u}_k(x) = \hat{u}_k(x_k + \mu_k x) + 4\log\mu_k$, $\tilde{u}_k(x) \le \tilde{u}_k(0)$ and $\Delta^2\tilde{u}_k = e^{\tilde{u}_k}$ in Ω_k. Note by Claim 3, $|\nabla^i\tilde{u}_k(x)| \le C(R)$, for all $x \in B_R(0)$. By standard regularity arguments, $\tilde{u}_k \to \tilde{u}$ in $C^4_{loc}(\mathbb{R}^4)$ where $\tilde{u}$ satisfies

$$\Delta^2\tilde{u} = e^{\tilde{u}},\ \tilde{u}(0) = 0, \int_{\mathbb{R}^4} e^{\tilde{u}}\, dx < +\infty. \tag{3.6}$$

Note that solutions to (3.6) are nonunique. To characterize $\tilde{u}$, we compute

$$\Delta\tilde{u}_k(x) = \int_\Omega \mu_k^2 \Delta_x G(x_k + \mu_k x, y) e^{\hat{u}_k(y)}\, dy$$

and for $x \in B_R(0)$,

$$\int_{B_R(0)} |\Delta \tilde{u}_k|\, dx \le C \int_\Omega e^{\hat{u}_k(y)} \left(\mu_k^2 \int_{B_R(0)} \frac{dx}{|x_k + \mu_k x - y|^2} \right) dy$$

$$\le CR^2 \int_\Omega e^{\hat{u}_k(y)}\, dy \le CR^2.$$

That is, for any $R > 0$, we have $\int_{B_R(0)} |\Delta \tilde{u}|\, dx \le CR^2$. It then follows from results of [25] and [37] that $\tilde{u}(x) = -4 \log \left(1 + \frac{|x|^2}{8\sqrt{6}}\right)$. Moreover, $\int_{B_{R\mu_k}(x_k)} e^{\hat{u}_k}\, dx = \int_{B_R(0)} e^{\tilde{u}_k}\, dx$ and hence

$$\lim_{R\to+\infty} \lim_{k\to+\infty} \int_{B_{R\mu_k}(x_k)} e^{\hat{u}_k}\, dx = 64\pi^2. \tag{3.7}$$

□

We say that the property $\mathcal{H}_p$ holds if there exists $(x_{k,1}, ..., x_{k,p}) \in \Omega^p$ such that, denoting $\mu_{k,i} := e^{-\frac{1}{4}\hat{u}_k(x_{k,i})}$, we have that

(i) $\lim_{k\to+\infty} \frac{|x_{k,i}-x_{k,j}|}{\mu_{k,i}} = +\infty$, $\forall i \neq j$,

(ii) $\lim_{k\to+\infty} \frac{d(x_{k,i},\partial\Omega)}{\mu_{k,i}} = +\infty$,

(iii) $\lim_{k\to+\infty} (\hat{u}_k(x_{k,i} + \mu_{k,i}x) + 4 \log \mu_{k,i}) = -4 \log(1 + \frac{|x|^2}{8\sqrt{6}})$ in $C^4_{loc}(\mathbb{R}^4)$.

By Claim 4, $\mathcal{H}_1$ holds.

Claim 5: Assume that $\mathcal{H}_p$ holds. Then either $\mathcal{H}_{p+1}$ holds, or there exists $C > 0$ such that

$$\inf_{i=1,\dots,p} \{|x - x_{k,i}|^4\} e^{\hat{u}_k(x)} \le C, \forall x \in \Omega. \tag{3.8}$$

