

# Classification of the stable solution to the fractional ( $2 < s < 3$ ) Lane-Emden equation\*

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## Abstract

We classify the stable solutions (positive or sign-changing, radial or not) to the following nonlocal Lane-Emden equation:

$$(-\Delta)^s u = |u|^{p-1} u \quad \text{in } \mathbb{R}^n$$

for  $2 < s < 3$ .

## 1 Introduction and Main results

Consider the stable solution of the following equation

$$(-\Delta)^s u = |u|^{p-1} u \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where  $(-\Delta)^s$  is the fractional Laplacian operator for  $2 < s < 3$ .

The motivation of studying such an equation is originated from the classical Lane-Emden equation

$$-\Delta u = |u|^{p-1} u \quad \text{in } \mathbb{R}^n \tag{1.2}$$

and its parabolic counterpart, which have played a crucial role in the development of nonlinear PDEs in the last decades. These arise in astrophysics and Riemannian geometry. The pioneering works on Eq.(1.2) were contributed by R. Fowler [12, 13]. Later, the ground-breaking result on equation (1.2) is the fundamental Liouville-type theorems established by Gidas and Spruck [14], they claimed that the Eq. (1.2) has no positive solution whenever  $p \in (1, 2^* - 1)$ , where  $2^* = 2n/(n - 2)$  if  $n \geq 3$  and  $2^* = \infty$  if  $n \leq 2$ . The critical case  $p = 2^* - 1$ , Eq.(1.2) has a unique positive solution

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up to translation and rescaling which is radial and explicitly formulated, see Caffarelli-Gidas-Spruck [1]. Since then many experts in partial differential equations devote to the above equations for various parameters  $s$  and  $p$ .

For the nonlocal case of  $0 < s < 1$ , a counterpart of the classification results of Gidas and Spruck [14], and Caffarelli-Gidas-Spruck [1] holds for the fractional Lane-Emden equation (1.1), see the works due to Li [19] and Chen-Li-Ou [5]. In these cases, the Sobolev exponent is given by  $P_S(n, s) = (n + 2s)/(n - 2s)$  if  $n > 2s$ , and otherwise  $P_S(n, s) = \infty$ .

Recently, for the nonlocal case of  $0 < s < 1$ , Davila, Dupaigne and Wei in [6] gave a complete classification of finite Morse index solution of (1.1); for the nonlocal case of  $1 < s < 2$ , Fazly and Wei in [17] gave a complete classification of finite Morse index solution of (1.1). For the local cases  $s = 1$  and  $s = 2$ , such kind of classification is proved by Farina in [10] and Davila, Dupaigne, Wang and Wei in [7], respectively. For the case  $s = 3$ , the Joseph-Lundgren exponent (for the triharmonic Lane-Emden equation) is obtained and classification is proved by in [21].

However, when  $2 < s < 3$ , the equation (1.1) has not been considered so far. In this paper we classify the stable solution of (1.1).

There are many ways of defining the fractional Laplacian  $(-\Delta)^s$ , where  $s$  is any positive, noninteger number. Caffarelli and Silvestre in [2] gave a characterization of the fractional Laplacian when  $0 < s < 1$  as the Dirichlet-to-Neumann map for a function  $u_e$  satisfying a higher order elliptic equation in the upper half space with one extra spatial dimension. This idea was later generalized by Yang in [27] when the  $s$  is being any positive, noninteger number. See also Chang-Gonzales [4] and Case-Chang [3] for general manifolds.

To introduce the fractional operator  $(-\Delta)^s$  for  $2 < s < 3$ , just like the case of  $1 < s < 2$ , via the Fourier transform, we can define

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s} \widehat{u}(\xi)$$

or equivalently define this operator inductively by  $(-\Delta)^s = (-\Delta)^{s-2} \circ (-\Delta)^2$ .

**Definition 1.1.** We say a solution  $u$  of (1.1) is stable outside a compact set if there exists  $R_0 > 0$  such that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\varphi(x) - \varphi(y))^2}{|x - y|^{n+2s}} dx dy - p \int_{\mathbb{R}^n} |u|^{p-1} \varphi^2 dx \geq 0 \quad (1.3)$$

for any  $\varphi \in C_c^\infty(\mathbb{R}^n \setminus \overline{B_{R_0}})$ .

Set

$$p_s(n) = \begin{cases} \infty & \text{if } n \leq 2s, \\ \frac{n+2s}{n-2s} & \text{if } n > 2s. \end{cases}$$

The first main result of the present paper is the following

**Theorem 1.1.** Suppose that  $n > 2s$  and  $2 < s < \delta < 3$ . Let  $u \in C^{2\delta}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n, (1 + |z|)^{n+2s} dz)$  be a solution of (1.1) which is stable outside a compact set. Assume

(1)  $1 < p < p_s(n)$  or

(2)  $p_s(n) < p$  and

$$p \frac{\Gamma(\frac{n}{2} - \frac{s}{p-1})\Gamma(s + \frac{s}{p-1})}{\Gamma(\frac{s}{p-1})\Gamma(\frac{n-2s}{2} - \frac{s}{p-1})} > \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2}, \quad (1.4)$$

then the solution  $u \equiv 0$ .

(3)  $p = p_s(n)$ , then  $u$  has finite energy, i.e.,

$$\|u\|_{\dot{H}^s(\mathbb{R}^n)}^2 := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} dx dy = \int_{\mathbb{R}^n} |u|^{p+1} < +\infty.$$

If in addition  $u$  is stable, then  $u \equiv 0$ .

**Remark 1.1.** In the Theorem 1.1 the condition (1.4) is optimal. In fact, the radial singular solution  $u = |x|^{-\frac{2s}{p-1}}$  is stable if

$$p \frac{\Gamma(\frac{n}{2} - \frac{s}{p-1})\Gamma(s + \frac{s}{p-1})}{\Gamma(\frac{s}{p-1})\Gamma(\frac{n-2s}{2} - \frac{s}{p-1})} \leq \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2}.$$

See [22].

**Remark 1.2.** The hypothesis (2) of Theorem 1.1 is equivalent to

$$p < p_c(n) := \begin{cases} +\infty & \text{if } n \leq n_0(s), \\ \frac{n+2s-2-2a_{n,s}\sqrt{n}}{n-2s-2-2a_{n,s}\sqrt{n}} & \text{if } n > n_0(s), \end{cases} \quad (1.5)$$

where  $n_0(s)$  is the largest root of  $n - 2s - 2 - 2a_{n,s}\sqrt{n} = 0$ , see [22]. More details and further sharp results about  $a_{n,s}$  and  $n_0(s)$  see [23].

**Remark 1.3.** In this remark, we further analyze the hypothesis (2) in Theorem 1.1. Recall that when  $s = 1$  the condition (1.4) gives a upper bounded of  $p$  (originated from Joseph and Lundgren [18]), it is

$$p < p_c(n) := \begin{cases} \infty & \text{if } n \leq 10, \\ \frac{(n-2)^2-4n+8\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n \geq 11. \end{cases} \quad (1.6)$$

For the case  $s = 2$ , (1.4) induce the upper bound of  $p$  which is given by the following formula (cf. Gazzola and Grunau [16]):

$$p < p_c(n) = \begin{cases} \infty & \text{if } n \leq 12, \\ \frac{n+2-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}}{n-6-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}} & \text{if } n \geq 13. \end{cases} \quad (1.7)$$

In the triharmonic case, the corresponding exponent given by see ([21]) is the following

$$p < p_c(n) = \begin{cases} \infty & \text{if } n \leq 14, \\ \frac{n+4-2D(n)}{n-8-2D(n)} & \text{if } n \geq 15, \end{cases}$$

where

$$D(n) := \frac{1}{6} \left( 9n^2 + 96 - \frac{1536 + 1152n^2}{D_0(n)} - \frac{3}{2} D_0(n) \right)^{1/2};$$

$$D_0(n) := -(D_1(n) + 36\sqrt{D_2(n)})^{1/3};$$

$$D_1(n) := -94976 + 20736n + 103104n^2 - 10368n^3 + 1296n^5 - 3024n^4 - 108n^6;$$

$$\begin{aligned} D_2(n) := & 6131712 - 16644096n^2 + 6915840n^4 - 690432n^6 - 3039232n \\ & + 4818944n^3 - 1936384n^5 + 251136n^7 - 30864n^8 - 4320n^9 \\ & + 1800n^{10} - 216n^{11} + 9n^{12}. \end{aligned}$$

## 2 Preliminary

Throughout this paper we denote  $b := 5 - 2s$  and define the operator

$$\Delta_b w := \Delta w + \frac{b}{y} w_y = y^{-b} \mathbf{div}(y^b \nabla w)$$

for a function  $w \in W^{3,2}(\mathbb{R}^{n+1}; y^b dx dy)$ . We firstly quote the following result.

**Theorem 2.1.** (See [27] ) Assume  $2 < s < 3$ . Let  $u_e \in W^{3,2}(\mathbb{R}^{n+1}; y^b dx dy)$  satisfy the equation

$$\Delta_b^3 u_e = 0 \tag{2.1}$$

on the upper half space for  $(x, y) \in \mathbb{R}^n \times \mathbb{R}_+$  (where  $y$  is the spacial direction) and the boundary conditions:

$$\begin{aligned} u_e(x, 0) &= f(x), \\ \lim_{y \rightarrow 0} y^b \frac{\partial u_e}{\partial y} &= 0, \\ \frac{\partial^2 u_e}{\partial y^2} |_{y=0} &= \frac{1}{2s} \Delta_x u_e |_{y=0}, \\ \lim_{y \rightarrow 0} C_{n,s} y^b \frac{\partial}{\partial y} \Delta_b^2 u_e &= (-\Delta)^s f(x), \end{aligned} \tag{2.2}$$

where  $f(x)$  is some function defined on  $H^s(\mathbb{R}^n)$ . Then we have

$$\int_{\mathbb{R}^n} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi = C_{n,s} \int_{\mathbb{R}_+^{n+1}} y^b |\nabla \Delta_b u_e(x, y)|^2 dx dy. \tag{2.3}$$

Applying the above theorem to solutions of (1.1), we conclude that the extended function  $u_e(x, y)$ , where  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  and  $y \in \mathbb{R}^+$ , satisfies

$$\begin{cases} \Delta_b^3 u_e = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \lim_{y \rightarrow 0} y^b \frac{\partial u_e}{\partial y} = 0 & \text{on } \partial \mathbb{R}_+^{n+1}, \\ \frac{\partial^2 u_e}{\partial y^2} \Big|_{y=0} = \frac{1}{2s} \Delta_x u_e \Big|_{y=0} & \text{on } \partial \mathbb{R}_+^{n+1}, \\ \lim_{y \rightarrow 0} y^b \frac{\partial}{\partial y} \Delta_b^2 u_e = -C_{n,s} |u_e|^{p-1} u_e & \text{in } \mathbb{R}_+^{n+1}. \end{cases} \quad (2.4)$$

Moreover,

$$\int_{\mathbb{R}^n} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi = C_{n,s} \int_{\mathbb{R}_+^{n+1}} y^b |\nabla \Delta_b u_e(x, y)|^2 dx dy$$

and  $u(x) = u_e(x, 0)$ .

Define

$$\begin{aligned} E(\lambda, x, u_e) = & \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \frac{1}{2} \theta_1^b |\nabla \Delta_b u_e^\lambda|^2 - \frac{C_{n,s}}{p+1} \int_{\partial \mathbb{R}_+^{n+1} \cap B_1} |u_e^\lambda|^{p+1} \\ & + \sum_{0 \leq i,j \leq 4, i+j \leq 5} C_{i,j}^1 \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \lambda^{i+j} \frac{d^i u_e^\lambda}{d\lambda^i} \frac{d^j u_e^\lambda}{d\lambda^j} \\ & + \sum_{0 \leq t,s \leq 2, t+s \leq 3} C_{t,s}^2 \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \lambda^{t+s} \nabla_{S^n} \frac{d^t u_e^\lambda}{d\lambda^t} \nabla_{S^n} \frac{d^s u_e^\lambda}{d\lambda^s} \\ & + \sum_{0 \leq l,k \leq 1, l+k \leq 1} C_{l,k}^3 \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \lambda^{l+k} \Delta_{S^n} \frac{d^l u_e^\lambda}{d\lambda^l} \Delta_{S^n} \frac{d^k u_e^\lambda}{d\lambda^k} \\ & + \left( \frac{s}{p-1} + 1 \right) \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b (\Delta_b u_e^\lambda)^2. \end{aligned} \quad (2.5)$$

The following is the monotonicity formula which will play an important role.

**Theorem 2.2.** *Let  $u_e$  satisfy the equation (2.1) with the boundary conditions (2.2), we have the following*

$$\begin{aligned} \frac{dE(\lambda, x, u_e)}{d\lambda} = & \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left( 3\lambda^5 \left( \frac{d^3 u_e^\lambda}{d\lambda^3} \right)^2 + A_1 \lambda^3 \left( \frac{d^2 u_e^\lambda}{d\lambda^2} \right)^2 + A_2 \lambda \left( \frac{du_e^\lambda}{d\lambda} \right)^2 \right) \\ & + \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left( 2\lambda^3 |\nabla_{S^n} \frac{d^2 u_e^\lambda}{d\lambda^2}|^2 + B_1 \lambda |\nabla_{S^n} \frac{du_e^\lambda}{d\lambda}| \right) \\ & + \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \lambda \left( \Delta_{S^n} \frac{du_e^\lambda}{d\lambda} \right)^2, \end{aligned} \quad (2.6)$$

where  $\theta_1 = \frac{y}{r}$  and

$$\begin{aligned} A_1 &:= 10\delta_1 - 2\delta_2 - 56 + \alpha_0^2 - 2\alpha_0 - 2\beta_0 - 4, \\ A_2 &:= -18\delta_1 + 6\delta_2 - 4\delta_3 + 2\delta_4 + 72 - \alpha_0^2 + \beta_0^2 + 2\alpha_0 + 2\beta_0, \\ B_1 &:= 8\alpha - 4\beta - 2\beta_0 + 4(n+b) - 14, \\ \alpha &:= n+b-2-\frac{4s}{p-1}, \beta := \frac{2s}{p-1}(3+\frac{2s}{p-1}-n-b), \\ \alpha_0 &:= n+b-\frac{4s}{p-1}, \quad \beta_0 := \frac{2s}{p-1}(1+\frac{2s}{p-1}-n-b) \end{aligned}$$

and

$$\begin{aligned} \delta_1 &= 2(n+b) - \frac{8s}{p-1}, \\ \delta_2 &= (n+b)(n+b-2) - (n+b)\frac{12s}{p-1} + \frac{12s}{p-1}(1+\frac{2s}{p-1}), \\ \delta_3 &= -\frac{8s}{p-1}(1+\frac{2s}{p-1})(2+\frac{2s}{p-1}) + 2(n+b)\frac{6s}{p-1}(1+\frac{2s}{p-1}) \\ &\quad - (n+b)(n+b-2)(1+\frac{4s}{p-1}), \\ \delta_4 &= (3+\frac{2s}{p-1})(2+\frac{2s}{p-1})(1+\frac{2s}{p-1})\frac{2s}{p-1} \\ &\quad - 2(n+b)(1+\frac{2s}{p-1})(2+\frac{2s}{p-1})\frac{2s}{p-1} \\ &\quad + (n+b)(n+b-2)(2+\frac{2s}{p-1})\frac{2s}{p-1}. \end{aligned} \tag{2.7}$$

We will give the proof of Theorem 2.2 in the next section. Now we would like to state a consequent result of Theorem 2.2. Recall that  $E(\lambda, x, u_e)$ , defined in (2.5), can be divided into two parts: the integral over the ball  $B_\lambda$  and the terms on the boundary  $\partial B_\lambda$ . We note that in our blow-down analysis, the coefficients (including positive or negative, big or small) of the boundary terms can be estimated in a unified way, therefore we may change some coefficients of the boundary terms in  $E(\lambda, x, u_e)$ . After such a change, we denote the new functional by  $E^c(\lambda, x, u_e)$ .

Define

$$p_m(n) := \begin{cases} +\infty & \text{if } n < 2s + 6 + \sqrt{73}, \\ \frac{5n+10s-\sqrt{15(n-2s)^2+120(n-2s)+370}}{5n-10s-\sqrt{15(n-2s)^2+120(n-2s)+370}} & \text{if } n \geq 2s + 6 + \sqrt{73}. \end{cases} \tag{2.8}$$

We have the following

**Theorem 2.3.** *Assume that  $\frac{n+2s}{n-2s} < p < p_m(n)$ , then  $E^c(\lambda, x, u_e)$  is a nondecreasing function of  $\lambda > 0$ . Furthermore,*

$$\frac{dE^c(\lambda, x, u_e)}{d\lambda} \geq C(n, s, p) \lambda^{2s\frac{p+1}{p-1}-6-n} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda(x_0)} y^b \left( \frac{2s}{p-1} u_e + \lambda \partial_r u_e \right)^2,$$

where  $C(n, s, p)$  is a constant independent of  $\lambda$ .

By carefully comparing  $\frac{n+2s}{n-2s} < p < p_m(n)$  with  $p > \frac{n+2s}{n-2s}$  and (1.4), we get the following (see the last section of the current paper) monotonicity formula for our blow down analysis.

**Theorem 2.4.** *Assume that  $p > \frac{n+2s}{n-2s}$  and (1.4), then  $E^c(\lambda, x, u_e)$  is a nondecreasing function of  $\lambda > 0$ . Furthermore,*

$$\frac{dE^c(\lambda, x, u_e)}{d\lambda} \geq C(n, s, p) \lambda^{2s\frac{p+1}{p-1}-6-n} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda(x_0)} y^b \left( \frac{2s}{p-1} u_e + \lambda \partial_r u_e \right)^2,$$

where  $C(n, s, p)$  is a constant independent of  $\lambda$ .

