

On finite Morse index solutions to the quadharmonic Lane-Emden equation *

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Abstract

In this paper, we compute the Joseph-Lundgren exponent for the quadharmonic Lane-Emden equation, derive a monotonicity formula and classify the finite Morse index solution to the following quadharmonic Lane-Emden equation:

$$\Delta^4 u = |u|^{p-1} u \quad \text{in } \mathbb{R}^n.$$

As a byproduct, we also get a monotonicity formula for the quadharmonic maps $\Delta^4 u = 0$.

1 Main results and Background

We study the finite Morse index solution of the following Lane-Emden equation

$$\Delta^4 u = |u|^{p-1} u \quad \text{in } \mathbb{R}^n. \tag{1.1}$$

The goal of the current paper is to complete the classification of the finite Morse index solution to Eq. (1.1) and to establish the corresponding Liouville-type theorems. It is well known that the Liouville-type theorems play a crucial role to obtain a priori L^∞ -bounds for solutions of semilinear elliptic and parabolic problems. See the monograph of Quittner and Souplet [19]. To the best of our knowledge, such a kind of work to Eq. (1.1) has not been accomplished previously. Postponing the background of this topic to the last part of this section, we would like to state the main results of the current article first. Recall that a solution u of (1.1) is said to be stable outside a compact $\Theta \subset \mathbb{R}^n$ if

$$\int_{\mathbb{R}^n} |\Delta^2 \varphi|^2 dx \geq p \int_{\mathbb{R}^n} |u|^{p-1} \varphi^2 dx, \quad \text{for any } \varphi \in H^4(\mathbb{R}^n \setminus \Theta).$$

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In particular, if $\Theta = \emptyset$, we say that u is stable on \mathbb{R}^n . On the other hand, the Morse index of the solution u of (1.1) is defined as the maximal dimension over all subspaces E of $H^4(\mathbb{R}^n)$ satisfying

$$\int_{\mathbb{R}^n} |\Delta^2 \varphi|^2 dx < p \int_{\mathbb{R}^n} |u|^{p-1} \varphi^2 dx, \quad \text{for any } \varphi \in E \setminus \{0\}.$$

It is known that if a solution u to (1.1) has finite Morse index, then u must be stable outside a compact of \mathbb{R}^n . The first main result of the present paper is the following

Theorem 1.1. *Let u be a stable solution of (1.1). If $1 < p < p_c(n)$, then $u \equiv 0$.*

Where $p_c(n) := p_{cQuadharmonic}(n)$ is the Joseph-Lundgren exponent for the quadharmonic Lane-Emden equation (1.1), which will be introduced shortly. Further, we have more general results for the finite Morse index solutions.

Theorem 1.2. *Let u be a finite Morse index solution of (1.1). Assume that either*

$$(1) \quad 1 < p < \frac{n+8}{n-8} \text{ or}$$

$$(2) \quad \frac{n+8}{n-8} < p < p_c(n),$$

then the solution $u \equiv 0$.

$$(3) \quad \text{If } p = \frac{n+8}{n-8}, \text{ then } u \text{ has a finite energy, i.e.,}$$

$$\int_{\mathbb{R}^n} |\Delta^2 u|^2 = \int_{\mathbb{R}^n} |u|^{p+1} < +\infty.$$

Remark 1.1. *In the above both Theorems, $p_c(n)$, given below, called the Joseph-Lundgren exponent. The condition $p < p_c(n)$ is optimal. In fact the radial singular solution is stable when $p \geq p_c(n)$ (See [17]).*

Now we give the Joseph-Lundgren exponent for the quadharmonic Lane-Emden equation, which is defined by

$$p_c(n) := p_{cQuadharmonic}(n) := \begin{cases} \infty & \text{if } n \leq 17, \\ \frac{n+6-2d(n)}{n-10-2d(n)} & \text{if } n \geq 18, \end{cases} \quad (1.2)$$

where

$$d(n) := \sqrt{\frac{1}{4}n^2 + 5 + \frac{1}{2}\sqrt{d_6}} - \frac{1}{2}\sqrt{d_7 + \frac{d_3}{\sqrt{d_6}}} \quad (1.3)$$

and

$$\begin{aligned} d_0 := & 2097152 - \frac{45}{4}n^{10} + 180n^9 - 396n^8 - 5184n^7 + 36928n^6 + 27648n^5 \\ & - 132096n^4 + 147456n^3 - 1572864n^2; \end{aligned}$$

$$\begin{aligned}
d_1 &:= \frac{3}{65536}n^{24} - \frac{9}{4096}n^{23} + \frac{81}{2048}n^{22} - \frac{33}{128}n^{21} - \frac{123}{128}n^{20} + \frac{303}{16}n^{19} + \frac{21}{8}n^{18} \\
&\quad - 1056n^{17} + 3888n^{16} + 25396n^{15} - 279456n^{14} + 947712n^{13} + 1979904n^{12} \\
&\quad - 48427008n^{11} + 135979008n^{10} + 677117952n^9 - 2620588032n^8 \\
&\quad - 3265265664n^7 + 14294188032n^6 + 2415919104n^5 - 16106127360n^4; \\
d_2 &:= (d_0 + 12\sqrt{d_1})^{\frac{1}{3}}; d_3 := 128n^2; \\
d_4 &:= -\frac{8192}{3} + \frac{1}{32}n^8 - \frac{1}{2}n^7 + n^6 + 16n^5 - \frac{584}{3}n^4 - 128n^3 + \frac{4096}{3}n^2; \\
d_5 &:= \frac{40}{3}n^2 + \frac{128}{3}, \quad d_6 := \frac{1}{2}d_5 + \frac{1}{6}d_2 - \frac{d_4}{d_2}, \quad d_7 := d_5 - \frac{1}{6}d_2 + \frac{d_4}{d_2}.
\end{aligned}$$

The expression of $d(n)$ seems very complicated, however we have that

$$d(n) < \sqrt{n} \quad \text{for } n \geq 18; \quad \lim_{n \rightarrow \infty} \frac{d(n)}{\sqrt{n}} = 1.$$

Remark 1.2. In the quadharmonic case, the above Joseph-Lundgren exponent $p_c(n)$ actually satisfies a 8-th order polynomial algebraic equation. It is interesting that we can obtain the explicit formulation here.

Remark 1.3. Theorems 1.1-1.2 are proved by Monotonicity Formula which we introduce in the next section.

Now we describe the background and development for lane-Emden equations. It is well known that the Lane-Emden equation

$$-\Delta u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n \tag{1.4}$$

and its parabolic counterpart have played a key role in the development of methods and applications of nonlinear PDEs in the last decades. The fundamental works on Eq.(1.4) are due to [9, 10]. Another ground-breaking result on equation (1.4) is the celebrated Liouville-type theorem due to Gidas and Spruck [3], they assert that the Eq. (1.4) has no positive solution whenever $p \in (1, 2^* - 1)$, where $2^* = 2n/(n - 2)$ if $n \geq 3$ and $2^* = \infty$ if $n \leq 2$. However, if $p = 2^* - 1$ the Eq. (1.4) has a unique positive solution (up to translation and rescaling) which is radial and explicit (see Caffarelli-Gidas-Spruck [1]). Since then there has been an extensive literature on such a type of equations or systems. Among them, the paper [8] by Farina in 2007 (see also [7]), the equation (1.4) is revisited for $p > \frac{n+2}{n-2}$. The author obtained some classification results and Liouville-type theorems for $\mathbb{C}^2(\mathbb{R}^n)$ smooth solutions including stable solutions, finite Morse index solutions, solutions which are stable outside a compact set, radial solutions and non-negative solutions. The results obtained in [8] were applied to subcritical, critical and supercritical values of the exponent p . Moreover, the critical stability exponent $p_c(n)$ (Joseph-Lundgren exponent) is determined which is larger

than the classical critical exponent $p_S = 2^* - 1$ in Sobolev imbedding theorems. Precisely, in [8], the $p_c(n)$ is given by

$$p_c(n) := p_{c\text{Harmonic}}(n) := \begin{cases} \infty & \text{if } n \leq 10, \\ \frac{(n-2)^2-4n+8\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n \geq 11 \end{cases} \quad (1.5)$$

which can be traced back to Joseph-Lundgren [13]. It is proved that the $\mathbb{C}^2(\mathbb{R}^n)$ stable solution of Eq.(1.4) is identically to zero if $p < p_c(n)$, while Eq.(1.4) admits a smooth positive, bounded, stable and radial solution if $p \geq p_c(n)$ ($n \geq 11$). In some sense, the Joseph-Lundgren exponent $p_c(n)$ is a critical threshold for obtaining the Liouville-type theorems for stable or finite Morse index solutions. The proof of Farina involves a delicate use of Nash-Moser's iteration technique, which is a classical tool for regularity of second order elliptic operators and falls short for higher order operators.

About the biharmonic equation

$$\Delta^2 u = |u|^{p-1} u \quad \text{in } \mathbb{R}^n, \quad (1.6)$$

the corresponding Joseph-Lundgren exponent (See Gazzola and Grunau [11], 2006) is

$$p_c := p_{c\text{Biharmonic}}(n) := \begin{cases} \infty & \text{if } n \leq 12, \\ \frac{n+2-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}}{n-6-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}} & \text{if } n \geq 13. \end{cases} \quad (1.7)$$

Further, Davila, Dupaigne, Wang and Wei in [6] obtain the Liouville-type theorem and give a complete characterization of all finite Morse index solutions (whether radial or not, whether positive or not). See Lin [14] and Wei-Xu [23] where it is shown that any nonnegative solution u of Eq.(1.6) is $\equiv 0$ for $1 < p < \frac{n+4}{n-2}$.

Very recently, in our previous manuscript [16], we derive a new monotonicity formula and classify completely all the finite Morse index solutions (positive or sign-changing, radial or not) to the triharmonic Lane-Emden equation:

$$(-\Delta)^3 u = |u|^{p-1} u \quad \text{in } \mathbb{R}^n, \quad (1.8)$$

where the corresponding Joseph-Lundgren exponent is determined by the following formula:

$$p_{c\text{TriHarmonic}}(n) := p_c := \begin{cases} \infty & \text{if } n \leq 14, \\ \frac{n+4-2D(n)}{n-8-2D(n)} & \text{if } n \geq 15, \end{cases}$$

where

$$D(n) := \frac{1}{6} \left(9n^2 + 96 - \frac{1536 + 1152n^2}{d_0(n)} - \frac{3}{2} d_0(n) \right)^{1/2};$$

$$D_0(n) := -(D_1(n) + 36\sqrt{D_2(n)})^{1/3};$$

$$D_1(n) := -94976 + 20736n + 103104n^2 - 10368n^3 + 1296n^5 - 3024n^4 - 108n^6;$$

$$\begin{aligned}
D_2(n) := & 6131712 - 16644096n^2 + 6915840n^4 - 690432n^6 - 3039232n \\
& + 4818944n^3 - 1936384n^5 + 251136n^7 - 30864n^8 - 4320n^9 \\
& + 1800n^{10} - 216n^{11} + 9n^{12}.
\end{aligned}$$

Obviously, the exponent $p_c(n)$ becomes more and more complex along with the order's increasing.

On the other hand, we note the nonlocal Lane-Emden equation:

$$(-\Delta)^s u = |u|^{p-1} u \quad \text{in } \mathbb{R}^n. \quad (1.9)$$

When $0 < s < 1$ and $1 < s < 2$, the complete classification of finite Morse index solution to Eq. (1.9) has been finished by Davila, Dupaigne, Wei in [5] and Fazly, Wei in [12] respectively.

Unfortunately, so far ones do not have a generic approach to deal with the general polyharmonic Lane-Emden case

$$(-\Delta)^m u = |u|^{p-1} u \quad \text{in } \mathbb{R}^n \quad (1.10)$$

or polyharmonic map $(-\Delta)^m u = 0$ in \mathbb{R}^n to obtain the Liouville-type theorem and a complete characterization of all finite Morse index solutions. This remains an interesting open problem.

Finally, we refer the readers to J. Serrin, H. Zou [20], P. Souplet [21] and E. Mitidieri [15] for the Lane-Emden systems and X. F. Wang [22] for the corresponding reaction-diffusion equations.

The paper is organized as follows: In Section 2, we introduce the monotonicity formula (Theorems 2.1 and 2.2). In Subsections 2.1-2.2, we give some preliminary calculations related to the functional of the monotonicity formula. In Section 3, we give the representations on the operators Δ^j , $j = 1, 2, 3$. In Section 4, we establish the differential by part formulas. In Sections 5-6, we calculate the derivatives of the functional of the monotonicity formula and prove Theorem 2.1. The Section 7 is devoted to prove the desired monotonicity formula, i.e., Theorem 2.2. In Section 8, we will show that the homogeneous stable solution must be zero. The Section 9 is on the energy estimates and blow-down analysis, we will prove Theorem 1.1. Finally, the Section 10 will study the finite Morse index solution and prove Theorem 1.2.

2 Monotonicity formula

Let (r, θ) be the spherical coordinates in \mathbb{R}^n , i.e., $r = |x| \in (0, \infty)$ and $\theta = x/|x|$ is a point of the unit sphere S^{n-1} . The symbols Δ_θ and ∇_θ refer respectively to the Laplace-Beltrami operator and the gradient on S^{n-1} . We denote $\partial_r u = \nabla u \cdot \frac{x}{r}$, $r = |x|$. Let

$$B_\lambda := \{y \in \mathbb{R}^n : |y - x| < \lambda\}$$

and

$$u^\lambda(x) := \lambda^{\frac{8}{p-1}} u(\lambda x), \quad \lambda > 0.$$

Define

$$\begin{aligned} E(\lambda, x, u) := & \int_{B_1} \frac{1}{2} |\Delta^2 u^\lambda|^2 - \frac{1}{p+1} |u^\lambda|^{p+1} \\ & + \int_{\partial B_1} \left(\sum_{i,j \geq 0, i+j \leq 7} C_{i,j}^0 \lambda^{i+j} \frac{d^i u^\lambda}{d\lambda^i} \frac{d^j u^\lambda}{d\lambda^j} + \sum_{i,j \geq 0, i+j \leq 5} C_{i,j}^1 \lambda^{i+j} \nabla_\theta \frac{d^i u^\lambda}{d\lambda^i} \nabla_\theta \frac{d^j u^\lambda}{d\lambda^j} \right. \\ & + \sum_{i,j \geq 0, i+j \leq 3} C_{i,j}^2 \lambda^{i+j} \Delta_\theta \frac{d^i u^\lambda}{d\lambda^i} \Delta_\theta \frac{d^j u^\lambda}{d\lambda^j} \\ & \left. + \sum_{i,j \geq 0, i+j \leq 1} C_{i,j}^3 \lambda^{i+j} \nabla_\theta \Delta_\theta \frac{d^i u^\lambda}{d\lambda^i} \nabla_\theta \Delta_\theta \frac{d^j u^\lambda}{d\lambda^j} \right). \end{aligned} \tag{2.1}$$

Theorem 2.1. Suppose that u is a solution of (1.1), then we have the following monotonicity formula

$$\begin{aligned} \frac{d}{d\lambda} E(\lambda, x, u) = & \int_{\partial B_1} \sum_{j=1}^4 (A_j + a_j) \lambda^{2j-1} \left(\frac{d^j u^\lambda}{d\lambda^j} \right)^2 + 2\lambda \int_{\partial B_1} \left(\nabla_\theta \Delta_\theta \frac{du^\lambda}{d\lambda} \right)^2 \\ & + \int_{\partial B_1} \sum_{s=1}^3 (B_s + b_s) \lambda^{2s-1} \left(\nabla_\theta \frac{d^s u^\lambda}{d\lambda^s} \right)^2 + \int_{\partial B_1} \sum_{l=1}^2 (C_l + c_l) \lambda^{2l-1} \left(\Delta_\theta \frac{d^l u^\lambda}{d\lambda^l} \right)^2 \\ & + 2 \int_{\partial B_1} \lambda |\nabla_\theta \frac{du^\lambda}{d\lambda}|^2, \end{aligned} \tag{2.2}$$

where the constants $C_{i,j}^s$ depending on n, p can be determined in our proofs below.

Remark 2.1. In our proof below, the constants $C_{i,j}^s$ (and the constants in the next Corollary) can be determined via n, p and may be negative, but we don't need the exact expressions of them. Furthermore, we don't need the positiveness of the constants in our proof of the Liouville type theorem. Therefore, we just represent them by the generic notations $C_{i,j}^s$.

Theorem 2.2. Suppose that u is a solution of (1.1). If $\frac{n+8}{n-8} < p < p_c(n)$, then there exists a constant $C(n, p) > 0$ such that

$$\begin{aligned} \frac{d}{d\lambda} E(\lambda, x, u) \geq & C(n, p) \int_{\partial B_1} \lambda \left(\frac{du^\lambda}{d\lambda} \right)^2 \\ = & C(n, p) \lambda^{8\frac{p+1}{p-1}-8-n} \int_{\partial B_\lambda} \left(\frac{8}{p-1} + \lambda \partial_r u \right)^2 \end{aligned} \tag{2.3}$$

The proof of Theorem 2.1 will be split in the Sections 2-6. Based on Theorem 2.1, by combining the algebraic and differential analysis in the Section 7, we can get Theorem 2.2.

By slightly modifying the proof of Theorem 2.2, we are able to get the monotonicity formula for the quadharmonic map, i.e.,

$$\Delta^4 u = 0.$$

Indeed, let $p \rightarrow +\infty$ in (2.1) and denote $E_\infty(\lambda, x, u) = \lim_{p \rightarrow \infty} E(\lambda, x, u)$, where the term $\frac{1}{p+1} \lambda^{8\frac{p+1}{p-1}-n} \int_{\partial B_\lambda} |u^\lambda|^{p+1}$ is understood vanished, then we have

Corollary 2.1. *Assume that $9 \leq n \leq 17$, then there exist c_{ij} such that $E_\infty(\lambda, x, u)$, is a nondecreasing function of $\lambda > 0$. Furthermore,*

$$\frac{dE_\infty(\lambda, x, u)}{d\lambda} \geq C(n) \lambda^{-n} \int_{\partial B_\lambda(x_0)} (\lambda \partial_r u)^2,$$

where $C(n) > 0$ is a constant independent of λ .

2.1 The calculation of $\frac{d}{d\lambda} \overline{E}(u, \lambda)$

Suppose that $x = 0$ in the functional $E(\lambda, x, u)$ and denote by B_λ the balls centered at zero with radius λ . Set

$$\overline{E}(u, \lambda) := \lambda^{8\frac{p+1}{p-1}-n} \left(\int_{B_\lambda} \frac{1}{2} |\Delta^2 u|^2 - \frac{1}{p+1} \int_{B_\lambda} |u|^{p+1} \right).$$

Set

$$v := \Delta u, w := \Delta v, z := \Delta w. \quad (2.4)$$

Define

$$\begin{aligned} u^\lambda(x) &:= \lambda^{\frac{8}{p-1}}(\lambda x), v^\lambda(x) := \lambda^{\frac{8}{p-1}+2} v(\lambda x), \\ w^\lambda(x) &:= \lambda^{\frac{8}{p-1}+4} w(\lambda x), z^\lambda(x) := \lambda^{\frac{8}{p-1}+6} z(\lambda x). \end{aligned} \quad (2.5)$$

Therefore,

$$\Delta u^\lambda(x) = v^\lambda(x), \Delta v^\lambda(x) = w^\lambda(x), \Delta w^\lambda(x) = z^\lambda. \quad (2.6)$$

Furthermore, differentiating (2.5) with respect to λ we have

$$\Delta \frac{du^\lambda}{d\lambda} = \frac{dv^\lambda}{d\lambda}, \Delta \frac{dv^\lambda}{d\lambda} = \frac{dw^\lambda}{d\lambda}, \Delta \frac{dw^\lambda}{d\lambda} = \frac{dz^\lambda}{d\lambda}.$$

Note that

$$\overline{E}(u, \lambda) = \overline{E}(u^\lambda, 1) = \int_{B_1} \frac{1}{2} |\Delta^2 u^\lambda|^2 - \frac{1}{p+1} \int_{B_1} |u^\lambda|^{p+1}.$$

Taking derivative of the energy with respect to λ and integrating by part we have

$$\begin{aligned}
\frac{d\bar{E}(u^\lambda, 1)}{d\lambda} &= \int_{B_1} \Delta^2 u^\lambda \Delta^2 \frac{du^\lambda}{d\lambda} - \int_{B_1} |u^\lambda|^{p-1} u^\lambda \frac{du^\lambda}{d\lambda} \\
&= \int_{\partial B_1} \Delta^2 u^\lambda \frac{\partial \Delta \frac{du^\lambda}{d\lambda}}{\partial n} - \int_{B_1} \nabla \Delta^2 u^\lambda \nabla \Delta \frac{du^\lambda}{d\lambda} - \int_{B_1} |u^\lambda|^{p-1} u^\lambda \frac{du^\lambda}{d\lambda} \\
&= \int_{\partial B_1} \Delta^2 u^\lambda \frac{\partial \Delta \frac{du^\lambda}{d\lambda}}{\partial n} - \int_{\partial B_1} \frac{\partial \Delta^2 u^\lambda}{\partial n} \Delta \frac{du^\lambda}{d\lambda} + \int_{B_1} \Delta^3 u^\lambda \Delta \frac{du^\lambda}{d\lambda} \\
&\quad - \int_{B_1} |u^\lambda|^{p-1} u^\lambda \frac{du^\lambda}{d\lambda} \\
&= \int_{\partial B_1} \Delta^2 u^\lambda \frac{\partial \Delta \frac{du^\lambda}{d\lambda}}{\partial n} - \int_{\partial B_1} \frac{\partial \Delta^2 u^\lambda}{\partial n} \Delta \frac{du^\lambda}{d\lambda} + \int_{\partial B_1} \Delta^3 u^\lambda \frac{\partial \frac{du^\lambda}{d\lambda}}{\partial n} \\
&\quad - \int_{B_1} \nabla \Delta u^\lambda \nabla \frac{du^\lambda}{d\lambda} - \int_{B_1} |u^\lambda|^{p-1} u^\lambda \frac{du^\lambda}{d\lambda} \\
&= \int_{\partial B_1} \Delta^2 u^\lambda \frac{\partial \Delta \frac{du^\lambda}{d\lambda}}{\partial n} - \int_{\partial B_1} \frac{\partial \Delta^2 u^\lambda}{\partial n} \Delta \frac{du^\lambda}{d\lambda} + \int_{\partial B_1} \Delta^3 u^\lambda \frac{\partial \frac{du^\lambda}{d\lambda}}{\partial n} \\
&\quad - \int_{\partial B_1} \frac{\partial \Delta^3 u^\lambda}{\partial n} \frac{du^\lambda}{d\lambda} + \int_{B_1} \Delta^4 u^\lambda \frac{du^\lambda}{d\lambda} - \int_{B_1} |u^\lambda|^{p-1} u^\lambda \frac{du^\lambda}{d\lambda}.
\end{aligned}$$

In view of (1.1) and (2.6), we have the following

$$\begin{aligned}
\frac{d\bar{E}(u^\lambda, 1)}{d\lambda} &= \int_{\partial B_1} \Delta^3 u^\lambda \frac{\partial}{\partial r} \frac{du^\lambda}{d\lambda} + \Delta^2 u^\lambda \frac{\partial}{\partial r} \frac{d}{d\lambda} \Delta u^\lambda \\
&\quad - \frac{\partial}{\partial r} \Delta^3 u^\lambda \frac{du^\lambda}{d\lambda} - \frac{\partial}{\partial r} \Delta^2 u^\lambda \frac{d}{d\lambda} \Delta u^\lambda \\
&= \int_{\partial B_1} z^\lambda \frac{\partial}{\partial r} \frac{du^\lambda}{d\lambda} + w^\lambda \frac{\partial}{\partial r} \frac{dv^\lambda}{d\lambda} \\
&\quad - \frac{\partial}{\partial r} z^\lambda \frac{du^\lambda}{d\lambda} - \frac{\partial}{\partial r} w^\lambda \frac{dv^\lambda}{d\lambda}.
\end{aligned}$$