Proof. Let $w_k(x) := \inf_{i=1,\dots,p} |x - x_{k,i}|^4 e^{\hat{u}_k(x)}$. Assume that $\|w_k\|_{L^\infty(\Omega)} \to +\infty$ when $k \to +\infty$. Let $y_k \in \Omega$ be such that $w_k(y_k) = \max_\Omega w_k$ and $\gamma_k := e^{-\frac{1}{4}\hat{u}_k(y_k)}$ and $v_k(x) := \hat{u}_k(y_k + \gamma_k x) + 4 \log \gamma_k$. Then v_k satisfies $\Delta^2 v_k = e^{v_k}$. Note that $w_k(y_k) = \inf_{i=1,\dots,p} \frac{|y_k - x_{k,i}|^4}{\gamma_k^4} \to +\infty$. Then $\lim_{k\to+\infty} \frac{|y_k - x_{k,i}|}{\gamma_k} \to +\infty$ for all $i = 1, ..., p$. Assume that there exists i such that $y_k - x_{k,i} = O(\mu_{k,i})$, Then $y_k = x_{k,i} + \mu_{k,i}\theta_{k,i}$ and

$$|y_k - x_{k,i}|^4 e^{\hat{u}_k(y_k)} = |\theta_{k,i}|^4 e^{\hat{u}_k(x_{k,i}+\mu_{k,i}\theta_{k,i})+4\log\mu_{k,i}} \to |\theta_{\infty,i}|^4 \frac{1}{(1 + \frac{|\theta_{\infty,i}|^2}{8\sqrt{6}})^4}$$

where $\theta_{\infty,i} = \lim_{k\to+\infty} \theta_{k,i}$. This implies that $w_k(y_k) = O(1)$. A contradiction. Thus $\frac{|y_k - x_{k,i}|}{\mu_{k,i}} \to +\infty$ for all $i = 1, ..., p$.

Let $x \in B_R(0)$ and let $\epsilon \in (0, 1)$. Then $w_k(y_k + \gamma_k x) \le w_k(y_k)$. That is, $\inf_{i=1,\dots,p} |y_k - x_{k,i} + \gamma_k x|^4 e^{\hat{u}_k(y_k+\gamma_k x)} \le \inf_{i=1,\dots,p} |y_k - x_{k,i}|^4 e^{\hat{u}_k(y_k)}$ and so

$$e^{v_k(x)} \le \frac{\inf_{i=1,\dots,p} |y_k - x_{k,i}|^4}{\inf_{i=1,\dots,p} |y_k - x_{k,i} + \gamma_k x|^4}.$$

Let $k \ge k(R)$ be such that $\frac{|y_k - x_{k,i}|}{\gamma_k} \ge \frac{R}{\epsilon}$ for all $i = 1, ..., p, k \ge k(R)$. Then for $i = 1, ..., p$, we have $|y_k - x_{k,i} + \gamma_k x| \ge |y_k - x_{k,i}|(1 - \epsilon)$ and $\inf_{i=1,\dots,p} |y_k - x_{k,i} +$

$\gamma_k x|^4 \geq \inf_{i=1,\dots,p} |y_k - x_{k,i}|^4 (1-\epsilon)^4$. This yields

$$e^{v_k(x)} \leq \frac{1}{(1-\epsilon)^4},\ x \in B_R(0), k \geq k(R).$$

Similar to Claim 3, we also have that

$$\lim_{k\to+\infty} \frac{d(y_k, \partial\Omega)}{\gamma_k} = +\infty \text{ and } \lim_{k\to+\infty} v_k(x) = -4\log\left(1 + \frac{|x|^2}{8\sqrt{6}}\right)$$

in $C^4_{loc}(\mathbb{R}^4)$. Letting $x_{k,p+1} = y_k$, then $\mathcal{H}_{p+1}$ holds. The claim is thus proved. □

Claim 6: There exists N such that $\mathcal{H}_N$ holds and there exists $C > 0$ such that

$$\inf_{i=1,\dots,p} |x - x_{k,i}|^4 e^{\hat{u}_k(x)} \leq C,\ \forall x \in \Omega. \tag{3.9}$$