### 3 Monotonicity formula and the proof of Theorem 2.2

The derivation of the monotonicity for the (1.1) when  $2 < s < 3$  is complicated in its process, we divide it into several subsections. In subsection 3.1, we derive  $\frac{d}{d\lambda} \bar{E}(u_e, \lambda)$ . In subsection 3.2, we calculate  $\frac{\partial^j}{\partial r^j} u_e^\lambda$  and  $\frac{\partial^i}{\partial \lambda^i} u_e^\lambda$ ,  $i, j = 1, 2, 3, 4$ . In subsection 3.3, the operator  $\Delta_b^2$  and its representation will be given. In subsection 3.4, we decompose  $\frac{d}{d\lambda} \bar{E}(u_e^\lambda, 1)$ . Finally, combine with the above four subsections, we can obtain the monotonicity formula, hence get the proof of Theorem 2.2.

Suppose that  $x_0 = 0$  and denote by  $B_\lambda$  the balls centered at zero with radius  $\lambda$ . Set

$$\bar{E}(u_e, \lambda) := \lambda^{2s\frac{p+1}{p-1}-n} \left( \int_{\mathbb{R}_+^{n+1} \cap B_\lambda} \frac{1}{2} y^b |\nabla \Delta_b u_e|^2 - \frac{C_{n,s}}{p+1} \int_{\partial \mathbb{R}_+^{n+1} \cap B_\lambda} |u_e|^{p+1} \right).$$

#### 3.1 The derivation of $\frac{d}{d\lambda} \bar{E}(u_e, \lambda)$

Define

$$\begin{aligned} v_e &:= \Delta_b u_e, u_e^\lambda(X) := \lambda^{\frac{2s}{p-1}} u_e(\lambda X), \quad w_e(X) := \Delta_b v_e \\ v_e^\lambda(X) &:= \lambda^{\frac{2s}{p-1}+2} v_e(\lambda X), \quad w_e^\lambda(X) := \lambda^{\frac{2s}{p-1}+4} w_e(\lambda X), \end{aligned} \tag{3.1}$$

where  $X = (x, y) \in \mathbb{R}_+^{n+1}$ . Therefore,

$$\Delta_b u_e^\lambda(X) = v_e^\lambda(X), \Delta_b v_e^\lambda(X) = w_e^\lambda(X). \tag{3.2}$$

Hence

$$\begin{aligned} \Delta_b w_e^\lambda &= 0, \\ \lim_{y \rightarrow 0} y^b \frac{\partial u_e}{\partial y} &= 0, \\ \frac{\partial^2 u_e}{\partial y^2} \Big|_{y=0} &= \frac{1}{2s} \Delta_x u_e \Big|_{y=0}, \\ \lim_{y \rightarrow 0} C_{n,s} y^b \frac{\partial}{\partial y} w_e^\lambda &= -C_{n,s} |u_e|^{p-1} u_e. \end{aligned}$$

In addition, differentiating (3.2) with respect to  $\lambda$  we have

$$\Delta_b \frac{du_e^\lambda}{d\lambda} = \frac{dv_e^\lambda}{d\lambda}, \quad \Delta_b \frac{dv_e^\lambda}{d\lambda} = \frac{dw_e^\lambda}{d\lambda}.$$

Note that

$$\bar{E}(u_e, \lambda) = \bar{E}(u_e^\lambda, 1) = \int_{\mathbb{R}_+^{n+1} \cap B_1} \frac{1}{2} y^b |\nabla v_e^\lambda|^2 - \frac{C_{n,s}}{p+1} \int_{\partial \mathbb{R}_+^{n+1} \cap B_1} |u_e^\lambda|^{p+1}.$$

Taking derivative of the energy  $\bar{E}(u_e^\lambda, 1)$  with respect to  $\lambda$  and integrating by part we have:

$$\begin{aligned} \frac{d\bar{E}(u_e^\lambda, 1)}{d\lambda} &= \int_{\mathbb{R}_+^{n+1} \cap B_1} y^b \nabla v_e^\lambda \nabla \frac{dv_e^\lambda}{d\lambda} - C_{n,s} \int_{\partial \mathbb{R}_+^{n+1} \cap B_1} |u_e^\lambda|^{p-1} u_e^\lambda \frac{du_e^\lambda}{d\lambda} \\ &= \int_{\partial(\mathbb{R}_+^{n+1} \cap B_1)} y^b \frac{\partial v_e^\lambda}{\partial n} \frac{dv_e^\lambda}{d\lambda} - \int_{\mathbb{R}_+^{n+1} \cap B_1} (y^b \Delta v_e^\lambda + b y^{b-1} \frac{\partial v_e^\lambda}{\partial y}) \frac{dv_e^\lambda}{d\lambda} \\ &\quad - C_{n,s} \int_{\partial \mathbb{R}_+^{n+1} \cap B_1} |u_e^\lambda|^{p-1} u_e^\lambda \frac{du_e^\lambda}{d\lambda} \\ &= - \int_{\partial \mathbb{R}^{n+1} \cap B_1} y^b \frac{\partial v_e^\lambda}{\partial y} \frac{dv_e^\lambda}{d\lambda} + \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} y^b \frac{\partial v_e^\lambda}{\partial r} \\ &\quad - \int_{\mathbb{R}_+^{n+1} \cap B_1} y^b \Delta v_e^\lambda \frac{dv_e^\lambda}{d\lambda} + b y^{b-1} \frac{\partial v_e^\lambda}{\partial y} \frac{v_e^\lambda}{d\lambda} - \int_{\partial \mathbb{R}_+^{n+1} \cap B_1} y^b \frac{w_e^\lambda}{\partial y} \frac{u_e^\lambda}{d\lambda}. \end{aligned} \tag{3.3}$$

Now note that from the definition of  $v_e^\lambda$  and by differentiating it with respect to  $\lambda$ , we get the following identity for  $X \in \mathbb{R}_+^{n+1}$ ,

$$r \frac{\partial v_e^\lambda}{\partial r} = \lambda \partial_\lambda v_e^\lambda - \left( \frac{2s}{p-1} + 2 \right) v_e^\lambda.$$

Hence,

$$\begin{aligned} \int_{\mathbb{R}_+^{n+1} \cap B_1} y^b \frac{\partial v_e^\lambda}{\partial r} \frac{dv_e^\lambda}{d\lambda} &= \int_{\mathbb{R}_+^{n+1} \cap B_1} y^b \left( \lambda \frac{dv_e^\lambda}{d\lambda} \frac{dv_e^\lambda}{d\lambda} - \left( \frac{2s}{p-1} + 2 \right) v_e^\lambda \frac{dv_e^\lambda}{d\lambda} \right) \\ &= \lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} y^b \left( \frac{dv_e^\lambda}{d\lambda} \right)^2 - \left( \frac{s}{p-1} + 1 \right) \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} y^b (v_e^\lambda)^2. \end{aligned}$$

Note that

$$\begin{aligned} - \int_{\mathbb{R}_+^{n+1} \cap B_1} y^b \Delta v_e^\lambda \frac{dv_e^\lambda}{d\lambda} &= \int_{\partial \mathbb{R}_+^{n+1} \cap B_1} y^b \frac{\partial v_e^\lambda}{\partial y} \frac{v_e^\lambda}{d\lambda} - \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} y^b \frac{\partial v_e^\lambda}{\partial r} \frac{v_e^\lambda}{d\lambda} \\ &\quad + \int_{\mathbb{R}_+^{n+1} \cap B_1} \nabla v_e^\lambda \nabla (y^b \frac{dv_e^\lambda}{d\lambda}). \end{aligned}$$

Integration by part we have

$$\begin{aligned}
& \int_{\mathbb{R}_+^{n+1} \cap B_1} y^b \nabla v_e^\lambda \nabla \frac{dv_e^\lambda}{d\lambda} \\
&= - \int_{\partial \mathbb{R}_+^{n+1} \cap B_1} y^b \frac{\partial v_e^\lambda}{\partial y} \frac{dv_e^\lambda}{d\lambda} + \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} y^b \frac{\partial v_e^\lambda}{\partial r} \frac{dv_e^\lambda}{d\lambda} \\
&\quad - \int_{\mathbb{R}_+^{n+1} \cap B_1} \nabla \cdot (y^b \nabla v_e^\lambda) \frac{dv_e^\lambda}{d\lambda} \\
&= - \int_{\partial \mathbb{R}_+^{n+1} \cap B_1} y^b \frac{\partial v_e^\lambda}{\partial y} \frac{dv_e^\lambda}{d\lambda} + \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} y^b \frac{\partial v_e^\lambda}{\partial r} \frac{dv_e^\lambda}{d\lambda} \\
&\quad - \int_{\mathbb{R}_+^{n+1} \cap B_1} y^b \Delta_b v_e^\lambda \Delta_b \frac{du_e^\lambda}{d\lambda}.
\end{aligned}$$

Now

$$\begin{aligned}
& - \int_{\mathbb{R}_+^{n+1} \cap B_1} y^b \Delta_b v_e^\lambda \Delta_b \frac{du_e^\lambda}{d\lambda} = - \int_{\mathbb{R}_+^{n+1} \cap B_1} y^b \Delta_b v_e^\lambda (\Delta \frac{du_e^\lambda}{d\lambda} + \frac{b}{y} \frac{\partial}{\partial y} \frac{du_e^\lambda}{d\lambda}) \\
&= - \int_{\partial(\mathbb{R}_+^{n+1} \cap B_1)} y^b \Delta_b v_e^\lambda \frac{\partial}{\partial n} \frac{du_e^\lambda}{d\lambda} + \int_{\mathbb{R}_+^{n+1} \cap B_1} \nabla(y^b \Delta_b v_e^\lambda) \nabla \frac{du_e^\lambda}{d\lambda} \\
&\quad - \int_{\mathbb{R}_+^{n+1} \cap B_1} b y^{b-1} \Delta_b v_e^\lambda \frac{\partial}{\partial y} \frac{du_e^\lambda}{d\lambda} \\
&= - \int_{\partial(\mathbb{R}_+^{n+1} \cap B_1)} y^b \Delta_b v_e^\lambda \frac{\partial}{\partial n} \frac{du_e^\lambda}{d\lambda} + \int_{\mathbb{R}_+^{n+1} \cap B_1} y^b \nabla \Delta_b v_e^\lambda \nabla \frac{du_e^\lambda}{d\lambda} \\
&= - \int_{\partial(\mathbb{R}_+^{n+1} \cap B_1)} y^b \Delta_b v_e^\lambda \frac{\partial}{\partial n} \frac{du_e^\lambda}{d\lambda} + \int_{\partial(\mathbb{R}_+^{n+1} \cap B_1)} y^b \frac{\partial \Delta_b v_e^\lambda}{\partial n} \frac{du_e^\lambda}{d\lambda} \\
&\quad - \int_{\mathbb{R}_+^{n+1} \cap B_1} y^b \Delta_b^2 v_e^\lambda \frac{du_e^\lambda}{d\lambda} \\
&= - \int_{\partial(\mathbb{R}_+^{n+1} \cap B_1)} y^b \Delta_b v_e^\lambda \frac{\partial}{\partial n} \frac{du_e^\lambda}{d\lambda} + \int_{\partial(\mathbb{R}_+^{n+1} \cap B_1)} y^b \frac{\partial \Delta_b v_e^\lambda}{\partial n} \frac{du_e^\lambda}{d\lambda}.
\end{aligned}$$

Here we have used that  $\Delta_b^2 v_e^\lambda = \Delta_b^3 u_e^\lambda = 0$ . Therefore, combine with the above arguments we get that

$$\begin{aligned}
\int_{\mathbb{R}_+^{n+1} \cap B_1} y^b \nabla v_e^\lambda \nabla \frac{dv_e^\lambda}{d\lambda} &= - \int_{\partial \mathbb{R}_+^{n+1} \cap B_1} y^b \frac{\partial v_e^\lambda}{\partial y} \frac{dv_e^\lambda}{d\lambda} + \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} y^b \frac{\partial v_e^\lambda}{\partial r} \frac{dv_e^\lambda}{d\lambda} \\
&\quad + \int_{\partial \mathbb{R}_+^{n+1} \cap B_1} y^b \Delta_b v_e^\lambda \frac{\partial}{\partial y} \frac{du_e^\lambda}{d\lambda} - \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} y^b \Delta v_e^\lambda \frac{\partial}{\partial r} \frac{du_e^\lambda}{d\lambda} \\
&\quad - \int_{\partial \mathbb{R}_+^{n+1} \cap B_1} y^b \frac{\partial}{\partial y} \Delta_b v_e^\lambda \frac{du_e^\lambda}{d\lambda} + \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} y^b \frac{\partial}{\partial r} \Delta_b v_e^\lambda \frac{du_e^\lambda}{d\lambda} \\
&= \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} y^b \frac{\partial v_e^\lambda}{\partial r} \frac{dv_e^\lambda}{d\lambda} - \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} y^b \Delta v_e^\lambda \frac{\partial}{\partial r} \frac{du_e^\lambda}{d\lambda} \\
&\quad - C(n, s) \int_{\partial \mathbb{R}_+^{n+1} \cap B_1} |u_e^\lambda|^{p-1} \frac{du_e^\lambda}{d\lambda} + \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} y^b \frac{\partial}{\partial r} \Delta_b v_e^\lambda \frac{du_e^\lambda}{d\lambda}.
\end{aligned} \tag{3.4}$$

Here, we have used that  $\frac{\partial \Delta u_e^\lambda(x, 0)}{\partial y} = 0$ ,  $\frac{\partial}{\partial y} \frac{du_e^\lambda}{d\lambda} = 0$  on  $\partial \mathbb{R}_+^{n+1}$  and  $\lim_{y \rightarrow 0} y^b \frac{\partial}{\partial y} \Delta_b v_e^\lambda = -C_{n,s} |u_e^\lambda|^{p-1} u_e^\lambda$ . By (3.3) and (3.4) we obtain that

$$\begin{aligned}
\frac{d}{d\lambda} \bar{E}(u_e^\lambda, 1) &= \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} y^b \frac{\partial v_e^\lambda}{\partial r} \frac{dv_e^\lambda}{d\lambda} + \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} y^b \frac{\partial w_e^\lambda}{\partial r} \frac{du_e^\lambda}{d\lambda} \\
&\quad - \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} y^b w_e^\lambda \frac{\partial}{\partial r} \frac{du_e^\lambda}{d\lambda}.
\end{aligned} \tag{3.5}$$

Recall (3.1) and differentiate it with respect to  $\lambda$ , we have

$$\begin{aligned}
\frac{du_e^\lambda(X)}{d\lambda} &= \frac{1}{\lambda} \left( \frac{2s}{p-1} u_e^\lambda(X) + r \partial_r u_e^\lambda(X) \right), \\
\frac{dv_e^\lambda(X)}{d\lambda} &= \frac{1}{\lambda} \left( \left( \frac{2s}{p-1} + 2 \right) v_e^\lambda(X) + r \partial_r v_e^\lambda(X) \right), \\
\frac{dw_e^\lambda(X)}{d\lambda} &= \frac{1}{\lambda} \left( \left( \frac{2s}{p-1} + 4 \right) w_e^\lambda(X) + r \partial_r w_e^\lambda(X) \right).
\end{aligned}$$

Differentiate the above equations with respect to  $\lambda$  again we get

$$\lambda \frac{d^2 u_e^\lambda(X)}{d\lambda^2} + \frac{du_e^\lambda(x)}{d\lambda} = \frac{2s}{p-1} \frac{du_e^\lambda(X)}{d\lambda} + r \partial_r \frac{du_e^\lambda}{d\lambda}.$$

Hence, for  $X \in \mathbb{R}_+^{n+1} \cap B_1$ , we have

$$\begin{aligned}\partial_r(u_e^\lambda(X)) &= \lambda \frac{du_e^\lambda}{d\lambda} - \frac{2s}{p-1} u_e, \\ \partial_r\left(\frac{du_e^\lambda(X)}{d\lambda}\right) &= \lambda \frac{d^2 u_e^\lambda(X)}{d\lambda^2} + \left(1 - \frac{2s}{p-1}\right) \frac{du_e^\lambda}{d\lambda}, \\ \partial_r(v_e^\lambda(X)) &= \lambda \frac{dv_e^\lambda}{d\lambda} - \left(\frac{2s}{p-1} + 2\right) v_e^\lambda, \\ \partial_r(w_e^\lambda(X)) &= \lambda \frac{dw_e^\lambda}{d\lambda} - \left(\frac{2s}{p-1} + 4\right) w_e^\lambda.\end{aligned}$$

Plugging these equations into (3.5), we get that

$$\begin{aligned}\frac{d}{d\lambda} \overline{E}(u_e^\lambda, 1) &= \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} y^b \left( \lambda \frac{dv_e^\lambda}{d\lambda} \frac{dv_e^\lambda}{d\lambda} - \left(\frac{2s}{p-1} + 2\right) v_e^\lambda \frac{dv_e^\lambda}{d\lambda} \right) \\ &\quad + y^b \left( \lambda \frac{dw_e^\lambda}{d\lambda} \frac{u_e^\lambda}{d\lambda} - \left(\frac{2s}{p-1} + 4\right) w_e^\lambda \frac{du_e^\lambda}{d\lambda} \right) \\ &\quad - y^b \left( \lambda w_e^\lambda \frac{d^2 u_e^\lambda}{d\lambda^2} + \left(1 - \frac{2s}{p-1}\right) w_e^\lambda \frac{du_e^\lambda}{d\lambda} \right) \\ &= \underbrace{\int_{\mathbb{R}_+^{n+1} \cap \partial B_1} y^b \left[ \lambda \frac{dv_e^\lambda}{d\lambda} \frac{dv_e^\lambda}{d\lambda} - \left(\frac{2s}{p-1} + 2\right) v_e^\lambda \frac{dv_e^\lambda}{d\lambda} \right]}_{(3.6)} \\ &\quad + \underbrace{y^b \left[ \lambda \frac{dw_e^\lambda}{d\lambda} \frac{du_e^\lambda}{d\lambda} - \lambda w_e^\lambda \frac{d^2 u_e^\lambda}{d\lambda^2} \right] - 5y^b w_e^\lambda \frac{du_e^\lambda}{d\lambda}}_{-5y^b w_e^\lambda \frac{du_e^\lambda}{d\lambda}} \\ &:= \overline{E}_{d1}(u_e^\lambda, 1) + \overline{E}_{d2}(u_e^\lambda, 2).\end{aligned}$$