Further, from (2.6), let $k := \frac{8}{p-1}$, we have that

$$\begin{aligned}
\frac{\partial}{\partial r} \frac{du^\lambda}{d\lambda} &= \lambda \frac{d^2 u^\lambda}{d\lambda^2} - (k-1) \frac{du^\lambda}{d\lambda}, \\
\frac{\partial}{\partial r} \frac{dv^\lambda}{d\lambda} &= \lambda \frac{d^2 v^\lambda}{d\lambda^2} - (k+1) \frac{du^\lambda}{d\lambda},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial r} z^\lambda &= \lambda \frac{dz^\lambda}{d\lambda} - (k+6) z^\lambda, \\
\frac{\partial}{\partial r} w^\lambda &= \lambda \frac{dw^\lambda}{d\lambda} - (k+4) w^\lambda.
\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned} \frac{d\bar{E}(u^\lambda, 1)}{d\lambda} &= \underbrace{\int_{\partial B_1} \lambda z^\lambda \frac{d^2 u^\lambda}{d\lambda^2} + 7z^\lambda \frac{du^\lambda}{d\lambda} - \lambda \frac{dz^\lambda}{d\lambda} \frac{du^\lambda}{d\lambda}}_{\text{Term 1}} \\ &\quad + \underbrace{\int_{\partial B_1} \lambda w^\lambda \frac{d^2 v^\lambda}{d\lambda^2} + 3w^\lambda \frac{dv^\lambda}{d\lambda} - \lambda \frac{dw^\lambda}{d\lambda} \frac{dv^\lambda}{d\lambda}}_{\text{Term 2}} \\ &:= \bar{E}_{d_1}(u^\lambda, 1) + \bar{E}_{d_2}(u^\lambda, 1). \end{aligned} \quad (2.7)$$

2.2 The computations of $\frac{\partial^j}{\partial r^j} u^\lambda$ by $\frac{\partial^i}{\partial \lambda^i} u^\lambda, j = 1, 2, 3, 4, 5, 6$

We start our derivation from the following

$$\lambda \frac{du^\lambda}{d\lambda} = \frac{8}{p-1} u^\lambda + r \frac{\partial}{\partial r} u^\lambda. \quad (2.8)$$

Differentiating (2.8) $j (j = 1, 2, 3, 4, 5)$ times with respect to λ we have

$$\lambda \frac{d^2 u^\lambda}{d\lambda^2} + \frac{du^\lambda}{d\lambda} = \frac{8}{p-1} \frac{du^\lambda}{d\lambda} + r \frac{\partial}{\partial r} \frac{du^\lambda}{d\lambda}, \quad (2.9)$$

$$\lambda \frac{d^3 u^\lambda}{d\lambda^3} + 2 \frac{d^2 u^\lambda}{d\lambda^2} = \frac{8}{p-1} \frac{d^2 u^\lambda}{d\lambda^2} + r \frac{\partial}{\partial r} \frac{d^2 u^\lambda}{d\lambda^2}, \quad (2.10)$$

$$\lambda \frac{d^4 u^\lambda}{d\lambda^4} + 3 \frac{d^3 u^\lambda}{d\lambda^3} = \frac{8}{p-1} \frac{d^3 u^\lambda}{d\lambda^3} + r \frac{\partial}{\partial r} \frac{d^3 u^\lambda}{d\lambda^3}, \quad (2.11)$$

$$\lambda \frac{d^5 u^\lambda}{d\lambda^5} + 4 \frac{d^4 u^\lambda}{d\lambda^4} = \frac{8}{p-1} \frac{d^4 u^\lambda}{d\lambda^4} + r \frac{\partial}{\partial r} \frac{d^4 u^\lambda}{d\lambda^4}, \quad (2.12)$$

$$\lambda \frac{d^6 u^\lambda}{d\lambda^6} + 5 \frac{d^5 u^\lambda}{d\lambda^5} = \frac{8}{p-1} \frac{d^5 u^\lambda}{d\lambda^5} + r \frac{\partial}{\partial r} \frac{d^5 u^\lambda}{d\lambda^5}. \quad (2.13)$$

Differentiating (2.8) $j (j = 1, 2, 3, 4, 5)$ times with respect to r we have

$$\lambda \frac{\partial}{\partial r} \frac{du^\lambda}{d\lambda} = \left(\frac{8}{p-1} + 1 \right) \frac{\partial}{\partial r} u^\lambda + r \frac{\partial^2}{\partial r^2} u^\lambda, \quad (2.14)$$

$$\lambda \frac{\partial^2}{\partial r^2} \frac{du^\lambda}{d\lambda} = \left(\frac{8}{p-1} + 2 \right) \frac{\partial^2}{\partial r^2} u^\lambda + r \frac{\partial^3}{\partial r^3} u^\lambda, \quad (2.15)$$

$$\lambda \frac{\partial^3}{\partial r^3} \frac{du^\lambda}{d\lambda} = \left(\frac{8}{p-1} + 3 \right) \frac{\partial^3}{\partial r^3} u^\lambda + r \frac{\partial^4}{\partial r^4} u^\lambda, \quad (2.16)$$

$$\lambda \frac{\partial^4}{\partial r^4} \frac{du^\lambda}{d\lambda} = \left(\frac{8}{p-1} + 4 \right) \frac{\partial^4}{\partial r^4} u^\lambda + r \frac{\partial^5}{\partial r^5} u^\lambda, \quad (2.17)$$

$$\lambda \frac{\partial^5}{\partial r^5} \frac{du^\lambda}{d\lambda} = \left(\frac{8}{p-1} + 5\right) \frac{\partial^5}{\partial r^5} u^\lambda + r \frac{\partial^6}{\partial r^6} u^\lambda. \quad (2.18)$$

From (2.8), on ∂B_1 , we have

$$\frac{\partial u^\lambda}{\partial r} = \lambda \frac{du^\lambda}{d\lambda} - \frac{8}{p-1} u^\lambda.$$

Next from (2.9), on ∂B_1 , we derive that

$$\frac{\partial}{\partial r} \frac{du^\lambda}{d\lambda} = \lambda \frac{d^2 u^\lambda}{d\lambda^2} + \left(1 - \frac{8}{p-1}\right) \frac{du^\lambda}{d\lambda}.$$

From (2.14), combining with the two equations above, on ∂B_1 , we get

$$\begin{aligned} \frac{\partial^2}{\partial r^2} u^\lambda &= \lambda \frac{\partial}{\partial r} \frac{du^\lambda}{d\lambda} - \left(1 + \frac{8}{p-1}\right) \frac{\partial}{\partial r} u^\lambda \\ &= \lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} - \lambda \frac{16}{p-1} \frac{du^\lambda}{d\lambda} + \left(1 + \frac{8}{p-1}\right) \frac{8}{p-1} u^\lambda. \end{aligned} \quad (2.19)$$

Differentiating (2.9) with respect to r , and combining with (2.9) and (2.10), we get that

$$\begin{aligned} \frac{\partial^2}{\partial r^2} \frac{du^\lambda}{d\lambda} &= \lambda \frac{\partial}{\partial r} \frac{d^2 u^\lambda}{d\lambda^2} - \frac{8}{p-1} \frac{\partial}{\partial r} \frac{du^\lambda}{d\lambda} \\ &= \lambda^2 \frac{d^3 u^\lambda}{d\lambda^3} + \left(2 - \frac{16}{p-1}\right) \lambda \frac{d^2 u^\lambda}{d\lambda^2} - \left(1 - \frac{8}{p-1}\right) \frac{8}{p-1} \frac{du^\lambda}{d\lambda}. \end{aligned} \quad (2.20)$$

From (2.15), on ∂B_1 , combining with (2.19) and (2.20), we have

$$\begin{aligned} \frac{\partial^3}{\partial r^3} u^\lambda &= \lambda \frac{\partial^2}{\partial r^2} \frac{du^\lambda}{d\lambda} - \left(2 + \frac{8}{p-1}\right) \frac{\partial^2}{\partial r^2} u^\lambda \\ &= \lambda^3 \frac{d^3 u^\lambda}{d\lambda^3} - \lambda^2 \frac{24}{p-1} \frac{d^2 u^\lambda}{d\lambda^2} + \lambda \left(\frac{24}{p-1} + \frac{192}{(p-1)^2}\right) \frac{du^\lambda}{d\lambda} \\ &\quad - \left(2 + \frac{8}{p-1}\right) \left(1 + \frac{8}{p-1}\right) \frac{8}{p-1} u^\lambda. \end{aligned} \quad (2.21)$$

Now differentiating (2.9) once with respect to r , we get

$$\lambda \frac{\partial^2}{\partial r^2} \frac{d^2 u^\lambda}{d\lambda^2} = \left(\frac{8}{p-1} + 1\right) \frac{\partial^2}{\partial r^2} \frac{du^\lambda}{d\lambda} + r \frac{\partial^3}{\partial r^3} \frac{du^\lambda}{d\lambda},$$

then on ∂B_1 , we have

$$\frac{\partial^3}{\partial r^3} \frac{du^\lambda}{d\lambda} = \lambda \frac{\partial^2}{\partial r^2} \frac{d^2 u^\lambda}{d\lambda^2} - \left(\frac{8}{p-1} + 1\right) \frac{\partial^2}{\partial r^2} \frac{du^\lambda}{d\lambda}. \quad (2.22)$$

Now differentiating (2.10) twice with respect to r , we get

$$\lambda \frac{\partial}{\partial r} \frac{d^3 u^\lambda}{d\lambda^3} = \left(\frac{8}{p-1} - 1\right) \frac{\partial}{\partial r} \frac{d^2 u^\lambda}{d\lambda^2} + r \frac{\partial^2}{\partial r^2} \frac{d^2 u^\lambda}{d\lambda^2},$$

hence on ∂B_1 , combining with (2.10) and (2.11) there holds

$$\begin{aligned} \frac{\partial^2}{\partial r^2} \frac{d^2 u^\lambda}{d\lambda^2} &= \lambda \frac{\partial}{\partial r} \frac{d^3 u^\lambda}{d\lambda^3} + \left(1 - \frac{8}{p-1}\right) \frac{\partial}{\partial r} \frac{d^2 u^\lambda}{d\lambda^2} \\ &= \lambda^2 \frac{d^4 u^\lambda}{d\lambda^4} + \lambda \left(4 - \frac{16}{p-1}\right) \frac{d^3 u^\lambda}{d\lambda^3} + \left(1 - \frac{8}{p-1}\right) \left(2 - \frac{8}{p-1}\right) \frac{d^2 u^\lambda}{d\lambda^2}. \end{aligned} \quad (2.23)$$

Now differentiating (2.9) with respect to r , we have

$$\lambda \frac{\partial}{\partial r} \frac{d^2 u^\lambda}{d\lambda^2} = \frac{8}{p-1} \frac{\partial}{\partial r} \frac{du^\lambda}{d\lambda} + r \frac{\partial^2}{\partial r^2} \frac{du^\lambda}{d\lambda}.$$

This combining with (2.9) and (2.10), on ∂B_1 , we have

$$\begin{aligned} \frac{\partial^2}{\partial r^2} \frac{du^\lambda}{d\lambda} &= \lambda \frac{\partial}{\partial r} \frac{d^2 u^\lambda}{d\lambda^2} - \frac{8}{p-1} \frac{\partial}{\partial r} \frac{du^\lambda}{d\lambda} \\ &= \lambda^2 \frac{d^3 u^\lambda}{d\lambda^3} + \lambda \left(2 - \frac{16}{p-1}\right) \frac{d^2 u^\lambda}{d\lambda^2} - \frac{8}{p-1} \left(1 - \frac{8}{p-1}\right) \frac{du^\lambda}{d\lambda}. \end{aligned} \quad (2.24)$$

Now from (2.22), combining with (2.23) and (2.24), we get

$$\begin{aligned} \frac{\partial^3}{\partial r^3} \frac{du^\lambda}{d\lambda} &= \lambda^3 \frac{d^4 u^\lambda}{d\lambda^4} + \lambda^2 \left(3 - \frac{24}{p-1}\right) \frac{d^3 u^\lambda}{d\lambda^3} - \lambda \left(1 - \frac{8}{p-1}\right) \frac{24}{p-1} \frac{d^2 u^\lambda}{d\lambda^2} \\ &\quad + \left(1 - \frac{8}{p-1}\right) \left(1 + \frac{8}{p-1}\right) \frac{8}{p-1} \frac{du^\lambda}{d\lambda}. \end{aligned} \quad (2.25)$$

From (2.16), on ∂B_1 , combining with (2.25) we obtain that

$$\begin{aligned} \frac{\partial^4}{\partial r^4} u^\lambda &= \lambda \frac{\partial^3}{\partial r^3} \frac{du^\lambda}{d\lambda} - \left(3 + \frac{8}{p-1}\right) \frac{\partial^3}{\partial r^3} u^\lambda \\ &= \lambda^4 \frac{d^4 u^\lambda}{d\lambda^4} - \lambda^3 \frac{32}{p-1} \frac{d^3 u^\lambda}{d\lambda^3} + \lambda^2 \left(2 + \frac{16}{p-1}\right) \frac{24}{p-1} \frac{d^2 u^\lambda}{d\lambda^2} \\ &\quad - \lambda \left(1 + \frac{8}{p-1}\right) \left(1 + \frac{4}{p-1}\right) \frac{64}{p-1} \frac{du^\lambda}{d\lambda} \\ &\quad + \left(3 + \frac{8}{p-1}\right) \left(2 + \frac{8}{p-1}\right) \left(1 + \frac{8}{p-1}\right) \frac{8}{p-1} u^\lambda. \end{aligned}$$

From (2.17), on ∂B_1 , we have

$$\frac{\partial^5}{\partial r^5} u^\lambda = \lambda \frac{\partial^4}{\partial r^4} \frac{du^\lambda}{d\lambda} - \left(\frac{8}{p-1} + 4\right) \frac{\partial^4}{\partial r^4} u^\lambda.$$

Now differentiating (2.9) three and four times with respect to r , we get

$$\lambda \frac{\partial^3}{\partial r^3} \frac{d^2 u^\lambda}{d\lambda^2} = \left(\frac{8}{p-1} + 2\right) \frac{\partial^3}{\partial r^3} \frac{du^\lambda}{d\lambda} + r \frac{\partial^4}{\partial r^4} \frac{du^\lambda}{d\lambda} \quad (2.26)$$

and

$$\lambda \frac{\partial^4}{\partial r^4} \frac{d^2 u^\lambda}{d\lambda^2} = \left(\frac{8}{p-1} + 3\right) \frac{\partial^4}{\partial r^4} \frac{du^\lambda}{d\lambda} + r \frac{\partial^5}{\partial r^5} \frac{du^\lambda}{d\lambda}. \quad (2.27)$$

Next differentiating (2.10) two, three times with respect to r , we have

$$\lambda \frac{\partial^2}{\partial r^2} \frac{d^3 u^\lambda}{d\lambda^3} = \frac{8}{p-1} \frac{\partial^2}{\partial r^2} \frac{d^2 u^\lambda}{d\lambda^2} + r \frac{\partial^3}{\partial r^3} \frac{d^2 u^\lambda}{d\lambda^2} \quad (2.28)$$

and

$$\lambda \frac{\partial^3}{\partial r^3} \frac{d^3 u^\lambda}{d\lambda^3} = \left(\frac{8}{p-1} + 1\right) \frac{\partial^3}{\partial r^3} \frac{d^2 u^\lambda}{d\lambda^2} + r \frac{\partial^4}{\partial r^4} \frac{d^2 u^\lambda}{d\lambda^2}. \quad (2.29)$$

Differentiating (2.11) with respect to r , we have

$$r \frac{\partial^2}{\partial r^2} \frac{d^3 u^\lambda}{d\lambda^3} = \lambda \frac{\partial}{\partial r} \frac{d^4 u^\lambda}{d\lambda^4} + \left(2 - \frac{8}{p-1}\right) \frac{\partial}{\partial r} \frac{d^3 u^\lambda}{d\lambda^3}. \quad (2.30)$$

From (2.11) and (2.12), on ∂B_1 , we have

$$\begin{aligned} \frac{\partial}{\partial r} \frac{d^3 u^\lambda}{d\lambda^3} &= \lambda \frac{d^4 u^\lambda}{d\lambda^4} + \left(3 - \frac{8}{p-1}\right) \frac{d^3 u^\lambda}{d\lambda^3}, \\ \frac{\partial}{\partial r} \frac{d^4 u^\lambda}{d\lambda^4} &= \lambda \frac{d^5 u^\lambda}{d\lambda^5} + \left(4 - \frac{8}{p-1}\right) \frac{d^4 u^\lambda}{d\lambda^4}. \end{aligned}$$

Hence on the boundary ∂B_1 , we have

$$\begin{aligned} \frac{\partial^2}{\partial r^2} \frac{d^3 u^\lambda}{d\lambda^3} &= \lambda^2 \frac{d^5 u^\lambda}{d\lambda^5} + \left(4 - \frac{8}{p-1}\right) \lambda \frac{d^4 u^\lambda}{d\lambda^4} + \left(2 - \frac{8}{p-1}\right) \lambda \frac{d^4 u^\lambda}{d\lambda^4} \\ &\quad + \left(3 - \frac{8}{p-1}\right) \left(2 - \frac{8}{p-1}\right) \frac{d^3 u^\lambda}{d\lambda^3}. \end{aligned} \quad (2.31)$$

Combining with (2.28), on the boundary ∂B_1 , we have

$$\begin{aligned} \frac{\partial^3}{\partial r^3} \frac{d^2 u^\lambda}{d\lambda^2} &= \lambda \frac{\partial^2}{\partial r^2} \frac{d^3 u^\lambda}{d\lambda^3} - \frac{8}{p-1} \frac{\partial^2}{\partial r^2} \frac{d^2 u^\lambda}{d\lambda^2} \\ &= \lambda^3 \frac{d^5 u^\lambda}{d\lambda^5} + \left(6 - \frac{8}{p-1}\right) \lambda^2 \frac{d^4 u^\lambda}{d\lambda^4} + \left(3\left(\frac{8}{p-1}\right)^2 - \frac{72}{p-1} + 6\right) \lambda \frac{d^3 u^\lambda}{d\lambda^3} \\ &\quad - \left(1 - \frac{8}{p-1}\right) \left(2 - \frac{8}{p-1}\right) \frac{8}{p-1} \frac{d^2 u^\lambda}{d\lambda^2}. \end{aligned} \quad (2.32)$$

Hence on the boundary ∂B_1 , we get that

$$\begin{aligned}
\frac{\partial^4}{\partial r^4} \frac{du^\lambda}{d\lambda} &= \lambda \frac{\partial^3}{\partial r^3} \frac{d^2 u^\lambda}{d\lambda^2} - \left(\frac{8}{p-1} + 2\right) \frac{\partial^3}{\partial r^3} \frac{du^\lambda}{d\lambda} \\
&= \lambda^4 \frac{d^5 u^\lambda}{d\lambda^5} + \left(4 - \frac{32}{p-1}\right) \lambda^3 \frac{d^4 u^\lambda}{d\lambda^4} + \left(5\left(\frac{8}{p-1}\right)^2 - 6\frac{8}{p-1}\right) \lambda^2 \frac{d^3 u^\lambda}{d\lambda^3} \\
&\quad - 4\frac{8}{p-1} \left(\frac{8}{p-1} - 1\right) \left(\frac{8}{p-1} + 1\right) \lambda \frac{d^2 u^\lambda}{d\lambda^2} \\
&\quad - \left(\frac{8}{p-1} + 2\right) \left(1 - \frac{8}{p-1}\right) \left(1 + \frac{8}{p-1}\right) \frac{8}{p-1} \frac{du^\lambda}{d\lambda}.
\end{aligned} \tag{2.33}$$

Therefore, we obtain that

$$\begin{aligned}
\frac{\partial^5}{\partial r^5} u_e^\lambda &= \lambda \frac{\partial^4}{\partial r^4} \frac{du^\lambda}{d\lambda} - \left(\frac{8}{p-1} + 4\right) \frac{\partial^4}{\partial r^4} u^\lambda \\
&= \lambda^5 \frac{d^5 u^\lambda}{d\lambda^5} - 5\frac{8}{p-1} \lambda^4 \frac{d^4 u^\lambda}{d\lambda^4} + 10\frac{8}{p-1} \left(\frac{8}{p-1} + 1\right) \lambda^3 \frac{d u^\lambda}{d\lambda^3} \\
&\quad - 10\frac{8}{p-1} \left(\frac{8}{p-1} + 1\right) \left(\frac{8}{p-1} + 2\right) \lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} \\
&\quad + 5\frac{8}{p-1} \left(\frac{8}{p-1} + 1\right) \left(\frac{8}{p-1} + 2\right) \left(\frac{8}{p-1} + 3\right) \lambda \frac{du^\lambda}{d\lambda} \\
&\quad - \frac{8}{p-1} \left(\frac{8}{p-1} + 1\right) \left(\frac{8}{p-1} + 2\right) \left(\frac{8}{p-1} + 3\right) \left(\frac{8}{p-1} + 4\right) u^\lambda.
\end{aligned} \tag{2.34}$$

Finally we compute $\frac{\partial^6}{\partial r^6} u^\lambda$. From (2.18), on the boundary ∂B_1 , we have

$$\frac{\partial^6}{\partial r^6} u^\lambda = \lambda \frac{\partial^5}{\partial r^5} \frac{du^\lambda}{d\lambda} - \left(\frac{8}{p-1} + 5\right) \frac{\partial^5}{\partial r^5} u^\lambda.$$

Differentiating (2.9) four times with respect to r , on the boundary ∂B_1 , we have

$$\frac{\partial^5}{\partial r^5} \frac{du^\lambda}{d\lambda} = \lambda \frac{\partial^4}{\partial r^4} \frac{d^3 u^\lambda}{d\lambda^2} - \left(\frac{8}{p-1} + 3\right) \frac{\partial^4}{\partial r^4} \frac{du^\lambda}{d\lambda}. \tag{2.35}$$

Differentiating (2.17) with respect to r , on the boundary ∂B_1 , we have

$$\frac{\partial^2}{\partial r^2} \frac{d^4 u^\lambda}{d\lambda^4} = \lambda \frac{\partial}{\partial r} \frac{d^5 u^\lambda}{d\lambda^5} + \left(3 - \frac{8}{p-1}\right) \frac{\partial}{\partial r} \frac{d^4 u^\lambda}{d\lambda^4}. \tag{2.36}$$

In a similar way, differentiating (2.10) with respect to r three times and differentiating (2.11) with respect to r two times, on the boundary ∂B_1 , we have

$$\frac{\partial^4}{\partial r^4} \frac{du^\lambda}{d\lambda^2} = \lambda \frac{\partial^3}{\partial r^3} \frac{d^3 u^\lambda}{d\lambda^3} - \left(\frac{8}{p-1} + 1\right) \frac{\partial^3}{\partial r^3} \frac{d^2 u^\lambda}{d\lambda^2} \tag{2.37}$$

and

$$\frac{\partial^3}{\partial r^3} \frac{d^3 u^\lambda}{d\lambda^3} = \lambda \frac{\partial^2}{\partial r^2} \frac{d^4 u^\lambda}{d\lambda^4} + \left(1 - \frac{8}{p-1}\right) \frac{\partial^2}{\partial r^2} \frac{d^3 u^\lambda}{d\lambda^3}. \tag{2.38}$$

From (2.36), combining with (2.12) and (2.13), on the boundary ∂B_1 , we get that

$$\frac{\partial^2}{\partial r^2} \frac{d^4 u^\lambda}{d\lambda^4} = \lambda^2 \frac{d^6 u^\lambda}{d\lambda^6} + (8 - 2 \frac{8}{p-1}) \lambda \frac{d^5 u^\lambda}{d\lambda^5} + (\frac{8}{p-1} - 4)(\frac{8}{p-1} - 3) \frac{d^4 u^\lambda}{d\lambda^4}.$$