Proof. Otherwise, since $\mathcal{H}_1$ holds, then $\mathcal{H}_p$ holds for all $p \geq 1$. Given $R > 0$, we have $B_{R\mu_{k,i}}(x_{k,i}) \cap B_{R\mu_{k,j}}(x_{k,j}) = \emptyset$ for all $i \neq j, k \geq k(R)$. Then

$$\rho_k = \int_\Omega e^{\hat{u}_k}\,dx \geq \int_{\cup_{i=1,\dots,p} B_{R\mu_{k,i}}(x_i)} = \sum_{i=1}^p \int_{B_{R\mu_{k,i}}(x_{k,i})} e^{\hat{u}_k(y)}\,dy \geq 64\pi^2 p + o(1)_R$$

where $\lim_{R\to+\infty}\lim_{k\to+\infty} o(1)_R = 0$. Since $\rho_k \leq \Lambda$, we derive that $p \leq \Lambda/64\pi^2$ for all p: a contradiction. Hence Claim 6 holds. □

Claim 7: For $p = 1, 2, 3$, there exists $C > 0$ such that

$$\inf_{i=1,\dots,p} |x - x_{k,i}|^p |\nabla^p \hat{u}_k(x)| \leq C,\ \forall x \in \Omega. \tag{3.10}$$

Proof. By Green's representation formula, we have

$$\nabla^p \hat{u}_k(x) = \int_\Omega \nabla^p_x G(x,y) e^{\hat{u}_k(y)}\,dy.$$

Hence

$$|\nabla^p \hat{u}_k(x)| \leq C \int_\Omega |x-y|^{-p} e^{\hat{u}_k(y)}\,dy. \tag{3.11}$$

Let $R_k(x) := \inf_{i=1,\dots,N} |x - x_{k,i}|, \Omega_{k,i} = \{x \in \Omega : |x - x_{k,i}| = R_k(x)\}$. Then

$$\begin{aligned}\int_{\Omega_{k,i}} |x-y|^{-p} e^{\hat{u}_k(y)}\,dy \quad &= \quad \int_{\Omega_{k,i} \cap B_{\frac{|x-x_{k,i}|}{2}}(x_{k,i})} |x-y|^{-p} e^{\hat{u}_k(y)}\,dy \\ &\quad + \int_{\Omega_{k,i} \setminus B_{\frac{|x-x_{k,i}|}{2}}(x_{k,i})} |x-y|^{-p} e^{\hat{u}_k(y)}\,dy.\end{aligned}$$

Note that for $y \in \Omega_{k,i} \setminus B_{\frac{|x-x_{k,i}|}{2}}(x_{k,i})$, $|x-y|^{-p} e^{\hat{u}_k(y)} \leq \frac{C}{|x-y|^p |y-x_{k,i}|^4}$. Then Claim 6 and easy computations show that

$$\int_{B_R(0) \setminus B_{\frac{|x-x_{k,i}|}{2}}(x_{k,i})} \frac{1}{|x-y|^p |y-x_{k,i}|^4}\,dy \leq \frac{C}{|x-x_{k,i}|^p}.$$

Thus

$$\left| \int_{\Omega_{k,i} \setminus B_{\frac{|x-x_{k,i}|}{2}}(x_{k,i})} |x-y|^{-p} e^{\hat{u}_k(y)}\,dy \right| \leq \frac{C}{|x-x_{k,i}|^p}. \tag{3.12}$$

On the other hand, for $y \in \Omega_{k,i} \cap B_{\frac{|x-x_{k,i}|}{2}}(x_{k,i})$, we have $|x-y| \geq |x-x_{k,i}| - |y - x_{k,i}| \geq \frac{1}{2}|x-x_{k,i}|$ and hence

$$\left| \int_{\Omega_{k,i} \cap B_{\frac{|x-x_{k,i}|}{2}}(x_{k,i})} |x-y|^{-p} e^{\hat{u}_k(y)}\, dy \right| \leq \frac{C}{|x-x_{k,i}|^p}. \tag{3.13}$$

Combining (3.12) and (3.13), we obtain the desired estimates. ☐

Claim 8: Let $x_i := \lim_{k\to+\infty} x_{k,i} \in \bar{\Omega}$ and $S := \{x_i, i = 1, ..., N\}$. Assume that $\lim_{k\to+\infty} \alpha_k = +\infty$. Then $\hat{u}_k \to -\infty$ uniformly in $\bar{\Omega}\backslash S$.