### 3.2 The calculations of $\frac{\partial^j}{\partial r^j} u_e^\lambda$ and $\frac{\partial^i}{\partial \lambda^i} u_e^\lambda$ , $i, j = 1, 2, 3, 4$

Note

$$\lambda \frac{du_e^\lambda}{d\lambda} = \frac{2s}{p-1} u_e^\lambda + r \frac{\partial}{\partial r} u_e^\lambda. \quad (3.7)$$

Differentiating (3.7) once, twice and thrice with respect to  $\lambda$  respectively, we have

$$\lambda \frac{d^2 u_e^\lambda}{d\lambda^2} + \frac{du_e^\lambda}{d\lambda} = \frac{2s}{p-1} \frac{du_e^\lambda}{d\lambda} + r \frac{\partial}{\partial r} \frac{du_e^\lambda}{d\lambda}, \quad (3.8)$$

$$\lambda \frac{d^3 u_e^\lambda}{d\lambda^3} + 2 \frac{d^2 u_e^\lambda}{d\lambda^2} = \frac{2s}{p-1} \frac{d^2 u_e^\lambda}{d\lambda^2} + r \frac{\partial}{\partial r} \frac{d^2 u_e^\lambda}{d\lambda^2}, \quad (3.9)$$

$$\lambda \frac{d^4 u_e^\lambda}{d\lambda^4} + 3 \frac{d^3 u_e^\lambda}{d\lambda^3} = \frac{2s}{p-1} \frac{d^3 u_e^\lambda}{d\lambda^3} + r \frac{\partial}{\partial r} \frac{d^3 u_e^\lambda}{d\lambda^3}. \quad (3.10)$$

Similarly, differentiating (3.7) once, twice and thrice with respect to  $r$  respectively we have

$$\lambda \frac{\partial}{\partial r} \frac{du_e^\lambda}{d\lambda} = \left(\frac{2s}{p-1} + 1\right) \frac{\partial}{\partial r} u_e^\lambda + r \frac{\partial^2}{\partial r^2} u_e^\lambda, \quad (3.11)$$

$$\lambda \frac{\partial^2}{\partial r^2} \frac{du_e^\lambda}{d\lambda} = \left(\frac{2s}{p-1} + 2\right) \frac{\partial^2}{\partial r^2} u_e^\lambda + r \frac{\partial^3}{\partial r^3} u_e^\lambda, \quad (3.12)$$

$$\lambda \frac{\partial^3}{\partial r^3} \frac{du_e^\lambda}{d\lambda} = \left(\frac{2s}{p-1} + 3\right) \frac{\partial^3}{\partial r^3} u_e^\lambda + r \frac{\partial^4}{\partial r^4} u_e^\lambda. \quad (3.13)$$

From (3.7), on  $\mathbb{R}_+^{n+1} \cap \partial B_1$ , we have

$$\frac{\partial u_e^\lambda}{\partial r} = \lambda \frac{du_e^\lambda}{d\lambda} - \frac{2s}{p-1} u_e^\lambda.$$

Next from (3.8), on  $\mathbb{R}_+^{n+1} \cap \partial B_1$ , we derive that

$$\frac{\partial}{\partial r} \frac{du_e^\lambda}{d\lambda} = \lambda \frac{d^2 u_e^\lambda}{d\lambda^2} + \left(1 - \frac{2s}{p-1}\right) \frac{du_e^\lambda}{d\lambda}.$$

From (3.11), combine with the two equations above, on  $\mathbb{R}_+^{n+1} \cap \partial B_1$ , we get

$$\begin{aligned} \frac{\partial^2}{\partial r^2} u_e^\lambda &= \lambda \frac{\partial}{\partial r} \frac{du_e^\lambda}{d\lambda} - \left(1 + \frac{2s}{p-1}\right) \frac{\partial}{\partial r} u_e^\lambda \\ &= \lambda^2 \frac{d^2 u_e^\lambda}{d\lambda^2} - \lambda \frac{4s}{p-1} \frac{du_e^\lambda}{d\lambda} + \left(1 + \frac{2s}{p-1}\right) \frac{2s}{p-1} u_e^\lambda. \end{aligned} \quad (3.14)$$

Differentiating (3.8) with respect to  $r$ , and combine with (3.8) and (3.9), we get that

$$\begin{aligned} \frac{\partial^2}{\partial r^2} \frac{du_e^\lambda}{d\lambda} &= \lambda \frac{\partial}{\partial r} \frac{d^2 u_e^\lambda}{d\lambda^2} - \frac{2s}{p-1} \frac{\partial}{\partial r} \frac{du_e^\lambda}{d\lambda} \\ &= \lambda^2 \frac{d^3 u_e^\lambda}{d\lambda^3} + \left(2 - \frac{4s}{p-1}\right) \lambda \frac{d^2 u_e^\lambda}{d\lambda^2} - \left(1 - \frac{2s}{p-1}\right) \frac{2s}{p-1} \frac{du_e^\lambda}{d\lambda}. \end{aligned} \quad (3.15)$$

From (3.12), on  $\mathbb{R}_+^{n+1} \cap \partial B_1$ , combine with (3.14) and (3.15), we have

$$\begin{aligned} \frac{\partial^3}{\partial r^3} u_e^\lambda &= \lambda \frac{\partial^2}{\partial r^2} \frac{du_e^\lambda}{d\lambda} - \left(2 + \frac{2s}{p-1}\right) \frac{\partial^2}{\partial r^2} u_e^\lambda \\ &= \lambda^3 \frac{d^3 u_e^\lambda}{d\lambda^3} - \lambda^2 \frac{6s}{p-1} \frac{d^2 u_e^\lambda}{d\lambda^2} + \lambda \left(\frac{6s}{p-1} + \frac{12s^2}{(p-1)^2}\right) \frac{du_e^\lambda}{d\lambda} \\ &\quad - \left(2 + \frac{2s}{p-1}\right) \left(1 + \frac{2s}{p-1}\right) \frac{2s}{p-1} u_e^\lambda. \end{aligned} \quad (3.16)$$

Now differentiating (3.8) once with respect to  $r$ , we get

$$\lambda \frac{\partial^2}{\partial r^2} \frac{d^2 u_e^\lambda}{d\lambda^2} = \left(\frac{2s}{p-1} + 1\right) \frac{\partial^2}{\partial r^2} \frac{du_e^\lambda}{d\lambda} + r \frac{\partial^3}{\partial r^3} \frac{du_e^\lambda}{d\lambda},$$

then on  $\mathbb{R}_+^{n+1} \cap \partial B_1$ , we have

$$\frac{\partial^3}{\partial r^3} \frac{du_e^\lambda}{d\lambda} = \lambda \frac{\partial^2}{\partial r^2} \frac{d^2 u_e^\lambda}{d\lambda^2} - \left(\frac{2s}{p-1} + 1\right) \frac{\partial^2}{\partial r^2} \frac{du_e^\lambda}{d\lambda}. \quad (3.17)$$

Now differentiating (3.9) twice with respect to  $r$ , we get

$$\lambda \frac{\partial}{\partial r} \frac{d^3 u_e^\lambda}{d\lambda^3} = \left( \frac{2s}{p-1} - 1 \right) \frac{\partial}{\partial r} \frac{d^2 u_e^\lambda}{d\lambda^2} + r \frac{\partial^2}{\partial r^2} \frac{d^2 u_e^\lambda}{d\lambda^2},$$

hence on  $\mathbb{R}_+^{n+1} \cap \partial B_1$ , combine with (3.9) and (3.10) there holds

$$\begin{aligned} \frac{\partial^2}{\partial r^2} \frac{d^2 u_e^\lambda}{d\lambda^2} &= \lambda \frac{\partial}{\partial r} \frac{d^3 u_e^\lambda}{d\lambda^3} + \left(1 - \frac{2s}{p-1}\right) \frac{\partial}{\partial r} \frac{d^2 u_e^\lambda}{d\lambda^2} \\ &= \lambda^2 \frac{d^4 u_e^\lambda}{d\lambda^4} + \lambda \left(4 - \frac{4s}{p-1}\right) \frac{d^3 u_e^\lambda}{d\lambda^3} + \left(1 - \frac{2s}{p-1}\right) \left(2 - \frac{2s}{p-1}\right) \frac{d^2 u_e^\lambda}{d\lambda^2}. \end{aligned} \quad (3.18)$$

Now differentiating (3.8) with respect to  $r$ , we have

$$\lambda \frac{\partial}{\partial r} \frac{d^2 u_e^\lambda}{d\lambda^2} = \frac{2s}{p-1} \frac{\partial}{\partial r} \frac{du_e^\lambda}{d\lambda} + r \frac{\partial^2}{\partial r^2} \frac{du_e^\lambda}{d\lambda}.$$

This combine with (3.8) and (3.9), on  $\mathbb{R}_+^{n+1} \cap \partial B_1$ , we have

$$\begin{aligned} \frac{\partial^2}{\partial r^2} \frac{du_e^\lambda}{d\lambda} &= \lambda \frac{\partial}{\partial r} \frac{d^2 u_e^\lambda}{d\lambda^2} - \frac{2s}{p-1} \frac{\partial}{\partial r} \frac{du_e^\lambda}{d\lambda} \\ &= \lambda^2 \frac{d^3 u_e^\lambda}{d\lambda^3} + \lambda \left(2 - \frac{4s}{p-1}\right) \frac{d^2 u_e^\lambda}{d\lambda^2} - \frac{2s}{p-1} \left(1 - \frac{2s}{p-1}\right) \frac{du_e^\lambda}{d\lambda}. \end{aligned} \quad (3.19)$$

Now from (3.17), combine with (3.18) and (3.19), we get

$$\begin{aligned} \frac{\partial^3}{\partial r^3} \frac{du_e^\lambda}{d\lambda} &= \lambda^3 \frac{d^4 u_e^\lambda}{d\lambda^4} + \lambda^2 \left(3 - \frac{6s}{p-1}\right) \frac{d^3 u_e^\lambda}{d\lambda^3} - \lambda \left(1 - \frac{2s}{p-1}\right) \frac{6s}{p-1} \frac{d^2 u_e^\lambda}{d\lambda^2} \\ &\quad + \left(1 - \frac{2s}{p-1}\right) \left(1 + \frac{2s}{p-1}\right) \frac{2s}{p-1} \frac{du_e^\lambda}{d\lambda}. \end{aligned} \quad (3.20)$$

From (3.13), on  $\mathbb{R}_+^{n+1} \cap \partial B_1$ , combine with (3.20) then

$$\begin{aligned} \frac{\partial^4}{\partial r^4} u_e^\lambda &= \lambda \frac{\partial^3}{\partial r^3} \frac{du_e^\lambda}{d\lambda} - \left(3 + \frac{2s}{p-1}\right) \frac{\partial^3}{\partial r^3} u_e^\lambda \\ &= \lambda^4 \frac{d^4 u_e^\lambda}{d\lambda^4} - \lambda^3 \frac{8s}{p-1} \frac{d^3 u_e^\lambda}{d\lambda^3} + \lambda^2 \left(2 + \frac{4s}{p-1}\right) \frac{6s}{p-1} \frac{d^2 u_e^\lambda}{d\lambda^2} \\ &\quad - \lambda \left(1 + \frac{2s}{p-1}\right) \left(1 + \frac{s}{p-1}\right) \frac{16s}{p-1} \frac{du_e^\lambda}{d\lambda} \\ &\quad + \left(3 + \frac{2s}{p-1}\right) \left(2 + \frac{2s}{p-1}\right) \left(1 + \frac{2s}{p-1}\right) \frac{2s}{p-1} u_e^\lambda. \end{aligned}$$

In summary, we have that

$$\begin{aligned} \frac{\partial^3}{\partial r^3} u_e^\lambda &= \lambda^3 \frac{d^3 u_e^\lambda}{d\lambda^3} - \lambda^2 \frac{6s}{p-1} \frac{d^2 u_e^\lambda}{d\lambda^2} + \lambda \left(\frac{6s}{p-1} + \frac{12s^2}{(p-1)^2}\right) \frac{du_e^\lambda}{d\lambda} \\ &\quad - \left(2 + \frac{2s}{p-1}\right) \left(1 + \frac{2s}{p-1}\right) \frac{2s}{p-1} u_e^\lambda \end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial r^2} u_e^\lambda &= \lambda^2 \frac{d^2 u_e^\lambda}{d\lambda^2} - \lambda \frac{4s}{p-1} \frac{du_e^\lambda}{d\lambda} + \left(1 + \frac{2s}{p-1}\right) \frac{2s}{p-1} u_e^\lambda \\ \frac{\partial u_e^\lambda}{\partial r} &= \lambda \frac{du_e^\lambda}{d\lambda} - \frac{2s}{p-1} u_e^\lambda.\end{aligned}$$

### 3.3 On the operator $\Delta_b^2$ and its representation

Note that

$$\Delta_b u = y^{-b} \nabla \cdot (y^b \nabla u) = u_{rr} + \frac{n+b}{r} u_r + \frac{1}{r^2} \theta_1^{-b} \mathbf{div}_{S^n}(\theta_1^b \nabla_{S^n} u),$$

where  $\theta_1 = \frac{y}{r}$ ,  $r = \sqrt{|x|^2 + y^2}$ . Set  $v = \Delta_b u$  and  $\Delta_b^2 u := w$ . Then

$$\begin{aligned}w &= \Delta_b v = v_{rr} + \frac{n+b}{r} v_r + \frac{1}{r^2} \theta_1^{-b} \mathbf{div}_{S^n}(\theta_1^b \nabla_{S^n} v) \\ &= \partial_{rrrr} u + \frac{2(n+b)}{r} \partial_{rrr} u + \frac{(n+b)(n+b-2)}{r^2} \partial_{rr} u - \frac{(n+b)(n+b-2)}{r^3} \partial_r u \\ &\quad + r^{-4} \theta_1^{-b} \mathbf{div}_{S^n}(\theta_1^b \nabla(\theta_1^{-b} \mathbf{div}_{S^n}(\theta_1^b \nabla_{S^n} u))) \\ &\quad + 2r^{-2} \theta_1^{-b} \mathbf{div}_{S^n}(\theta_1^b \nabla_{S^n}(u_{rr} + \frac{n+b-2}{r} u_r)) \\ &\quad - 2(n+b-3)r^{-4} \theta_1^{-b} \mathbf{div}_{S^n}(\theta_1^b \nabla_{S^n} u).\end{aligned}$$

On  $\mathbb{R}_+^{n+1} \cap \partial B_1$ , we have

$$\begin{aligned}w &= \underbrace{\partial_{rrrr} u + 2(n+b) \partial_{rrr} u + (n+b)(n+b-2) \partial_{rr} u - (n+b)(n+b-2) \partial_r u}_{+ \theta_1^{-b} \mathbf{div}_{S^n}(\theta_1^b \nabla(\theta_1^{-b} \mathbf{div}_{S^n}(\theta_1^b \nabla_{S^n} u)))} \\ &\quad + \underbrace{2\theta_1^{-b} \mathbf{div}_{S^n}(\theta_1^b \nabla_{S^n}(u_{rr} + \frac{n+b-2}{r} u_r))}_{- 2(n+b-3)\theta_1^{-b} \mathbf{div}_{S^n}(\theta_1^b \nabla_{S^n} u)} \\ &:= I(u) + J(u) + K(u) + L(u).\end{aligned}$$