Therefore, combining with (2.38), we get

$$\begin{aligned} \frac{\partial^3}{\partial r^3} \frac{d^3 u^\lambda}{d\lambda^3} &= \lambda^3 \frac{d^6 u^\lambda}{d\lambda^6} + (9 - 3 \frac{8}{p-1}) \lambda^2 \frac{d^5 u^\lambda}{d\lambda^5} + 3(\frac{8}{p-1} - 2)(\frac{8}{p-1} - 3) \lambda \frac{d^4 u^\lambda}{d\lambda^4} \\ &\quad + (3 - \frac{8}{p-1})(2 - \frac{8}{p-1})(1 - \frac{8}{p-1}) \frac{d^3 u^\lambda}{d\lambda^3}. \end{aligned}$$

This above combining with (2.37) yields

$$\begin{aligned} \frac{\partial^4}{\partial r^4} \frac{d^2 u^\lambda}{d\lambda^2} &= \lambda \frac{\partial^3}{\partial r^3} \frac{d^3 u^\lambda}{d\lambda^3} - (\frac{8}{p-1} + 1) \frac{\partial^3}{\partial r^3} \frac{d^2 u^\lambda}{d\lambda^2} \\ &= \lambda^4 \frac{d^6 u^\lambda}{d\lambda^6} + (8 - 4 \frac{8}{p-1}) \lambda^3 \frac{d^5 u^\lambda}{d\lambda^5} + 6(\frac{8}{p-1} - 2)(\frac{8}{p-1} - 1) \lambda^2 \frac{d^4 u^\lambda}{d\lambda^4} \\ &\quad - 4 \frac{8}{p-1} (\frac{8}{p-1} - 1) (\frac{8}{p-1} - 2) \lambda \frac{d^3 u^\lambda}{d\lambda^3} \\ &\quad + \frac{8}{p-1} (\frac{8}{p-1} - 1) (\frac{8}{p-1} - 2) (\frac{8}{p-1} + 1) \frac{d^2 u^\lambda}{d\lambda^2}. \end{aligned}$$

Combining with (2.35), we get that

$$\begin{aligned} \frac{\partial^5}{\partial r^5} \frac{du^\lambda}{d\lambda} &= \lambda \frac{\partial^4}{\partial r^4} \frac{d^2 u^\lambda}{d\lambda^2} - (\frac{8}{p-1} + 3) \frac{\partial^4}{\partial r^4} \frac{du^\lambda}{d\lambda} \\ &= \lambda^5 \frac{d^6 u^\lambda}{d\lambda^6} + (5 - 5 \frac{8}{p-1}) \lambda^4 \frac{d^5 u^\lambda}{d\lambda^5} + 10 \frac{8}{p-1} (\frac{8}{p-1} - 1) \lambda^3 \frac{d^4 u^\lambda}{d\lambda^4} \\ &\quad - 10 \frac{8}{p-1} (\frac{8}{p-1} - 1) (\frac{8}{p-1} + 1) \lambda^2 \frac{d^3 u^\lambda}{d\lambda^3} \\ &\quad + 5 \frac{8}{p-1} (\frac{8}{p-1} - 1) (\frac{8}{p-1} + 1) (\frac{8}{p-1} + 2) \lambda \frac{d^2 u^\lambda}{d\lambda^2} \\ &\quad + (\frac{8}{p-1} + 3) (\frac{8}{p-1} + 2) (1 - \frac{8}{p-1}) (\frac{8}{p-1} + 1) \frac{8}{p-1} \frac{du^\lambda}{d\lambda}. \end{aligned}$$

Therefore, from (2.18), we obtain that

$$\begin{aligned}
\frac{\partial^6}{\partial r^6} u^\lambda &= \lambda \frac{\partial^5}{\partial r^5} \frac{du^\lambda}{d\lambda} - \left(\frac{8}{p-1} + 5 \right) \frac{\partial^5}{\partial r^5} u^\lambda \\
&= \lambda^6 \frac{d^6 u^\lambda}{d\lambda^6} - 6 \frac{8}{p-1} \lambda^5 \frac{d^5 u^\lambda}{d\lambda^5} + 15 \frac{8}{p-1} \left(\frac{8}{p-1} + 1 \right) \lambda^4 \frac{d^4 u^\lambda}{d\lambda^4} \\
&\quad - 20 \frac{8}{p-1} \left(\frac{8}{p-1} + 1 \right) \left(\frac{8}{p-1} + 2 \right) \lambda^3 \frac{d^3 u^\lambda}{d\lambda^3} \\
&\quad + 15 \frac{8}{p-1} \left(\frac{8}{p-1} + 1 \right) \left(\frac{8}{p-1} + 2 \right) \left(\frac{8}{p-1} + 3 \right) \lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} \\
&\quad - 6 \frac{8}{p-1} \left(\frac{8}{p-1} + 1 \right) \left(\frac{8}{p-1} + 2 \right) \left(\frac{8}{p-1} + 3 \right) \left(\frac{8}{p-1} + 4 \right) \lambda \frac{du^\lambda}{d\lambda} \\
&\quad + \frac{8}{p-1} \left(\frac{8}{p-1} + 1 \right) \left(\frac{8}{p-1} + 2 \right) \left(\frac{8}{p-1} + 3 \right) \left(\frac{8}{p-1} + 4 \right) \left(\frac{8}{p-1} + 5 \right) u^\lambda.
\end{aligned}$$

In a summary, let $k := \frac{8}{p-1}$, we have the following

$$\begin{aligned}
\frac{\partial u^\lambda}{\partial r} &= \lambda \frac{du^\lambda}{d\lambda} - ku^\lambda, \\
\frac{\partial^2 u^\lambda}{\partial r^2} &= \lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} - 2k\lambda \frac{du^\lambda}{d\lambda} + k(k+1)u^\lambda,
\end{aligned} \tag{2.39}$$

$$\begin{aligned}
\frac{\partial^3 u^\lambda}{\partial r^3} &= \lambda^3 \frac{d^3 u^\lambda}{d\lambda^3} - 3k\lambda^2 \frac{d^3 u^\lambda}{d\lambda^2} + 3k(k+1)\lambda \frac{du^\lambda}{d\lambda} - k(k+1)(k+2)u^\lambda, \\
\frac{\partial^4 u^\lambda}{\partial r^4} &= \lambda^4 \frac{d^4 u^\lambda}{d\lambda^4} - 4k\lambda^3 \frac{d^3 u^\lambda}{d\lambda^3} + 6k(k+1)\lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} \\
&\quad - 4k(k+1)(k+2)\lambda \frac{du^\lambda}{d\lambda} + k(k+1)(k+2)(k+3)u^\lambda,
\end{aligned} \tag{2.40}$$

$$\begin{aligned}
\frac{\partial^5 u^\lambda}{\partial r^5} &= \lambda^5 \frac{d^5 u^\lambda}{d\lambda^5} - 5k\lambda^4 \frac{d^4 u^\lambda}{d\lambda^4} + 10k(k+1)\lambda^3 \frac{d^3 u^\lambda}{d\lambda^3} - 10k(k+1)(k+2)\lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} \\
&\quad + 5k(k+1)(k+2)(k+3)\lambda \frac{du^\lambda}{d\lambda} \\
&\quad - k(k+1)(k+2)(k+3)(k+4)u^\lambda,
\end{aligned} \tag{2.41}$$

$$\begin{aligned}
\frac{\partial^6 u^\lambda}{\partial r^6} &= \lambda^6 \frac{d^6 u^\lambda}{d\lambda^6} - 6k\lambda^5 \frac{d^5 u^\lambda}{d\lambda^5} + 15k(k+1)\lambda^4 \frac{d^4 u^\lambda}{d\lambda^4} - 20k(k+1)(k+2)\lambda^3 \frac{d^3 u^\lambda}{d\lambda^3} \\
&\quad + 15k(k+1)(k+2)(k+3)\lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} \\
&\quad - 6k(k+1)(k+2)(k+3)(k+4)\lambda \frac{du^\lambda}{d\lambda} \\
&\quad + k(k+1)(k+2)(k+3)(k+4)(k+5)u^\lambda.
\end{aligned} \tag{2.42}$$

3 On the operators $\Delta^j, j = 1, 2, 3$

Let us recall that

$$\Delta u = (\partial_{rr} + \frac{n-1}{r}\partial_r)u + \Delta_\theta(r^{-2}u),$$

$$\begin{aligned} \Delta^2 u &= (\partial_{rr} + \frac{n-1}{r}\partial_r)^2 + \Delta_\theta\left((\partial_{rr} + \frac{n-1}{r}\partial_r)(r^{-2}u) + r^{-2}(\partial_{rr} + \frac{n-1}{r}\partial_r)u\right) \\ &\quad + \Delta_\theta^2(r^{-4}u) \end{aligned}$$

and

$$\begin{aligned} \Delta^3 u &= (\partial_{rr} + \frac{n-1}{r}\partial_r)^3 + \Delta_\theta\left((\partial_{rr} + \frac{n-1}{r}\partial_r)(r^{-2}(\partial_{rr} + \frac{n-1}{r}\partial_r)u)\right. \\ &\quad \left.+ (\partial_{rr} + \frac{n-1}{r}\partial_r)^2(r^{-2}u) + r^{-2}(\partial_{rr} + \frac{n-1}{r}\partial_r)^2\right) + \Delta_\theta^2\left((\partial_{rr} + \frac{n-1}{r}\partial_r)(r^{-4}u) + r^{-4}(\partial_{rr} + \frac{n-1}{r}\partial_r)u + r^{-2}(\partial_{rr} + \frac{n-1}{r}\partial_r)(r^{-2}u)\right) \\ &\quad + \Delta_\theta^3(r^{-6}u) \\ &:= F_0(u) + \Delta_\theta F_1(u) + \Delta_\theta^2 F_2(u) + \Delta_\theta^3 F_3(u). \end{aligned} \tag{3.1}$$

By a direct calculation, if we denote that $a = n - 1$, on the boundary ∂B_1 , then we have

$$\begin{aligned} F_0(u) &:= (\partial_{rr} + \frac{n-1}{r}\partial_r)^3 u = \left(\partial_{r^6} + 3a\partial_{r^5} + 3a(a-2)\partial_{r^4} + a(a-2)(a-7)\partial_{r^3} - 3a(a-2)(a-4)\partial_{r^2} + 3a(a-2)(a-4)\partial_r\right)u \\ &:= \sum_{j=1}^6 a_j \partial_{r^j} u, \end{aligned}$$

$$\begin{aligned} F_1(u) &:= \left(3\partial_{r^4} + (6a-12)\partial_{r^3} + (3a^2-24a+42)\partial_{r^2} + (60a-9a^2-96)\partial_r + (8a^2-64a+120)\right)u \\ &:= \sum_{j=0}^4 b_j \partial_{r^j} u, \end{aligned}$$

and

$$F_3(u) := \left(3\partial_{r^2} + (3a-12)\partial_r + 26 - 6a\right)u = \sum_{j=0}^2 v_j \partial_{r^j},$$

here $\partial_{r^3} := \partial_{rrr}$ and so on.

Let us recall that (2.39), (2.40), (2.41) and (2.42) in the end of the previous section, we can turn the differential with respect to r into with respect to λ . Thus we have the

following

$$F_0(u) = \sum_{j=0}^6 k_j \lambda^j \frac{d^j u^\lambda}{d\lambda^j}, F_1(u) = \sum_{j=0}^4 (-t_j) \lambda^j \frac{d^j u^\lambda}{d\lambda^j}, F_2(u) = \sum_{j=0}^2 e_j \lambda^j \frac{d^j u^\lambda}{d\lambda^j}. \quad (3.2)$$

For the simplicity, we let

$$\begin{aligned} a_6 &= 1, a_5 = 3a, a_4 = 3a(a-2), a_3 = a(a-2)(a-7), a_2 = -3a(a-2)(a-4), \\ a_1 &= 3a(a-2)(a-4). \end{aligned}$$

Then we have k_j determined by

$$\begin{aligned} k_6 &= 1, k_5 = -6ka_6 + a_5, k_4 = 15k(k+1)a_6 - 5ka_5 + a_4, \\ k_3 &= -20k(k+1)(k+2)a_6 + 10k(k+1)a_5 - 4ka_4 + a_3, \\ k_2 &= 15k(k+1)(k+2)(k+3)a_6 - 10k(k+1)(k+2)a_5 + 6k(k+1)a_4 - 3ka_3 + a_2, \\ k_1 &= -6k(k+1)(k+2)(k+3)(k+4)a_6 + 5k(k+1)(k+2)(k+3)a_5 \\ &\quad - 4k(k+1)(k+2)a_4 + 3k(k+1)a_3 - 2ka_2 + a_1, \\ k_0 &= k(k+1)(k+2)(k+3)(k+4)(k+5)a_6 - k(k+1)(k+2)(k+3)(k+4)a_5 \\ &\quad + k(k+1)(k+2)(k+3)a_4 - k(k+1)(k+2)a_3 + k(k+1)a_2 - ka_1; \end{aligned} \quad (3.3)$$

and t_j are determined by

$$\begin{aligned} t_4 &= -b_4, t_3 = 4b_4k - b_3, t_2 = -6b_4k(k+1) + 3b_3k - b_2, \\ t_1 &= 4b_4k(k+1)(k+2) - 3b_3k(k+1) + 2b_2k - b_1, \\ t_0 &= -b_4k(k+1)(k+2)(k+3) + b_3k(k+1)(k+2) - b_2k(k+1) + b_1k - b_0 \end{aligned} \quad (3.4)$$

and e_j are determined by

$$e_2 = 3, e_1 = -6k + 3a - 12, e_0 = 3k(k+1) - (3a-12)k + 26 - 6a. \quad (3.5)$$

4 Differentiating by part formulas

In all this sections, we denote that $f^{(j)} = \frac{d^j f}{d\lambda^j}$ and respectively.

Lemma 4.1. *We have the following type-1 (i.e., $\lambda^j f^{(j)} f^{(1)}$) differentiating by part formulas:*

$$\begin{aligned} ff^{(1)} &= \frac{d}{d\lambda} \left(\frac{1}{2} f^2 \right), \\ \lambda^2 f^{(2)} f^{(1)} &= -\lambda(f^{(1)})^2 + \frac{d}{d\lambda} \left(\frac{1}{2} \lambda^2 f^{(1)} f^{(1)} \right), \\ \lambda^3 f^{(3)} f^{(1)} &= 3\lambda(f^{(1)})^2 - \lambda^3(f^{(2)})^2 + \frac{d}{d\lambda} \left(\lambda^3 f^{(2)} f^{(1)} \right), \end{aligned}$$

$$\begin{aligned}\lambda^4 f^{(4)} f^{(1)} = & -12\lambda(f^{(1)})^2 + 6\lambda^3(f^{(2)})^2 + \frac{d}{d\lambda}(\lambda^4 f^{(3)} f^{(1)} - \frac{1}{2}\lambda^4 f^{(2)} f^{(2)} \\ & - 4\lambda^3 f^{(2)} f^{(1)} + 6\lambda^2 f^{(1)} f^{(1)}),\end{aligned}$$

$$\begin{aligned}\lambda^5 f^{(5)} f^{(1)} = & 60\lambda(f^{(1)})^2 - 40\lambda^3(f^{(2)})^2 + \lambda^5(f^{(3)})^2 + \frac{d}{d\lambda}(\lambda^5 f^{(4)} f^{(1)} - \lambda^5 f^{(3)} f^{(2)} \\ & - 5\lambda^4 f^{(3)} f^{(1)} + 5\lambda^4 f^{(2)} f^{(2)} + 20\lambda^3 f^{(2)} f^{(1)} - 30\lambda^2 f^{(1)} f^{(1)}),\end{aligned}$$

$$\begin{aligned}\lambda^6 f^{(6)} f^{(1)} = & -360\lambda(f^{(1)})^2 + 300\lambda^3(f^{(2)})^2 - 14\lambda^5(f^{(3)})^2 + \frac{d}{d\lambda}(\lambda^6 f^{(5)} f^{(1)} \\ & - 6\lambda^5 f^{(4)} f^{(1)} + 12\lambda^5 f^{(3)} f^{(2)} + 30\lambda^4 f^{(3)} f^{(1)} - 45\lambda^4 f^{(2)} f^{(2)} \\ & - 120\lambda^3 f^{(2)} f^{(1)} + 180\lambda^2 f^{(1)} f^{(1)} - \lambda^6 f^{(4)} f^{(2)} + \frac{1}{2}\lambda^6 f^{(3)} f^{(3)}),\end{aligned}$$

$$\begin{aligned}\lambda^7 f^{(7)} f^{(1)} = & 2520\lambda(f^{(1)})^2 - 2520\lambda^3(f^{(2)})^2 + 189\lambda^5(f^{(3)})^2 - \lambda^7(f^{(4)})^2 \\ & + \frac{d}{d\lambda}(\lambda^7 f^{(6)} f^{(1)} - 7\lambda^6 f^{(5)} f^{(1)} + 42\lambda^5 f^{(4)} f^{(1)} - 84\lambda^5 f^{(3)} f^{(2)} \\ & - 210\lambda^4 f^{(3)} f^{(1)} + 315\lambda^4 f^{(2)} f^{(2)} + 840\lambda^3 f^{(2)} f^{(1)} \\ & - 1260\lambda^2 f^{(1)} f^{(1)} + 7\lambda^6 f^{(4)} f^{(2)} - 7\lambda^6 f^{(3)} f^{(3)} \\ & - \lambda^7 f^{(5)} f^{(2)} + \lambda^7 f^{(4)} f^{(3)}).\end{aligned}$$

Proof. These formulas above can be checked directly. \square

Remark 4.1. We see that we can decompose the term $\lambda^j f^{(j)} f^{(1)}$ into two parts, the quadratic form and derivative term, i.e.,

$$\lambda^j f^{(j)} f^{(1)} = \sum_{s \leq \frac{j+1}{2}, s \in N} b_{j,s} \lambda^{2s-1} (f^{(s)})^2 + \frac{d}{d\lambda} \left(\sum_{i,l} c_{i,l} \lambda^{i+l} f^{(i)} f^{(l)} \right).$$

Lemma 4.2. We have the following type-2 (i.e., $\lambda^{j+1} f^{(j)} f^{(2)}$) differential by part for-

mulas:

$$\begin{aligned}
\lambda f f^{(2)} &= -\lambda(f^{(1)})^2 + \frac{d}{d\lambda}(\lambda f f^{(1)} - \frac{1}{2}f^2), \\
\lambda^2 f^{(1)} f^{(2)} &= -\lambda(f^{(1)})^2 + \frac{d}{d\lambda}(\frac{1}{2}\lambda^2 f^{(1)} f^{(1)}), \\
\lambda^4 f^{(3)} f^{(2)} &= -2\lambda^3(f^{(2)})^2 + \frac{d}{d\lambda}(\frac{1}{2}\lambda^4 f^{(2)} f^{(2)}), \\
\lambda^5 f^{(4)} f^{(2)} &= 10\lambda^3(f^{(2)})^2 - \lambda^5(f^{(3)})^2 + \frac{d}{d\lambda}(\lambda^5 f^{(3)} f^{(2)} - \frac{5}{2}\lambda^4 f^{(2)} f^{(2)}), \\
\lambda^6 f^{(5)} f^{(2)} &= -60\lambda^3(f^{(2)})^2 + 9\lambda^5(f^{(3)})^2 + \frac{d}{d\lambda}(\lambda^6 f^{(4)} f^{(2)} - 6\lambda^5 f^{(3)} f^{(2)} \\
&\quad + 15\lambda^4 f^{(2)} f^{(2)} - \frac{1}{2}\lambda^6 f^{(3)} f^{(3)}), \\
\lambda^7 f^{(6)} f^{(2)} &= 420\lambda^3(f^{(2)})^2 - 84\lambda^5(f^{(3)})^2 + \lambda^7(f^{(4)})^2 + \frac{d}{d\lambda}(\lambda^7 f^{(5)} f^{(2)} \\
&\quad - \lambda^7 f^{(4)} f^{(3)} + \frac{7}{2}\lambda^6 f^{(3)} f^{(3)} - 7\lambda^6 f^{(4)} f^{(2)} + 42\lambda^5 f^{(3)} f^{(2)} \\
&\quad - 105\lambda^4 f^{(2)} f^{(2)} + \frac{7}{2}\lambda^6 f^{(3)} f^{(3)}).
\end{aligned}$$

Proof. These formulas can be verified directly. \square

Remark 4.2. We note that we can decompose the term $\lambda^{j+1} f^{(j)} f^{(2)}$ into two parts, the quadratic form and derivative term, i.e.,

$$\lambda^{j+1} f^{(j)} f^{(2)} = \sum_{s \leq \frac{j+2}{2}, s \in N} a_{j,s} \lambda^{2s-1} (f^{(s)})^2 + \frac{d}{d\lambda} \left(\sum_{i,l} c_{i,l} \lambda^{i+l} f^{(i)} f^{(l)} \right).$$

5 The term $\overline{E}_{d_2}(u^\lambda, 1)$

Recall the definition in (2.7)

$$\overline{E}_{d_2}(u^\lambda, 1) = \int_{\partial B_1} (\lambda w^\lambda \frac{d^2 v^\lambda}{d\lambda^2} + 3w^\lambda \frac{dv^\lambda}{d\lambda} - \lambda \frac{dw^\lambda}{d\lambda} \frac{dv^\lambda}{d\lambda}). \quad (5.1)$$

Since $w^\lambda = \Delta v^\lambda = \partial_{rr} v^\lambda + \frac{n-1}{r} \partial_r v^\lambda + r^{-2} \operatorname{div}_\theta(\nabla_\theta v^\lambda)$, in view of (2.39), on the boundary ∂B_1 , we have

$$\begin{aligned}
w^\lambda &= \lambda^2 \frac{d^2 v^\lambda}{d\lambda^2} + \lambda \frac{dv^\lambda}{d\lambda} (n-1 - \frac{16}{p-1}) \\
&\quad + u^\lambda \frac{8}{p-1} (2 + \frac{8}{p-1} - n) + \operatorname{div}_\theta(\nabla_\theta v^\lambda) \\
&:= \lambda^2 \frac{d^2 v^\lambda}{d\lambda^2} + \alpha \lambda \frac{dv^\lambda}{d\lambda} + \beta v^\lambda + \operatorname{div}_\theta(\nabla_\theta v^\lambda).
\end{aligned}$$

Integrate by part with suitable times we have the following

$$\begin{aligned}
\overline{E}_{d_2}(u^\lambda, 1) &= \int_{\partial B_1} [2\lambda^3(\frac{d^2v^\lambda}{d\lambda^2})^2 + (2\alpha - 2\beta - 4)\lambda(\frac{dv^\lambda}{d\lambda})^2] \\
&\quad \frac{d}{d\lambda} \int_{\partial B_1} \left[\frac{\beta}{2} \frac{d}{d\lambda} (\lambda(v^\lambda)^2) - \frac{1}{2} \lambda^3 \frac{d}{d\lambda} (\frac{dv^\lambda}{d\lambda})^2 + (\frac{\beta}{2} + 2)(v^\lambda)^2 \right. \\
&\quad \left. - \frac{1}{2} \frac{d}{d\lambda} (\lambda|\nabla_\theta v^\lambda|^2) - \frac{1}{2} |\nabla_\theta v^\lambda|^2 \right] \\
&\quad + 2 \int_{\partial B_1} \lambda |\nabla_\theta \frac{dv^\lambda}{d\lambda}|^2.
\end{aligned} \tag{5.2}$$

Let us further investigate the inner structure of $\overline{E}_{d_2}(u^\lambda, 1)$, we can obtain more and crucial information for our construction of the monotonicity formula under the desired condition. Since $v^\lambda = \Delta u^\lambda$, on the boundary ∂B_1 , by a direct calculation we have the following

$$\begin{aligned}
\frac{dv^\lambda}{d\lambda} &= \lambda^2 \frac{d^3 u^\lambda}{d\lambda^3} + (\alpha + 2)\lambda \frac{d^2 u^\lambda}{d\lambda^2} + (\alpha + \beta) \frac{du^\lambda}{d\lambda} + \Delta_\theta \frac{du^\lambda}{d\lambda}, \\
\frac{d^2 v^\lambda}{d\lambda^2} &= \lambda^2 \frac{d^4 u^\lambda}{d\lambda^4} + (\alpha + 4)\lambda \frac{d^3 u^\lambda}{d\lambda^3} + (2\alpha + \beta + 2) \frac{d^2 u^\lambda}{d\lambda^2} + \Delta_\theta \frac{d^2 u^\lambda}{d\lambda^2},
\end{aligned}$$

therefore differential by part, we can get that

$$\begin{aligned}
\int_{\partial B_1} (2\alpha - 2\beta - 4)\lambda(\frac{dv^\lambda}{d\lambda})^2 + 2\lambda^3(\frac{d^2v^\lambda}{d\lambda^2})^2 &= \int_{\partial B_1} (2\alpha^2 - 2\alpha - 6\beta - 28)\lambda^5(\frac{d^3u^\lambda}{d\lambda^3})^2 \\
&\quad + (2\alpha^3 + (-16 - 2\beta)\alpha^2 + 16\alpha + 6\beta^2 + 32\beta + 40)\lambda^3(\frac{d^2u^\lambda}{d\lambda^2})^2 + 2\lambda^7(\frac{d^4u^\lambda}{d\lambda^4})^2 \\
&\quad + (-2\alpha^3 + (2\beta + 8)\alpha^2 + (2\beta^2 - 8)\alpha - 2\beta^3 - 8\beta^2 - 8\beta)\lambda(\frac{du^\lambda}{d\lambda})^2 \\
&\quad + \frac{d}{d\lambda} \left(\sum_{i,j} c_{i,j} \lambda^{i+j} \frac{d^i u^\lambda}{d\lambda^i} \frac{d^j u^\lambda}{d\lambda^j} \right) + (2\alpha - 2\beta - 4)\lambda(\Delta_\theta \frac{du^\lambda}{d\lambda})^2 \\
&\quad + 2\lambda^5(\nabla_\theta \frac{d^3u^\lambda}{d\lambda^3})^2 + (14 - 2\beta)\lambda^3(\nabla_\theta \frac{d^2u^\lambda}{d\lambda^2})^2 + (-2 - 2\beta)\lambda(\nabla_\theta \frac{du^\lambda}{d\lambda})^2 \\
&\quad + \frac{d}{d\lambda} \left(\sum_{i,j} e_{i,j} \lambda^{i+j} \nabla_\theta \frac{d^i u^\lambda}{d\lambda^i} \nabla_\theta \frac{d^j u^\lambda}{d\lambda^j} \right) + 2\lambda^3(\Delta_\theta \frac{d^2u^\lambda}{d\lambda^2})^2 \\
&:= \int_{\partial B_1} \sum_{j=1}^4 a_j \lambda^{2j-1} (\frac{d^j u^\lambda}{d\lambda^j})^2 + \left(\sum_{s=1}^3 b_s \lambda^{2s-1} (\nabla_\theta \frac{d^s u^\lambda}{d\lambda^s})^2 \right) + \sum_{l=1}^2 c_l \lambda^{2l-1} (\Delta_\theta \frac{d^l u^\lambda}{d\lambda^l})^2 \\
&\quad + \frac{d}{d\lambda} \left(\sum_{i,j} c_{i,j} \lambda^{i+j} \frac{d^i u^\lambda}{d\lambda^i} \frac{d^j u^\lambda}{d\lambda^j} \right) \\
&\quad + \frac{d}{d\lambda} \left(\sum_{i,j} e_{i,j} \lambda^{i+j} \nabla_\theta \frac{d^i u^\lambda}{d\lambda^i} \nabla_\theta \frac{d^j u^\lambda}{d\lambda^j} \right),
\end{aligned} \tag{5.3}$$

where $c_{i,j}$ and $e_{i,j}$ may have exact expressions of α, β , but we do not intend to give them here since the key term is the quadratic form.