Proof. Let $\delta > 0$ small such that $\Omega_\delta := \Omega \backslash \cup_{i=1}^N \overline{B}_\delta(x_i)$ is connected. Then $|\nabla \hat{u}_k(x)| \leq C(\Omega_\delta)$ for $x \in \Omega_\delta$ by the representation formula (3.3). Let $x_\delta \in \partial\Omega_\delta \cap \partial\Omega$, then we have $\hat{u}_k(x) = -\alpha_k$ and hence $|\hat{u}_k(x) + \alpha_k| \leq C$ for all $x \in \Omega_\delta$. This implies that $\hat{u}_k \to -\infty$ uniformly. ☐

Claim 9: Assume that $\lim_{k\to+\infty} \alpha_k = +\infty$. Then there exists $\gamma_1, ..., \gamma_N \geq 64\pi^2$ such that

$$\lim_{k\to+\infty} u_k(x) = \sum_{i=1}^N \gamma_i G(\cdot, x_i) \text{ in } C^4_{loc}(\bar{\Omega}\backslash S).$$

Proof. Since u_k satisfies

$$\Delta^2 u_k = e^{-\alpha_k} e^{u_k}$$

and u_k is bounded in $C^0_{loc}(\bar{\Omega}\backslash S)$ by Claim 8, by standard regularity arguments we deduce that $u_k \to \psi$ in $C^4(\bar{\Omega}\backslash S)$, where $\psi \in C^4(\bar{\Omega}\backslash S)$. Thus, for $\delta > 0$ small enough,

$$u_k(x) = \int_\Omega G(x,y) e^{\hat{u}_k(y)}\, dy = \sum_{i=1}^N \int_{B_\delta(x_i)\cap\Omega} G(x,y) e^{\hat{u}_k(y)}\, dy + o(1).$$

Since $G(x, \cdot)$ is continuous in $\bar{\Omega}\backslash\{x\}$, we get that

$$\lim_{k\to+\infty} u_k(x) = \sum_{i=1}^N \gamma_i G(x, x_i)$$

where $\gamma_i := \lim_{\delta\to 0} \lim_{k\to+\infty} \int_{B_\delta(x_i)\cap\Omega} e^{\hat{u}_k(y)}\, dy$. By Claims 4 and 5, $\gamma_i \geq 64\pi^2$. Then $\psi = \sum_{i=1}^N \gamma_i G(x, x_i)$. So we get the result. ☐

Claim 10: Let $x_i := \lim_{k\to+\infty} x_{k,i} \in \bar{\Omega}$ and $S := \{x_i, i = 1, ..., N\}$. Assume that $\lim_{k\to+\infty} \alpha_k = \alpha_\infty \in \mathbb{R}$. Then $S \subset \partial\Omega$ and there exists $u \in C^4(\overline{\Omega})$ such that $\Delta^2 u = e^{-\alpha_\infty} e^u$ in Ω, $u = \partial_\nu u = 0$ in $\partial\Omega$ and

$$\lim_{k\to+\infty} u_k = u \text{ in } C^4_{loc}(\overline{\Omega} \setminus S).$$

Proof. Indeed, with (3.4), we get that $\|\hat{u}_k\|_{L^1(\Omega)} \leq C$ for all $k \in \mathbb{N}$. It then follows from Theorem 1.2 of [34] that there exists $\hat{u} \in C^4(\Omega)$ such that $\lim_{k\to+\infty} \hat{u}_k = \hat{u}$ in $C^3_{loc}(\Omega)$. Therefore $S \subset \partial\Omega$. It then follows from Claims 6 and 7 and standard elliptic theory that there exists $u \in C^4(\overline{\Omega} \setminus S)$ such that

$$\lim_{k\to+\infty} u_k = u \text{ in } C^4_{loc}(\overline{\Omega} \setminus S).$$

Moreover, passing to the limit $k \to +\infty$ in Claim 7, we get that

$$\inf_{i=1,\dots,N} |x - x_i||\nabla u(x)| \leq C \text{ for all } x \in \Omega \setminus S.$$