By these notations, we can rewrite the term  $\overline{E}_{d2}(u_e^\lambda, 1)$  appear in (3.6) as following

$$\begin{aligned}
& \overline{E}_{d2}(u_e^\lambda, 1) \\
&= \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b (\lambda \frac{dw_e^\lambda}{d\lambda} \frac{du_e^\lambda}{d\lambda} - \lambda w_e^\lambda \frac{d^2 u_e^\lambda}{d\lambda^2}) - 5\theta_1^b w_e^\lambda \frac{du_e^\lambda}{d\lambda} \\
&= \underbrace{\int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \lambda \theta_1^b \frac{d}{d\lambda} I(u_e^\lambda) \frac{du_e^\lambda}{d\lambda} - \lambda \theta_1^b I(u_e^\lambda) \frac{d^2 u_e^\lambda}{d\lambda^2} - 5\theta_1^b I(u_e^\lambda) \frac{du_e^\lambda}{d\lambda}}_{\lambda \theta_1^b \frac{d}{d\lambda} J(u_e^\lambda) \frac{du_e^\lambda}{d\lambda} - \lambda \theta_1^b J(u_e^\lambda) \frac{d^2 u_e^\lambda}{d\lambda^2} - 5\theta_1^b J(u_e^\lambda) \frac{du_e^\lambda}{d\lambda}} \\
&\quad + \underbrace{\int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \lambda \theta_1^b \frac{d}{d\lambda} K(u_e^\lambda) \frac{du_e^\lambda}{d\lambda} - \lambda \theta_1^b K(u_e^\lambda) \frac{d^2 u_e^\lambda}{d\lambda^2} - 5\theta_1^b K(u_e^\lambda) \frac{du_e^\lambda}{d\lambda}}_{\lambda \theta_1^b \frac{d}{d\lambda} L(u_e^\lambda) \frac{du_e^\lambda}{d\lambda} - \lambda \theta_1^b L(u_e^\lambda) \frac{d^2 u_e^\lambda}{d\lambda^2} - 5\theta_1^b L(u_e^\lambda) \frac{du_e^\lambda}{d\lambda}} \\
&\quad + \underbrace{\int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \lambda \theta_1^b \frac{d}{d\lambda} L(u_e^\lambda) \frac{du_e^\lambda}{d\lambda} - \lambda \theta_1^b L(u_e^\lambda) \frac{d^2 u_e^\lambda}{d\lambda^2} - 5\theta_1^b L(u_e^\lambda) \frac{du_e^\lambda}{d\lambda}}, \tag{3.21}
\end{aligned}$$

we define as

$$\begin{aligned}
\overline{E}_{d2}(u_e^\lambda, 1) &:= \mathcal{I} + \mathcal{J} + \mathcal{K} + \mathcal{L} \\
&:= I_1 + I_2 + I_3 + J_1 + J_2 + J_3 + K_1 + K_2 + K_3 + L_1 + L_2 + L_3.
\end{aligned}$$

where  $I_1, I_2, I_3, J_1, J_2, J_3, K_1, K_2, K_3, L_1, L_2, L_3$  are corresponding successively to the 12 terms in (3.21). By the conclusions of subsection 2.2, we have

$$\begin{aligned}
I(u_e^\lambda) &= \partial_{rrrr} u_e^\lambda + 2(n+b)\partial_{rrr} u_e^\lambda \\
&\quad + (n+b)(n+b-2)\partial_{rr} u_e^\lambda - (n+b)(n+b-2)\partial_r u_e^\lambda \\
&= \lambda^4 \frac{d^4 u_e^\lambda}{d\lambda^4} + \lambda^3 \left(2(n+b) - \frac{8s}{p-1}\right) \frac{d^3 u_e^\lambda}{d\lambda^3} \\
&\quad + \lambda^2 \left[ \frac{12s}{p-1} \left(1 + \frac{2s}{p-1}\right) - (n+b) \frac{12s}{p-1} + (n+b)(n+b-2) \right] \frac{d^2 u_e^\lambda}{d\lambda^2} \\
&\quad + \lambda \left[ -\frac{8s}{p-1} \left(1 + \frac{2s}{p-1}\right) \left(2 + \frac{2s}{p-1}\right) + 2(n+b) \frac{6s}{p-1} \left(1 + \frac{2s}{p-1}\right) \right. \\
&\quad \left. + (n+b)(n+b-2) \left(-\frac{4s}{p-1} - 1\right) \right] \frac{du_e^\lambda}{d\lambda} \\
&\quad + \left[ \left(1 + \frac{2s}{p-1}\right) \left(2 + \frac{2s}{p-1}\right) \left(3 + \frac{2s}{p-1}\right) \frac{2s}{p-1} \right. \\
&\quad \left. - (n+b) \left(1 + \frac{2s}{p-1}\right) \left(2 + \frac{2s}{p-1}\right) \frac{4s}{p-1} \right. \\
&\quad \left. + (n+b)(n+b-2) \left(\frac{2s}{p-1} + 2\right) \frac{2s}{p-1} \right] u_e^\lambda. \tag{3.22}
\end{aligned}$$

For convenience, we denote that

$$I(u_e^\lambda) = \lambda^4 \frac{d^4 u_e^\lambda}{d\lambda^4} + \lambda^3 \delta_1 \frac{d^3 u_e^\lambda}{d\lambda^3} + \lambda^2 \delta_2 \frac{d^2 u_e^\lambda}{d\lambda^2} + \lambda \delta_3 \frac{du_e^\lambda}{d\lambda} + \delta_4 u_e^\lambda, \quad (3.23)$$

where  $\delta_i$  are the corresponding coefficients of  $\lambda^i \frac{d^i u_e^\lambda}{d\lambda^i}$  appeared in (3.22) for  $i = 1, 2, 3, 4$ . Now taking the derivative of (3.23) with respect to  $\lambda$ , we get

$$\begin{aligned} \frac{d}{d\lambda} I(u_e^\lambda) &= \lambda^4 \frac{d^5 u_e^\lambda}{d\lambda^5} + \lambda^3 (\delta_1 + 4) \frac{d^4 u_e^\lambda}{d\lambda^4} + \lambda^2 (3\delta_1 + \delta_2) \frac{d^3 u_e^\lambda}{d\lambda^3} \\ &\quad + \lambda (2\delta_2 + \delta_3) \frac{d^2 u_e^\lambda}{d\lambda^2} + (\delta_3 + \delta_4) \frac{du_e^\lambda}{d\lambda} \end{aligned} \quad (3.24)$$

and

$$\begin{aligned} &\partial_{rr} u_e^\lambda + (n + b - 2) \partial_r u_e^\lambda \\ &= \lambda^2 \frac{d^2 u_e^\lambda}{d\lambda^2} + \lambda (n + b - 2 - \frac{4s}{p-1}) \frac{du_e^\lambda}{d\lambda} + \frac{2s}{p-1} (3 + \frac{2s}{p-1} - n - b) u_e^\lambda \\ &:= \lambda^2 \frac{d^2 u_e^\lambda}{d\lambda^2} + \lambda \alpha \frac{du_e^\lambda}{d\lambda} + \beta u_e^\lambda. \end{aligned} \quad (3.25)$$

Hence,

$$\frac{d}{d\lambda} [\partial_{rr} u_e^\lambda + (n + b - 2) \partial_r u_e^\lambda] = \lambda^2 \frac{d^3 u_e^\lambda}{d\lambda^3} + \lambda (\alpha + 2) \frac{d^2 u_e^\lambda}{d\lambda^2} + (\alpha + \beta) \frac{du_e^\lambda}{d\lambda}, \quad (3.26)$$

here  $\alpha = n + b - 2 - \frac{4s}{p-1}$  and  $\beta = \frac{2s}{p-1} (3 + \frac{2s}{p-1} - n - b)$ .

### 3.4 The computations of $I_1, I_2, I_3$ and $\mathcal{I}$

$$\begin{aligned}
I_1 &:= \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \lambda \theta_1^b \frac{d}{d\lambda} I(u_e^\lambda) \frac{du_e^\lambda}{d\lambda} \\
&= \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left( \lambda^5 \frac{d^5 u_e^\lambda}{d\lambda^5} + \lambda^4 (4 + \delta_1) \frac{d^4 u_e^\lambda}{d\lambda^4} + \lambda^3 (3\delta_1 + \delta_2) \frac{d^3 u_e^\lambda}{d\lambda^3} \right. \\
&\quad \left. + \lambda^2 (2\delta_2 + \delta_3) \frac{d^2 u_e^\lambda}{d\lambda^2} + \lambda (\delta_3 + \delta_4) \frac{du_e^\lambda}{d\lambda} \right) \frac{du_e^\lambda}{d\lambda} \\
&= \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left[ \lambda^5 \frac{d^4 u_e^\lambda}{d\lambda^4} \frac{du_e^\lambda}{d\lambda} - \lambda^5 \frac{d^3 u_e^\lambda}{d\lambda^3} \frac{d^2 u_e^\lambda}{d\lambda^2} + (\delta_1 - 1) \lambda^4 \frac{d^3 u_e^\lambda}{d\lambda^3} \frac{du_e^\lambda}{d\lambda} \right. \\
&\quad \left. + (4 - \delta_1 + \delta_2) \lambda^3 \frac{d^2 u_e^\lambda}{d\lambda^2} \frac{du_e^\lambda}{d\lambda} + \frac{3\delta_1 - \delta_2 + \delta_3 - 12}{2} \lambda^2 \left( \frac{du_e^\lambda}{d\lambda} \right)^2 \right] \\
&\quad + \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left[ (12 - 3\delta_1 + \delta_2 + \delta_4) \lambda \left( \frac{du_e^\lambda}{d\lambda} \right)^2 \right. \\
&\quad \left. + (\delta_1 - 4 - \delta_2) \lambda^3 \left( \frac{d^2 u_e^\lambda}{d\lambda^2} \right)^2 + \lambda^5 \left( \frac{d^3 u_e^\lambda}{d\lambda^3} \right)^2 \right] \\
&\quad + \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b (6 - \delta_1) \lambda^4 \frac{d^3 u_e^\lambda}{d\lambda^3} \frac{d^2 u_e^\lambda}{d\lambda^2}, \tag{3.27}
\end{aligned}$$

where  $\delta_i (i = 1, 2, 3, 4)$  are defined in (3.22) and (3.23). Denote  $f = u_e^\lambda, f' := \frac{du_e^\lambda}{d\lambda}$ , we have used the following differential identities:

$$\begin{aligned}
\lambda^5 f''''' f' &= [\lambda^5 f''''' f' - \lambda^5 f''' f'' - 5\lambda^4 f''' f' + 20\lambda^3 f'' f' - 30\lambda^2 f' f']' \\
&\quad + 60\lambda(f')^2 - 20\lambda^3(f'')^2 + \lambda^5(f''')^2 + 10\lambda^4 f''' f'', 
\end{aligned}$$

$$\begin{aligned}
\lambda^4 f''''' f' &= [\lambda^4 f''''' f' - 4\lambda^3 f'' f' + 6\lambda^2 f' f']' - 12\lambda(f')^2 + 4\lambda^3(f'')^2 - \lambda^4 f''' f'', \\
\lambda^3 f''''' f' &= [\lambda^3 f'' f' - \frac{3\lambda^2}{2} f' f']' + 3\lambda(f')^2 - \lambda^3(f'')^2,
\end{aligned}$$

and

$$\lambda^2 f'' f' = [\frac{\lambda^2}{2} f' f']' - \lambda(f')^2.$$

$$\begin{aligned}
I_2 &:= -\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b I(u_e^\lambda) \frac{d^2 u_e^\lambda}{d\lambda^2} \\
&= -\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left( \lambda^4 \frac{d^4 u_e^\lambda}{d\lambda^4} + \lambda^3 \delta_1 \frac{d^3 u_e^\lambda}{d\lambda^3} + \lambda^2 \delta_2 \frac{d^2 u_e^\lambda}{d\lambda^2} + \lambda \delta_3 \frac{du_e^\lambda}{d\lambda} + \delta_4 u_e^\lambda \right) \frac{d^2 u_e^\lambda}{d\lambda^2} \\
&= \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left[ -\lambda^5 \frac{d^3 u_e^\lambda}{d\lambda^3} \frac{d^2 u_e^\lambda}{d\lambda^2} - \delta_4 \lambda \frac{du_e^\lambda}{d\lambda} u_e^\lambda \right] \\
&\quad + \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left[ \lambda^5 \left( \frac{d^3 u_e^\lambda}{d\lambda^3} \right)^2 - \delta_2 \lambda^3 \left( \frac{d^2 u_e^\lambda}{d\lambda^2} \right)^2 + \delta_4 \lambda \left( \frac{du_e^\lambda}{d\lambda} \right)^2 \right] \\
&\quad + \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left[ (5 - \delta_1) \lambda^4 \frac{d^3 u_e^\lambda}{d\lambda^3} \frac{d^2 u_e^\lambda}{d\lambda^2} - \delta_3 \lambda^2 \frac{d^2 u_e^\lambda}{d\lambda^2} \frac{du_e^\lambda}{d\lambda} + \delta_4 \frac{du_e^\lambda}{d\lambda} u_e^\lambda \right].
\end{aligned} \tag{3.28}$$

Here we have used that

$$-\lambda^5 f'''' f'' = [-\lambda^5 f''' f'']' + 5\lambda^4 f''' f'' + \lambda^5 (f''')^2$$

and

$$-\lambda f'' f = [-\lambda f' f]' + f' f + \lambda (f')^2.$$

$$\begin{aligned}
I_3 &:= -5 \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b I(u_e^\lambda) \frac{du_e^\lambda}{d\lambda} \\
&= -5 \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left[ \lambda^4 \frac{d^4 u_e^\lambda}{d\lambda^4} + \lambda^3 \delta_1 \frac{d^3 u_e^\lambda}{d\lambda^3} + \lambda^2 \delta_2 \frac{d^2 u_e^\lambda}{d\lambda^2} + \lambda \delta_3 \frac{du_e^\lambda}{d\lambda} + \delta_4 u_e^\lambda \right] \frac{du_e^\lambda}{d\lambda} \\
&= \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left[ -5 \frac{d^3 u_e^\lambda}{d\lambda^3} \frac{du_e^\lambda}{d\lambda} + (20 - 5\delta_1) \lambda^3 \frac{d^2 u_e^\lambda}{d\lambda^2} \frac{du_e^\lambda}{d\lambda} \right] \\
&\quad + \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left[ (5\delta_1 - 20) \lambda^3 \left( \frac{d^2 u_e^\lambda}{d\lambda^2} \right)^2 - 5\delta_3 \lambda \left( \frac{du_e^\lambda}{d\lambda} \right)^2 \right] \\
&\quad + \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left[ 5\lambda^4 \frac{d^3 u_e^\lambda}{d\lambda^3} \frac{d^2 u_e^\lambda}{d\lambda^2} + (15\delta_1 - 60 - 5\delta_2) \frac{d^2 u_e^\lambda}{d\lambda^2} \frac{du_e^\lambda}{d\lambda} - 5\delta_4 \frac{du_e^\lambda}{d\lambda} u_e^\lambda \right].
\end{aligned} \tag{3.29}$$

Here we have use that

$$-\lambda^4 f'''' f' = [-5\lambda^4 f''' f' + 20\lambda^3 f'' f']' - 20\lambda^3 (f'')^2 - 60\lambda^2 f'' f' + 5\lambda^4 f''' f''$$

and

$$-\lambda^3 f'''' f' = [-\lambda^3 f''' f']' + 3\lambda^2 f'' f' + \lambda^3 (f'')^2.$$

Now we add up  $I_1, I_2, I_3$  and further integrate by part, we can get the term  $\mathcal{I}$ .

$$\begin{aligned}
\mathcal{I} &:= I_1 + I_2 + I_3 \\
&= \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left[ \lambda^5 \frac{d^4 u_e^\lambda}{d\lambda^4} \frac{du_e^\lambda}{d\lambda} - 2\lambda^5 \frac{d^3 u_e^\lambda}{d\lambda^3} \frac{d^2 u_e^\lambda}{d\lambda^2} \right. \\
&\quad + (\delta_1 - 6)\lambda^4 \frac{d^3 u_e^\lambda}{d\lambda^3} \frac{du_e^\lambda}{d\lambda} + (24 - 6\delta_1 + \delta_2)\lambda^3 \frac{d^2 u_e^\lambda}{d\lambda^2} \frac{du_e^\lambda}{d\lambda} \\
&\quad + (9\delta_1 - 3\delta_2 - 36)\lambda^2 \frac{du_e^\lambda}{d\lambda} \frac{du_e^\lambda}{d\lambda} \\
&\quad + (8 - \delta_1)\lambda^4 \left( \frac{d^2 u_e^\lambda}{d\lambda^2} \right)^2 - \delta_4 \lambda \frac{du_e^\lambda}{d\lambda} u_e^\lambda - 2\delta_4 (u_e^\lambda)^2 \Big] \\
&\quad + \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left( 2\lambda^5 \left( \frac{d^3 u_e^\lambda}{d\lambda^3} \right)^2 + (10\delta_1 - 2\delta_2 - 56)\lambda^3 \left( \frac{d^2 u_e^\lambda}{d\lambda^2} \right)^2 \right. \\
&\quad \left. + (-18\delta_1 + \delta_2 - 4\delta_3 + 2\delta_4 + 72)\lambda \left( \frac{du_e^\lambda}{d\lambda} \right)^2 \right). \tag{3.30}
\end{aligned}$$

Since  $u_e^\lambda(X) = \lambda^{\frac{2s}{p-1}} u_e(\lambda X)$ , we have the following

$$\begin{aligned}
\lambda^4 \frac{d^4 u_e^\lambda}{d\lambda^4} &= \lambda^{\frac{2s}{p-1}} \left[ \frac{2s}{p-1} \left( \frac{2s}{p-1} - 1 \right) \left( \frac{2s}{p-1} - 2 \right) \left( \frac{2s}{p-1} - 3 \right) u_e(\lambda X) \right. \\
&\quad + \frac{8s}{p-1} \left( \frac{2s}{p-1} - 1 \right) \left( \frac{2s}{p-1} - 2 \right) r \lambda \partial_r u_e(\lambda X) \\
&\quad + \frac{12s}{p-1} \left( \frac{2s}{p-1} - 1 \right) r^2 \lambda^2 \partial_{rr} u_e(\lambda X) \\
&\quad \left. + \frac{8s}{p-1} r^3 \lambda^3 \partial_{rrr} u_e(\lambda X) + r^4 \lambda^4 \partial_{rrrr} u_e(\lambda X) \right],
\end{aligned}$$

and

$$\begin{aligned}
\lambda^3 \frac{d^3 u_e^\lambda}{d\lambda^3} &= \lambda^{\frac{2s}{p-1}} \left[ \frac{2s}{p-1} \left( \frac{2s}{p-1} - 1 \right) \left( \frac{2s}{p-1} - 2 \right) u_e(\lambda X) \right. \\
&\quad + \frac{6s}{p-1} \left( \frac{2s}{p-1} - 1 \right) r \lambda \partial_r u_e(\lambda X) \\
&\quad \left. + \frac{6s}{p-1} r^2 \lambda^2 \partial_{rr} u_e(\lambda X) + r^3 \lambda^3 \partial_{rrr} u_e(\lambda X) \right],
\end{aligned}$$

$$\lambda^2 \frac{d^2 u_e^\lambda}{d\lambda^2}$$

$$= \lambda^{\frac{2s}{p-1}} \left[ \frac{2s}{p-1} \left( \frac{2s}{p-1} - 1 \right) u_e(\lambda X) + \frac{4s}{p-1} r \lambda \partial_r u_e(\lambda X) + r^2 \lambda^2 \partial_{rr} u_e(\lambda X) \right]$$

and

$$\lambda \frac{du_e^\lambda}{d\lambda} = \lambda^{\frac{2s}{p-1}} \left[ \frac{2s}{p-1} u_e(\lambda X) + r \lambda \partial_r u_e(\lambda X) \right].$$