Remark 5.1. From (5.2), the first term of the above integral is positive. Recall that $v^\lambda = \Delta u^\lambda$ now we have

$$\begin{aligned} \bar{E}_{d_2}(u^\lambda, 1) &\geq \frac{d}{d\lambda} \int_{\partial B_1} \left[\frac{\beta}{2} \frac{d}{d\lambda} (\lambda(\Delta u^\lambda)^2)^2 - \frac{1}{2} \lambda^3 \frac{d}{d\lambda} \left(\frac{d\Delta u^\lambda}{d\lambda} \right)^2 + \frac{\beta}{2} (\Delta_b u^\lambda)^2 \right. \\ &\quad \left. - \frac{1}{2} \frac{d}{d\lambda} (\lambda |\nabla_\theta \Delta u^\lambda|^2) - \frac{1}{2} |\nabla_\theta \Delta u^\lambda|^2 \right]. \end{aligned} \quad (5.4)$$

If we use this estimate alone, we can not construct the desired monotonicity formula for all n with $\frac{n+8}{n-8} < p < p_c(n)$. More precisely, when $n \in [15, 27]$, it seems that, under the condition $\frac{n+8}{n-8} < p < p_c(n)$, the desired monotonicity formula can not hold.

6 The term $\bar{E}_{d_1}(u^\lambda, 1)$

Recall that (2.7), we have

$$\bar{E}_{d_1}(u^\lambda, 1) = \int_{\partial B_1} \left(\lambda z^\lambda \frac{d^2 u^\lambda}{d\lambda^2} + 7z^\lambda \frac{du^\lambda}{d\lambda} - \lambda \frac{dz^\lambda}{d\lambda} \frac{du^\lambda}{d\lambda} \right).$$

6.1 The integral corresponding to the operator F_0

First, we consider the operator F_0 as defined in (3.2). We split the integral into two parts, we denote that

$$\begin{aligned} F_{01} &:= \int_{\partial B_1} \left(\lambda F_0(u^\lambda) \frac{d^2 u^\lambda}{d\lambda^2} \right), \\ F_{02} &:= \int_{\partial B_1} \left(7F_0(u^\lambda) - \lambda \frac{dF_0(u^\lambda)}{d\lambda} \right) \frac{du^\lambda}{d\lambda}. \end{aligned}$$

Recall that (3.2), if we denote that $f = u^\lambda, f' = \frac{du^\lambda}{d\lambda}$, we have

$$F_0(u^\lambda) = k_6 \lambda^6 \frac{d^6 u^\lambda}{d\lambda^6} + k_5 \frac{d^5 u^\lambda}{d\lambda^5} + k_4 \lambda^4 \frac{d^4 u^\lambda}{d\lambda^4} + k_3 \frac{d^3 u^\lambda}{d\lambda^3} + k_2 \frac{d^2 u^\lambda}{d\lambda^2} + k_1 \lambda \frac{du^\lambda}{d\lambda} + k_0 u \lambda.$$

Hence,

$$\begin{aligned} 7F_0(u^\lambda) - \lambda \frac{dF_0(u^\lambda)}{d\lambda} &= -k_6 \lambda^7 \frac{d^7 u^\lambda}{d\lambda^7} + (k_6 - k_5) \lambda^6 \frac{d^6 u^\lambda}{d\lambda^6} + (2k_5 - k_4) \lambda^5 \frac{d^5 u^\lambda}{d\lambda^5} \\ &\quad + (3k_4 - k_3) \lambda^4 \frac{d^4 u^\lambda}{d\lambda^4} + (4k_3 - k_2) \lambda^3 \frac{d^3 u^\lambda}{d\lambda^3} \\ &\quad + (5k_2 - k_1) \lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} + (6k_1 - k_0) \lambda \frac{du^\lambda}{d\lambda} + 7k_0 u^\lambda. \end{aligned}$$

By the following differential identity, combining with the differential by part formulas of Section 4, we have

$$\begin{aligned}
& \lambda^7 f'''''f'' + k_5\lambda^6 f'''''f'' + k_4\lambda^5 f'''''f'' + k_3\lambda^4 f'''''f'' \\
& + k_2\lambda^3 f'''''f'' + k_1\lambda^2 f'''''f'' + k_0\lambda f'''''f'' \\
& = \left[\lambda^7 f'''''f'' - \lambda^7 f'''''f''' + (k_5 - 7)\lambda^6 f'''''f'' + (7 - \frac{k_5}{2})\lambda^6(f''')^2 \right. \\
& + (-6k_5 + k_4 + 42)\lambda^5 f'''''f'' + (15k_5 - \frac{5}{2}k_4 + \frac{1}{2}k_3 - 105)\lambda^4(f'')^2 \\
& \left. + k_0\lambda f'''''f'' - \frac{1}{2}k_0 f''^2 \right]' + \lambda^7(f''')^2 + (9k_5 - k_4 - 84)\lambda^5(f''')^2 \\
& + (-60k_5 + 10k_4 - 2k_3 + k_2 + 420)\lambda^3(f'')^2 - k_0\lambda(f')^2 + k_1\lambda^2 f'f''
\end{aligned}$$

and

$$\begin{aligned}
& -\lambda^7 f'''''''f' + (1 - k_5)\lambda^6 f'''''''f' + (2k_5 - k_4)\lambda^5 f'''''''f' + (3k_4 - k_3)\lambda^4 f'''''''f' \\
& + (4k_3 - k_2)\lambda^3 f'''''''f' + (5k_2 - k_1)\lambda^2 f'''''f' + (6k_1 - k_0)\lambda f'f' + 7k_0 f'f' \\
& = \left[-\lambda^7 f'''''''f' + \lambda^7 f'''''''f'' - \lambda^7 f'''''''f''' + (8 - k_5)\lambda^6 f'''''''f' + (k_5 - 15)\lambda^6 f'''''''f'' \right. \\
& - (14k_5 - k_4 - 138)\lambda^5 f'''''''f'' + (55k_5 - \frac{13}{2}k_4 + \frac{1}{2}k_3 - 480)\lambda^4(f'')^2 \\
& + (8k_5 - k_4 - 48)\lambda^5 f'''''''f' + (-40k_5 + 8k_4 - k_3 + 240)\lambda^4 f'''''''f' \\
& \left. + (160k_5 - 32k_4 + 8k_3 - k_2 - 960)\lambda^3 f'''''f' + \frac{7}{2}k_0 f''^2 \right]' \\
& + \lambda^7(f''')^2 + (14k_5 - k_4 - 138)\lambda^5(f''')^2 + (22 - k_5)\lambda^6 f'''''f''' \\
& + (-380k_5 + 58k_4 - 10k_3 + k_2 + 2820)\lambda^3(f'')^2 + (6k_1 - k_0)\lambda(f')^2 \\
& + (-480k_5 + 96k_4 - 24k_3 + 8k_2 - k_1 + 2880)\lambda^2 f''f'.
\end{aligned}$$

Then by the above two identities ($f = u^\lambda$, $f' = \frac{d}{d\lambda}u^\lambda$), we get the following integral corresponding to the operator A :

$$\begin{aligned}
\mathcal{F}_0 &= \int_{\partial B_1} \left(\lambda F_0(u^\lambda) \frac{d^2 u^\lambda}{d\lambda^2} + 7F_0(u^\lambda) \frac{du^\lambda}{d\lambda} - \lambda \frac{dF_0(u^\lambda)}{d\lambda} \frac{du^\lambda}{d\lambda} \right) \\
&= \mathcal{F}_{01} + \int_{\partial B_1} \left[A_4 \lambda^7 \left(\frac{d^4 u^\lambda}{d\lambda^4} \right)^2 + A_3 \lambda^5 \left(\frac{d^3 u^\lambda}{d\lambda^3} \right)^2 + A_2 \lambda^3 \left(\frac{d^2 u^\lambda}{d\lambda^2} \right)^2 + A_1 \lambda \left(\frac{du^\lambda}{d\lambda} \right)^2 \right],
\end{aligned}$$

where

$$\begin{aligned}
A_4 &= 2, \\
A_3 &= 26k_5 - 2k_4 - 288k_6, \\
A_2 &= -440k_5 + 68k_4 - 12k_3 + 2k_2 + 3240k_6, \\
A_1 &= 480k_5 - 96k_4 + 24k_3 - 8k_2 + 6k_1 - 2k_0 - 2280k_6,
\end{aligned} \tag{6.1}$$

and the part \mathcal{F}_{01} denotes the differential term, exactly, it is

$$\begin{aligned}\mathcal{F}_{01} := & \frac{d}{d\lambda} \int_{\partial B_1} \left[-\lambda^7 \frac{d^6 u^\lambda}{d\lambda^6} \frac{du^\lambda}{d\lambda} + 2\lambda^7 \frac{d^5 u^\lambda}{d\lambda^5} \frac{d^2 u^\lambda}{d\lambda^2} - 2\lambda^7 \frac{d^4 u^\lambda}{d\lambda^4} \frac{d^3 u^\lambda}{d\lambda^3} \right. \\ & + (8 - k_5) \lambda^6 \frac{d^5 u^\lambda}{d\lambda^5} \frac{du^\lambda}{d\lambda} + (-20k_5 + 2k_4 + 180) \lambda^5 \frac{d^3 u^\lambda}{d\lambda^3} \frac{d^2 u^\lambda}{d\lambda^2} \\ & + (70k_5 - 9k_4 + k_3 - 585) \lambda^4 \left(\frac{d^2 u^\lambda}{d\lambda^2} \right)^2 + (8k_5 - k_4 - 48) \lambda^5 \frac{d^4 u^\lambda}{d\lambda^4} \frac{du^\lambda}{d\lambda} \\ & + (-40k_5 + 8k_4 - k_3 + 240) \lambda^4 \frac{d^3 u^\lambda}{d\lambda^3} \frac{du^\lambda}{d\lambda} \\ & + (160k_5 - 32k_4 + 8k_3 - k_2 - 960) \lambda^3 \frac{d^2 u^\lambda}{d\lambda^2} \frac{du^\lambda}{d\lambda} + 3k_0 (u^\lambda)^2 \\ & + (2k_5 - 22) \lambda^6 \frac{d^4 u^\lambda}{d\lambda^4} \frac{d^2 u^\lambda}{d\lambda^2} + (18 - k_5) \lambda^6 \left(\frac{d^3 u^\lambda}{d\lambda^3} \right)^2 + k_0 \lambda u_e^\lambda \frac{du^\lambda}{d\lambda} \\ & \left. + (-240k_5 + 48k_4 - 12k_3 + 4k_2 + \frac{1}{2}k_1 + 1440) \lambda^2 \left(\frac{du^\lambda}{d\lambda} \right)^2 \right].\end{aligned}$$

6.2 The integrals corresponding to the operator $\Delta_\theta F_1(u^\lambda)$

Let us define

$$\begin{aligned}I_1(F_1) := & \int_{\partial B_1} \lambda^1 \Delta_\theta F_1(u^\lambda) \frac{d^2 u^\lambda}{d\lambda^2} = \int_{\partial B_1} \lambda^{j+1} \sum_{j=0}^4 (-t_j) \Delta_\theta \frac{d^j u^\lambda}{d\lambda^j} \frac{d^2 u^\lambda}{d\lambda^2} \\ = & \int_{\partial B_1} \sum_{j=0}^4 t_j \lambda^{j+1} \nabla_\theta \frac{d^j u^\lambda}{d\lambda^j} \nabla_\theta \frac{d^2 u^\lambda}{d\lambda^2},\end{aligned}$$

and

$$\begin{aligned}I_2(F_1) := & \int_{\partial B_1} \left(7\Delta_\theta F_1(u^\lambda) - \lambda \frac{d}{d\lambda} \Delta_\theta F_1(u) \right) \frac{du^\lambda}{d\lambda} \\ = & \int_{\partial B_1} \left(7\Delta_\theta \left(\sum_{j=0}^4 (-t_j) \lambda^j \frac{d^j u^\lambda}{d\lambda^j} \right) - \lambda \frac{d}{d\lambda} \Delta_\theta \left(\sum_{j=0}^4 (-t_j) \lambda^j \frac{d^j u^\lambda}{d\lambda^j} \right) \right) \frac{du^\lambda}{d\lambda} \\ = & \int_{\partial B_1} \sum_{j=1}^5 t_{0j} \lambda^j \nabla_\theta \frac{d^j u^\lambda}{d\lambda^j} \nabla_\theta \frac{du^\lambda}{d\lambda},\end{aligned}$$

where

$$t_{05} = -t_4, t_{04} = 3t_4 - t_3, t_{03} = 4t_3 - t_2, t_{02} = 5t_2 - t_1, t_{01} = 6t_1 - t_0,$$

and t_j is defined in (3.4). Recall the formulas in the Lemma 4.1-4.2, we regard that $f = \nabla_\theta u^\lambda$, then we can obtain that

$$\begin{aligned} I_1(F_1) + I_2(F_1) &= \int_{\partial B_1} B_1 \lambda (\nabla_\theta \frac{du^\lambda}{d\lambda})^2 + B_2 \lambda^3 (\nabla_\theta \frac{d^2 u^\lambda}{d\lambda^2})^2 + B_3 \lambda^5 (\nabla_\theta \frac{d^3 u^\lambda}{d\lambda^3})^2 \\ &\quad + \frac{d}{d\lambda} \int_{\partial B_1} \left(\sum_{0 \leq i, j \leq 2} b_{i,j} \lambda^{i+j} \nabla_\theta \frac{d^i u^\lambda}{d\lambda^i} \nabla_\theta \frac{d^j u^\lambda}{d\lambda^j} \right), \end{aligned}$$

where

$$B_1 = -2t_0 + 6t_1 - 8t_2 + 24t_3 - 96t_4, B_2 = 2t_2 - 12t_3 + 68t_4, B_3 = -2t_4 = 6. \quad (6.2)$$

The coefficient $b_{i,j}$ can be determined by t_j but we do not give the precise form since the constant is not important for our estimate below.

6.3 The integral corresponding to the operator $\Delta_\theta^2 F_2(u^\lambda)$

Recall that the sphere representation of triple-harmonic operator, i.e., (3.1) and (3.2). Let us define the following

$$\begin{aligned} I_1(F_2) &:= \int_{\partial B_1} \lambda^{j+1} \Delta_\theta^2 F_2(u^\lambda) \frac{d^2 u^\lambda}{d\lambda^2} = \int_{\partial B_1} \lambda^1 \sum_{j=0}^2 e_j \Delta_\theta^2 \frac{d^2 u^\lambda}{d\lambda^2} \frac{d^2 u^\lambda}{d\lambda^2} \\ &= \int_{\partial B_1} \lambda^{j+1} \Delta_\theta \frac{d^j u^\lambda}{d\lambda^j} \Delta_\theta \frac{d^2 u^\lambda}{d\lambda^2} \end{aligned}$$

and

$$\begin{aligned} I_2(F_2) &:= \int_{\partial B_1} \left(7\Delta_\theta F_2(u^\lambda) - \lambda \frac{d}{d\lambda} \Delta_\theta^2 F_2(u^\lambda) \right) \frac{du^\lambda}{d\lambda} \\ &= \int_{\partial B_1} \left(7\Delta_\theta \left(\sum_{j=0}^2 e_j \lambda^j \Delta_\theta \frac{d^j u^\lambda}{d\lambda^j} \right) - \lambda \frac{d}{d\lambda} \Delta_\theta^2 \left(\sum_{j=0}^2 e_j \lambda^j \Delta_\theta \frac{d^j u^\lambda}{d\lambda^j} \right) \right) \frac{du^\lambda}{d\lambda} \\ &= \sum_{j=1}^3 \int_{\partial B_1} e_{0j} \Delta_\theta \frac{d^j u^\lambda}{d\lambda^j} \Delta_\theta \frac{du^\lambda}{d\lambda}, \end{aligned}$$

where

$$e_{03} = -e_2, e_{02} = 5e_2 - e_1, e_{01} = 6e_1 - e_0$$

and e_j is defined in (3.5).

Recall the formulas in the Lemma 4.1-4.2, this time we regard that $f = \Delta_\theta u^\lambda$, then we can obtain that

$$\begin{aligned} I_1(F_2) + I_2(F_2) &= \int_{\partial B_1} C_1 \lambda (\Delta_\theta \frac{du^\lambda}{d\lambda})^2 + C_2 \lambda^3 (\Delta_\theta \frac{d^2 u^\lambda}{d\lambda^2})^2 \\ &\quad + \frac{d}{d\lambda} \int_{\partial B_1} \left(\sum_{0 \leq i, j \leq 1} C_{i,j} \lambda^{i+j} \Delta_\theta \frac{d^i u^\lambda}{d\lambda^i} \Delta_\theta \frac{d^j u^\lambda}{d\lambda^j} \right), \end{aligned}$$

where

$$C_1 = -2e_0 + 6e_1 - 8e_2, C_2 = 2e_2 \quad (6.3)$$

and $C_{i,j}$ are determined by e_j , we don't need to outline the precise expressions of them since the constant will not affect our applying the monotonicity formula.

6.4 The integral corresponding to the operator $\Delta_\theta^3 F_3(u^\lambda)$

Recall the formulas in the Lemma 4.1-4.2 and set $f = \nabla_\theta \Delta_\theta u^\lambda$, then we can obtain that

$$\begin{aligned} I(F_3) &:= \int_{\partial B_1} \left(\lambda \Delta_\theta^3 F_3(u^\lambda) \frac{d^2 u^\lambda}{d\lambda^2} + 7 \Delta_\theta^3 F_3(u^\lambda) \frac{du^\lambda}{d\lambda} - \lambda \frac{d\Delta_\theta^3 F_3(u^\lambda)}{d\lambda} \frac{du^\lambda}{d\lambda} \right) \\ &= \int_{\partial B_1} -\lambda \nabla_\theta \Delta_\theta u^\lambda \nabla_\theta \Delta_\theta \frac{d^2 u^\lambda}{d\lambda^2} - 7 \nabla_\theta \Delta_\theta u^\lambda \nabla_\theta \Delta_\theta \frac{du^\lambda}{d\lambda} + \lambda |\nabla_\theta \Delta_\theta u^\lambda|^2 \\ &= \frac{d}{d\lambda} \left[\int_{\partial B_1} -\lambda \nabla_\theta \Delta_\theta u^\lambda \nabla_\theta \Delta_\theta \frac{du^\lambda}{d\lambda} - 3(\nabla_\theta \Delta_\theta u^\lambda)^2 \right. \\ &\quad \left. + 2\lambda \int_{\partial B_1} (\nabla_\theta \Delta_\theta \frac{du^\lambda}{d\lambda})^2 \right]. \end{aligned}$$

6.5 The monotonicity formula and the proof of Theorem 2.1

We sum up the terms \bar{E}_{d_1} and \bar{E}_{d_2} . Then

$$\begin{aligned} E(\lambda, x, u) &:= \int_{B_1} \frac{1}{2} |\Delta^2 u^\lambda|^2 - \frac{1}{p+1} |u^\lambda|^{p+1} \\ &\quad + \int_{\partial B_1} \left(\sum_{i,j \geq 0, i+j \leq 7} C_{i,j}^0 \lambda^{i+j} \frac{d^i u^\lambda}{d\lambda^i} \frac{d^j u^\lambda}{d\lambda^j} + \sum_{i,j \geq 0, i+j \leq 5} C_{i,j}^1 \lambda^{i+j} \nabla_\theta \frac{d^i u^\lambda}{d\lambda^i} \nabla_\theta \frac{d^j u^\lambda}{d\lambda^j} \right. \\ &\quad \left. + \sum_{i,j \geq 0, i+j \leq 3} C_{i,j}^2 \lambda^{i+j} \Delta_\theta \frac{d^i u^\lambda}{d\lambda^i} \Delta_\theta \frac{d^j u^\lambda}{d\lambda^j} \right. \\ &\quad \left. + \sum_{i,j \geq 0, i+j \leq 1} C_{i,j}^3 \lambda^{i+j} \nabla_\theta \Delta_\theta \frac{d^i u^\lambda}{d\lambda^i} \nabla_\theta \Delta_\theta \frac{d^j u^\lambda}{d\lambda^j} \right), \end{aligned}$$

where the constant $C_{i,j}^k$ can be determined by the calculation of \bar{E}_{d_1} and \bar{E}_{d_2} in the above three subsections. Then we obtain the Theorem 2.1.