We are left with proving that u can be smoothly extended to S. We fix $x_0 \in S$ and we let $\delta > 0$ small enough such that

$$|x - x_0||\nabla u(x)| \leq C \text{ for all } x \in \Omega \cap B_\delta(x_0) \setminus \{x_0\}.$$

Therefore, there exists $C' > 0$ such that for all $x, y \in \Omega \cap B_\delta(x_0) \setminus \{x_0\}$ such that $|x - x_0| = |y - x_0|$, we have that

$$|u(x) - u(y)| \leq C'.$$

Taking $y \in \partial\Omega$, we then get $|u(x)| \leq C'$ for all $x \in \Omega \cap B_\delta(x_0) \setminus \{x_0\}$. Proceeding similarly for all the points of S, we get that there exists $C > 0$ such that $|u(x)| \leq C$ for all $x \in \Omega \setminus S$.

We let $w \in H_0^2(\Omega)$ such that $\Delta^2 w = e^{-\alpha_\infty} e^u$. (Since $|u| \leq C$, we may simply put $e^u = 1$ when $x = x_0$.) It follows from standard theory that $w \in C^3(\overline{\Omega})$ and that

$$w(x) = \int_\Omega G(x, y) e^{-\alpha_\infty} e^{u(y)}\, dy$$

for all $x \in \Omega$. For $\delta > 0$ small enough and $x \in \bar{\Omega} \backslash S$,

$$u_k(x) = \int_\Omega G(x, y) e^{\hat{u}_k(y)}\, dy = \int_{(\bigcup_{i=1}^N B_\delta(x_i))^c \cap \Omega} G(x, y) e^{\hat{u}_k(y)}\, dy + O(\delta). \tag{3.14}$$

Passing to the limit (first in k and then in δ) in (3.14) and noting that $|u| \leq C$, we get that

$$u(x) = \int_\Omega G(x, y) e^{-\alpha_\infty} e^{u(y)}\, dy$$

for all $x \in \bar{\Omega} \backslash S$. Therefore, $u \equiv w$ in $\bar{\Omega} \backslash S$ and u can be extended smoothly as a C^3-function on $\overline{\Omega}$. Coming back to the definition of w, we get that w is C^4 and then $u \in C^4(\overline{\Omega})$. This ends the proof of Claim 10. As a remark, let us note that if the concentration points were isolated (that is $x_i \neq x_j$ for all $i \neq j$), the argument above would prove that (u_k) is bounded uniformly near the boundary, which would immediately exclude boundary blow-up. □

Now, we exclude the boundary blow-up in case $\lim_{k \to +\infty} \alpha_k = +\infty$:

Claim 11: Assume that $\lim_{k \to +\infty} \alpha_k = +\infty$. Let $x_0 \in \partial\Omega$. Then

$$\lim_{r \to 0} \lim_{k \to +\infty} \int_{B_r(x_0) \cap \Omega} e^{\hat{u}_k}\, dx = 0.$$

In particular, $S \cap \partial\Omega = \emptyset$.

Proof. We argue by contradiction and we let $x_0 \in \partial\Omega \cap S$. Then (3.7) yields

$$\lim_{r \to 0} \lim_{k \to +\infty} \int_{B_r(x_0) \cap \Omega} e^{\hat{u}_k}\, dx \geq 64\pi^2.$$

Thus for all $\delta > 0$, we have that

$$\int_{B_\delta(x_0) \cap \Omega} e^{\hat{u}_k}\, dx \geq 32\pi^2 \tag{3.15}$$

for all $k \in \mathbb{N}$ large enough. Furthermore, we may assume that $S \cap B_\delta(x_0) = \{x_0\}$. Let $y_k := x_0 + \rho_{k,r}\nu(x_0)$ with

$$\rho_{k,r} = \frac{\int_{\partial\Omega\cap B_r(x_0)}(x - x_0, \nu)(\Delta u_k)^2\, dx}{\int_{\partial\Omega\cap B_r(x_0)}(\nu(x_0), \nu)(\Delta u_k)^2\, dx} \tag{3.16}$$

where $r << r_1$ such that $\frac{1}{2} \leq (\nu(x_0)\cdot\nu) \leq 1$ for $x \in \bar{B}_r(x_0)\cap\Omega$. Here $\nu(x)$ is the outer normal vector to $T_{x_0}\partial\Omega$ at x. Then it is easy to see that $|\rho_{k,r}| \leq 2r$ and

$$\int_{\partial\Omega\cap B_r(x_0)}(x - y_k, \nu)(\Delta u_k)^2\, dx = 0. \tag{3.17}$$