Hence, by scaling we have the following derivatives:

$$\begin{aligned}
& \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \lambda^5 \frac{d^4 u_e^\lambda}{d\lambda^4} \frac{du_e^\lambda}{d\lambda} \\
&= \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda} \lambda^{2s \frac{p+1}{p-1} - n - 5} y^b \left[ \frac{2s}{p-1} \left( \frac{2s}{p-1} - 1 \right) \left( \frac{2s}{p-1} - 2 \right) \left( \frac{2s}{p-1} - 3 \right) u_e \right. \\
&\quad + \frac{8s}{p-1} \left( \frac{2s}{p-1} - 1 \right) \left( \frac{2s}{p-1} - 2 \right) \lambda \partial_r u_e + \frac{12s}{p-1} \left( \frac{2s}{p-1} - 1 \right) \lambda^2 \partial_{rr} u_e \\
&\quad \left. + \frac{8s}{p-1} \lambda^3 \partial_{rrr} u_e + \lambda^4 \partial_{rrrr} u_e \right] \left[ \frac{2s}{p-1} u_e + r \lambda \partial_r u_e \right]; \\
& \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \lambda^5 \frac{d^3 u_e^\lambda}{d\lambda^3} \frac{d^2 u_e^\lambda}{d\lambda^2} \\
&= \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda} \lambda^{2s \frac{p+1}{p-1} - n - 5} y^b \left[ \frac{2s}{p-1} \left( \frac{2s}{p-1} - 1 \right) \left( \frac{2s}{p-1} - 2 \right) u_e \right. \\
&\quad + \frac{6s}{p-1} \left( \frac{2s}{p-1} - 1 \right) \lambda \partial_r u_e + \frac{6s}{p-1} \lambda^2 \partial_{rr} u_e + \lambda^3 \partial_{rrr} u_e \\
&\quad \left. \left[ \frac{2s}{p-1} \left( \frac{2s}{p-1} - 1 \right) u_e + \frac{4s}{p-1} \lambda \partial_r u_e + \lambda^2 \partial_{rr} u_e \right] \right]; \tag{3.31} \\
& \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \lambda^4 \frac{d^3 u_e^\lambda}{d\lambda^3} \frac{du_e^\lambda}{d\lambda} \\
&= \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda} \lambda^{2s \frac{p+1}{p-1} - n - 5} y^b \left[ \frac{2s}{p-1} \left( \frac{2s}{p-1} - 1 \right) \left( \frac{2s}{p-1} - 2 \right) u_e \right. \\
&\quad + \frac{6s}{p-1} \left( \frac{2s}{p-1} - 1 \right) \lambda \partial_r u_e + \frac{6s}{p-1} \lambda^2 \partial_{rr} u_e + \lambda^3 \partial_{rrr} u_e \\
&\quad \left. \left[ \frac{2s}{p-1} u_e + \lambda \partial_r u_e \right] \right]; \\
& \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \lambda^3 \frac{d^2 u_e^\lambda}{d\lambda^2} \frac{du_e^\lambda}{d\lambda} \\
&= \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda} \lambda^{2s \frac{p+1}{p-1} - n - 5} y^b \left[ \frac{2s}{p-1} \left( \frac{2s}{p-1} - 1 \right) u_e \right. \\
&\quad + \frac{4s}{p-1} \lambda \partial_r u_e + \lambda^2 \partial_{rr} u_e \left. \left[ \frac{2s}{p-1} u_e + \lambda \partial_r u_e \right] \right].
\end{aligned}$$

and

$$\begin{aligned}
& \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \lambda^2 \frac{du_e^\lambda}{d\lambda} \frac{du_e^\lambda}{d\lambda} \\
&= \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda} \lambda^{2s \frac{p+1}{p-1} - n - 5} y^b \left[ \frac{2s}{p-1} u_e + \lambda \partial_r u_e \right]^2.
\end{aligned}$$

Further,

$$\begin{aligned}
& \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \lambda^4 \frac{d^2 u_e^\lambda}{d\lambda^2} \frac{d^2 u_e^\lambda}{d\lambda^2} \\
&= \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda} \lambda^{2s \frac{p+1}{p-1} - n - 5} y^b \left[ \frac{2s}{p-1} \left( \frac{2s}{p-1} - 1 \right) u_e \right. \\
&\quad \left. + \frac{4s}{p-1} \lambda \partial_r u_e + \lambda^2 \partial_{rr} u_e \right]^2, \\
& \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \lambda \frac{du_e^\lambda}{d\lambda} u_e^\lambda \\
&= \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda} \lambda^{2s \frac{p+1}{p-1} - n - 5} y^b \left[ \frac{2s}{p-1} u_e + \lambda \partial_r u_e \right] u_e,
\end{aligned}$$

and

$$\frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b u_e^\lambda = \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda} \lambda^{2s \frac{p+1}{p-1} - n - 5} y^b u_e^2.$$

### 3.5 The computations of $J_i, K_i, L_i (i = 1, 2, 3)$ and $\mathcal{J}, \mathcal{K}, \mathcal{L}$

Firstly,

$$\begin{aligned}
J_1 &:= \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \lambda \theta_1^b \frac{d}{d\lambda} J(u_e^\lambda) \frac{du_e^\lambda}{d\lambda} = \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \lambda \theta_1^b J\left(\frac{du_e^\lambda}{d\lambda}\right) \frac{du_e^\lambda}{d\lambda} \\
&= \lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \operatorname{div}_{S^n} \left( \theta_1^b \nabla_{S^n} \left( \theta_1^{-b} \operatorname{div}_{S^n} \left( \theta_1^b \nabla_{S^n} \frac{du_e^\lambda}{d\lambda} \right) \right) \right) \frac{du_e^\lambda}{d\lambda} \\
&= -\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \nabla_{S^n} \left( \theta_1^{-b} \operatorname{div}_{S^n} \left( \theta_1^b \nabla_{S^n} \frac{du_e^\lambda}{d\lambda} \right) \right) \nabla_{S^n} \frac{du_e^\lambda}{d\lambda} \quad (3.32) \\
&= \lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^{-b} \left[ \operatorname{div}_{S^n} \left( \theta_1^b \nabla_{S^n} \frac{du_e^\lambda}{d\lambda} \right) \right]^2 \\
&= \lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b (\Delta_{S^n} \frac{du_e^\lambda}{d\lambda})^2,
\end{aligned}$$

here we have used integrate by part formula on the unit sphere  $S^n$ .

$$\begin{aligned}
J_2 &:= -\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b J(u_e^\lambda) \frac{d^2 u_e^\lambda}{d\lambda^2} \\
&= -\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \mathbf{div}_{S^n}(\theta_1^b \nabla_{S^n}(\theta_1^{-b} \mathbf{div}_{S^n}(\theta_1^b \nabla_{S^n} u_e^\lambda))) \frac{d^2 u_e^\lambda}{d\lambda^2} \\
&= \lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \nabla_{S^n}(\theta_1^{-b} \mathbf{div}_{S^n}(\theta_1^b \nabla_{S^n} u_e^\lambda)) \nabla_{S^n} \frac{d^2 u_e^\lambda}{d\lambda^2} \\
&= -\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^{-b} \mathbf{div}_{S^n}(\theta_1^b \nabla_{S^n} u_e^\lambda) \frac{d^2}{d\lambda^2} \mathbf{div}_{S^n}(\theta_1^b \nabla_{S^n} u_e^\lambda) \\
&= \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} -\lambda \theta_1^{-b} [\mathbf{div}_{S^n}(\theta_1^b \nabla_{S^n} u_e^\lambda)] \frac{d}{d\lambda} [\mathbf{div}_{S^n}(\theta_1^b \nabla_{S^n} u_e^\lambda)] \quad (3.33) \\
&\quad + \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^{-b} \mathbf{div}_{S^n}(\theta_1^b \nabla_{S^n} u_e^\lambda) \cdot \frac{d}{d\lambda} \mathbf{div}_{S^n}(\theta_1^b \nabla_{S^n} u_e^\lambda) \\
&\quad + \lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^{-b} \left[ \frac{d}{d\lambda} \mathbf{div}_{S^n}(\theta_1^b \nabla_{S^n} u_e^\lambda) \right]^2 \\
&= \frac{d}{d\lambda} \left( \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left( -\frac{1}{2} \lambda \frac{d}{d\lambda} (\Delta_{S^n} u_e^\lambda)^2 + \frac{1}{2} (\Delta_{S^n} u_e^\lambda)^2 \right) \right) \\
&\quad + \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \lambda (\Delta_{S^n} \frac{du_e^\lambda}{d\lambda})^2,
\end{aligned}$$

here we denote that  $g = \mathbf{div}_{S^n}(\theta_1^b \nabla_{S^n} u_e^\lambda)$ ,  $g' = \frac{d}{d\lambda} \mathbf{div}_{S^n}(\theta_1^b \nabla_{S^n} u_e^\lambda)$  and we have used that

$$-\lambda g g' = [-gg']' + gg' + \lambda(g')^2 = [-gg' + \frac{1}{2}g^2]' + \lambda(g')^2.$$

Further,

$$\begin{aligned}
J_3 &:= -5 \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b J(u_e^\lambda) \frac{du_e^\lambda}{d\lambda} \\
&= -5 \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \mathbf{div}_{S^n}(\theta_1^b \nabla_{S^n}(\theta_1^{-b} \mathbf{div}_{S^n}(\theta_1^b \nabla_{S^n} u))) \frac{du_e^\lambda}{d\lambda} \\
&= 5 \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \nabla_{S^n}(\theta_1^{-b} \mathbf{div}_{S^n}(\theta_1^b \nabla_{S^n} u_e^\lambda)) \nabla_{S^n} \frac{du_e^\lambda}{d\lambda} \quad (3.34) \\
&= -5 \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^{-b} \mathbf{div}_{S^n}(\theta_1^b \nabla_{S^n} u_e^\lambda) \frac{d}{d\lambda} \mathbf{div}_{S^n}(\theta_1^b \nabla_{S^n} u_e^\lambda) \\
&= -\frac{5}{2} \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b (\Delta_{S^n} u_e^\lambda)^2.
\end{aligned}$$

Therefore, combine with (3.32), (3.33) and (3.34), we get that

$$\begin{aligned}
\mathcal{J} &:= J_1 + J_2 + J_3 \\
&= 2\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^{-b} \left[ \frac{d}{d\lambda} \operatorname{div}_{S^n} (\theta_1^b \nabla_{S^n} u_e^\lambda) \right]^2 \\
&\quad - 4 \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^{-b} \operatorname{div}_{S^n} (\theta_1^b \nabla_{S^n} u) \frac{d}{d\lambda} \operatorname{div}_{S^n} (\theta_1^b \nabla_{S^n} u) \\
&\quad + \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} -\lambda \theta_1^b [\operatorname{div}_{S^n} (\theta_1^b \nabla_{S^n} u_e^\lambda)] \frac{d}{d\lambda} [\operatorname{div}_{S^n} (\theta_1^b \nabla_{S^n} u_e^\lambda)] \\
&= 2\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^{-b} \left[ \frac{d}{d\lambda} \operatorname{div}_{S^n} (\theta_1^b \nabla_{S^n} u_e^\lambda) \right]^2 \\
&\quad - 2 \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^{-b} \operatorname{div}_{S^n} (\theta_1^b \nabla_{S^n} u_e^\lambda) \operatorname{div}_{S^n} (\theta_1^b \nabla_{S^n} u_e^\lambda) \\
&\quad + \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} -\lambda \theta_1^{-b} [\operatorname{div}_{S^n} (\theta_1^b \nabla_{S^n} u_e^\lambda)] \frac{d}{d\lambda} [\operatorname{div}_{S^n} (\theta_1^b \nabla_{S^n} u_e^\lambda)] \\
&= \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left( -2(\Delta_{S^n} u_e^\lambda)^2 - \frac{1}{2} \lambda \frac{d}{d\lambda} (\Delta_{S^n} u_e^\lambda)^2 \right) \\
&\quad + 2\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b (\Delta_{S^n} u_e^\lambda)^2.
\end{aligned} \tag{3.35}$$

Note that

$$\begin{aligned}
&\frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b [\theta_1^{-b} \operatorname{div}_{S^n} (\theta_1^b \nabla_{S^n} u_e^\lambda)]^2 \\
&= \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda} \lambda^{2s \frac{p+1}{p-1} - n - 5} (\lambda^2 \Delta_b u_e - \lambda^2 \partial_{rr} u_e - (n+b)\lambda \partial_r u_e)^2,
\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}
&\frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \lambda \frac{d}{d\lambda} [\theta_1^{-b} \operatorname{div}_{S^n} (\theta_1^b \nabla_{S^n} u_e^\lambda)]^2 \\
&= \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda} \lambda^{2s \frac{p+1}{p-1} - n - 4} \frac{d}{d\lambda} (\lambda^2 \Delta_b u_e - \lambda^2 \partial_{rr} u_e - (n+b)\lambda \partial_r u_e)^2.
\end{aligned}$$

Next we compute  $K_1, K_2, K_3$  and  $\mathcal{K}$ .

$$\begin{aligned}
& K_1 \\
:= & \lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \frac{d}{d\lambda} K(u_e^\lambda) \frac{du_e^\lambda}{d\lambda} \\
= & 2\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \mathbf{div}_{S^n} \left( \theta_1^b \nabla_{S^n} \left( \frac{d}{d\lambda} (\partial_{rr} + (n+b-2)\partial_r) u_e^\lambda \right) \right) \frac{du_e^\lambda}{d\lambda} \\
= & 2\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \mathbf{div}_{S^n} \left( \theta_1^b \nabla_{S^n} \left( \lambda^3 \frac{d^3 u_e^\lambda}{d\lambda^3} + \lambda^2(\alpha+2) \frac{d^2 u_e^\lambda}{d\lambda^2} + \lambda(\alpha+\beta) \frac{du_e^\lambda}{d\lambda} \right) \right) \frac{du_e^\lambda}{d\lambda} \\
= & -2\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \nabla_{S^n} \left( \lambda^3 \frac{d^3 u_e^\lambda}{d\lambda^3} + \lambda^2(\alpha+2) \frac{d^2 u_e^\lambda}{d\lambda^2} + \lambda(\alpha+\beta) \frac{du_e^\lambda}{d\lambda} \right) \nabla_{S^n} \frac{du_e^\lambda}{d\lambda} \\
= & \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} -\lambda^3 \theta_1^b \frac{d}{d\lambda} \left( \frac{d}{d\lambda} \nabla_{S^n} u_e^\lambda \right)^2 + (2-2\alpha)\lambda^2 \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left( \frac{d}{d\lambda} \nabla_{S^n} u_e^\lambda \right) \\
& \cdot \left( \frac{d^2}{d\lambda^2} \nabla_{S^n} u_e^\lambda \right) + 2\lambda^3 \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left( \frac{d^2}{d\lambda^2} \nabla_{S^n} u_e^\lambda \right)^2 \\
& - (2\alpha+2\beta)\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left( \frac{d}{d\lambda} \nabla_{S^n} u_e^\lambda \right)^2.
\end{aligned} \tag{3.37}$$

Here we denote that  $h = \nabla_{S^n} u_e^\lambda, h' = \frac{d}{d\lambda} \nabla_{S^n} u_e^\lambda$ , and have used that

$$-\lambda^3 h' h''' = \left[ -\frac{\lambda^3}{2} \frac{d}{d\lambda} (h')^2 \right]' + 3\lambda^2 h' h'' + \lambda^3 (h'')^2.$$

Next,

$$\begin{aligned}
K_2 := & -\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b K(u_e^\lambda) \frac{d^2 u_e^\lambda}{d\lambda^2} \\
= & -2\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \mathbf{div}_{S^n} \left( \theta_1^b \nabla_{S^n} \left( \lambda^2 \frac{d^2 u_e^\lambda}{d\lambda^2} + \lambda\alpha \frac{du_e^\lambda}{d\lambda} + \beta u_e^\lambda \right) \right) \frac{d^2 u_e^\lambda}{d\lambda^2} \\
= & 2\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \nabla_{S^n} \left( \lambda^2 \frac{d^2 u_e^\lambda}{d\lambda^2} + \lambda\alpha \frac{du_e^\lambda}{d\lambda} + \beta u_e^\lambda \right) \nabla_{S^n} \frac{d^2 u_e^\lambda}{d\lambda^2} \\
= & \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left[ 2\beta\lambda \nabla_{S^n} u_e^\lambda \frac{d}{d\lambda} \nabla_{S^n} u_e^\lambda - \beta (\nabla_{S^n} u_e^\lambda)^2 \right] \\
& + 2\lambda^3 \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left( \frac{d^2}{d\lambda^2} \nabla_{S^n} u_e^\lambda \right)^2 - 2\lambda\beta \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left( \frac{d}{d\lambda} \nabla_{S^n} u_e^\lambda \right)^2 \\
& + 2\lambda^2 \alpha \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \frac{d}{d\lambda} \nabla_{S^n} u_e^\lambda \frac{d^2}{d\lambda^2} \nabla_{S^n} u_e^\lambda.
\end{aligned} \tag{3.38}$$

Here we have used that

$$2\lambda h h'' = [2\lambda h h' - h^2]' - 2\lambda(h')^2.$$

Further,

$$\begin{aligned}
K_3 &:= -5 \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b K(u_e^\lambda) \frac{du_e^\lambda}{d\lambda} \\
&= -10 \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \operatorname{div}_{S^n} (\theta_1^b \nabla_{S^n} (\lambda^2 \frac{d^2 u_e^\lambda}{d\lambda^2} + \lambda \alpha \frac{du_e^\lambda}{d\lambda} + \beta u_e^\lambda)) \frac{du_e^\lambda}{d\lambda} \\
&= 10 \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \nabla_{S^n} (\lambda^2 \frac{d^2 u_e^\lambda}{d\lambda^2} + \lambda \alpha \frac{du_e^\lambda}{d\lambda} + \beta u_e^\lambda) \nabla_{S^n} \frac{du_e^\lambda}{d\lambda} \\
&= \frac{d}{d\lambda} \left[ 5\beta \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \nabla_{S^n} u_e^\lambda \nabla_{S^n} u_e^\lambda \right] + 10\lambda \alpha \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left( \frac{d}{d\lambda} \nabla_{S^n} u_e^\lambda \right)^2 \\
&\quad + 10\lambda^2 \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \frac{d}{d\lambda} \nabla_{S^n} u_e^\lambda \frac{d^2}{d\lambda^2} \nabla_{S^n} u_e^\lambda.
\end{aligned} \tag{3.39}$$