7 The desired monotonicity formula: The proof of Theorem 2.2

In this section, we construct the desired monotonicity formula via the blow-down analysis. We start from the Lemma 2.3 in the previous section. Firstly, we have

Lemma 7.1. If $p > \frac{n+8}{n-8}$, then

$$\sum_{l=1}^2 (C_l + c_l) \lambda^{2l-1} (\Delta_\theta \frac{d^l u^\lambda}{d\lambda^l})^2 \geq 0.$$

Proof. We known from (6.3), (3.5) and (5.3) that

$$C_1 = -6k^2 + (-72 + 6n)k - 178 + 30n; \quad C_2 + c_2 = 8,$$

where $k =: \frac{8}{p-1}$. By this natation we observe that $p > \frac{n+8}{n-8}$ is equivalent to $0 < k < \frac{n-8}{2}$. By finding the roots (denoted by $r_1(n), r_2(n)$) of the equation

$$-6k^2 + (-72 + 6n)k - 178 + 30n = 0$$

about variable k , we get that

$$\begin{aligned} r_1(n) &:= \frac{1}{2}n - 6 - \frac{1}{6}\sqrt{9n^2 - 36n + 228} \\ &= \frac{1}{6}(3n - 36 - \sqrt{9n^2 - 36n + 228}) \\ &= \frac{1}{6} \frac{(3n - 36)^2 - (9n^2 - 36n + 228)}{3n - 36 + \sqrt{9n^2 - 36n + 228}} \\ &= \frac{-30n + 178}{3n - 36 + \sqrt{9n^2 - 36n + 228}} < 0 \text{ for } n \geq 6 \end{aligned}$$

and

$$r_2(n) := \frac{1}{2}n - 6 + \frac{1}{6}\sqrt{9n^2 - 36n + 228} > \frac{1}{2}(n - 8),$$

therefore we obtain that $C_1 > 0$ if $0 < k < \frac{n-8}{2}$. Recall that $c_2 = 2\alpha - 2\beta - 4 > 0$, then the conclusion follows. \square

Theorem 7.1. If $\frac{n+8}{n-8} < p < p_c(n)$, then there exist constants $b_{i,j}$ such that

$$\sum_{s=1}^3 (B_s + b_s) \lambda^{2s-1} (\nabla_\theta \frac{d^s u^\lambda}{d\lambda^s})^2 \geq \frac{d}{d\lambda} \left(\sum_{0 \leq i,j \leq 2, i+j \leq 3} b_{i,j} \nabla_\theta \frac{d^i u^\lambda}{d\lambda^i} \nabla_\theta \frac{d^j u^\lambda}{d\lambda^j} \right).$$

Proof. To see this, from (6.2), (3.4) and (5.3), we get that

$$\begin{aligned} Bb_1 &:= B_1 + b_1 = 6k^4 + (144 - 12n)k^3 + (6n^2 - 204n + 994)k^2 \\ &\quad + (60n^2 - 850n + 2732)k + 94n^2 - 1012n + 2644, \end{aligned}$$

and

$$\begin{aligned} Bb_2 &:= B_2 + b_2 = -38k^2 + (-292 + 38n)k - 6n^2 + 132n - 544, \\ Bb_3 &:= B_3 + b_3 = 8. \end{aligned}$$

Lemma 7.2. If $p > \frac{n+8}{n-8}$, then we have $Bb_1 > 0$.

Proof. To show this, we first see that if $n \in [9, 29]$, under the condition $0 < k < \frac{n-8}{2}$, by a direct case by case calculation we can prove that $Bb_1 > 0$.

For $n \geq 29$, let us introduce the transform $k = \frac{n-8}{2}a$, hence $0 < a < 1$, then we have

$$\begin{aligned} Bb_1 &= \frac{3}{8}(n-8)^4a^4 - \frac{3}{2}(n-12)(n-8)^3a^3 + \frac{1}{2}(3n^2 - 102n + 497)(n-8)^2a^2 \\ &\quad + (n-8)(30n^2 - 425n + 1366)a + 94n^2 - 1012n + 2644 \\ &= f_4(n)a^4 - f_3(n)a^3 + f_2(n)a^2 + f_1(n)a + f_0(n). \end{aligned}$$

We can see that if $n \geq 29$, then $f_j(n) > 0$ for $j = 1, 2, 3, 4$. Since $0 < a < 1$, we have

$$\begin{aligned} Bb_1 &= f_4(n)a^4 - f_3(n)a^3 + f_2(n)a^2 + f_1(n)a + f_0(n) \\ &\geq f_4(n)a^4 + (f_2(n) + f_1(n) + f_0(n) - f_3(n))a^3 \\ &= f_4(n)a^4 + (-1596 + 9n^3 - \frac{261}{2}n^2 + 738n)a^3. \end{aligned}$$

By a direct calculation we can show that

$$-1596 + 9n^3 - \frac{261}{2}n^2 + 738n > 0 \text{ if } n \geq 6.$$

Hence we derive that $Bb_1 > 0$ when $n \geq 29$. \square

Lemma 7.3. Assume that $\frac{n+8}{n-8} < p < p_c(n)$ and $n \geq 9$, then $Bb_2 > 0$ except for $n = 17, 18$.

Proof. Firstly, we recall that $\frac{n+8}{n-8} < p < p_c(n)$ then we have $\frac{n-10}{2} - \sqrt{n} < k < \frac{n-8}{2}$ for $n \geq 18$. By solving the equation $Bb_2 = 0$, we find the roots

$$\begin{aligned} r_1(n) &:= \frac{1}{2}n - \frac{73}{19} - \frac{1}{38}\sqrt{133n^2 - 532n + 664}, \\ r_2(n) &:= \frac{1}{2}n - \frac{73}{19} + \frac{1}{38}\sqrt{133n^2 - 532n + 664}. \end{aligned}$$

Notice that $r_1(n) < \frac{n-10}{2} - \sqrt{n}$ is equivalent to

$$133n^2 - 1976n - 3344\sqrt{n} - 1292 > 0,$$

the above inequality holds whenever $n \geq 21$. The strict inequality $r_2(n) > \frac{n-8}{2}$ can be proved by a direct calculation.

Therefore, the conclusion holds when $n \geq 21$. For the remaining case $9 \leq n \leq 20$ except for $n = 17, 18$, we can show that the conclusion also holds. \square

Combining with Lemmas 7.2 and 7.3, we immediately get the following lemma.

Lemma 7.4. For $n \in [9, +\infty), n \in \mathbb{N}^+$, except for $n = 17, 18$, we have that

$$\sum_{s=1}^3 (B_s + b_s) \lambda^{2s-1} (\nabla_\theta \frac{d^s u^\lambda}{d\lambda^s})^2 \geq 0.$$

Next, we have to consider the remaining case when $n = 17, 18$. In view of Lemmas 4.1-4.2, we have the following differential identity (denote that $f' := \nabla_\theta \frac{du^\lambda}{d\lambda}$):

$$\begin{aligned} \sum_{s=1}^3 (B_s + b_s) \lambda^{2s-1} (\nabla_\theta \frac{d^s u^\lambda}{d\lambda^s})^2 &= Bb_1 \lambda (f')^2 + (Bb_2 + 4Bb_3) \lambda^3 (f'')^2 \\ &\quad + Bb_3 \lambda (\lambda^2 f''' + 2\lambda f'')^2 + \frac{d}{d\lambda} (-2Bb_3 \lambda^4 (f'')^2) \\ &\geq (Bb_2 + 4Bb_3) \lambda^3 (f'')^2 + \frac{d}{d\lambda} (-2Bb_3 \lambda^4 (f'')^2). \end{aligned} \tag{7.1}$$

A direct calculation shows that under the condition $\frac{n+8}{n-8} < p < p_c(n)$ and $n = 18$, we have that $Bb_2 + 4Bb_3 > 0$ (this way fails when $n = 17$). Hence, when $n = 18$, we get that

$$\begin{aligned} \sum_{s=1}^3 (B_s + b_s) \lambda^{2s-1} (\nabla_\theta \frac{d^s u^\lambda}{d\lambda^s})^2 &\geq (Bb_2 + 4Bb_3) \lambda^3 (f'')^2 + \frac{d}{d\lambda} (-2Bb_3 \lambda^4 (f'')^2) \\ &\geq \frac{d}{d\lambda} (-2Bb_3 \lambda^4 (f'')^2). \end{aligned}$$

We need different method to handle with $n = 17$ in the following. Notice that by the mean value inequality we have the following differential identity:

$$\begin{aligned} \epsilon B_{10} \lambda (f')^2 + B_{30} \lambda (\lambda^2 f''' + 2\lambda f'')^2 &\geq -2\sqrt{\epsilon \cdot B_{30} \cdot B_{10}} (\lambda^3 f''' f' + 2\lambda f'' f') \\ &= 2\sqrt{\epsilon \cdot B_{30} \cdot B_{10}} (\lambda^3 (f'')^2 - \lambda (f')^2) + 2\sqrt{\epsilon \cdot B_{30} \cdot B_{10}} \frac{d}{d\lambda} \left(\lambda^3 f'' f' - \frac{1}{2} \lambda^2 (f')^2 \right), \end{aligned}$$

here we have used the Lemmas 4.1-4.2; $\epsilon \in (0, 1)$ is to be determined later.

Combining with (7.1), we have

$$\begin{aligned} \sum_{s=1}^3 (B_s + b_s) \lambda^{2s-1} (\nabla_\theta \frac{d^s u^\lambda}{d\lambda^s})^2 &= Bb_1 \lambda (f')^2 + (Bb_2 + 4Bb_3) \lambda^3 (f'')^2 \\ &\quad + Bb_3 \lambda (\lambda^2 f''' + 2\lambda f'')^2 + \frac{d}{d\lambda} (-2Bb_3 \lambda^4 (f'')^2) \\ &\geq \left(B_{20} + 4B_{30} + 2\sqrt{\epsilon \cdot B_{30} \cdot B_{10}} \right) \lambda^3 (f'')^2 \\ &\quad + \left((1-\epsilon) B_{10} - 2\sqrt{\epsilon \cdot B_{30} \cdot B_{10}} \right) \lambda (f')^2 \\ &\quad + 2\sqrt{\epsilon \cdot B_{30} \cdot B_{10}} \frac{d}{d\lambda} \left(\lambda^3 f'' f' - \frac{1}{2} \lambda^2 (f')^2 \right). \end{aligned} \tag{7.2}$$

To avoid confusion, we denote that $B_{j0} := Bb_j$ for $j = 1, 2, 3$ in the above and following. Thus, our conclusion holds once we can prove

$$B_{20} + 4B_{30} + 2\sqrt{\epsilon \cdot B_{30} \cdot B_{10}} \geq 0, \tag{7.3}$$

and

$$(1 - \epsilon)B_{10} - 2\sqrt{\epsilon \cdot B_{30} \cdot B_{10}} \geq 0. \quad (7.4)$$

To make sure (7.4) holds, we select $\epsilon \in (0, 1)$ satisfying

$$\frac{4\epsilon \cdot B_{30}}{(1 - \epsilon)^2} \leq \min_{0 \leq k \leq \frac{n-8}{2}} B_{10}.$$

Notice that when $n = 17$, $\min_{0 \leq k \leq \frac{n-8}{2}} B_{10}|_{n=17} = 12606$, hence $\epsilon \leq 0.9508$. Thus we select that $\epsilon = 0.9508$. Now we consider the inequality (7.3). Note that

if $B_{20} + 4B_{30} \geq 0$, then the inequality (7.3) holds immediately;
 if $B_{20} + 4B_{30} < 0$, then (7.3) is equivalent to

$$4\epsilon \cdot B_{30} \cdot B_{10} - (B_{20} + B_{30})^2 \geq 0. \quad (7.5)$$

By a direct calculation we can prove that when $n = 17$, under the condition $\frac{n+8}{n-8} < p < p_c(n)$, i.e., $\max\{0, R_1(n)\} < k < \frac{n-8}{2}$ ($R_1(n)$ see (8.10)), we have that

$$\begin{aligned} & \sum_{s=1}^3 (B_s + b_s) \lambda^{2s-1} (\nabla_\theta \frac{d^s u^\lambda}{d\lambda^s})^2 \geq (B_{20} + 4B_{30} + 2\sqrt{\epsilon \cdot B_{30} \cdot B_{10}}) \lambda^3 (f'')^2 \\ & + ((1 - \epsilon)B_{10} - 2\sqrt{\epsilon \cdot B_{30} \cdot B_{10}}) \lambda (f')^2 \\ & + 2\sqrt{\epsilon \cdot B_{30} \cdot B_{10}} \frac{d}{d\lambda} \left(\lambda^3 f'' f' - \frac{1}{2} \lambda^2 (f')^2 \right) \\ & \geq 2\sqrt{\epsilon \cdot B_{30} \cdot B_{10}} \frac{d}{d\lambda} \left(\lambda^3 f'' f' - \frac{1}{2} \lambda^2 (f')^2 \right). \end{aligned} \quad (7.6)$$

Summing up, for all cases, we have proved Theorem 7.1. \square

Theorem 7.2. If $\frac{n+8}{n-8} < p < p_c(n)$ and $n \geq 9$, then there exist constants $a_{i,j}$ such that

$$\sum_{j=1}^4 (A_j + a_j) \lambda^{2j-1} \left(\frac{d^j u^\lambda}{d\lambda^j} \right)^2 \geq \frac{d}{d\lambda} \left(\sum_{0 \leq i, j \leq 2, i+j \leq 5} a_{i,j} \frac{d^i u^\lambda}{d\lambda^i} \frac{d^j u^\lambda}{d\lambda^j} \right).$$

Firstly, from (6.1), (5.3) and (3.3) we have that

$$\begin{aligned} Aa_1 := A_1 + a_1 &= -4k^6 + (-88 + 12n)k^5 + (-12n^2 + 208n - 860)k^4 \\ &+ (4n^3 - 152n^2 + 1544n - 4304)k^3 + (32n^3 - 776n^2 + 5408n - 11196)k^2 \\ &+ (92n^3 - 1576n^2 + 8428n - 14040)k + 64n^3 - 940n^2 + 4368n - 6372, \end{aligned}$$

$$\begin{aligned} Aa_2 := A_2 + a_2 &= 28k^4 + (464 - 56n)k^3 + (32n^2 - 656n + 2668)k^2 \\ &+ (-4n^3 + 232n^2 - 2364n + 6456)k - 16n^3 + 380n^2 - 2640n + 5556, \end{aligned} \quad (7.7)$$

$$Aa_3 := A_3 + a_3 = -28k^2 + (-216 + 28n)k - 4n^2 + 96n - 408$$

and $Aa_4 := A_4 + a_4 = 4$.

We separate the proofs into several Lemmas.

Lemma 7.5. If $p > \frac{n+8}{n-8}$ and $n \geq 9$, then $A_1 + a_1 > 0$.

Proof. To see this, in fact from (6.1), (5.3) and (3.3) we have that

$$\begin{aligned} A_1 = & -2k^6 + (-72 + 6n)k^5 + (-6n^2 + 174n - 818)k^4 + (2n^3 - 132n^2 + \\ & 1492n - 4272)k^3 + (30n^3 - 768n^2 + 5412n - 11222)k^2 + (94n^3 - 1596n^2 \\ & + 8486n - 14088)k + 66n^3 - 954n^2 + 4398n - 6390, \end{aligned}$$

and

$$\begin{aligned} a_1 = & -2k^6 + (-16 + 6n)k^5 + (-6n^2 + 34n - 42)k^4 + (2n^3 - 20n^2 + 52n \\ & - 32)k^3 + (2n^3 - 8n^2 - 4n + 26)k^2 + (-2n^3 + 20n^2 - 58n + 48)k \\ & - 2n^3 + 14n^2 - 30n + 18. \end{aligned}$$

It is can be observed that $-3, -1$ and $n - 3, n - 5$ are the roots of $A_1 = 0$, hence we have that

$$A_1 = (k + 3)(k + 1)(k - (n - 3))(k - (n - 5))(-2k^2 + (2n - 48)k + 22n - 142),$$

from this, it is not difficult to see that $A_1 > 0$ if $0 < k < \frac{n-8}{2}$. For $a_1 = 0$, we have observed that $-1, -1, 1, n - 3, n - 3, n - 1$ are the roots, hence

$$a_1 = (k + 1)^2(k - 1)(k - (n - 3))^2(k - (n - 1)),$$

thus $a_1 > 0$ if $1 < k < n - 1$. For $n \geq 18$, we have that $\frac{n-10}{2} - \sqrt{n} > 1$, then $a_1 > 0$ if $\frac{n+8}{n-8} < p$, i.e., $0 < k < \frac{n-8}{2}$. Once $a_1 > 0$, then $A_1 + a_1 > 0$. For the case $n \in [9, 17]$, we calculate case by case showing that $A_1 + a_1 > 0$ if $0 < k < \frac{n-8}{2}$. This completes the proof of Lemma 7.5. \square

Lemma 7.6. If $\frac{n+8}{n-8} < p < p_c(n)$ and $n \geq 9$, then $A_3 + a_3 > 0$.

Proof. By solving the roots of $A_3 + a_3 = 0$, we get that

$$\begin{aligned} r_{10}(n) &:= \frac{1}{2}n - \frac{27}{7} - \frac{1}{14}\sqrt{21n^2 - 84n + 60}, \\ r_{20}(n) &:= \frac{1}{2}n - \frac{27}{7} + \frac{1}{14}\sqrt{21n^2 - 84n + 60}. \end{aligned}$$

Notice that $r_{10}(n) < \frac{n-10}{2} - \sqrt{n}$ is equivalent to

$$21n^2 - 85n - 32\sqrt{n} - 196 > 0.$$

The above inequality holds if $n \geq 7$. The root $r_{20}(n) > \frac{n-8}{2}$ can be seen immediately. Thus, for $n \geq 18$ we know that $A_3 + a_3 > 0$ if $R_1(n) < k < \frac{n-8}{2}$. However, when $n \in [9, 17]$, we may compute case by case to show that $A_3 + a_3 > 0$ if $\frac{n+8}{n-8} < p < p_c(n)$. This finishes the proof of Lemma 7.6. \square

Lemma 7.7. If $\frac{n+8}{n-8} < p < p_c(n)$, for $n \in [9, 13]$ or $n \geq 21$, we have that $A_2 + a_2 > 0$.

Proof. To prove Lemma 7.7, we introduce the transformation $n := t^2$, $k := \frac{n-10}{2} - a \cdot t$, hence $0 < a < 1$. From (7.7), we have that

$$\begin{aligned} A_2 + a_2 &= -524 + \frac{3}{4}t^8 + (-10a^2 - 6)t^6 - 24at^5 + (28a^4 + 40a^2 - 5)t^4 \\ &\quad + (96a^3 + 96a)t^3 + (-92a^2 + 68)t^2 - 576at \\ &\geq -524 + \frac{3}{4}t^8 - 16t^6 - 24t^5 - 5t^4 - 24t^2 - 576t > 0 \text{ if } n = t^2 \geq 28. \end{aligned}$$

For the cases $n \in [9, 13]$ and $n \in [21, 27]$, we may calculate case by case to show that $A_2 + a_2 > 0$ if $\frac{n+8}{n-8} < p < p_c(n)$. This finishes the proof of Lemma 7.7. \square

Combining with Lemmas 7.5, 7.6 and 7.7, we have that

Lemma 7.8. *If $\frac{n+8}{n-8} < p < p_c(n)$, for $n \in [9, 13]$ and $n \geq 21$, we have*

$$\sum_{j=1}^4 (A_j + a_j) \lambda^{2j-1} \left(\frac{d^j u^\lambda}{d\lambda^j} \right)^2 \geq 0.$$

Remark 7.1. *To establish the Theorem 7.2, we need to deal with the case $n \in [14, 20]$ by some other methods.*

Now we turn to the case $n \in [14, 20]$. To deal with these cases, firstly we establish the following differential identity (denote that $f' = \frac{du^\lambda}{d\lambda}$):

$$\begin{aligned} \sum_{j=1}^4 (A_j + a_j) \lambda^{2j-1} \left(\frac{d^j u^\lambda}{d\lambda^j} \right)^2 &:= \sum_{j=1}^4 d_j \lambda^{2j-1} \left(\frac{d^j u^\lambda}{d\lambda^j} \right)^2 \\ &= d_1 \lambda(f')^2 + (d_2 + 4d_3) \lambda^3 (f'')^2 + d_3 \lambda(\lambda^2 f'''' + 2\lambda f'')^2 \\ &\quad + d_4 \lambda^7 (f''''')^2 + \frac{d}{d\lambda} (-2d_3 \lambda^4 (f'')^2), \end{aligned} \tag{7.8}$$

where we have used Lemma 4.2. The term $d_2 + 4d_3$ is not nonnegative. Now we invoke the mean value inequality. For the parameter $x \in [0, 1]$ to be determined later, we have that

$$\begin{aligned} d_1 \lambda(f')^2 + d_4 \lambda^7 (f''''')^2 &= x \cdot d_1 \lambda(f')^2 + d_4 \lambda^7 (f''''')^2 + (1-x) \cdot d_1 \lambda(f')^2 \\ &\geq 2\sqrt{x \cdot d_1 d_4} \lambda^4 f''' f' + (1-x) d_1 \lambda(f')^2 \\ &= 2\sqrt{x \cdot d_1 d_4} \left(-12\lambda(f')^2 + 6\lambda^3 (f'')^2 \right) + (1-x) d_1 \lambda(f')^2 \\ &\quad + \frac{d}{d\lambda} \left(2\sqrt{x \cdot d_1 d_4} (\lambda^4 f''' f' - \frac{1}{2}\lambda^4 (f'')^2 - 4\lambda^3 f'' f' + 6\lambda^2 (f')^2) \right) \\ &= \left((1-x)d_1 - 24\sqrt{x \cdot d_1 \cdot d_4} \right) \lambda(f')^2 + 12\sqrt{x \cdot d_1 \cdot d_4} \lambda^3 (f'')^2 \\ &\quad + \frac{d}{d\lambda} \left(\sum_{1 \leq i, j \leq 3, i+j \leq 4} d_{i,j} \lambda^{i+j} f^{(i)} f^{(j)} \right), \end{aligned} \tag{7.9}$$

where we have used Lemma 4.2. The constants in the derivative terms, namely, $d_{i,j}$ can be determined but we do not need the exactly expressions. In particular, $d_{i,j}$ may be changed in the following derivation, but we still denote as $d_{i,j}$. Combine (7.8) and (7.9), we obtain that

$$\begin{aligned}
& \sum_{j=1}^4 (A_j + a_j) \lambda^{2j-1} \left(\frac{d^j u^\lambda}{d\lambda^j} \right)^2 := \sum_{j=1}^4 d_j \lambda^{2j-1} \left(\frac{d^j u^\lambda}{d\lambda^j} \right)^2 \\
& = d_1 \lambda(f')^2 + (d_2 + 4d_3) \lambda^3 (f'')^2 + d_3 \lambda(\lambda^2 f''' + 2\lambda f'')^2 + d_4 \lambda^7 (f''')^2 \\
& \quad + \frac{d}{d\lambda} (-2d_3 \lambda^4 (f'')^2), \\
& \geq \left((1-x)d_1 - 24\sqrt{x \cdot d_1 \cdot d_4} \right) \lambda(f')^2 + (d_2 + 4d_3 + 12\sqrt{x \cdot d_1 \cdot d_4}) \lambda^3 (f'')^2 \\
& \quad + d_3 \lambda(\lambda^2 f''' + 2\lambda f'')^2 + \frac{d}{d\lambda} \left(\sum_{1 \leq i, j \leq 3, i+j \leq 4} d_{i,j} \lambda^{i+j} f^{(i)} f^{(j)} \right) \\
& \geq \left((1-x)d_1 - 24\sqrt{x \cdot d_1 \cdot d_4} \right) \lambda(f')^2 + (d_2 + 4d_3 + 12\sqrt{x \cdot d_1 \cdot d_4}) \lambda^3 (f'')^2 \\
& \quad + \frac{d}{d\lambda} \left(\sum_{1 \leq i, j \leq 3, i+j \leq 4} d_{i,j} \lambda^{i+j} f^{(i)} f^{(j)} \right).
\end{aligned} \tag{7.10}$$

If

$$(1-x)d_1 - 24\sqrt{x \cdot d_1 \cdot d_4} > 0 \tag{7.11}$$

and

$$d_2 + 4d_3 + 12\sqrt{x \cdot d_1 \cdot d_4} \geq 0 \tag{7.12}$$

hold simultaneously, then we have Theorem 7.2. By this method, we have the following

Lemma 7.9. *If $\frac{n+8}{n-8} < p < p_c(n)$, and $n \in [14, 20]$ except for $n = 17$, then there exist constant $d_{i,j}$ such that*

$$\sum_{j=1}^4 (A_j + a_j) \lambda^{2j-1} \left(\frac{d^j u^\lambda}{d\lambda^j} \right)^2 \geq \frac{d}{d\lambda} \left(\sum_{1 \leq i, j \leq 3, i+j \leq 4} d_{i,j} \lambda^{i+j} \frac{d^i u^\lambda}{d\lambda^i} \frac{d^j u^\lambda}{d\lambda^j} \right).$$

Proof. To prove this Lemma, we just need to check that the conditions (7.11) and (7.12) hold simultaneously when $n \in [14, 20]$ except for $n = 17$. We perform these case by case. To make sure that (7.11) hold, we need to select that $x \in [0, 1]$ satisfying

$$\frac{24^2 d_4 \cdot x}{(1-x)^2} < \min_{0 \leq k \leq \frac{n-8}{2}} d_1. \tag{7.13}$$

On the other hand, condition (7.12) can be obtained by the following two cases:

If $d_2 + 4d_3 \geq 0$, then (7.12) holds immediately;

If $d_2 + 4d_3 < 0$, then (7.12) is equivalent to $12^2 x \cdot d_1 \cdot d_4 - (d_2 + 4d_3)^2 > 0$.