Now applying the Pohozaev's identity in $\Omega\cap B_r(x_0)$ with $y = y_k$, $f(u) = e^{-\alpha_k}e^{u_k}$ and $F(u) = e^{-\alpha_k}(e^{u_k} - 1)$, and using Dirichlet boundary condition and (3.17), we obtain that

$$\begin{aligned}
4\int_{\Omega\cap B_r(x_0)}(e^{\hat{u}_k} - e^{-\alpha_k})\, dx = &\int_{\Omega\cap\partial B_r(x_0)}\langle x - y_k, \nu\rangle(e^{-\alpha_k}e^{u_k} - e^{-\alpha_k})\, d\sigma\\
&-2\int_{\Omega\cap\partial B_r(x_0)}\frac{\partial u_k}{\partial\nu}\Delta u_k\, d\sigma\\
&+\int_{\Omega\cap\partial B_r(x_0)}\left[\frac{1}{2}\langle x - y_k, \nu\rangle(\Delta u_k)^2 + \frac{\partial(-\Delta u_k)}{\partial\nu} < x - y_k, \nabla u_k >\right] d\sigma\\
&+\int_{\Omega\cap\partial B_r(x_0)}\left[-\frac{\partial}{\partial\nu}u_k\langle x - y_k, \nabla\Delta u_k\rangle + < \nabla u_k, \nabla\Delta u_k >< x - y_k, \nu >\right] d\sigma.
\end{aligned}$$

Note that $u_k \to \Psi = \sum_{i=1}^N \gamma_i G(x, x_i)$ in $C^3(\bar{\Omega}\backslash S)$, where $G(x, x_0) = 0$. Thus we obtain that all the terms in the last three integrals are of the form

$$\lim_{k\to+\infty}\int_{\Omega\cap\partial B_r(x_0)}\left[O(1)\right] dx = O(r^3)$$

while

$$\lim_{k\to+\infty}\int_{\partial\Omega\cap B_r(x_0)}(x - y_k, \nu)(e^{-\alpha_k}e^{u_k} - e^{-\alpha_k})\, d\sigma = O(r^4).$$

Since $\lim_{k\to+\infty}\alpha_k = +\infty$, we thus obtain that

$$\left|\int_{\Omega\cap B_r(x_0)} e^{\hat{u}_k}\, dx\right| \leq Cr^3 \tag{3.18}$$

for $k \in \mathbb{N}$ large enough. Therefore,

$$\lim_{r\to 0}\lim_{k\to+\infty}\int_{\Omega\cap B_r(x_0)} e^{\hat{u}_k}\, dx = 0.$$

A contradiction with (3.15). This proves Claim 11. □

Claim 12: We have that

$$\lim_{k\to+\infty}\alpha_k = +\infty.$$

Proof. We argue by contradiction and assume that, up to extracting a subsequence, $\lim_{k\to+\infty}\alpha_k=\alpha_\infty\in\mathbb{R}$. We let $x_0\in S\subset\partial\Omega$ (this follows from Claim 10). Arguing as in Claim 11, we get that