Now combine with (3.37), (3.38) and (3.39), we get that

$$\begin{aligned}
\mathcal{K} &:= K_1 + K_2 + K_3 \\
&= \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left[ -\lambda^3 \frac{d}{d\lambda} \left( \frac{d}{d\lambda} \nabla_{S^n} u_e^\lambda \right)^2 \right. \\
&\quad \left. + 2\beta \lambda \nabla_{S^n} u_e^\lambda \frac{d}{d\lambda} \nabla_{S^n} u_e^\lambda + 4\beta (\nabla_{S^n} u_e^\lambda)^2 + 6\lambda^2 (\nabla_{S^n} \frac{du_e^\lambda}{d\lambda})^2 \right] \\
&\quad + 4\lambda^3 \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left( \frac{d^2}{d\lambda^2} \nabla_{S^n} u_e^\lambda \right)^2 \\
&\quad + (8\alpha - 4\beta - 12)\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left( \frac{d}{d\lambda} \nabla_{S^n} u_e^\lambda \right)^2.
\end{aligned} \tag{3.40}$$

Notice that by scaling we have

$$\begin{aligned}
&\frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b (\nabla_{S^n} u_e^\lambda)^2 \\
&= \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda} \lambda^{2s_{\frac{p+1}{p-1}} - n - 5} y^b [\lambda^2 |\nabla u_e|^2 - \lambda^2 |\partial_r u_e|^2]. \\
&\frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \lambda \frac{d}{d\lambda} (\nabla_{S^n} u_e^\lambda)^2 \\
&= \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda} \lambda^{2s_{\frac{p+1}{p-1}} - n - 4} y^b \frac{d}{d\lambda} [\lambda^2 |\nabla u_e|^2 - \lambda^2 |\partial_r u_e|^2]
\end{aligned} \tag{3.41}$$

and

$$\begin{aligned}
&\frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \lambda^3 \frac{d}{d\lambda} \left( \frac{d}{d\lambda} \nabla_{S^n} u_e^\lambda \right)^2 \\
&= \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda} \lambda^{2s_{\frac{p+1}{p-1}} - n - 4} y^b \frac{d}{d\lambda} \left[ \frac{2s}{p-1} \lambda \nabla u_e + \lambda^2 \nabla \partial_r u_e \right]^2.
\end{aligned}$$

Finally, we compute  $\mathcal{L}$ .

$$\begin{aligned}
L_1 &:= \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \lambda \theta_1^b \frac{d}{d\lambda} L(u_e^\lambda) \frac{du_e^\lambda}{d\lambda} \\
&= -2(n+b-3)\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \operatorname{div}_{S^n}(\theta_1^b \nabla_{S^n} \frac{du_e^\lambda}{d\lambda}) \frac{du_e^\lambda}{d\lambda} \\
&= 2(n+b-3)\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b (\nabla_{S^n} \frac{du_e^\lambda}{d\lambda})^2; \\
L_2 &:= \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} -\lambda \theta_1^b L(u_e^\lambda) \frac{d^2 u_e^\lambda}{d\lambda^2} \\
&= 2(n+b-3)\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \operatorname{div}_{S^n}(\theta_1^b \nabla_{S^n} u_e^\lambda) \frac{d^2 u_e^\lambda}{d\lambda^2} \\
&= -2(n+b-3) \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \lambda \theta_1^b \nabla_{S^n} u_e^\lambda \frac{d^2}{d\lambda^2} \nabla_{S^n} u_e^\lambda \\
&= -(n+b-3) \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \frac{d}{d\lambda} [2\lambda \nabla_{S^n} u_e^\lambda \nabla_{S^n} \frac{du_e^\lambda}{d\lambda} - (\nabla_{S^n} u_e^\lambda)^2] \\
&\quad + 2(n+b-3)\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b |\nabla_{S^n} \frac{du_e^\lambda}{d\lambda}|^2; \\
L_3 &:= \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} -5\theta_1^b L(u_e^\lambda) \frac{du_e^\lambda}{d\lambda} \\
&= 10(n+b-3) \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \operatorname{div}_{S^n}(\theta_1^b \nabla_{S^n} u_e^\lambda) \frac{du_e^\lambda}{d\lambda} \\
&= -10(n+b-3) \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \nabla_{S^n} u_e^\lambda \nabla_{S^n} \frac{du_e^\lambda}{d\lambda} \\
&= -5(n+b-3) \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b [\nabla_{S^n} u_e^\lambda]^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathcal{L} &:= L_1 + L_2 + L_3 \\
&= -(n+b-3) \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left[ \lambda \frac{d}{d\lambda} (\nabla_{S^n} u_e^\lambda)^2 \right] \\
&\quad - 4(n+b-3) \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b [\nabla_{S^n} u_e^\lambda]^2 \\
&\quad + 4(n+b-4)\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b |\nabla_{S^n} \frac{du_e^\lambda}{d\lambda}|^2 \\
&= -(n+b-3) \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda} \lambda^{2s \frac{p+1}{p-1} - n - 4} y^b \frac{d}{d\lambda} [\lambda^2 |\nabla u_e|^2 - \lambda^2 |\partial_r u_e|^2] \\
&\quad - 4(n+b-3) \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda} \lambda^{2s \frac{p+1}{p-1} - n - 5} y^b [\lambda^2 |\nabla u_e|^2 - \lambda^2 |\partial_r u_e|^2].
\end{aligned}$$

By rescaling, we have

$$\begin{aligned}
&\frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b \left[ \lambda \frac{d}{d\lambda} (\nabla_{S^n} u_e^\lambda)^2 \right] \\
&= \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda} \lambda^{2s \frac{p+1}{p-1} - n - 4} y^b \frac{d}{d\lambda} [\lambda^2 |\nabla u_e|^2 - \lambda^2 |\partial_r u_e|^2]; \\
&\frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^b [\nabla_{S^n} u_e^\lambda]^2 \\
&= \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda} \lambda^{2s \frac{p+1}{p-1} - n - 5} y^b [\lambda^2 |\nabla u_e|^2 - \lambda^2 |\partial_r u_e|^2].
\end{aligned} \tag{3.42}$$

### 3.6 The term $\bar{E}_{d_1}$

Notice that on the boundary  $\partial B_1$ ,

$$\begin{aligned}
v_e^\lambda &= \Delta_b u_e^\lambda \\
&= \lambda^2 \frac{d^2 u_e^\lambda}{d\lambda^2} + (n+b - \frac{4s}{p-1})\lambda \frac{du_e^\lambda}{d\lambda} + \frac{2s}{p-1}(1 + \frac{2s}{p-1} - n - b)u_e^\lambda + \Delta_\theta u_e^\lambda \\
&:= \lambda^2 \frac{d^2 u_e^\lambda}{d\lambda^2} + \alpha_0 \lambda \frac{du_e^\lambda}{d\lambda} + \beta_0 u_e^\lambda + \Delta_\theta u_e^\lambda.
\end{aligned}$$

Integrate by part, it follows that

$$\begin{aligned}
\int_{\partial B_1} y^b \lambda \left( \frac{dv_e^\lambda}{d\lambda} \right)^2 &= \int_{\partial B_1} \left( \lambda^5 \left( \frac{d^3 u_e^\lambda}{d\lambda^3} \right)^2 + (\alpha_0^2 - 2\alpha_0 - 2\beta_0 - 4) \lambda^3 \left( \frac{d^2 u_e^\lambda}{d\lambda^2} \right)^2 \right. \\
&\quad \left. + (-\alpha_0^2 + \beta_0^2 + 2\alpha_0 + 2\beta_0) \lambda \left( \frac{du_e^\lambda}{d\lambda} \right)^2 \right) \\
&+ \int_{\partial B_1} \left( -2\lambda^3 (\nabla_\theta \frac{d^2 u_e^\lambda}{d\lambda^2})^2 + (10 - 2\beta_0) \lambda (\nabla_\theta \frac{du_e^\lambda}{d\lambda})^2 \right) + \int_{\partial B_1} \lambda (\Delta_\theta \frac{du_e^\lambda}{d\lambda})^2 \\
&+ \frac{d}{d\lambda} \left( \int_{\partial B_1} \sum_{0 \leq i, j \leq 2, i+j \leq 2} c_{i,j}^1 \lambda^{i+j} \frac{d^i u_e^\lambda}{d\lambda^i} \frac{d^j u_e^\lambda}{d\lambda^j} + \sum_{0 \leq s, t \leq 2, s+t \leq 2} c_{s,t}^2 \lambda^{s+t} \frac{d^s u_e^\lambda}{d\lambda^s} \frac{d^t u_e^\lambda}{d\lambda^t} \right), 
\end{aligned} \tag{3.43}$$

where  $c_{i,j}^1, c_{s,t}^2$  depending on  $a, b$  hence on  $p, n$ .

**Proof of Theorem 2.2.** Notice that the equation (3.6), combine with the estimates on  $\mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{L}$  and (3.43), we obtain Theorem 2.2.  $\square$

## 4 Energy estimates and Blow down analysis

In this section, we do some energy estimates for the solutions of (1.1), which are important when we perform a blow-down analysis in the next section.

### 4.1 Energy estimates

**Lemma 4.1.** *Let  $u$  be a solution of (1.1) and  $u_e$  satisfy (2.4), then there exists a positive constant  $C$  such that*

$$\begin{aligned}
&\int_{\partial \mathbb{R}_+^{n+1}} |u_e|^{p+1} \eta^6 + \int_{\mathbb{R}_+^{n+1}} y^b |\nabla \Delta_b u_e|^2 \eta^6 \\
&\leq C \left[ \int_{\mathbb{R}_+^{n+1}} y^b |\Delta_b u_e|^2 \eta^4 |\nabla \eta|^2 + \int_{\mathbb{R}_+^{n+1}} y^b |\nabla u_e|^2 \frac{|\Delta_b \eta^6|^2}{\eta^6} + \int_{\mathbb{R}_+^{n+1}} y^b u_e^2 \frac{|\nabla \Delta_b \eta^6|^2}{\eta^6} \right. \\
&\quad \left. + \int_{\mathbb{R}_+^{n+1}} y^b |\nabla u_e|^2 \eta^2 |\nabla \eta|^4 + \int_{\mathbb{R}_+^{n+1}} y^b |\nabla^2 u_e|^2 \eta^4 |\nabla \eta|^2 \right. \\
&\quad \left. + \int_{\mathbb{R}_+^{n+1}} y^b |\nabla u_e|^2 \eta^4 |\nabla^2 \eta|^2 \right]. 
\end{aligned} \tag{4.1}$$

*Proof.* Multiply the equation (2.4) with  $y^b u_e \eta^6$ , where  $\eta$  is a test function, we get that

$$\begin{aligned}
0 &= \int_{\mathbb{R}_+^{n+1}} y^b u_e \eta^6 \Delta_b^3 u_e = \int_{\mathbb{R}_+^{n+1}} u_e \eta^6 \operatorname{div}(y^b \nabla \Delta_b^2 u_e) \\
&= - \int_{\partial \mathbb{R}_+^{n+1}} u_e \eta^6 \frac{\partial}{\partial y} \Delta_b^2 u_e - \int_{\mathbb{R}_+^{n+1}} y^b \nabla(u_e \eta^6) \nabla \Delta_b^2 u_e \\
&= C_{n,s} \int_{\partial \mathbb{R}_+^{n+1}} |u_e|^{p+1} \eta^6 - \int_{\partial \mathbb{R}_+^{n+1}} y^b \frac{\partial(u_e \eta^6)}{\partial y} \Delta_b^2 u_e + \int_{\mathbb{R}_+^{n+1}} y^b \Delta_b(u_e \eta^6) \Delta_b^2 u_e \\
&= C_{n,s} \int_{\partial \mathbb{R}_+^{n+1}} |u_e|^{p+1} \eta^6 + \int_{\mathbb{R}_+^{n+1}} y^b \Delta_b(u_e \eta^6) \Delta_b^2 u_e \\
&= C_{n,s} \int_{\partial \mathbb{R}_+^{n+1}} |u_e|^{p+1} \eta^6 - \int_{\partial \mathbb{R}_+^{n+1}} \Delta_b(u_e \eta^6) y^b \frac{\partial \Delta_b u_e}{\partial y} \\
&\quad - \int_{\mathbb{R}_+^{n+1}} y^b \nabla(\Delta_b(u_e \eta^6)) \nabla \Delta_b u_e \\
&= C_{n,s} \int_{\partial \mathbb{R}_+^{n+1}} |u_e|^{p+1} \eta^6 - \int_{\mathbb{R}_+^{n+1}} y^b \nabla(\Delta_b(u_e \eta^6)) \nabla \Delta_b u_e.
\end{aligned} \tag{4.2}$$

Hence, we have

$$C_{n,s} \int_{\partial \mathbb{R}_+^{n+1}} |u_e|^{p+1} \eta^6 = \int_{\mathbb{R}_+^{n+1}} y^b \nabla(\Delta_b(u_e \eta^6)) \nabla \Delta_b u_e. \tag{4.3}$$

Since  $\Delta_b(\xi \eta) = \eta \Delta_b \xi + \xi \Delta_b \eta + 2 \nabla \xi \nabla \eta$ , we have

$$\Delta_b(u_e \eta^6) = \eta^6 \Delta_b u_e + u_e \Delta_b \eta^6 + 12 \eta^5 \nabla u_e \nabla \eta,$$

therefore,

$$\begin{aligned}
\nabla \Delta_b(u_e \eta^6) \nabla \Delta_b u_e &= 6 \eta^5 \Delta_b u_e \nabla \eta \nabla \Delta_b u_e + (\eta)^6 (\nabla \Delta_b u_e)^2 + \Delta_b \eta^6 \nabla u_e \nabla \Delta_b u_e \\
&\quad + u_e \nabla \Delta_b \eta^6 \nabla \Delta_b u_e + 60 \eta^4 (\nabla \eta \nabla \Delta_b u_e) (\nabla u_e \nabla \eta) \\
&\quad + 12 \eta^5 \sum_{i,j} \partial_{ij} u_e \partial_i \eta \partial_j \Delta_b u_e + 12 \eta^5 \sum_{i,j} \partial_i u_e \partial_{ij} \eta \partial_j \Delta_b u_e.
\end{aligned} \tag{4.4}$$

here  $\partial_j$  ( $j = 1, \dots, n, n+1$ ) denote the derivatives with respect to  $x_1, \dots, x_n, y$  respectively. A similar way can be applied to deal with the following term  $|\nabla \Delta_b(u_e \eta^3)|^2$ . On the other hand, by the stability condition, we have

$$p \int_{\mathbb{R}^n} |u|^{p+1} \eta^6 \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) \eta^3(x) - u(y) \eta^3(y))^2}{|x-y|^{n+2s}} = \frac{1}{C_{n,s}} \int_{\mathbb{R}_+^{n+1}} y^b |\nabla \Delta_b(u_e \eta^3)|^2. \tag{4.5}$$

(Here we notice that  $u_e(x, 0) = u(x)$ , see Theorem 2.1, (2.3))

Combine with (4.3), (4.4) and (4.5), we have

$$\begin{aligned}
& \int_{\mathbb{R}_+^{n+1}} y^b |\nabla \Delta_b u_e|^2 \eta^6 \\
& \leq C\varepsilon \int_{\mathbb{R}_+^{n+1}} y^b (\nabla \Delta_b u_e)^2 \eta^6 + C(\varepsilon) \left[ \int_{\mathbb{R}_+^{n+1}} y^b |\Delta_b u_e|^2 \eta^4 |\nabla \eta|^2 \right. \\
& \quad + \int_{\mathbb{R}_+^{n+1}} y^b |\nabla u_e|^2 \left( \frac{|\Delta_b \eta|^2}{\eta^6} + \eta^4 |\nabla^2 \eta|^2 \right) \\
& \quad \left. + \int_{\mathbb{R}_+^{n+1}} y^b u_e^2 \frac{|\nabla \Delta_b \eta|^2}{\eta^6} + \int_{\mathbb{R}_+^{n+1}} y^b |\nabla u_e|^2 \eta^2 |\nabla \eta|^4 + \int_{\mathbb{R}_+^{n+1}} y^b |\nabla^2 u_e|^2 \eta^4 |\nabla \eta|^2 \right],
\end{aligned}$$

we can select  $\varepsilon$  so small that  $C\varepsilon \leq \frac{1}{2}$ . Combine with (4.3) and (4.4), we obtain our conclusion.  $\square$

**Corollary 4.1.** *Let  $u$  be a solution of (1.1) and  $u_e$  satisfy (2.4), then*

$$\begin{aligned}
& \int_{\partial \mathbb{R}_+^{n+1} \cap B_{R/2}} |u_e|^{p+1} + \int_{\mathbb{R}_+^{n+1} \cap B_{R/2}} y^b (\nabla \Delta_b u_e)^2 \\
& \leq C \left[ R^{-6} \int_{\mathbb{R}_+^{n+1} \cap B_R} y^b u_e^2 + R^{-4} \int_{\mathbb{R}_+^{n+1} \cap B_R} y^b |\nabla u_e|^2 \right. \\
& \quad \left. + R^{-2} \int_{\mathbb{R}_+^{n+1} \cap B_R} y^b (|\Delta_b u_e|^2 + |\nabla^2 u_e|^2) \right].
\end{aligned}$$

*Proof.* We let  $\eta = \xi^m$  where  $m > 1$  in the estimate (4.1). We have

$$\begin{aligned}
& \int_{\partial \mathbb{R}_+^{n+1}} |u_e|^{p+1} \xi^{6m} + \int_{\mathbb{R}_+^{n+1}} y^b |\nabla \Delta_b u_e|^2 \xi^{6m} \\
& \leq C \left[ \int_{\mathbb{R}_+^{n+1}} y^b (|\Delta_b u_e|^2 + |\nabla^2 u_e|^2) \xi^{6m-2} |\nabla \xi|^2 \right. \\
& \quad \left. + \int_{\mathbb{R}_+^{n+1}} y^b |\nabla u_e|^2 \xi^{6m-4} (|\nabla^2 \xi|^2 + |\nabla \xi|^4) + \int_{\mathbb{R}_+^{n+1}} y^b u_e^2 \xi^{6m-6} |\nabla^3 \xi|^2 \right].
\end{aligned}$$