(7.14)

For the simplicity, we denote that $d := d(k, n, x) = 12^2 x \cdot d_1 \cdot d_4 - (d_2 + 4d_3)^2$. Now we turn to consider the inequalities (7.13) and (7.14). Let

$$R_1(n) = \frac{n-10}{2} - d(n), \quad (7.15)$$

where $d(n)$ is defined in (1.3). Note that $p < p_c(n)$, then $\max\{R_1(n), 0\} < k$. Recall that $k := \frac{8}{p-1}$.

For $n = 14$, $\min_{0 \leq k \leq \frac{n-8}{2}|_{n=14}} d_1 = 46156$, then from (7.13) we get that $x \leq 0.8001464380$, thus we select $x = 0.8001464380$. Thus, $d_2 + 4d_3 < 0$ if $0 < k < 0.02572910109$ and $d(k, n, x)_{n=14, x=0.8001464380} > 0$ if $0 < k < 0.7919464848$. Hence, by selecting the parameter $x = 0.8001464380$, we can make sure that inequalities (7.13) and (7.14) hold simultaneously. Hence, Lemma 7.9 holds for the case $n = 14$.

Similarly we may prove Lemma 7.9 for $n = 15, 16, 17, 18, 19$. We omit these details. But we would like to give the proof for the case of $n = 20$ for reader's convenience.

Let $n = 20$. Then $\min_{0 \leq k \leq \frac{n-8}{2}|_{n=20}} d_1 = 216988$. From (7.13) we get that $x \leq 0.9021282144$. Thus we may select $x = 0.9021282144$. Hence, $d_2 + 4d_3 < 0$ if $0.9244642513 \approx R_1(n=20) < k < 1.026523007$ and $d(k, n, x)_{n=20, x=0.9021282144} > 0$ if $0.9244642513 \approx R_1(n=20) < k < 1.894875455$. So, by selecting the parameter $x = 0.9021282144$, then both inequalities (7.13) and (7.14) hold simultaneously. Therefore, Lemma 7.9 holds when $n = 20$. \square

Remark 7.2. For the case $n = 17$, this method of Lemma 7.9 does not work. But if $k > 0.02175341614$, then we have Lemma 7.9 hold. Since, when $n = 17$,

$$\min_{0 \leq k \leq \frac{n-8}{2}|_{n=17}} d_1 = 110656,$$

then from (7.13) we get that $x \leq 0.8657397553$, thus we select $x = 0.8657397553$. It follows that $d_2 + 4d_3 < 0$ if $0 < k < 0.5256119817$ and that

$$d(k, n, x)_{n=17, x=0.8657397553} > 0 \text{ if } 0.02175341614 < k < 1.358050900.$$

Thus, by selecting the parameter $x = 0.8657397553$ and $k > 0.02175341614$, then the inequalities (7.13) and (7.14) hold simultaneously, hence the Lemma 7.9 holds.

Remark 7.3. We note that the dimension $n = 17$ is critical dimension when we define the Joseph-Lundgren exponent in (1.2), it is the most thorny dimension when we establish Theorem 7.2.

But by getting more information from (7.10), we still have the following

Lemma 7.10. If $\frac{n+8}{n-8} < p < p_c(n)$ and $n = 17$, then there exist constant $d_{i,j}$ such that

$$\sum_{j=1}^4 (A_j + a_j) \lambda^{2j-1} \left(\frac{d^j u^\lambda}{d\lambda^j} \right)^2 \geq \frac{d}{d\lambda} \left(\sum_{1 \leq i, j \leq 3, i+j \leq 4} d_{i,j} \lambda^{i+j} \frac{d^i u^\lambda}{d\lambda^i} \frac{d^j u^\lambda}{d\lambda^j} \right).$$

Proof. In view of Remark 7.2, we only need to consider the case of $0 < k < 0.04$. From (7.10), we discard the term $d_3\lambda(\lambda^2 f''' + 2\lambda f'')^2$ (which is nonnegative term hence a "good" term) directly in our estimate, now we "pick up" and make full use of this term. To achieve this, let us select parameters $x_1, x_2, y \in [0, 1]$ whose exact values are to be determined later. We have

$$\begin{aligned} & y \cdot d_1 \cdot \lambda(f')^2 + d_3 \cdot \lambda(\lambda^2 f'' + 2\lambda f')^2 \\ &= x_1 \cdot y \cdot d_1 \cdot \lambda(f')^2 + d_3 \cdot \lambda(\lambda^2 f'' + 2\lambda f')^2 + (1 - x_1) \cdot y \cdot d_1 \cdot \lambda(f')^2 \\ &\geq 2\sqrt{x_1 \cdot y \cdot d_1 \cdot d_3} \lambda^3 (f'')^2 + \left((1 - x_1) \cdot y \cdot d_1 - 2\sqrt{x_1 \cdot y \cdot d_1 \cdot d_3} \right) \lambda(f')^2 \\ &\quad + \frac{d}{d\lambda} \left(2\sqrt{x_1 \cdot y \cdot d_1 \cdot d_3} (\lambda^3 f'' f' - \frac{1}{2}\lambda^2 (f')^2) \right) \end{aligned}$$

and

$$\begin{aligned} & (1 - y) \cdot d_1 \lambda(f')^2 + d_4 \lambda^7 (f''')^2 \\ &= x_2 \cdot (1 - y) \cdot d_1 \lambda(f')^2 + d_4 \lambda^7 (f''')^2 + (1 - x_2) \cdot (1 - y) \cdot d_1 \lambda(f')^2 \\ &\geq 2\sqrt{x_2 \cdot (1 - y) \cdot d_1 \cdot d_4} \lambda^4 f''' f' + (1 - x_2)(1 - y) d_1 \lambda(f')^2 \\ &= 2\sqrt{x_2(1 - y)d_1d_4} \left(-12\lambda(f')^2 + 6\lambda^3(f'')^2 \right) + (1 - x_2)(1 - y) d_1 \lambda(f')^2 \\ &\quad + \frac{d}{d\lambda} \left(2\sqrt{x_2(1 - y)d_1d_4} (\lambda^4 f''' f' - \frac{1}{2}\lambda^4 (f'')^2 - 4\lambda^3 f'' f' + 6\lambda^2 (f')^2) \right) \\ &= \left((1 - x_2)(1 - y)d_1 - 24\sqrt{x_2(1 - y) \cdot d_1 \cdot d_4} \right) \lambda(f')^2 \\ &\quad + 12\sqrt{x_2(1 - y) \cdot d_1 \cdot d_4} \lambda^3 (f'')^2 + \frac{d}{d\lambda} \left(\sum_{1 \leq i, j \leq 3, i+j \leq 4} d_{i,j} \lambda^{i+j} f^{(i)} f^{(j)} \right). \end{aligned}$$

Therefore from (7.8), combine with the above two inequalities, we get that

$$\begin{aligned} & \sum_{j=1}^4 (A_j + a_j) \lambda^{2j-1} \left(\frac{d^j u^\lambda}{d\lambda^j} \right)^2 := \sum_{j=1}^4 d_j \lambda^{2j-1} \left(\frac{d^j u^\lambda}{d\lambda^j} \right)^2 \\ &= d_1 \lambda(f')^2 + (d_2 + 4d_3) \lambda^3 (f'')^2 + d_3 \lambda(\lambda^2 f''' + 2\lambda f'')^2 \\ &\quad + d_4 \lambda^7 (f''')^2 + \frac{d}{d\lambda} (-2d_3 \lambda^4 (f'')^2) \\ &\geq (d_2 + 4d_3 + 2\sqrt{x_1 \cdot y \cdot d_1 \cdot d_3} + 12\sqrt{x_2(1 - y) \cdot d_1 \cdot d_4}) \lambda^3 (f'')^2 \\ &\quad + \left((1 - x_1) \cdot y \cdot d_1 + (1 - x_2)(1 - y) d_1 - 2\sqrt{x_1 \cdot y \cdot d_1 \cdot d_3} \right. \\ &\quad \left. - 24\sqrt{x_2(1 - y) \cdot d_1 \cdot d_4} \right) \lambda(f')^2 + \frac{d}{d\lambda} \left(\sum_{1 \leq i, j \leq 3, i+j \leq 4} d_{i,j} \lambda^{i+j} f^{(i)} f^{(j)} \right). \end{aligned}$$

We may establish the Lemma (7.10) if we have

$$(1 - x_1)y d_1 + (1 - x_2)(1 - y) d_1 - 2\sqrt{x_1 y d_1 d_3} - 24\sqrt{x_2(1 - y) d_1 d_4} > 0 \quad (7.16)$$

and

$$d_2 + 4d_3 + 2\sqrt{x_1 \cdot y \cdot d_1 \cdot d_3} + 12\sqrt{x_2(1-y) \cdot d_1 \cdot d_4} \geq 0 \quad (7.17)$$

hold simultaneously. We will select proper parameters $x_1, x_2, y \in [0, 1]$ to make sure the inequalities (7.16) and (7.17) hold under the condition $0 < k < 0.04$. For simplicity, we denote that

$$\begin{aligned} f_1 &:= \left((1-x_1)y + (1-x_2)(1-y) \right)^2 d_1 - 4x_1 y d_3 - 24^2 x_2 (1-y) d_4, \\ f_2 &:= f_1^2 - (4 \times 24)^2 x_1 y x_2 (1-y) d_3 d_4, \\ h_1 &:= (d_2 + 4d_3)^2 - 4x_1 y d_1 d_3 - 12^2 x_2 (1-y) d_1 d_4, \\ h_2 &:= 48^2 x_1 y x_2 (1-y) d_3 d_4 d_1^2 - h_1^2. \end{aligned}$$

Hence, (7.16) is equivalent to $f_1 > 0$ and $f_2 > 0$. As for (7.17) we observe that

$$\begin{aligned} \text{if } h_1 \leq 0, \text{ then (7.17) holds immediately,} \\ \text{if } h_1 > 0, \text{ then } h_2 > 0. \end{aligned} \quad (7.18)$$

How can we select the proper parameter x_1, x_2, y ? Numerically, by considering the end-point case, namely $y = 0$ and $y = 1$, then as in (7.13) to determine x_1, x_2 , we find that we may select y closing to 0 and x_1, x_2 closing to 1. In fact, let us select that $y = 0.1, x_1 = x_2 = 0.8$, we have that

$$\begin{aligned} h_1 &= 748.16k^8 - 25955.84k^7 + 2.5965440 \cdot 10^5 k^6 + 1.8144000 \cdot 10^5 k^5 \\ &\quad - 1.370264514 \cdot 10^7 k^4 + 2.783143502 \cdot 10^7 k^3 + 1.905355534 \cdot 10^8 k^2 \\ &\quad - 2.946605150 \cdot 10^8 k + 1.316646282 \cdot 10^7, \\ h_2 &= 2.689086 \cdot 10^{14} + 3.883824251 \cdot 10^7 k^{15} - 5.597433856 \cdot 10^5 k^{16} \\ &\quad + 1.045307292 \cdot 10^{16} k - 1.881709367 \cdot 10^{16} k^5 + 8.22093182 \cdot 10^{12} k^{10} \\ &\quad + 8.52692786 \cdot 10^{14} k^7 + 7.5489982 \cdot 10^{11} k^9 - 3.79196866 \cdot 10^{10} k^{12} \\ &\quad + 1.322360547 \cdot 10^{10} k^{13} + 1.138472739 \cdot 10^{17} k^3 - 1.062469518 \cdot 10^9 k^{14} \\ &\quad - 8.42043516 \cdot 10^{11} k^{11} - 1.943781276 \cdot 10^{16} k^4 - 3.114952088 \cdot 10^{14} k^8 \\ &\quad - 8.750277873 \cdot 10^{16} k^2 + 4.523393231 \cdot 10^{15} k^6, \\ f_1 &= -0.1600k^6 + 4.6400k^5 - 31.6800k^4 - 93.2800k^3 + 556.6400k^2 \\ &\quad + 4947.5200k + 2745.6000, \\ f_2 &= -40.20787k^8 + 31.6672860k^{10} + 7.3936205 \cdot 10^6 + 9492.1751k^7 \\ &\quad + 2.66134182 \cdot 10^7 k + 0.02560000k^{12} - 1.48480312k^{11} - 3.9194703 \cdot 10^5 k^5 \\ &\quad + 4.99650206 \cdot 10^6 k^3 - 7.8719259 \cdot 10^5 k^4 + 2.75922586 \cdot 10^7 k^2 \\ &\quad + 18460.169k^6 - 264.138939k^9. \end{aligned}$$

We can verify that $f_1 > 0$ if $0 < k < 13.82353260$; $f_2 > 0$ if $0 < k < 9.306459393$; $h_1 > 0$ if $0 < k < 0.04606463340$; $h_2 > 0$ if $0 < k < 0.1757218049$.

Thus, when $0 < k < 0.04$, we have that f_1, f_2, h_1, h_2 all are positive. Hence, combine with (7.18), we know that (7.16) and (7.17) hold. This proves the Lemma 7.10. \square

8 Homogeneous stable solutions must be zero solution

Firstly, we establish the representations of the harmonic, biharmonic, triharmonic and quadharmonic operators in terms of the spherical coordinates. We continue from the Section 4. By a direct calculation we have

$$\begin{aligned} \Delta^4 u = & (\partial_{rr} + \frac{n-1}{r}\partial_r)^4 + \Delta_\theta^4(\partial_{rr} + \frac{n-1}{r}\partial_r)(r^{-8}u) + \Delta_\theta\left((\partial_{rr} + \frac{n-1}{r}\partial_r)^2\right. \\ & \left(r^{-2}(\partial_{rr} + \frac{n-1}{r}\partial_r)u\right) + (\partial_{rr} + \frac{n-1}{r}\partial_r)^3(r^{-2}u) + (\partial_{rr} + \frac{n-1}{r}\partial_r) \\ & \left(r^{-2}(\partial_{rr} + \frac{n-1}{r}\partial_r)^2u\right) + r^{-2}(\partial_{rr} + \frac{n-1}{r}\partial_r)^3u \\ & + \Delta_\theta^2\left((\partial_{rr} + \frac{n-1}{r}\partial_r)^2(r^{-4}u) + (\partial_{rr} + \frac{n-1}{r}\partial_r)(r^{-4}(\partial_{rr} + \frac{n-1}{r}\partial_r)u)\right) \\ & + (\partial_{rr} + \frac{n-1}{r}\partial_r)(r^{-2}(\partial_{rr} + \frac{n-1}{r}\partial_r)r^{-2}u) + r^{-2}(\partial_{rr} + \frac{n-1}{r}\partial_r) \\ & \left(r^{-2}(\partial_{rr} + \frac{n-1}{r}\partial_r)u\right) + r^{-2}(\partial_{rr} + \frac{n-1}{r}\partial_r)^2(r^{-2}u) + r^{-4}(\partial_{rr} + \frac{n-1}{r}\partial_r)^2u \\ & + \Delta_\theta^3\left((\partial_{rr} + \frac{n-1}{r}\partial_r)(r^{-6}u) + r^{-2}(\partial_{rr} + \frac{n-1}{r}\partial_r)(r^{-4}u)\right. \\ & \left.+ r^{-6}(\partial_{rr} + \frac{n-1}{r}\partial_r)u + r^{-4}(\partial_{rr} + \frac{n-1}{r}\partial_r)(r^{-2}u)\right). \end{aligned}$$

Then, let $u = r^{-k}w(\theta)$, recall that $k = \frac{8}{p-1}$, by a direct calculation, the function w satisfy

$$\Delta_\theta^4 w - J_3 \Delta_\theta^3 w + J_2 \Delta_\theta^2 w - J_1 \Delta_\theta w + J_0 w = |w|^{p-1}w, \quad (8.1)$$

where

$$\begin{aligned} J_0 &= k(k+2-n)(k+2)(k+4-n)(k+4)(k+6-n)(k+6)(k+8-n), \\ -J_1 &= k(k+2-n)(k+4)(k+6-n)(k+6)(k+8-n) \\ &+ (k+2)(k+4-n)(k+4)(k+6-n)(k+6)(k+8-n) \\ &+ k(k+2-n)(k+2)(k+4-n)(k+6)(k+8-n) \\ &+ k(k+2-n)(k+2)(k+4-n)(k+4)(k+6-n), \end{aligned} \quad (8.2)$$

$$\begin{aligned} J_2 &= (k+4)(k+6-n)(k+6)(k+8-n) + k(k+2-n)(k+6)(k+8-n) \\ &+ (k+2)(k+4-n)(k+6)(k+8-n) + k(k+2-n)(k+4)(k+6-n) \\ &+ (k+2)(k+4-n)(k+4)(k+6-n) + k(k+2-n)(k+2)(k+4-n), \end{aligned} \quad (8.3)$$

and

$$\begin{aligned} -J_3 &= (k+6)(k+8-n) + (k+4)(k+6-n) + k(k+2-n) \\ &\quad + (k+2)(k+4-n). \end{aligned} \quad (8.4)$$

Since $w \in W^{4,2}(\mathbb{S}^{n-1}) \cap L^{p+1}(\mathbb{S}^{n-1})$, $r^{-\frac{n-8}{2}}w(\theta)\eta_\varepsilon(r)$ can be approximated by $C_0^\infty(B_{4/\varepsilon} \setminus B_{\varepsilon/4})$ functions in $W^{4,2}(B_{2/\varepsilon} \setminus B_{\varepsilon/2}) \cap L^{p+1}(B_{2/\varepsilon} \setminus B_{\varepsilon/2})$. Hence, here we may insert the stability condition of u and choose a test function of the form $\varphi = r^{-\frac{n-8}{2}}w(\theta)\eta_\varepsilon(r)$. Note that

$$\begin{aligned} \Delta^2\varphi &= r^{-\frac{n}{2}}\eta_\varepsilon(r)\left(q(q+2-n)(q+2)(q+4-n)w(\theta) + ((q+2)(q+4-n)\right. \\ &\quad \left.+ q(q+2-n))\Delta_\theta w + \Delta_\theta^2 w\right) + \sum_{j=1}^4 c_j r^{-\frac{n}{2}+j}\eta_\varepsilon^{(j)}(r)w(\theta), \end{aligned}$$

where $q = \frac{n-8}{2}$, c_j are some constants, $\eta^{(j)} := \frac{d^j}{dr^j}\eta_\varepsilon$ for $j \geq 1$ and $\eta_\varepsilon^{(0)} := \eta_\varepsilon$. Hence we have that

$$\begin{aligned} \int_{\mathbb{R}^n} |\Delta^2\varphi|^2 &= \left(\int_0^\infty r^{-1}\eta_\varepsilon^2(r)dr\right)\left(\int_{\mathbb{S}^{n-1}} q_0 w^2(\theta) + q_1 |\nabla_\theta w|^2 + q_2 |\Delta_\theta w|^2\right. \\ &\quad \left.+ q_3 |\nabla_\theta \Delta_\theta w|^2 + |\Delta_\theta^2 w|^2\right) + \int_0^\infty \sum_{1 \leq i+j \leq 8, i,j \geq 0} c_{i,j} r^{i+j-1} \eta_\varepsilon^{(i)} \eta_\varepsilon^{(j)} dr \\ &\quad \cdot \left(\int_{\mathbb{S}^{n-1}} w^2(\theta) + |\nabla_\theta w|^2 + |\Delta_\theta w|^2 + |\nabla_\theta \Delta_\theta w|^2 + |\Delta_\theta^2 w|^2\right), \end{aligned} \quad (8.5)$$

where

$$\begin{aligned} q_0 &= \left(q(q+2-n)(q+2)(q+4-n)\right)^2, \\ q_1 &= 2\left((q+2)(n-q-4) + q(n-q-2)\right)q(q+2-n)(q+2)(q+4-n) \\ q_2 &= \left((q+2)(q+4-n) + q(q+2-n)\right)^2 + 2q(q+2-n)(q+2)(q+4-n) \\ q_3 &= 2\left((q+2)(n-q-4) + q(n-q-2)\right) \end{aligned} \quad (8.6)$$

Substituting this into the stability condition for u , we get that

$$p\left(\int_{\mathbb{S}^{n-1}} |w|^{p+1} d\theta\right) \cdot \left(\int_0^\infty r^{-1}\eta_\varepsilon^2(r)dr\right) \leq \int_{\mathbb{R}^n} |\Delta^2\varphi|^2. \quad (8.7)$$

Notice that

$$\begin{aligned} \int_0^\infty r^{-1}\eta_\varepsilon^2(r)dr &\geq |\log \varepsilon| \rightarrow +\infty, \quad \text{as } \varepsilon \rightarrow 0, \\ \int_0^\infty \sum_{1 \leq i+j \leq 8, i,j \geq 0} r^{i+j-1} |\eta_\varepsilon^{(i)} \eta_\varepsilon^{(j)}| dr &\leq C, \quad \text{for any } i, j. \end{aligned}$$

Combine with (8.5) and (8.7), we obtain that

$$\begin{aligned} p \int_{\mathbb{S}^{n-1}} |w|^{p+1} d\theta &\leq \int_{\mathbb{S}^{n-1}} q_0 w^2(\theta) + q_1 |\nabla_\theta w|^2 + q_2 |\Delta_\theta w|^2 \\ &\quad + q_3 |\nabla_\theta \Delta_\theta w|^2 + |\Delta_\theta^2 w|^2, \end{aligned}$$

Combining this with (8.1), we have the following estimate,

$$\begin{aligned} &\int_{\mathbb{S}^{n-1}} (pJ_0 - q_0) w^2(\theta) + (pJ_1 - q_1) |\nabla_\theta w|^2 + (pJ_2 - q_2) |\Delta_\theta w|^2 \\ &\quad + (pJ_3 - q_3) |\nabla_\theta \Delta_\theta w|^2 + (p-1) |\Delta_\theta^2 w|^2 \leq 0. \end{aligned} \tag{8.8}$$

Notice that $pJ_0 - q_0 = 0$ is equivalent to

$$p \frac{\Gamma(\frac{n}{2} - \frac{4}{p-1}) \Gamma(4 + \frac{4}{p-1})}{\Gamma(\frac{4}{p-1}) \Gamma(\frac{n-8}{2} - \frac{4}{p-1})} = \frac{\Gamma(\frac{n+8}{4})^2}{\Gamma(\frac{n-8}{4})^2}, \tag{8.9}$$

see [17]. To solve this, let us the transform $k := \frac{8}{p-1}$ and $k := \frac{n-10}{2} - a$, then we can reduce the equation $pJ_0 - q_0 = 0$ to the following

$$\begin{aligned} &n^4 a^8 + (-n^5 - 20n^3)a^6 + (\frac{3}{8}n^6 + 5n^4 + 118n^2)a^4 \\ &+ (-\frac{1}{16}n^7 + \frac{5}{4}n^5 - 2n^3 - 180n)a^2 \\ &+ 81 + \frac{1}{16}n^7 - \frac{7}{16}n^6 - 2n^5 + \frac{115}{8}n^4 + 16n^3 - 109n^2 = 0. \end{aligned}$$

We further let that $a^2 = t$ then the above equation is reduced to a fourth-order equation. Find the roots, they are Let

$$R_1(n) = \frac{n-10}{2} - d(n), \quad R_2(n) = \frac{n-10}{2} + d(n), \tag{8.10}$$

where $d(n)$ is defined in (1.3). Transform back (note that $p > \frac{n+8}{n-8}$, hence $0 < k < \frac{n-8}{2}$ and thus only root satisfy this), we get the so called Joseph-Lundgren exponent, see (1.2). We have the following lemma after a direct calculation.