$$4\int_{\Omega\cap B_r(x_0)}(e^{\hat{u}_k}-e^{-\alpha_k})\,dx=\int_{\Omega\cap\partial B_r(x_0)}\langle x-y_k,\nu\rangle(e^{-\alpha_k}e^{u_k}-e^{-\alpha_k})\,d\sigma$$
$$-2\int_{\Omega\cap\partial B_r(x_0)}\frac{\partial u_k}{\partial\nu}\Delta u_k\,d\sigma$$
$$+\int_{\Omega\cap\partial B_r(x_0)}\left[\frac{1}{2}\langle x-y_k,\nu\rangle(\Delta u_k)^2+\frac{\partial(-\Delta u_k)}{\partial\nu}<x-y_k,\nabla u_k>\right]d\sigma$$
$$+\int_{\Omega\cap\partial B_r(x_0)}\left[-\frac{\partial}{\partial\nu}u_k\langle x-y_k,\nabla\Delta u_k\rangle+<\nabla u_k,\nabla\Delta u_k><x-y_k,\nu>\right]d\sigma.$$

Letting $k\to+\infty$, we then get with Claim 10 that

$$4\times 32\pi^2\le 4\int_{\Omega\cap B_r(x_0)}e^{-\alpha_k}\,dx+\int_{\Omega\cap\partial B_r(x_0)}\langle x-y_\infty,\nu\rangle(e^{u-\alpha_\infty}-e^{-\alpha_\infty})\,d\sigma$$
$$-2\int_{\Omega\cap\partial B_r(x_0)}\frac{\partial u}{\partial\nu}\Delta u\,d\sigma$$
$$+\int_{\Omega\cap\partial B_r(x_0)}\left[\frac{1}{2}\langle x-y_\infty,\nu\rangle(\Delta u)^2+\frac{\partial(-\Delta u)}{\partial\nu}<x-y_\infty,\nabla u>\right]d\sigma$$
$$+\int_{\Omega\cap\partial B_r(x_0)}\left[-\frac{\partial}{\partial\nu}u\langle x-y_\infty,\nabla\Delta u\rangle+<\nabla u,\nabla\Delta u><x-y_\infty,\nu>\right]d\sigma$$

for all $r>0$ small enough, where $y_\infty:=\lim_{k\to+\infty}y_k$ depends on r with $|y_\infty-x_0|\le 2r$. With Claim 10, we know that $u\in C^4(\overline{\Omega})$. Passing to the limit $r\to 0$ above, we get that the RHS goes to zero. A contradiction. Then $\lim_{k\to+\infty}\alpha_k=+\infty$, and Claim 12 is proved. □

Claim 13: $\gamma_i=64\pi^2, i=1,...,N$.

Proof. Since $x_i\in\Omega$, the same proof as in Lemma 3.5 of Lin-Wei [38] gives the claim. We also refer to Druet-Robert [17]. □

Claim 14: The identity (1.8) holds.

Proof. The proof is exactly the same as that of Theorem 1.2 of Lin-Wei [38] and as in Druet-Robert [17]. □

Theorem 1.1 follows form Claims 9-14.

4. Proof of Theorem 1.2

By Theorem 1.1, there are no boundary bubbles for (1.1). The proof of Theorem 1.2 follows along the lines of Sections 3 and 4 of [26]: we just need to change the Navier boundary condition to Dirichlet boundary condition. Let us sketch the changes. We first choose a good approximate function: fix $P\in\Omega$ and let

$$U_{\epsilon,P}(x):=\log\frac{\gamma\epsilon^4}{(\epsilon^2+|x-P|^2)^4},\tag{4.1}$$

where $\gamma := 3 \cdot 2^7 = 384$. We consider the projection of $U_{\epsilon,P}$:

$$\begin{cases} \Delta^2 \mathcal{P}_\Omega U_{\epsilon,P} - e^{U_{\epsilon,P}} = 0 \text{ in } \Omega, \\ \mathcal{P}_\Omega U_{\epsilon,P} = \partial_\nu \mathcal{P}_\Omega U_{\epsilon,P} = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.2}$$

Set

$$\mathcal{P}_\Omega U_{\epsilon,P} = U_{\epsilon,P} - \varphi_{\epsilon,P} \tag{4.3}$$