Let  $\xi = 1$  in  $B_{R/2}$  and  $\xi = 0$  in  $B_R^C$ , satisfying  $|\nabla \xi| \leq \frac{C}{R}$ , then we have the desired estimates.  $\square$

**Lemma 4.2.** *Suppose that  $u$  is a solution of (1.1) which is stable outside some ball  $B_{R_0} \subset \mathbb{R}^n$ . For  $\eta \in C_c^\infty(\mathbb{R}^n \setminus \overline{B_{R_0}})$  and  $x \in \mathbb{R}^n$ , define*

$$\rho(x) = \int_{\mathbb{R}^n} \frac{(\eta(x) - \eta(y))^2}{|x - y|^{n+2s}} dy. \quad (4.6)$$

*Then*

$$\int_{\mathbb{R}^n} |u|^{p+1} \eta^2 dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)\eta(x) - u(y)\eta(y)|^2}{|x - y|^{n+2s}} dx dy \leq C \int_{\mathbb{R}^n} u^2 \rho dx. \quad (4.7)$$

**Lemma 4.3.** Let  $m > n/2$  and  $x \in \mathbb{R}^n$ . Set

$$\rho(x) := \int_{\mathbb{R}^n} \frac{(\eta(x) - \eta(y))^2}{|x - y|^{n+2s}} dy \text{ where } \eta(x) = (1 + |x|^2)^{-m/2}. \quad (4.8)$$

Then there is a constant  $C = C(n, s, m) > 0$  such that

$$C^{-1}(1 + |x|^2)^{-n/2-s} \leq \rho(x) \leq C(1 + |x|^2)^{-n/2-s}. \quad (4.9)$$

**Corollary 4.2.** Suppose that  $m > n/2$ ,  $\eta$  is given by (4.8) and  $R > R_0 > 1$ . Define

$$\rho_R(x) = \int_{\mathbb{R}^n} \frac{(\eta_R(x) - \eta_R(y))^2}{|x - y|^{n+2s}} dy, \text{ where } \eta_R(x) = \eta(x/R)\psi(x/R) \quad (4.10)$$

for a standard test function  $\psi$  that  $\psi \in C^\infty(\mathbb{R}^n)$ ,  $0 \leq \psi \leq 1$ ,  $\psi = 0$  on  $B_1$  and  $\psi = 1$  on  $\mathbb{R}^n \setminus B_2$ . Then there exists a constant  $C > 0$  such that

$$\rho_R(x) \leq C\eta^2(x/R)|x|^{-(n+2s)} + R^{-2s}\rho(x/R).$$

**Lemma 4.4.** Suppose that  $u$  is a solution of (1.1) which is stable outside a ball  $B_{R_0}$ . Consider  $\rho_R$  which is defined in (4.10) for  $n/2 < m < n/2 + s(p+1)/2$ . Then there exists a constant  $C > 0$  such that

$$\int_{\mathbb{R}^n} u^2 \rho_R \leq C \left( \int_{B_{3R_0}} u^2 \rho_R + R^{n-2s\frac{p+1}{p-1}} \right)$$

for any  $R > 3R_0$ .

The proofs of Lemma 4.2, Corollary 4.2, Lemma 4.3 and Lemma 4.4 are similar to that of Lemmas 2.1, 2.2, 2.4 in [6] and we omit the details here.

**Lemma 4.5.** Suppose that  $p \neq \frac{n+2s}{n-2s}$ . Let  $u$  be a solution of (1.1) which is stable outside a ball  $B_{R_0}$  and  $u_e$  satisfy (2.4). Then there exists a constant  $C > 0$  such that

$$\int_{B_R} y^b u_e^2 \leq CR^{n+6-2s\frac{p+1}{p-1}}, \quad \int_{B_R} y^b |\nabla u_e|^2 \leq CR^{n+4-2s\frac{p+1}{p-1}},$$

$$\int_{B_R} y^b (|\nabla^2 u_e|^2 + |\Delta_b u_e|^2) \leq CR^{n+2-2s\frac{p+1}{p-1}}.$$

*Proof.* Recall that the Possion formula for the fractional equation for the case  $0 < s < 1$  (see [2]), we can generalize the expression formula to the general case with non-integer positive real number. Therefore,

$$u_e(x, y) = C_{n,s} \int_{\mathbb{R}^n} u(z) \frac{y^{2s}}{(|x - z|^2 + y^2)^{\frac{n+2s}{2}}} dz.$$

Then we have

$$|u_e(x, y)|^2 \leq C \int_{\mathbb{R}^n} u^2(z) \frac{y^{2s}}{(|x - z|^2 + y^2)^{\frac{n+2s}{2}}} dz, \quad (4.11)$$

and

$$\partial_y u_e(x, y) = C_{n,s} \int_{\mathbb{R}^n} u(z) \left[ \frac{2sy^{2s-1}}{(|x-z|^2 + y^2)^{\frac{n+2s}{2}}} - \frac{(n+2s)y^{2s+1}}{(|x-z|^2 + y^2)^{\frac{n+2s+2}{2}}} \right] dz,$$

also

$$\partial_{x_j} u_e(x, y) = -C_{n,s} \int_{\mathbb{R}^n} u(z) \frac{(n+2s)(x_j - z_j)y^{2s}}{(|x-z|^2 + y^2)^{\frac{n+2s+2}{2}}} dz,$$

for  $j = 1, 2, \dots, n$ . Hence by Hölder's inequality we have

$$|\nabla u_e(x, y)|^2 \leq C \int_{\mathbb{R}^n} \frac{u^2(z)y^{2s-2}}{(|x-z|^2 + y^2)^{\frac{n+2s}{2}}} dz. \quad (4.12)$$

By a straightforward calculation we have

$$\begin{aligned} \partial_{x_j x_j} u_e(x, y) &= C_{n,s} \int_{\mathbb{R}^n} u(z) \left[ \frac{(n+2s)(n+2s+2)(x_j - z_j)^2 y^{2s}}{(|x-z|^2 + y^2)^{\frac{n+2s+4}{2}}} \right. \\ &\quad \left. - \frac{(n+2s)y^{2s}}{(|x-z|^2 + y^2)^{\frac{n+2s+2}{2}}} \right] dz, \end{aligned}$$

$$\begin{aligned} \partial_{x_j y} u_e(x, y) &= C_{n,s} \int_{\mathbb{R}^n} u(z) \left[ \frac{(n+2s)(n+2s+2)(x_j - z_j)^2 y^{2s+1}}{(|x-z|^2 + y^2)^{\frac{n+2s+4}{2}}} \right. \\ &\quad \left. - \frac{2s(n+2s)(x_j - z_j)^2 y^{2s-1}}{(|x-z|^2 + y^2)^{\frac{n+2s+2}{2}}} \right], \end{aligned}$$

and

$$\begin{aligned} \partial_{yy} u_e(x, y) &= C_{n,s} \int_{\mathbb{R}^n} u(z) \left[ \frac{2s(2s-1)y^{2s-2}}{(|x-z|^2 + y^2)^{\frac{n+2s}{2}}} \right. \\ &\quad \left. - \frac{(n+2s)(4s+1)y^{2s}}{(|x-z|^2 + y^2)^{\frac{n+2s+2}{2}}} + \frac{(n+2s)(n+2s+2)y^{2s+2}}{(|x-z|^2 + y^2)^{\frac{n+2s+4}{2}}} \right]. \end{aligned}$$

Therefore, we have

$$|\nabla^2 u_e(x, y)| + |\Delta_b u_e(x, y)| \leq C \int_{\mathbb{R}^n} |u(z)| \frac{y^{2s-2}}{(|x-z|^2 + y^2)^{\frac{n+2s}{2}}} dz.$$

Hence,

$$|\nabla^2 u_e(x, y)|^2 + |\Delta_b u_e(x, y)|^2 \leq C \int_{\mathbb{R}^n} u^2(z) \frac{y^{2s-4}}{(|x-z|^2 + y^2)^{\frac{n+2s}{2}}} dz. \quad (4.13)$$

Now we turn to estimate the following integration, which provides a unify way to deal with our desired estimates.

Define

$$\begin{aligned}
A_k &:= \int_{|x| \leq R, z \in \mathbb{R}^n} u^2(z) \left[ \int_0^R \frac{y^{2k+1}}{(|x-z|^2 + y^2)^{\frac{n+2s}{2}}} dy \right] dz dx \\
&= \int_{|x| \leq R, z \in \mathbb{R}^n} u^2(z) \left[ \int_0^{R^2} \frac{\alpha^k}{(|x-z|^2 + \alpha)^{\frac{n+2s}{2}}} d\alpha \right] dz dx \\
&\leq \frac{1}{2} \int_{|x| \leq R, z \in \mathbb{R}^n} u^2(z) \left[ \int_0^{R^2} \frac{d\alpha}{(|x-z|^2 + \alpha)^{\frac{n+2s}{2}} - k} \right] dz dx \\
&= \frac{1}{2} \left( \frac{n+2s}{2} - k \right) \int_{|x| \leq R, z \in \mathbb{R}^n} u^2(z) \left[ (|x-z|^2)^{k-\frac{n+2s}{2}+1} \right. \\
&\quad \left. - (|x-z|^2 + R^2)^{k-\frac{n+2s}{2}+1} \right], \tag{4.14}
\end{aligned}$$

where  $k = 0, 1, 2$ . Split the integral to  $|x-z| \leq 2R$  and  $|x-z| > 2R$ , for the case of  $|x-z| \leq 2R$ , we see that

$$\begin{aligned}
&\int_{|x| \leq R, |x-z| \leq 2R} u^2(z) \left[ (|x-z|^2)^{k-\frac{n+2s}{2}+1} - (|x-z|^2 + R^2)^{k-\frac{n+2s}{2}+1} \right] \\
&\leq \int_{|x| \leq R, |x-z| \leq 2R} u^2(z) \left[ (|x-z|^2)^{k-\frac{n+2s}{2}+1} \right] \\
&\leq CR^{2k-2s+2} \int_{|z| \leq 3R} u^2(z) dz \\
&\leq R^{2k-2s+2} \left( \int_{B_{3R}} |u|^{p+1} \eta_R^2 \right)^{2/(p+1)} \left( \int_{B_{3R}} \eta_R^{-4/(p-1)} \right)^{(p-1)/(p+1)} \\
&\leq CR^{2k-2s+2} \left( \int_{B_{3R}} u^2(z) \rho_R(z) \right)^{2/(p+1)} \\
&\leq CR^{n+2k+2-2s \frac{p+1}{p-1}}.
\end{aligned}$$

Here we have used Lemma 4.2 and 4.4. For the case of  $|x-z| > 2R$ , by the mean value theorem, we have

$$\begin{aligned}
&\int_{|x| \leq R, |x-z| > 2R} u^2(z) \left[ (|x-z|^2)^{k-\frac{n+2s}{2}+1} - (|x-z|^2 + R^2)^{k-\frac{n+2s}{2}+1} \right] \\
&\leq R^2 \int_{|x| \leq R, |x-z| > 2R} u^2(z) \left[ (|x-z|^2)^{k-\frac{n+2s}{2}} \right] \\
&\leq CR^{n+2} \int_{|z| \geq R} u_e^2(z) |z|^{2k-n-2s} dz \\
&\leq CR^{n+2} \left[ \int_{|z| \geq R} (u_e^{p+1}(z)) \right]^{2/(p+1)} \left( \int_{|z| \geq R} |z|^{(2k-n-2s) \frac{p+1}{p-1}} \right)^{(p-1)/(p+1)} \\
&\leq CR^{n+2k+2-2s \frac{p+1}{p-1}},
\end{aligned}$$

here we have used Lemma 4.2. Hence, we obtain that

$$A_k \leq CR^{n+2k+2-2s \frac{p+1}{p-1}}, \tag{4.15}$$

where  $C = C(n, s, p)$  independent of  $R$ . Now, combine with (4.11), (4.12) and (4.13), recall that  $b = 5 - 2s$ , we have

$$\begin{aligned} \int_{B_R} y^b u_e^2 dx dy &\leq A_2, \quad \int_{B_R} y^b |\nabla u_e|^2 dx dy \leq A_1, \\ \int_{B_R} y^b (|\nabla^2 u_e|^2 + |\Delta_b u_e|^2) dx dy &\leq A_0. \end{aligned}$$

Apply (4.15), we finish our proof.  $\square$

Combine Corollary 4.1 and Lemma 4.4, we have the following lemma.

**Lemma 4.6.** *Let  $u$  be a solution of (1.1) which is stable outside a ball  $B_{R_0}$  and  $u_e$  satisfy (2.4). Then there exists a positive constant  $C$  such that*

$$\begin{aligned} &\int_{\partial \mathbb{R}_+^{n+1} \cap B_R} |u_e|^{p+1} + R^{-6} \int_{\mathbb{R}_+^{n+1} \cap B_R} y^b |u_e|^2 + R^{-4} \int_{\mathbb{R}_+^{n+1} \cap B_R} y^b |\nabla u_e|^2 \\ &+ R^{-2} \int_{\mathbb{R}_+^{n+1} \cap B_R} y^b (|\Delta_b u_e|^2 + |\nabla^2 u_e|^2) + \int_{\mathbb{R}_+^{n+1} \cap B_R} y^b |\nabla \Delta_b u_e|^2 \leq CR^{n-2s\frac{p+1}{p-1}}. \end{aligned}$$

## 4.2 Blow down analysis and the proof of Theorem 1.1

**The proof of Theorem 1.1.** Suppose that  $u$  is a solution of (1.1) which is stable outside the ball of radius  $R_0$  and suppose that  $u_e$  satisfies (2.4). In the subcritical case, i.e.,  $1 < p < p_s(n)$ , Lemma 4.2 implies that  $u \in \dot{H}^s(\mathbb{R}^n) \cap L^{p+1}(\mathbb{R}^n)$ . Multiplying (1.1) with  $u$  and integrate, we obtain that

$$\int_{\mathbb{R}^n} |u|^{p+1} = \|u\|_{\dot{H}^s(\mathbb{R}^n)}^2. \quad (4.16)$$

Multiplying (1.1) with  $u^\lambda(x) = u(\lambda x)$  yields

$$\int_{\mathbb{R}^n} |u|^{p-1} u^\lambda = \int_{\mathbb{R}^n} (-\Delta)^{s/2} u (-\Delta)^{s/2} u^\lambda = \lambda^s \int_{\mathbb{R}^n} w w_\lambda,$$

where  $w = (-\Delta)^{s/2} u$ . Following the ideas provided in [24, 25] and using the change of variable  $z = \sqrt{\lambda}x$ , we can get the following Pohozaev identity

$$\begin{aligned} -\frac{n}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} &= \frac{2s-n}{2} \int_{\mathbb{R}^n} |w|^2 + \frac{d}{d\lambda} \int_{\mathbb{R}^n} w^{\sqrt{\lambda}} w^{1/\sqrt{\lambda}} dz \Big|_{\lambda=1} \\ &= \frac{2s-n}{2} \|u\|_{\dot{H}^s(\mathbb{R}^n)}^2. \end{aligned}$$

Hence, we have the following Pohozaev identity

$$\frac{n}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} = \frac{n-2s}{2} \|u\|_{\dot{H}^s(\mathbb{R}^n)}^2.$$

For  $p < p_s(n)$ , this equality above together with (4.16) proves that  $u \equiv 0$ . For  $p = p_s(n)$ , this equality above means that the energy is finite. Further, since  $u \in \dot{H}^s(\mathbb{R}^n)$ , apply the stability inequality with test function  $\psi = u\eta^2(\frac{x}{R})$ , and let  $R \rightarrow +\infty$  (where  $\eta$  is cutoff function), then we get that

$$p \int_{\mathbb{R}^n} |u|^{p+1} \leq \|u\|_{\dot{H}^s(\mathbb{R}^n)}^2.$$

This together with (4.16) gives that  $u \equiv 0$ .

Now we consider the supercritical case, i.e.,  $p > \frac{n+2s}{n-2s}$ , we perform the proof via a few steps.