Lemma 8.1. *For the $d(n)$ appears in the Joseph-Lundgren exponent, see (1.2), we have*

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{d(n)}{\sqrt{n}} &= 1, \\ d(n) &< \sqrt{n} \text{ for } n \geq 18. \end{aligned}$$

Remark 8.1. *This Lemma gives optimal bound for $d(n)$ in the sense that if find the optimal constant B satisfy*

$$d(n) < B\sqrt{n} \text{ for } n \geq n_0$$

Lemma 8.2. If $\frac{n-10}{2} - \sqrt{n} < k < \frac{n-10}{2} + \sqrt{n}$ for $n \geq 118$, we have

$$pJ_1 - q_1 > 0.$$

Proof. Equivalently, we consider the function $W_1 := k(pJ_1 - q_1)$. From (8.2) and (8.6), we have that

$$\begin{aligned} W_1 = & -4k^7 + (-128 + 12n)k^6 + (-12n^2 + 324n - 1696)k^5 + (4n^3 - 264n^2 \\ & + 3488n - 12032)k^4 + (68n^3 - 2168n^2 + 19056n - 49216)k^3 + (376n^3 \\ & - 8176n^2 + 55104n - 115712)k^2 + (-144384 - \frac{1}{16}n^6 + \frac{3}{4}n^5 + 732n^3 \\ & - 13520n^2 + 78336n)k + 384n^3 - 6912n^2 + 39936n - 73728. \end{aligned}$$

Set $k = \frac{n-10}{2} + a\sqrt{n}$, $n = t^2$ (this is a key point). Hence, by the assumption we have that $-1 \leq a \leq 1$. Thus

$$\begin{aligned} W_1 = & -108 + (\frac{1}{2} - \frac{3}{8}a^2)t^{12} + (\frac{3}{4}a^3 - \frac{3}{4}a)t^{11} + (-\frac{51}{8} + \frac{3}{2}a^4 + \frac{9}{4}a^2)t^{10} \\ & + (-3a^5 + \frac{9}{4}a)t^9 + (\frac{3}{4} - 2a^6 - 9a^4 + 11a^2)t^8 + (4a^7 + 2a^3 + 28a)t^7 \\ & + (\frac{351}{2} + 12a^6 - 2a^4 - 10a^2)t^6 + (-44a^5 - 32a^3 - 47a)t^5 \\ & + (-132a^4 - 150a^2 - 157)t^4 + (76a^3 - 224a)t^3 + (228a^2 - 1126)t^2 - 36at. \end{aligned}$$

For the case of $0 \leq a \leq 1$, we get from the above identity that

$$\begin{aligned} W_1 \geq & -108 + \frac{1}{8}t^{12} - \frac{3}{4}t^{11} - \frac{51}{8}t^{10} - \frac{9}{4}t^9 + \frac{3}{4}t^8 + \frac{351}{2}t^6 - 123t^5 - 439t^4 \\ & - 224t^3 - 1126t^2 - 36t > 0 \text{ if } n = t^2 \geq 117 (t \geq 10.7725). \end{aligned}$$

For the case of $0 \leq a \leq 1$, we get that

$$\begin{aligned} W_1 \geq & -108 + \frac{1}{8}t^{12} - \frac{3}{4}t^{11} - \frac{51}{8}t^{10} - \frac{9}{4}t^9 + \frac{3}{4}t^8 - 34t^7 + \frac{351}{2}t^6 - 439t^4 \\ & - 76t^3 - 1354t^2 > 0 \text{ if } n = t^2 \geq 118 (t \geq 10.8562627). \end{aligned}$$

□

Lemma 8.3. If $\frac{n-10}{2} - \sqrt{n} < k < \frac{n-10}{2} + \sqrt{n}$ for $n \geq 30$, we have

$$pJ_2 - q_2 > 0.$$

Proof. Equivalently, we consider the function $W_2 := k(pJ_2 - q_2)$. From (8.3) and (8.6), we have that

$$\begin{aligned} W_2 = & 6k^5 + (144 - 12n)k^4 + (6n^2 - 228n + 1352)k^3 + (84n^2 - 1544n \\ & + 6272)k^2 + (14464 - \frac{3}{8}n^4 + 3n^3 + 338n^2 - 4512n)k + 352n^2 \\ & - 4480n + 13824. \end{aligned}$$

Set $k = \frac{n-10}{2} + a\sqrt{n}$, $n = t^2$, hence $-1 \leq a \leq 1$. It follows that

$$\begin{aligned} W_2 = & 554 + (3 - \frac{3}{2}a^2)t^8 + (3a^3 - 3a)t^7 + (-\frac{51}{2} + 3a^4 + 3a^2)t^6 \\ & + (-6a^5 - 3a)t^5 + (-6a^4 + 34a^2 - 51)t^4 + (28a^3 + 80a)t^3 \\ & + (92a^2 + 427)t^2 + 106at. \end{aligned}$$

For the case of $0 \leq a \leq 1$, from the above inequality we have that

$$\begin{aligned} W_2 \geq & 554 + \frac{3}{2}t^8 - 3t^7 - \frac{51}{2}t^6 - 9t^5 - 57t^4 + 427t^2 \\ & > 0 \text{ if } n = t^2 \geq 30 (t \geq 5.475795). \end{aligned}$$

For the case of $-1 \leq a \leq 0$, we observe that

$$\begin{aligned} W_2 \geq & 554 + \frac{3}{2}t^8 - 3t^7 - \frac{51}{2}t^6 - 57t^4 - 108t^3 + 427t^2 - 106t \\ & > 0 \text{ if } n = t^2 \geq 28 (t \geq 5.257108771). \end{aligned}$$

□

Lemma 8.4. If $\frac{n-10}{2} - \sqrt{n} < k < \frac{n-10}{2} + \sqrt{n}$ for $n \geq 11$, we have

$$pJ_3 - q_3 > 0.$$

Proof. We consider the function $W_3 := k(pJ_3 - q_3)$. From (8.4) and (8.6), we have that

$$W_3 = -4k^3 + (-64 + 4n)k^2 + (-n^2 + 48n - 320)k - 640 + 96n.$$

Set $k = \frac{n-10}{2} + a\sqrt{n}$, $n = t^2$, then $-1 \leq a \leq 1$. Thus,

$$W_3 = -140 + (-2a^2 + 8)t^4 + (4a^3 - 4a)t^3 + (-4a^2 - 34)t^2 - 20at.$$

Since $a \in [-1, 1]$, from the above inequality we get that

$$W_3 \geq -140 + 6t^4 - 4t^3 - 38t^2 - 20t > 0 \text{ if } n = t^2 \geq 12 \text{ (i.e., } t \geq 3.40511).$$

□

For the lower dimension case, we can calculate numerically and thus we have the following Lemma

Lemma 8.5. Consider the supercritical case $p > \frac{n+8}{n-8}$, i.e., $0 < k < \frac{n-8}{2}$. We have the following facts.

- (1) If $0 < k < \frac{n-8}{2}$ and $n \leq 17$, then $pJ_1 - q_1, pJ_2 - q_2, pJ_3 - q_3 > 0$;
- (2) If $18 \leq n \leq 120$ and $R_1(n) < k < R_2(n)$, then $pJ_1 - q_1 > 0$;
- (3) If $18 \leq n \leq 29$ and $R_1(n) < k < R_2(n)$, then $pJ_2 - q_2 > 0$;

where $R_1(n), R_2(n)$ are given in (8.10).

Theorem 8.1. Let $u \in W_{loc}^{4,2}(\mathbb{R} \setminus \{0\})$ be a homogeneous and stable solution of (1.1) with $\frac{n+8}{n-8} < p < p_c(n)$. Assume that $|u|^{p+1} \in L^{p+1}(\mathbb{R}^n \setminus \{0\})$, then $u \equiv 0$.

Proof. From the inequality (8.8), combine with Lemmas 8.2, 8.3, 8.4 and 8.5, we may get the conclusion easily. \square

9 Energy estimates and Blow down analysis

In this section, we finish the energy estimates for the solutions to (1.1), which are important when we perform a blow-down analysis in the next section.

9.1 Energy estimates

Lemma 9.1. Let u be a stable solution of (1.1), then there exists a positive constant C independent of R such that

$$\begin{aligned} & \int_{\mathbb{R}^n} |u|^{p+1} \eta^2 + \int_{\mathbb{R}^n} |\Delta^2 u|^2 \eta^2 \\ & \leq C \left[\int_{\mathbb{R}^n} |\nabla \Delta u|^2 \left| \frac{\nabla \eta^2}{\eta} \right|^2 + |\Delta u|^2 \left(\left(\frac{\Delta \eta^2}{\eta} \right)^2 \right. \right. \\ & \quad \left. \left. + \left(\frac{\nabla^2 \eta^2}{\eta} \right)^2 \right) + |\nabla u|^2 \left(\frac{\nabla \Delta \eta^2}{\eta} \right)^2 + u^2 \left(\frac{\Delta^2 \eta^2}{\eta} \right)^2 \right] \end{aligned} \quad (9.1)$$

Proof. Multiply the equation (1.1) with $u\eta^2$, where η is a test function, we get that

$$\int_{\mathbb{R}^n} |u|^{p+1} \eta^2 = \int_{\mathbb{R}^n} (\Delta^4 u) u \eta^2 = \int_{\mathbb{R}^n} (\Delta^2 u) \Delta^2 (u \eta^2). \quad (9.2)$$

Since $\Delta(\xi\eta) = \eta\Delta\xi + \xi\Delta\eta + 2\nabla\xi\nabla\eta$, we have

$$\begin{aligned} \Delta^2 u \Delta^2 (u \eta^2) &= (\Delta^2 u)^2 \eta^2 + 2\Delta^2 u \Delta u \Delta \eta^2 + 2\Delta^2 u \nabla \eta^2 \nabla \Delta u \\ &\quad + u \Delta u \Delta^2 \eta^2 + 2\Delta^2 u \nabla u \nabla \Delta \eta^2 + 2\Delta^2 u \Delta (\nabla u \nabla \eta^2) \\ &= (\Delta^2 u)^2 \eta^2 + \text{the combination of terms with lower order than } (\Delta^2 u)^2. \end{aligned}$$

Further

$$\begin{aligned} |\Delta^2(u\eta)|^2 &= (\Delta^2 u)^2 \eta^2 + 4(\Delta u)^2 (\Delta \eta)^2 + u^2 (\Delta^2 \eta)^2 + 4(\nabla u \nabla \Delta \eta)^2 \\ &\quad + 4(\nabla \eta \nabla \Delta u)^2 + 4|\Delta(\nabla u \nabla \eta)|^2 + 4\Delta^2 u \eta \Delta u \Delta \eta + 2(\Delta^2 u) \eta u \Delta^2 \eta \\ &\quad + 4(\Delta^2 u) \eta \nabla u \nabla \Delta \eta + 4(\Delta^2 u) \eta \nabla \eta \nabla \Delta u + 4(\Delta^2 u) u \eta \Delta (\nabla u \nabla \eta) \\ &\quad + 4\Delta u (\Delta \eta) u \Delta^2 \eta + 8\Delta u \Delta \eta \nabla u \nabla \Delta \eta + 8\Delta u \Delta \eta \nabla \eta \nabla \Delta u \\ &\quad + 8\Delta u \Delta \eta \Delta (\nabla u \nabla \eta) + 4u \Delta^2 \eta \nabla u \nabla \Delta \eta + 4u \Delta^2 \eta \nabla \eta \nabla \Delta u \\ &\quad + 4u \Delta^2 \eta \Delta (\nabla u \nabla \eta) + 8(\nabla u \nabla \Delta \eta) (\nabla \eta \nabla \Delta u) \\ &\quad + 8\nabla u \nabla \Delta \eta \Delta (\nabla u \nabla \eta) + 8\nabla \eta \nabla \Delta u \Delta (\nabla u \nabla \eta) \\ &= (\Delta^2 u)^2 \eta^2 + \text{the combination of terms with lower order than } (\Delta^2 u)^2. \end{aligned} \quad (9.3)$$

On the other hand, by the stability condition, we have

$$p \int_{\mathbb{R}^n} |u|^{p+1} \eta^2 \leq \int_{\mathbb{R}^n} |\Delta^2(u\eta)|^2. \quad (9.4)$$

Combining with (9.2), (9.3) and (9.4), via the basic Cauchy inequality, we prove the lemma. \square

Lemma 9.2. *Let u be a stable solution of (1.1). Then*

$$\int_{B_R} |u|^{p+1} + \int_{B_R} |\Delta^2 u|^2 \leq CR^{-8} \int_{B_{2R}} u^2, \quad (9.5)$$

$$\int_{B_R} |u|^{p+1} + \int_{B_R} |\Delta^2 u|^2 \leq CR^{n-8\frac{p+1}{p-1}}. \quad (9.6)$$

Proof. Now let $\eta = \xi^m$, where $m > 4$, $\xi = 1$ in $B_{R/2}$ and $\xi = 0$ in $B_R^c C$ satisfying $|\nabla \xi| \leq \frac{C}{R}$. Plug into the estimates in the previous lemma. Then

$$\begin{aligned} \int_{\mathbb{R}^n} |\Delta^2 u|^2 \xi^{8m} + \int_{\mathbb{R}^n} |u|^{p+1} \xi^{8m} &\leq C \left(\int_{\mathbb{R}^n} u^2 g_0(\xi) + \int_{\mathbb{R}^n} |\nabla u|^2 g_1(\xi) \right. \\ &\quad \left. + \int_{\mathbb{R}^n} |\Delta u|^2 g_2(\xi) + \int_{\mathbb{R}^n} |\nabla \Delta u|^2 g_3(\xi) \right), \end{aligned} \quad (9.7)$$

where

$$\begin{aligned} g_0(\xi) &:= \xi^{8m-8} \sum_{0 \leq i+j+k+s+t+u+v+w=8} |\nabla^i \xi| |\nabla^j \xi| |\nabla^k \xi| |\nabla^s \xi| |\nabla^t \xi| |\nabla^u \xi| |\nabla^v \xi| |\nabla^w \xi|, \\ g_1(\xi) &:= \xi^{8m-6} \sum_{0 \leq i+j+k+s+t+u=6} |\nabla^i \xi| |\nabla^j \xi| |\nabla^k \xi| |\nabla^s \xi| |\nabla^t \xi| |\nabla^u \xi|, \\ g_2(\xi) &:= \xi^{8m-4} \sum_{0 \leq i+j+k+s=4} |\nabla^i \xi| |\nabla^j \xi| |\nabla^k \xi| |\nabla^s \xi|, \\ g_3(\xi) &:= \xi^{8m-2} \sum_{0 \leq i+j=2} |\nabla^i \xi| |\nabla^j \xi|, \end{aligned}$$

here $\nabla^0 \xi := \xi$ and notice that $g_m(\xi) \geq 0$ for $m = 0, 1, 2, 3$. Now we claim that

$$\begin{aligned} g_1^2(\xi) &\leq C g_0(\xi) g_2(\xi), \quad g_2^2(\xi) \leq C g_1(\xi) g_3(\xi), \quad g_3^2(\xi) \leq C \xi^{8m} g_2(\xi), \\ |\nabla^2 g_3(\xi)| &\leq C g_2(\xi), \quad |\nabla^2 g_2(\xi)| \leq C g_1(\xi), \quad |\nabla^2 g_1(\xi)| \leq C g_0(\xi). \end{aligned}$$

This claim can be checked directly and will be frequently applied to our estimates below. In the next, we evaluate every term in the right hand side of (9.7). By a integrate by part, we have

$$|\nabla \Delta u|^2 = \frac{1}{2} \Delta(\Delta u)^2 - \Delta^2 u \Delta u.$$

It follows that

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla \Delta u|^2 g_3(\xi) &= \frac{1}{2} \int_{\mathbb{R}^n} \Delta(\Delta u)^2 g_3(\xi) - \int_{\mathbb{R}^n} \Delta^2 u \Delta u g_3(\xi) \\
&= \frac{1}{2} \int_{\mathbb{R}^n} (\Delta u)^2 \Delta g_3(\xi) - \int_{\mathbb{R}^n} \Delta^2 u \Delta u g_3(\xi) \\
&\leq \varepsilon_3 \int_{\mathbb{R}^n} |\Delta^2 u|^2 \xi^{8m} + C(\varepsilon_3) \int_{\mathbb{R}^n} |\Delta u|^2 g_2(\xi),
\end{aligned} \tag{9.8}$$

where ε_3 is a parameter to be determined later. Integrating by part again we have

$$(\Delta u)^2 = \sum_{j,k} (u_j u_k)_{jk} - \sum_{j,k} (u_{jk})^2 - 2 \nabla \Delta u \nabla u,$$

hence $(\Delta u)^2 \leq \sum_{j,k} (u_j u_k)_{jk} - 2 \nabla \Delta u \nabla u$. By which we get

$$\begin{aligned}
\int_{\mathbb{R}^n} (\Delta u)^2 g_2(\xi) &\leq \int_{\mathbb{R}^n} \sum_{j,k} (u_j u_k)_{jk} g_2(\xi) - 2 \int_{\mathbb{R}^n} \nabla \Delta u \nabla u g_2(\xi) \\
&= \int_{\mathbb{R}^n} \sum_{j,k} u_j u_k g_2(\xi)_{jk} - 2 \int_{\mathbb{R}^n} \nabla \Delta u \nabla u g_2(\xi) \\
&\leq C(\varepsilon_2) \int_{\mathbb{R}^n} |\nabla u|^2 g_1(\xi) + \varepsilon_2 \int_{\mathbb{R}^n} |\nabla \Delta u|^2 g_3(\xi),
\end{aligned} \tag{9.9}$$

where ε_2 is a parameter to be determined later. From the differential identity, $|\nabla u|^2 = \frac{1}{2} \Delta u^2 - u \Delta u$, we get that

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla u|^2 g_1(\xi) &= \frac{1}{2} \int_{\mathbb{R}^n} \Delta u^2 g_1(\xi) - \int_{\mathbb{R}^n} u \Delta u g_1(\xi) \\
&= \frac{1}{2} \int_{\mathbb{R}^n} u^2 g_1(\xi) - \int_{\mathbb{R}^n} u \Delta u g_1(\xi) \\
&\leq C(\varepsilon_1) \int_{\mathbb{R}^n} u^2 g_0(\xi) + \varepsilon_1 \int_{\mathbb{R}^n} (\Delta u)^2 g_2(\xi).
\end{aligned} \tag{9.10}$$

Combining with (9.8), (9.9) and (9.10), by selecting the parameters $\varepsilon_1, \varepsilon_2$ small enough, we can obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\nabla u|^2 g_1(\xi) + \int_{\mathbb{R}^n} |\Delta u|^2 g_2(\xi) + \int_{\mathbb{R}^n} |\nabla \Delta u|^2 g_3(\xi) \\
&\leq C(\varepsilon_3) \int_{\mathbb{R}^n} u^2 g_0(\xi) + \varepsilon_3 \int_{\mathbb{R}^n} |\Delta^2 u|^2 \xi^{8m}.
\end{aligned}$$

Combining the above estimate with (9.7) and selecting ε_3 small enough, we have that

$$\int_{\mathbb{R}^n} |\Delta^2 u|^2 \xi^{8m} + \int_{\mathbb{R}^n} |u|^{p+1} \xi^{8m} \leq C \int_{\mathbb{R}^n} u^2 g_0(\xi).$$

This proves (9.5). Further, we let $\xi = 1$ in B_R and $\xi = 0$ in B_{2R}^c , satisfying $|\nabla \xi| \leq \frac{C}{R}$, then we have

$$\begin{aligned} & \int_{\mathbb{R}^n} |\Delta^2 u|^2 \xi^{8m} + \int_{\mathbb{R}^n} |u|^{p+1} \xi^{8m} \leq C \int_{\mathbb{R}^n} u^2 g_0(\xi) \leq CR^{-8} \int_{\mathbb{R}^n} u^2 \xi^{8m-8} \\ & \leq CR^{-8} \int_{\mathbb{R}^n} (|u|^{p+1} \xi^{(4m-4)(p+1)})^{\frac{2}{p+1}} R^{n(1-\frac{2}{p+1})}. \end{aligned}$$

By choosing $m = \frac{p+1}{p-1}$, hence $(4m-4)(p+1) = 8m$, it follows that (9.6) holds. \square

9.2 Blow-down analysis and the proof of Theorem 1.1

The proof of Theorem 1.1. Firstly, we consider $1 < p \frac{n+8}{n-8}$. If $p < \frac{n+8}{n-8}$, we can let $R \rightarrow +\infty$ in (9.6) to get $u \equiv 0$ directly. For $p = \frac{n+8}{n-8}$, hence $n = 8 \frac{p+1}{p-1}$, (9.6) gives that

$$\int_{\mathbb{R}^n} |\Delta^2 u|^2 + |u|^{p+1} < +\infty.$$

Hence

$$\lim_{R \rightarrow +\infty} \int_{B_{2R}(x) \setminus B_R(x)} |\Delta^2 u|^2 + |u|^{p+1} = 0.$$

Then by Lemma 9.2 and noting that now $n = 8 \frac{p+1}{p-1}$, we have

$$\begin{aligned} & \int_{B_R(x)} |\Delta^2 u|^2 + |u|^{p+1} \leq CR^{-8} \int_{B_{2R} \setminus B_R(x)} u^2 \\ & \leq CR^{-8} \left(\int_{B_{2R} \setminus B_R(x)} |u|^{p+1} \right)^{\frac{2}{p+1}} R^{n(1-\frac{2}{p+1})} \leq \left(\int_{B_{2R} \setminus B_R(x)} |u|^{p+1} \right)^{\frac{2}{p+1}}, \end{aligned}$$

letting $R \rightarrow +\infty$, we get that $u \equiv 0$.

Secondly, we consider the supercritical case, i.e., $p > \frac{n+8}{n-8}$. We divide the proof into several steps.

Step 1: $\lim_{\lambda \rightarrow +\infty} E(u, 0, \lambda) < +\infty$. From Theorem 2.2, we know that E is non-decreasing w.r.t. λ , note that

$$E(u, 0, \lambda) \leq \frac{1}{\lambda} \int_{\lambda}^{2\lambda} E(u, 0, t) dt \leq \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} \int_t^{t+\lambda} E(u, 0, \gamma) d\gamma dt,$$

where $C > 0$ is constant independent of γ . From Lemma 9.2, we have that

$$\frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} \int_t^{t+\lambda} \gamma^{8 \frac{p+1}{p-1} - n} \left[\int_{B_\gamma} \frac{1}{2} |\Delta^2 u|^2 dx - \frac{1}{p+1} \int_{B_\gamma} |u|^{p+1} dx \right] d\gamma dt \leq C,$$

where $C > 0$ is independent of γ . Further,

$$\begin{aligned}
& \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} \int_t^{t+\lambda} \int_{\partial B_\gamma} \gamma^{8\frac{p+1}{p-1}-n-7} [C_0 u + C_1 \gamma \partial_r u + C_2 \gamma^2 \partial_{rr} u + C_3 \gamma^3 \partial_{rrr} u] \\
& \quad [C_0^1 u + C_1^1 \gamma \partial_r u + C_2^1 \gamma^2 \partial_{rr} u] \\
& \leq C \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} \int_t^{t+\lambda} t^{8\frac{p+1}{p-1}-n-8} \int_{\partial B_\gamma} [u^2 + \gamma^2 (\partial_r u)^2 + \gamma^4 (\partial_{rr} u)^2 + \gamma^6 (\partial_{rrr} u)^2] \\
& \leq C \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} t^{8\frac{p+1}{p-1}-n-8} \int_{B_{3\lambda}} [u^2 + \gamma^2 (\partial_r u)^2 + \gamma^4 (\partial_{rr} u)^2 + \gamma^6 (\partial_{rrr} u)^2] \\
& \leq C \lambda^{n-8\frac{p+1}{p-1}+8} \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} t^{8\frac{p+1}{p-1}-n-7} dt \\
& \leq C.
\end{aligned} \tag{9.11}$$

Integrating by part if necessary, the remaining terms can be treated similarly as the estimate of (9.11).