Then $\varphi_{\epsilon,P}$ satisfies

$$\begin{cases} \Delta^2 \varphi_{\epsilon,P} = 0 \text{ in } \Omega, \\ \varphi_{\epsilon,P} = U_{\epsilon,P}, \ \partial_\nu \varphi_{\epsilon,P} = \partial_\nu U_{\epsilon,P} \text{ on } \ \partial\Omega. \end{cases} \tag{4.4}$$

On $\partial\Omega$, we have for ϵ sufficiently small

$$U_{\epsilon,P}(x) = \log(\gamma\epsilon^4) - 8\log|x-P| - \frac{4\epsilon^2}{|x-P|^2} + O(\epsilon^4)$$

uniformly in $C^4(\partial\Omega)$. Comparing (4.4) with (1.9) and (1.15), we have

$$\varphi_{\epsilon,P} = \log(\gamma\epsilon^4) - 64\pi^2 R(x,P) - \epsilon^2 R_1(x,P) + O(\epsilon^4), \text{ in } \Omega. \tag{4.5}$$

We now use $\mathcal{P}_\Omega U_{\epsilon,P}$ as a test function to compute an upper bound for $c_{64\pi^2}$. Let Q_0 be such that $R(Q_0,Q_0) = \max_{Q\in\Omega} R(Q,Q)$. Similar computations in [page 799,[26]] yield

$$J_{64\pi^2}[\mathcal{P}_\Omega U_{\epsilon,Q_0}] = A_0 - \frac{1}{2}(64\pi^2)^2 \max_{P\in\Omega} R(P,P)$$
$$-\frac{\epsilon^2}{2}\left[64\pi^2 R_1(Q_0,Q_0) + \frac{(64\pi^2)^2}{4}\Delta_x R(Q_0,Q_0)\right] + o(\epsilon^2)$$

where A_0 is a generic constant. By our assumption (1.14), we have

$$c_{64\pi^2} < A_0 - \frac{1}{2}(64\pi^2)^2 \max_{P\in\Omega} R(P,P). \tag{4.6}$$

On the other hand, let u_ρ be a minimizer of J_ρ for $\rho < 64\pi^2$. If u_ρ blows up as $\rho \to 64\pi^2$, then a lower bound can be obtained by following exactly the same computation in [26]:

$$c_{64\pi^2} \geq A_0 - \frac{1}{2}(64\pi^2)^2 \max_{P\in\Omega} R(P,P). \tag{4.7}$$

From (4.6) and (4.7), we deduce that blow-up does not occur. Then u_ρ is uniformly bounded from above. It then follows from Lemma 3.1 that u_ρ converges to a minimizer of $J_{64\pi^2}$ when $\rho \to 64\pi^2$.

Finally, when Ω is a ball, (without loss of generality, we may take $\Omega = B_1(0)$), by the result of Berchio, Gazzola and Weth [4], u is radially symmetric and strictly decreasing. Here $Q_0 = 0$. Now, by the so-called Boggio's formula [8], we have

$$G(x,y) = \frac{1}{8\pi^2}\int_1^{\frac{[x,y]}{|x-y|}} \frac{(v^2-1)}{v^3}\,dv, \quad \text{where } \ [x,y]^2 = |x-y|^2 + (1-|x|^2)(1-|y|^2),$$

for $x,y \in B_1(0)$. Thus

$$G(x,0) = \frac{1}{8\pi^2}\left(\log\frac{1}{|x|} + \frac{|x|^2}{2} - \frac{1}{2}\right), R(x,0) = \frac{1}{8\pi^2}\left(\frac{|x|^2}{2} - \frac{1}{2}\right),$$

and hence

$$\Delta_x R(0,0) = \frac{1}{2\pi^2} > 0.$$

It is easy to compute $R_1(x,0) = 4(2 - |x|^2)$ and hence

$$R_1(0,0) = 8 > 0.$$

This shows that condition (1.14) is satisfied. Theorem 1.2 is thus proved.

ACKNOWLEDGMENTS

The research of the second author is partially supported by a Competitive Earmarked Grant from RGC of HK. This work was carried out while the first author was visiting the Chinese University of Hong-Kong: he thanks this institution for its support and its hospitality.

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