**Step 1.**  $\lim_{\lambda \rightarrow \infty} E(u_e, 0, \lambda) < \infty$ .

From Theorem 2.4 we know that  $E$  is nondecreasing w.r.t.  $\lambda$ , so we only need to show that  $E(u_e, 0, \lambda)$  is bounded. Note that

$$E(u_e, 0, \lambda) \leq \frac{1}{\lambda} \int_{\lambda}^{2\lambda} E(u_e, 0, t) dt \leq \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} \int_t^{t+\lambda} E(u_e, 0, \gamma) d\gamma dt.$$

From Lemma 4.6, we have that

$$\begin{aligned} & \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} \int_t^{t+\lambda} \gamma^{2s \frac{p+1}{p-1} - n} \left[ \int_{\mathbb{R}_+^{n+1} \cap B_\gamma} \frac{1}{2} y^b |\nabla \Delta_b u_e|^2 dy dx \right. \\ & \quad \left. - \frac{C_{n,s}}{p+1} \int_{\partial \mathbb{R}_+^{n+1} \cap B_\gamma} |u_e|^{p+1} dx \right] d\gamma dt \leq C, \end{aligned}$$

where  $C > 0$  is independent of  $\gamma$ .

$$\begin{aligned} & \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} \int_t^{t+\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\gamma} \gamma^{2s \frac{p+1}{p-1} - n - 5} y^b \left[ \frac{2s}{p-1} \left( \frac{2s}{p-1} - 1 \right) \left( \frac{2s}{p-1} - 2 \right) u_e \right. \\ & \quad + \frac{6s}{p-1} \left( \frac{2s}{p-1} - 1 \right) \gamma \partial_r u_e + \frac{6s}{p-1} \gamma^2 \partial_{rr} u_e + \gamma^3 \partial_{rrr} u_e \left. \right] \\ & \quad \left[ \frac{2s}{p-1} \left( \frac{2s}{p-1} - 1 \right) u_e + \frac{4s}{p-1} \gamma \partial_r u_e + \gamma^2 \partial_{rr} u_e \right] \\ & \leq C \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} \int_t^{t+\lambda} t^{2s \frac{p+1}{p-1} - n - 5} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\gamma} \\ & \quad y^b [u_e^2 + \gamma^2 (\partial_r u_e)^2 + \gamma^4 (\partial_{rr} u_e)^2 + \gamma^6 (\partial_{rrr} u_e)^2] \quad (4.17) \\ & \leq C \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} t^{2s \frac{p+1}{p-1} - n - 5} \int_{\mathbb{R}_+^{n+1} \cap B_{3\lambda}} \\ & \quad y^b [u_e^2 + \gamma^2 (\partial_r u_e)^2 + \gamma^4 (\partial_{rr} u_e)^2 + \gamma^6 (\partial_{rrr} u_e)^2] \\ & \leq C \lambda^{n-2s \frac{p+1}{p-1} + 6} \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} t^{2s \frac{p+1}{p-1} - n - 5} dt \\ & \leq C \end{aligned}$$

and

$$\begin{aligned}
& \left| \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} \int_t^{t+\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\gamma} \gamma^{2s \frac{p+1}{p-1} - n - 4} y^b \frac{d}{d\gamma} (\gamma^2 \Delta_b u_e - \gamma^2 \partial_{rr} u_e - (n+b)\gamma \partial_r u_e)^2 \right| \\
& \leq \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} t^{2s \frac{p+1}{p-1} - n - 5} \int_t^{t+\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\gamma} y^b [2\gamma^2 \Delta_b u_e - 2\gamma^2 \partial_{rr} u_e - (n+b)\gamma \partial_r u_e] \\
& \quad [\gamma^2 \Delta_b u_e - \gamma^2 \partial_{rr} u_e - (n+b)\gamma \partial_r u_e] \\
& \leq \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} t^{2s \frac{p+1}{p-1} - n - 5} \int_t^{t+\lambda} \int_{\mathbb{R}_+^{n+1} \cap B_{3\lambda}} y^b [2\gamma^2 \Delta_b u_e - 2\gamma^2 \partial_{rr} u_e - (n+b)\gamma \partial_r u_e] \\
& \quad [\gamma^2 \Delta_b u_e - \gamma^2 \partial_{rr} u_e - (n+b)\gamma \partial_r u_e] \\
& \leq C \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} t^{2s \frac{p+1}{p-1} - n - 5} \int_t^{t+\lambda} \int_{\mathbb{R}_+^{n+1} \cap B_{3\lambda}} y^b [\gamma^4 (\Delta_b u_e)^2 + \gamma^4 (\partial_{rr} u_e)^2 + \gamma^2 \partial_r u_e] \\
& \leq C \lambda^{n-2s \frac{p+1}{p-1} + 6} \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} t^{2s \frac{p+1}{p-1} - n - 5} dt \\
& \leq C.
\end{aligned} \tag{4.18}$$

Integrate by part, by the scaling identity of section 3, for example (3.31), (3.36), (3.41) and (3.42), we can treat the remaining terms by a similar way as the estimates (4.17) and (4.18).

**Step 2.** There exists a sequence  $\lambda_i \rightarrow \infty$  such that  $(u_e^{\lambda_i})$  converges weakly to a function  $u_e^\infty$  in  $H_{loc}^3(\mathbb{R}^n; y^b dx dy)$ , this is a direct consequence of Lemma 4.6.

**Step 3. The function  $u_e^\infty$  is homogeneous.** Due to the scaling invariance of  $E$  (i.e.,  $E(u_e, 0, R\lambda) = E(u_e^\lambda, 0, R)$ ) and the monotonicity formula, for any given  $R_2 > R_1 > 0$ , we see that

$$\begin{aligned}
0 &= \lim_{i \rightarrow \infty} (E(u_e, 0, R_2 \lambda_i) - E(u_e, 0, R_1 \lambda_i)) \\
&= \lim_{i \rightarrow \infty} (E(u_e^{\lambda_i}, 0, R_2) - E(u_e^{\lambda_i}, 0, R_1)) \\
&\geq \liminf_{i \rightarrow \infty} \int_{(B_{R_2} \setminus B_{R_1}) \cap \mathbb{R}_+^{n+1}} y^b r^{2s \frac{p+1}{p-1} - n - 6} \left( \frac{2s}{p-1} u_e^{\lambda_i} + r \frac{\partial u_e^{\lambda_i}}{\partial r} \right)^2 dy dx \\
&\geq \int_{(B_{R_2} \setminus B_{R_1}) \cap \mathbb{R}_+^{n+1}} y^b r^{2s \frac{p+1}{p-1} - n - 6} \left( \frac{2s}{p-1} u_e^\infty + r \frac{\partial u_e^\infty}{\partial r} \right)^2 dy dx.
\end{aligned}$$

In the last inequality we have used the weak convergence of the sequence  $(u_e^{\lambda_i})$  to the function  $u_e^\infty$  in  $H_{loc}^3(\mathbb{R}^n; y^b dx dy)$ . This implies that

$$\frac{2s}{p-1} \frac{u_e^\infty}{r} + \frac{\partial u_e^\infty}{\partial r} = 0 \text{ a.e. in } \mathbb{R}_+^{n+1}.$$

Therefore,  $u_e^\infty$  is homogeneous.

**Step 4.**  $u_e^\infty = 0$ . This is a direct consequence of Theorem 3.1 in [17].

**Step 5.**  $(u_e^{\lambda_i})$  converges strongly to zero in  $H^3(B_R \setminus B_\varepsilon; y^b dx dy)$  and  $(u_e^{\lambda_i})$  converges strongly in  $L^{p+1}(\partial \mathbb{R}_+^{n+1} \cap (B_R \setminus B_\varepsilon))$  for all  $R > \varepsilon > 0$ . These are consequent results of Lemma 4.6 and Theorem 1.5 in [9].

**Step 6.**  $u_e = 0$ . Note that

$$\begin{aligned} \bar{E}(u_e, \lambda) &= \bar{E}(u_e^\lambda, 1) \\ &= \frac{1}{2} \int_{\mathbb{R}_+^{n+1} \cap B_1} y^b |\nabla \Delta_b u_e^\lambda|^2 dx dy - \frac{C_{n,s}}{p+1} \int_{\partial \mathbb{R}_+^{n+1} \cap B_1} |u_e^\lambda|^{p+1} dx \\ &= \frac{1}{2} \int_{\mathbb{R}_+^{n+1} \cap B_\varepsilon} y^b |\nabla \Delta_b u_e^\lambda|^2 dx dy - \frac{C_{n,s}}{p+1} \int_{\partial \mathbb{R}_+^{n+1} \cap B_\varepsilon} |u_e^\lambda|^{p+1} dx \\ &\quad + \frac{1}{2} \int_{\mathbb{R}_+^{n+1} \cap (B_1 \setminus B_\varepsilon)} y^b |\nabla \Delta_b u_e^\lambda|^2 dx dy - \frac{C_{n,s}}{p+1} \int_{\partial \mathbb{R}_+^{n+1} \cap (B_1 \setminus B_\varepsilon)} |u_e^\lambda|^{p+1} dx \\ &= \varepsilon^{n-2s \frac{p+1}{p-1}} \bar{E}(u_e, \lambda\varepsilon) + \frac{1}{2} \int_{\mathbb{R}_+^{n+1} \cap (B_1 \setminus B_\varepsilon)} y^b |\nabla \Delta_b u_e^\lambda|^2 dx dy \\ &\quad - \frac{C_{n,s}}{p+1} \int_{\partial \mathbb{R}_+^{n+1} \cap (B_1 \setminus B_\varepsilon)} |u_e^\lambda|^{p+1} dx. \end{aligned}$$

Letting  $\lambda \rightarrow +\infty$  and then  $\varepsilon \rightarrow 0$ , we deduce that  $\lim_{\lambda \rightarrow +\infty} \bar{E}(u_e, \lambda) = 0$ . Using the monotonicity of  $E$ ,

$$\begin{aligned} E(u_e, \lambda) &\leq \frac{1}{\lambda} \int_\lambda^{2\lambda} E(t) dt \leq \sup_{[\lambda, 2\lambda]} \bar{E} + C \frac{1}{\lambda} \int_\lambda^{2\lambda} [E - \bar{E}] \\ &\leq \sup_{[\lambda, 2\lambda]} \bar{E} + C \frac{1}{\lambda} \int_\lambda^{2\lambda} \lambda^{2s \frac{p+1}{p-1} - n - 5} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda} y^b [(u_e)^2 + \lambda^2 |\nabla u_e|^2 \\ &\quad + \lambda^4 (|\Delta_b u_e|^2 + |\nabla^2 u_e|^2) + \lambda^6 |\nabla \Delta_b u_e|^2] \\ &= \sup_{[\lambda, 2\lambda]} \bar{E} + C \frac{1}{\lambda} \lambda^{2s \frac{p+1}{p-1} - n - 5} \int_{\mathbb{R}_+^{n+1} \cap (B_{2\lambda} \setminus B_\lambda)} y^b [(u_e)^2 + \lambda^2 |\nabla u_e|^2 \\ &\quad + \lambda^4 (|\Delta_b u_e|^2 + |\nabla^2 u_e|^2) + \lambda^6 |\nabla \Delta_b u_e|^2] \\ &= \sup_{[\lambda, 2\lambda]} \bar{E} + C \frac{1}{\lambda} \int_{\mathbb{R}_+^{n+1} \cap (B_2 \setminus B_1)} y^b [(u_e^\lambda)^2 + |\nabla u_e^\lambda|^2 \\ &\quad + |\Delta_b u_e^\lambda|^2 + |\nabla^2 u_e^\lambda|^2 + |\nabla \Delta_b u_e^\lambda|^2] \end{aligned} \tag{4.19}$$

and so  $\lim_{\lambda \rightarrow \infty} E(u_e, \lambda) = 0$ . Since  $u$  is smooth, we also have  $E(u_e, 0) = 0$ . Since  $E$  is monotone,  $E \equiv 0$  and so  $\bar{u}$  must be homogenous, a contradiction unless  $u_e \equiv 0$ .

## 5 Algebraic analysis: The proof of Theorem 2.3

Let  $k := \frac{2s}{p-1}$  and  $m := n - 2s$ . By a direct calculation, we obtain that

$$\begin{aligned} A_1 &= -10k^2 + 10mk - m^2 + 12m + 25, \\ A_2 &= 3k^4 - 6mk^3 + (3m^2 - 12m - 30)k^2 + (12m^2 + 30m)k + 9m^2 + 36m + 27, \\ B_1 &= -6k^2 + 6mk + 12m + 30. \end{aligned} \tag{5.1}$$

Notice that our supercritical condition  $p > \frac{n+2s}{n-2s}$  is equivalent to  $0 < k < \frac{n-2s}{2} = \frac{m}{2}$ . Next, we have the following lemma which yields the sign of  $A_2$  and  $B_1$ .

**Lemma 5.1.** *If  $p > \frac{n+2s}{n-2s}$ , then  $A_2 > 0$  and  $B_1 > 0$ .*

*Proof.* From (5.1), we derive that

$$A_2 = 3(k+1)(k+3)(k-(m+1))(k-(m+3)), \tag{5.2}$$

and the roots of  $B_1 = 0$  are

$$\frac{1}{2}m - \frac{1}{2}\sqrt{m^2 + 8m + 20}, \quad \frac{1}{2}m + \frac{1}{2}\sqrt{m^2 + 8m + 20}.$$

Recall that  $p > \frac{n+2s}{n-2s}$  is equivalent to  $0 < k < \frac{m}{2}$ , we get the conclusion.  $\square$

To show monotonicity formula, we proceed to prove the following inequality. That is, there exist real numbers  $c_{i,j}$  and positive real number  $\epsilon$  such that

$$\begin{aligned} &3\lambda^5\left(\frac{d^3u^\lambda}{d\lambda^3}\right)^2 + A_1\lambda^3\left(\frac{d^2u^\lambda}{d\lambda^2}\right)^2 + A_2\lambda\left(\frac{du^\lambda}{d\lambda}\right)^2 \\ &\geq \epsilon\lambda\left(\frac{du^\lambda}{d\lambda}\right)^2 + \frac{d}{d\lambda}\left(\sum_{0 \leq i,j \leq 2} c_{i,j}\lambda^{i+j}\frac{d^i u^\lambda}{d\lambda^i}\frac{d^j u^\lambda}{d\lambda^j}\right). \end{aligned} \tag{5.3}$$

To deal with the rest of the dimensions, we employ the second idea: we find nonnegative constants  $d_1, d_2$  and constants  $c_1, c_2$  such that we have the following Jordan form decomposition:

$$\begin{aligned} 3\lambda^5(f''')^2 + A_1\lambda^3(f'')^2 + A_2\lambda(f')^2 &= 3\lambda(\lambda^2 f''' + c_1\lambda f'')^2 + d_1\lambda(\lambda f'' + c_2 f')^2 \\ &\quad + d_2\lambda(f')^2 + \frac{d}{d\lambda}\left(\sum_{i,j} e_{i,j}\lambda^{i+j}f^{(i)}f^{(j)}\right), \end{aligned} \tag{5.4}$$

where the unknown constants are to be determined.

**Lemma 5.2.** *Let  $p > \frac{n+2s}{n-2s}$  and  $A_1$  satisfy*

$$A_1 + 12 > 0, \tag{5.5}$$

*then there exist nonnegative numbers  $d_1, d_2$ , and real numbers  $c_1, c_2, e_{i,j}$  such that the differential inequality (5.4) holds.*

*Proof.* Since

$$4\lambda^4 f'''f'' = \frac{d}{d\lambda}(2\lambda^4(f'')^2) - 8\lambda^3(f'')^2$$

and

$$2\lambda^2 f'' f' = \frac{d}{d\lambda}(\lambda^2(f')^2) - 2\lambda(f')^2,$$

by comparing the coefficients of  $\lambda^3(f'')^2$  and  $\lambda(f')^2$ , we have that

$$d_1 = A_1 - 3c_1^2 + 12c_1, \quad d_2 = A_2 - (c_2^2 - 2c_2)(A_1 - 3c_1^2 + 12c_1).$$

In particular,

$$\max_{c_1} d_1(c_1) = A_1 + 12 \text{ and the critical point is } c_1 = 2.$$

Since  $A_2 > 0$ , we select that  $c_1 = 2, c_2 = 0$ . Hence, in this case, by a direct calculation we see that  $d_1 = A_1 + 12 > 0$ . Then we get the conclusion.  $\square$

We conclude from Lemma 5.2 that if  $A_1 + 12 > 0$  then (5.3) holds. This implies that when  $m < 6 + \sqrt{73}$ ,  $p > \frac{n+2s}{n-2s}$  or  $m \geq 6 + \sqrt{73}$  and

$$\frac{n+2s}{n-2s} < p < \frac{5m + 20s - \sqrt{15m^2 + 120m + 370}}{5m - \sqrt{15m^2 + 120m + 370}}, \quad (5.6)$$

then (5.3) holds.

Let

$$p_m(n) := \begin{cases} +\infty & \text{if } n < 2s + 6 + \sqrt{73}, \\ \frac{5n+10s-\sqrt{15(n-2s)^2+120(n-2s)+370}}{5n-10s-\sqrt{15(n-2s)^2+120(n-2s)+370}} & \text{if } n \geq 2s + 6 + \sqrt{73}. \end{cases} \quad (5.7)$$

Combining all the lemmas of this section, we obtain the Theorem 2.3.

Now we proceed to prove Theorem 2.4. From Corollary 1.1 of [22], we know that if  $n > 2s, s > 0, p > \frac{n+2s}{n-2s}$ , then there exists  $n_0(s)$ , where  $\frac{1}{\sqrt{n}} < a_{n,s} < \frac{1}{2}\frac{n-2s}{\sqrt{n}} + \frac{1}{\sqrt{n}}$ , such that the inequality (1.4) always holds whenever  $n \leq n_0(s)$ ; while when  $n > n_0(s)$ , then the inequality (1.4) is true if and only if

$$p < p_2 := \frac{n+2s-2-2a_{n,s}\sqrt{n}}{n-2s-2-2a_{n,s}\sqrt{n}},$$

where  $n_0(s)$  is in fact the largest  $n$  satisfying  $n - 2s - 2 - 2a_{n,s}\sqrt{n} \leq 0$ . In particular,  $\frac{n+2s}{n-2s} < \frac{n+2s-4}{n-2s-4} < p_2 < +\infty$ . Therefore, we introduce

$$p_c(n) := \begin{cases} +\infty & \text{if } n \leq n_0(s), \\ \frac{n+2s-2-2a_{n,s}\sqrt{n}}{n-2s-2-2a_{n,s}\sqrt{n}} & \text{if } n > n_0(s). \end{cases} \quad (5.8)$$

From [23], we use the sharp estimate  $n_0(s) < 2s + 8.998$  for  $2 < s < 3$ , then

$$n_0(s) \leq 2s + 8.998 < 2s + 6 + \sqrt{73} \simeq 2s + 14.544. \quad (5.9)$$

On the other hand, via the sharp estimate  $a_{n,s} < 1$  from [23]

$$\frac{5n + 10s - \sqrt{15(n-2s)^2 + 120(n-2s) + 370}}{5n - 10s - \sqrt{15(n-2s)^2 + 120(n-2s) + 370}} > \frac{n + 2s - 2 - 2a_{n,s}\sqrt{n}}{n - 2s - 2 - 2a_{n,s}\sqrt{n}} \quad (5.10)$$

provided that  $s \in (2, 3)$  and that

$$225m^4 - 720m^3 - 17244m^2 - 29088m + 7236 > 0, \text{ where } m = n - 2s. \quad (5.11)$$

The (5.11) holds whenever  $m > 11.12$ , that is  $n > 2s + 11.12$ . This combine with (5.9) we obtain that  $p_c(n) < p_m(n)$ . Therefore we get Theorem 2.4.  $\square$

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