Step 2: For any $\lambda > 0$, recall the definition

$$u^\lambda(x) := \lambda^{\frac{8}{p-1}} u(\lambda x)$$

and u^λ is also a smooth stable solution of (1.1) in \mathbb{R}^n . By rescaling the estimate (9.6) in Lemma 9.2, for any $\lambda > 0$ and balls $B_r(x) \subset \mathbb{R}^n$, we have that

$$\int_{B_r(x)} |\Delta^2 u^\lambda|^2 + |u^\lambda|^{p+1} \leq C r^{n-8\frac{p+1}{p-1}}.$$

In particular, u^λ are uniformly bounded in $L_{loc}^{p+1}(\mathbb{R}^n)$ and $\Delta^2 u^\lambda$ are uniformly bounded in $L_{loc}^2(\mathbb{R}^n)$. By elliptic estimates, u^λ are also uniformly bounded in $W_{loc}^{4,2}(\mathbb{R}^n)$. Hence, up to a sequence of $\lambda \rightarrow +\infty$, we can assume that $u^\lambda \rightarrow u^\infty$ weakly in $W_{loc}^{4,2} \cap L_{loc}^{p+1}(\mathbb{R}^n)$. By the Sobolev embedding, $u^\lambda \rightarrow u^\infty$ in $W_{loc}^{3,2}(\mathbb{R}^n)$. Then for any ball $B_R(0)$, by the interpolation theorem and the estimate (9.6), for any $q \in [1, p+1]$ as $\lambda \rightarrow +\infty$, we obtain that

$$\|u^\lambda - u^\infty\|_{L^q(B_R(0))} \leq \|u^\lambda - u^\infty\|_{L^1(B_R(0))}^t \|u^\lambda - u^\infty\|_{L^{p+1}(B_R(0))}^{1-t}, \tag{9.12}$$

where $t \in (0, 1]$ satisfying $\frac{1}{q} = t + \frac{1-t}{q+1}$. That is, $u^\lambda \rightarrow u^\infty$ in $L_{loc}^{q+1}(\mathbb{R}^n)$ for any $q \in [1, p+1]$. For any $\varphi \in C_0^\infty(\mathbb{R}^n)$, we have that

$$\begin{aligned}
\int_{\mathbb{R}^n} \Delta^2 u^\infty \Delta^2 \varphi - |u^\infty|^{p-1} u^\infty \varphi &= \lim_{\lambda \rightarrow +\infty} \int_{\mathbb{R}^n} \Delta^2 u^\lambda \Delta^2 \varphi - |u^\lambda|^{p-1} u^\lambda \varphi, \\
\int_{\mathbb{R}^n} |\Delta^2 \varphi|^2 - p|u^\infty|^{p-1} \varphi^2 &= \lim_{\lambda \rightarrow +\infty} \int_{\mathbb{R}^n} |\Delta^2 \varphi|^2 - p|u^\lambda|^{p-1} \varphi^2.
\end{aligned}$$

Therefore, $u^\infty \in W_{loc}^{4,2} \cap L_{loc}^{p+1}(\mathbb{R}^n)$ is a stable solution of (1.1) in \mathbb{R}^n .

Step 3: We claim that the function u^∞ is homogeneous. Due to the scaling invariance of the functional E (i.e., $E(u, 0, R\lambda) = E(u^\lambda, 0, R)$) and the monotonicity formula, for any given $R_2 > R_1 > 0$, we have that

$$\begin{aligned} 0 &= \lim_{i \rightarrow +\infty} \left(E(u, 0, R_2 \lambda_i) - E(u, 0, R_1 \lambda_i) \right) \\ &= \lim_{i \rightarrow +\infty} \left(E(u^{\lambda_i}, 0, R_2) - E(u^{\lambda_i}, 0, R_1) \right) \\ &\geq C(n, p) \liminf_{i \rightarrow +\infty} \int_{B_{R_2} \setminus B_{R_1}} r^{8\frac{p+1}{p-1}-n-8} \left(\frac{8}{p-1} u^{\lambda_i} + r \frac{\partial u^{\lambda_i}}{\partial r} \right)^2 \\ &\geq C(n, p) \int_{B_{R_2} \setminus B_{R_1}} r^{8\frac{p+1}{p-1}-n-8} \left(\frac{8}{p-1} u^\infty + r \frac{\partial u^\infty}{\partial r} \right)^2. \end{aligned}$$

In the last inequality we have used the weak convergence of the sequence (u^{λ_i}) to the function u^∞ in $W_{loc}^{1,2}(\mathbb{R}^n)$ as $i \rightarrow +\infty$. This equality above implies that

$$\frac{8}{p-1} \frac{u^\infty}{r} + \frac{\partial u^\infty}{\partial r} = 0, \quad \text{a.e. in } \mathbb{R}^n.$$

Integrating in r shows that

$$u^\infty(x) = |x|^{-\frac{8}{p-1}} u^\infty\left(\frac{x}{|x|}\right).$$

That is, u^∞ is homogeneous.

Step 4: $u^\infty = 0$. This is a direct consequence of Theorem 8.1 since u^∞ is homogeneous. Since this holds for the limit of any sequence $\lambda \rightarrow +\infty$, by (9.12) we get that

$$\lim_{\lambda \rightarrow +\infty} u^\lambda \text{ strongly in } L^2(B_4(0)).$$

Step 5: $u \equiv 0$. For all $\lambda \rightarrow +\infty$, we see that

$$\lim_{\lambda \rightarrow +\infty} \int_{B_4(0)} (u^\lambda)^2 = 0.$$

By (9.5) in Lemma 9.2, we have that

$$\lim_{\lambda \rightarrow +\infty} \int_{B_3(0)} |\Delta^2 u^\lambda|^2 + |u^\lambda|^{p+1} \leq \lim_{\lambda \rightarrow +\infty} \int_{B_4(0)} (u^\lambda)^2 = 0. \quad (9.13)$$

By the elliptic interior L^2 -estimate, we get that

$$\lim_{\lambda \rightarrow +\infty} \int_{B_2(0)} \sum_{j \leq 4} |\nabla^j u^\lambda|^2 = 0.$$

In particular, we can choose a sequence $\lambda_i \rightarrow +\infty$ such that

$$\int_{B_2(0)} \sum_{j \leq 4} |\nabla^j u^{\lambda_i}|^2 \leq 2^{-i}.$$

Hence we have

$$\int_1^2 \sum_{i=1}^{+\infty} \int_{\partial B_r} \sum_{j \leq 4} |\nabla^j u^{\lambda_i}|^2 dr \leq \sum_{i=1}^{+\infty} \int_1^2 \int_{\partial B_r} \sum_{j \leq 4} |\nabla^j u^{\lambda_i}|^2 \leq 1.$$

Therefore, the function

$$g(r) := \sum_{i=1}^{\infty} \int_{\partial B_r} \sum_{j \leq 4} |\nabla^j u^{\lambda_i}|^2 \in L^1(1, 2).$$

Then there exists an $r_0 \in (1, 2)$ such that $g(r_0) < +\infty$, by which we get that

$$\lim_{i \rightarrow +\infty} \|u^{\lambda_i}\|_{W^{4,2}(\partial B_{r_0})} = 0.$$

Combine with (9.13) and the scaling invariance of $E(u, 0, \lambda)$, we have

$$\lim_{i \rightarrow +\infty} E(\lambda r_0, 0, u) = \lim_{i \rightarrow +\infty} E(r_0, 0, u^{\lambda_i}) = 0.$$

Since $\lambda_i r_0 \rightarrow +\infty$ and $E(r, 0, u)$ is nondecreasing in r , we get that

$$\lim_{i \rightarrow +\infty} E(\lambda r_0, 0, u) = 0.$$

By the smoothness of u , $\lim_{i \rightarrow 0} E(\lambda r_0, 0, u) = 0$. Again by the monotonicity of $E(r, 0, u)$ and Step 4, we obtain that

$$E(r, 0, u) = 0 \text{ for all } r > 0.$$

Therefore by the monotonicity formula (i.e., Theorem 2.2) we known that u is homogeneous, then $u \equiv 0$ by Theorem 8.1. \square

10 Finite Morse index solution

In this section, we prove Theorem 1.2. Firstly, we have

Lemma 10.1. *Let u be a smooth (positive or sing changing) solution of (1.1) with finite Morse index, then there exist constant $C > 0$ and $R_0 > 0$ such that*

$$|u(x)| \leq C|x|^{-\frac{8}{p-1}}, \text{ for any } x \in B_{R_0}^c.$$

Proof. Assume that u is stable outside $B_{R_0}^c$. For any $x \in B_{R_0}^c$, let $M(x) := |u(x)|^{\frac{p-1}{8}}$ and $d(x) = |x| - R_0$. Assume that the conclusion does not holds, then there exists a sequence of $x_k \in B_{R_0}^c$ such that

$$M(x_k)d(x_k) \geq 2k.$$

Since u is bounded on any compact set of \mathbb{R}^n , $d(x_k) \rightarrow +\infty$. By the doubling Lemma (see [18]), there exists another sequence $y_k \in B_{R_0}^c$ such that

$$\begin{aligned} M(y_k)d(y_k) &\geq 2k, \quad M(y_k) \geq M(x_k), \\ M(z) &\leq 2M(y_k) \quad \text{for any } z \in B_{R_0}^c \text{ such that } |z - y_k| \leq \frac{k}{M(y_k)}. \end{aligned}$$

Now we define

$$u_k(x) := M(y_k)^{-\frac{8}{p-1}} u(y_k + M(y_k)^{-1}x) \quad \text{for } x \in B_R(0).$$

This and the above arguments give that $u_k(0) = 1$, $|u_k| \leq 2^{\frac{8}{p-1}}$ in $B_R(0)$. Further, $B_{k/M(y_k)} \cap B_{R_0} = \emptyset$, which implies that u is a stable solution in $B_{k/M(y_k)}(y_k)$, hence u_k is stable in $B_R(0)$. By elliptic regularity theory, u_k are uniformly bounded in $C_{loc}^9(\mathbb{R}^n)$, up to a subsequence, u_k converges to u_∞ in $C_{loc}^8(\mathbb{R}^n)$. By the above conclusions on u_k , we have

- (1) $|u_\infty(0)| = 1$;
- (2) $|u_\infty| \leq 2^{\frac{8}{p-1}}$ in \mathbb{R}^n ;
- (3) u_∞ is a smooth stable solution of (1.1) in \mathbb{R}^n .

By the Liouville theorem for stable solution, i.e., Theorem 1.1, we get that $u_\infty \equiv 0$, this is a contradiction. \square

Corollary 10.1. *Under the same assumptions in the above Lemma 10.1, there exist constant $C > 0$ and R_0 such that for all $x \in B_{R_0}^c$,*

$$\sum_{0 \leq j \leq 7} |x|^{\frac{8}{p-1}+j} |\nabla^j u(x)| \leq C.$$

Proof. For any x_0 with $|x_0| > R_0$, take $\lambda = \frac{|x_0|}{2}$ and define

$$\bar{u}(x) := \lambda^{\frac{8}{p-1}} u(x_0 + \lambda x).$$

By the previous Lemma, $\bar{u}(x) \leq C$ in $B_1(0)$. By the elliptic regularity theory we have

$$\sum_{0 \leq j \leq 7} |\nabla^k \bar{u}(0)| \leq C.$$

Scaling back we get the conclusion. \square

10.1 The proof of Theorem 1.2-(1)

This is about the subcritical case, i.e., $1 < p < \frac{n+8}{n-8}$. Firstly, we cite the following Pohozaev identity (see [23]).

Lemma 10.2. *For any function u satisfying (1.1), there holds*

$$\left(\frac{n-8}{2} - \frac{n}{p+1}\right) \int_{B_R} |u|^{p+1} = \int_{\partial B_R} B_4(u) d\sigma,$$

where

$$\begin{aligned} B_4(u) &= (2 - \frac{n}{2}) \sum_{k=1}^2 (-\Delta)^{4-k} u \frac{\partial(-\Delta u)^{k-1}}{\partial n} - \frac{R}{p+1} |u|^{p+1} \\ &\quad - (2 - \frac{n}{2}) \sum_{k=1}^2 \frac{\partial(-\Delta)^{4-k} u}{\partial n} (-\Delta)^{k-1} u + \frac{1}{2} (-\Delta)^4 u R + 2(-\Delta)^3 u \frac{\partial u}{\partial n} \\ &\quad - 2 \frac{\partial(-\Delta)^3 u}{\partial n} u + \sum_{k=1}^2 \langle x, \nabla(-\Delta)^{k-1} u \rangle \frac{\partial(-\Delta)^{4-k} u}{\partial n} \\ &\quad - \sum_{k=1}^2 (-\Delta)^{4-k} u \frac{\partial \langle x, \nabla(-\Delta)^k u \rangle}{\partial n}. \end{aligned}$$

The proof of Theorem 1.2-(1). By Corollary 10.1, for any $R > R_0$, noting that $p < \frac{n+8}{n-8}$ (hence $n - 8 \frac{p+1}{p-1} < 0$), we have the following estimate:

$$\int_{\partial B_R} |B_4(u)| d\sigma \leq C \int_{\partial B_R} R^{-\frac{16}{p-1}-7} d\sigma \leq CR^{n-8\frac{p+1}{p-1}} \rightarrow 0 \text{ as } R \rightarrow +\infty.$$

Letting $R \rightarrow +\infty$ in the above Pohozaev identity, we get that

$$\left(\frac{n-8}{2} - \frac{n}{p+1}\right) \int_{\mathbb{R}^n} |u|^{p+1} = 0.$$

Since $\frac{n-8}{2} - \frac{n}{p+1} < 0$, we see that $u \equiv 0$. \square

10.2 The proof of Theorem 1.2-(3)

Recall the assumption $p = \frac{n+8}{n-8}$ (critical case) in Theorem 1.2-(3). Since u is stable outside B_{R_0} , Lemma 9.2 holds if the support of η is outside B_{R_0} . Take $\varphi \in C_0^\infty(B_{2R_0} \setminus B_{R_0})$ such that $\varphi = 1$ in $B_R \setminus B_{3R_0}$ and $\sum_{0 \leq j \leq 7} |x|^j |\nabla^j \varphi| \leq 1000$. Then by choosing $\eta = \varphi^m$, where m is bigger than 1, we get that

$$\int_{B_R \setminus B_{3R_0}} |\Delta^2 u|^2 + |u|^{p+1} \leq C.$$

Letting $R \rightarrow +\infty$, we have

$$\int_{\mathbb{R}^n} |\Delta^2 u|^2 + |u|^{p+1} < +\infty. \tag{10.1}$$

By the interior elliptic estimates and the Holder's inequality, we have

$$\begin{aligned} R^{-6} \int_{B_{2R} \setminus B_R} |\nabla u|^2 &\leq C \int_{B_{3R} \setminus B_{R/2}} |\Delta^2 u|^2 + C \left(\int_{B_{3R} \setminus B_{R/2}} |u|^{p+1} \right)^{\frac{2}{p+1}}, \\ R^{-4} \int_{B_{2R} \setminus B_R} |\Delta u|^2 &\leq C \int_{B_{3R} \setminus B_{R/2}} |\Delta^2 u|^2 + C \left(\int_{B_{3R} \setminus B_{R/2}} |u|^{p+1} \right)^{\frac{2}{p+1}}, \\ R^{-2} \int_{B_{2R} \setminus B_R} |\nabla \Delta u|^2 &\leq C \int_{B_{3R} \setminus B_{R/2}} |\Delta^2 u|^2 + C \left(\int_{B_{3R} \setminus B_{R/2}} |u|^{p+1} \right)^{\frac{2}{p+1}}, \\ R^{-8} \int_{B_{2R} \setminus B_R} |u|^2 &\leq C \int_{B_{3R} \setminus B_{R/2}} |\Delta^2 u|^2 + C \left(\int_{B_{3R} \setminus B_{R/2}} |u|^{p+1} \right)^{\frac{2}{p+1}}, \end{aligned}$$

where C is a universal constant independent of R . Therefore, we have that

$$\max \left\{ R^{-6} \int_{B_{2R} \setminus B_R} |\nabla u|^2, R^{-4} \int_{B_{2R} \setminus B_R} |\Delta u|^2, R^{-2} \int_{B_{2R} \setminus B_R} |\nabla \Delta u|^2, R^{-8} \int_{B_{2R} \setminus B_R} |u|^2 \right\} \rightarrow 0 \text{ as } R \rightarrow +\infty.$$

On the other hand, testing (1.1) with $u\eta^2$, we get that

$$\int_{\mathbb{R}^n} |\Delta^2 u|^2 \eta^2 - |u|^{p+1} \eta^2 = - \int_{\mathbb{R}^n} \Delta^2 u (\Delta^2 (u\eta^2) - \Delta^2 u\eta^2)$$

and

$$\begin{aligned} \Delta^2 (u\eta^2) - \Delta^2 u\eta^2 &= 2\Delta u \Delta \eta^2 + 2\nabla \Delta u \nabla \eta^2 \\ &\quad + u \Delta^2 \eta^2 + 2\nabla u \nabla \Delta \eta^2 + 2\Delta (\nabla u \nabla \eta^2). \end{aligned}$$

Notice that the highest order derivative about u of the above expression is $\nabla \Delta u$. By selecting $\eta(x) = \xi(\frac{x}{R})^{4m}$, $m > 1$ and $\xi \in C_0^\infty(B_2)$, $\xi = 1$ in B_1 and $\sum_{1 \leq j \leq 5} |\nabla^j u| \leq 1000$, we get that

$$\begin{aligned} \left| \int_{\mathbb{R}^n} |\Delta^2 u|^2 \xi(\frac{x}{R})^{8m} - |u|^{p+1} \xi(\frac{x}{R})^{8m} \right| &\leq C \left(R^{-6} \int_{B_{2R} \setminus B_R} |\nabla u|^2 \right. \\ &\quad \left. + R^{-4} \int_{B_{2R} \setminus B_R} |\Delta u|^2 + R^{-2} \int_{B_{2R} \setminus B_R} |\nabla \Delta u|^2 + R^{-8} \int_{B_{2R} \setminus B_R} |u|^2 \right). \end{aligned}$$

Now letting $R \rightarrow +\infty$, we obtain that

$$\int_{\mathbb{R}^n} |\Delta^2 u|^2 - |u|^{p+1} = 0,$$

Combining this with (10.1) we get the conclusion. \square

10.3 The proof of Theorem 1.2-(2)

This is the supercritical case: $p > \frac{n+8}{n-8}$. Firstly, we have

Lemma 10.3. *There exists a constant $C > 0$ such that $E(r, 0, u) \leq C$ for all $r > R_0$.*

Proof. From the monotonicity formula, combine with the derivative estimate, i.e., Corollary 10.1, we have the following estimates:

$$\begin{aligned} E(r, 0, u) &\leq Cr^{8\frac{p+1}{p-1}-n}\left(|\Delta^2 u|^2 + |u|^{p+1}\right) \\ &+ C\left(\sum_{s,t \leq 5, s+t \leq 7} r^{8\frac{p+1}{p-1}-n-7+s+t} \int_{\partial B_r} |\nabla^s u| |\nabla^t u|\right. \\ &\leq C. \end{aligned}$$

□

As a consequence, we have the following

Corollary 10.2.

$$\int_{B_{3R_0}^c} \frac{\left(\frac{8}{p-1}u(x) + |x|\frac{\partial u(x)}{\partial r}\right)^2}{|x|^{n-8\frac{p+1}{p-1}}} dx < +\infty.$$

As before, we define the blowing down sequence,

$$u^\lambda(x) := \lambda^{\frac{8}{p-1}} u(\lambda x).$$

By Lemma 10.1 and Corollary 10.1, we know that u^λ are uniformly bounded in $C^9(B_r(0) \setminus B_{1/r}(0))$ for any fixed $r > 1$ and moreover, u^λ is stable outside $B_{R_0/\lambda}$. And there exists a function $u^\infty \in C^8(\mathbb{R}^n \setminus \{0\})$, such that up to a subsequence of $\lambda \rightarrow +\infty$, u^λ convergent to u^∞ in $C^8(\mathbb{R}^n \setminus \{0\})$, u^∞ is a stable solution of (1.1) in $\mathbb{R}^n \setminus \{0\}$. For any $r > 1$, by the previous Corollary, we have

$$\begin{aligned} &\int_{B_r \setminus B_{1/r}} \frac{\left(\frac{8}{p-1}u^\infty(x) + |x|\frac{\partial u^\infty(x)}{\partial r}\right)^2}{|x|^{n-8\frac{p+1}{p-1}}} dx \\ &= \lim_{\lambda \rightarrow +\infty} \int_{B_r \setminus B_{1/r}} \frac{\left(\frac{8}{p-1}u^\lambda(x) + |x|\frac{\partial u^\lambda(x)}{\partial r}\right)^2}{|x|^{n-8\frac{p+1}{p-1}}} dx \\ &= \lim_{\lambda \rightarrow +\infty} \int_{B_{\lambda r} \setminus B_{\lambda/r}} \frac{\left(\frac{8}{p-1}u(x) + |x|\frac{\partial u(x)}{\partial r}\right)^2}{|x|^{n-8\frac{p+1}{p-1}}} dx \\ &= 0. \end{aligned}$$

Hence,

$$\frac{8}{p-1}u^\infty(x) + |x|\frac{\partial u^\infty(x)}{\partial r} = 0 \quad a.e.,$$

that is, u is homogeneous, by Theorem 8.1, we get that $u^\infty \equiv 0$ if $p < p_c(n)$. Since this holds for any limit of u^λ as $\lambda \rightarrow +\infty$, then we have

$$\lim_{|x| \rightarrow +\infty} |x|^{\frac{8}{p-1}} |u(x)| = 0.$$

Then as the proof of Corollary 10.1, we have

$$\lim_{|x| \rightarrow +\infty} \sum_{0 \leq j \leq 7} |x|^{\frac{8}{p-1}+j} |\nabla^j u(x)| = 0.$$

Therefor, for any $\varepsilon > 0$, take an R_0 such that for $|x| > R_0$, there holds

$$\sum_{0 \leq j \leq 7} |x|^{\frac{8}{p-1}+j} |\nabla^j u(x)| \leq \varepsilon.$$

Then for any $r \gg R_0$, we have

$$\begin{aligned} E(r, 0, u) &\leq Cr^{8\frac{p+1}{p-1}-n} \int_{B_R(0)} |\Delta^2 u|^2 + |u|^{p+1} \\ &\quad + C\varepsilon r^{8\frac{p+1}{p-1}-n} \left(\int_{B_r(0) \setminus B_{R_0}(0)} |x|^{-8\frac{p+1}{p-1}} + r \int_{\partial B_r(0)} |x|^{-8\frac{p+1}{p-1}} \right) \\ &\leq C(R_0) \left(r^{8\frac{p+1}{p-1}-n} + \varepsilon \right). \end{aligned}$$

Therefore, we obtain that

$$\lim_{r \rightarrow +\infty} E(r, 0, u) = 0$$

since $8\frac{p+1}{p-1} + 1 - n < 0$ and ε can be arbitrarily small. On the other hand, since u is smooth, we have $\lim_{r \rightarrow 0} E(r, 0, u) = 0$. Thus $E(r, 0, u) = 0$ for all $r > 0$. By the monotonicity formula we get that u is homogeneous and hence by Theorem 8.1, we derive that $u \equiv 0$. This completes the proof. \square

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