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LIOUVILLE TYPE RESULTS AND REGULARITY OF THE EXTREMAL SOLUTIONS OF BIHARMONIC EQUATION WITH NEGATIVE EXPONENTS

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Abstract We first obtain Liouville type results for stable entire solutions of the biharmonic equation $-\Delta^2 u = u^{-p}$ in $\mathbb{R}^N$ for $p > 1$ and $3 \le N \le 12$. Then we consider the Navier boundary value problem for the corresponding equation and improve the known results on the regularity of the extremal solution for $3 \le N \le 12$. As a consequence, in the case of $p = 2$, we show that the extremal solution u^* is regular when $N = 7$. This improves earlier results of Guo-Wei [20] ($N \le 4$), Cowan-Esposito-Ghoussoub [2] ($N = 5$), Cowan-Ghoussoub [4] ($N = 6$).

1. **Introduction.**

Consider the biharmonic equation

$$\Delta^2 u = -u^{-p}, \quad u > 0 \ \text{ in } \mathbb{R}^N \tag{1.1}$$

where $N \ge 3$ and $p > 1$. Let

$$\Lambda(\phi) := \int_{\mathbb{R}^N} |\Delta \phi|^2 dx - p \int_{\mathbb{R}^N} u^{-(p+1)} \phi^2 dx, \quad \forall \phi \in H^2(\mathbb{R}^N). \tag{1.2}$$

A solution u is said to be *stable* if $\Lambda(\phi) \ge 0$ for any test function $\phi \in H^2(\mathbb{R}^N)$. The main aim of this paper is to classify the stable solutions.

We first consider the stability issue for radial entire solutions. It is known from [7] that for $N = 3$ and $1 < p < 3$; $N \ge 4$ and $p > 1$; and any $a > 0$, there is a unique $b := b(a) > 0$ such that the problem

$$\begin{cases} \Delta^2 u = -u^{-p} \quad \text{in } \mathbb{R}^N, \\ u(0) = a, \ \ u'(0) = 0, \ \ u''(0) = b, \ \ u'''(0) = 0 \end{cases} \tag{1.3}$$

has a unique positive radial solution $u_a(r)$ such that

$$r^{-\alpha} u_a(r) = [Q_4(\alpha)]^{-1/(p+1)} \quad \text{as } r \to \infty,$$

where

$$\alpha = \frac{4}{p+1}, \quad Q_4(\alpha) := \alpha(2-\alpha)(\alpha + N - 2)(\alpha + N - 4). \tag{1.4}$$

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It is also known from [7] that for any $\tilde{b} > b(a)$, the entire radial solution $\tilde{u}_a$ of (1.3) with the initial values $u(0) = a$, $u'(0) = 0$, $u''(0) = \tilde{b}$ and $u'''(0) = 0$ admits the growth rate $O(r^2)$ at $r = \infty$. A comparison principle (Lemma 3.2 in [24]) ensures that $\tilde{u}_a > u_a$. We can easily see that $\tilde{u}_a$ is stable if u_a is stable. This implies that the set of stable radial entire solutions of (1.1) with growth rate $O(r^2)$ at $r = \infty$ is richer than the set of stable radial entire solutions of (1.1) with growth rate $O(r^\alpha)$ at $r = \infty$. This is a big difference between the equations with "positive exponents" and "negative exponents". (In fact, it is known from [19, 21, 29] that all the radial entire solutions of the equation $\Delta^2 u = u^p$ have the same decay rate $O(r^{-\frac{4}{p-1}})$ at $r = \infty$.)

By arguments similar to those in the proof of Theorem 1 of [23], the radial entire solutions $\{u_a\}_{a>0}$ to (1.1) given in (1.3) is unstable if and only if

$$\frac{N^2(N-4)^2}{16} < pQ_4(\alpha). \tag{1.5}$$

The left hand side of (1.5) is the best constant of the Hardy-Rellich inequality (see [27]): Let $N \geq 3$,

$$\int_{\mathbb{R}^N} |\Delta\phi|^2 dx \geq \frac{N^2(N-4)^2}{16} \int_{\mathbb{R}^N} \frac{\phi^2}{|x|^4} dx, \quad \forall \phi \in H^2(\mathbb{R}^N),$$

while the right hand side of (1.5) comes from the weak radial solution $w(x) = (Q(\alpha))^{-\frac{1}{p+1}} |x|^\alpha$.

In appendix A, we shall prove that (1.5) holds if and only if p satisfies

$$\begin{cases} p_0^1(3) < p < p_0^2(3) & \text{if } N = 3, \\ p > 1 & \text{if } N = 4, \\ p > p_0(N) & \text{if } 5 \leq N \leq 12 \end{cases} \tag{1.6}$$

where $p_0^1(3), p_0^2(3)$ and $p_0(N)$ are defined as follows

$$p_0^1(3) = \frac{5 - \sqrt{13 - 3\sqrt{17}}}{3 + \sqrt{13 - 3\sqrt{17}}}, \ p_0^2(3) = \frac{5 + \sqrt{13 - 3\sqrt{17}}}{3 - \sqrt{13 - 3\sqrt{17}}} \tag{1.7}$$

$$p_0(N) = \frac{N + 2 - \sqrt{4 + N^2 - 4\sqrt{N^2 + H_N}}}{6 - N + \sqrt{4 + N^2 - 4\sqrt{N^2 + H_N}}}, \text{ with } H_N = (N(N-4)/4)^2. \tag{1.8}$$

When p satisfies (1.6), the radial solutions $\{u_a\}_{a>0}$ to (1.1) given in (1.3) are unstable. The study for radial solutions to (1.1) suggests the following conjecture:

Conjecture: A smooth stable solution to (1.1) *with growth rate* $O(|x|^\alpha)$ *at* ∞ *does* **NOT** *exist if and only if p satisfies (1.6).*

In this paper, we partially solve the above conjecture. To this end, we need to define some exponents.

Let

$$H_N^* := \frac{N^2(N-4)^2}{4} + \frac{(N-2)^2}{2} - 1, \tag{1.9}$$

$$p_*(N) = \begin{cases} \frac{N+2-\sqrt{4+N^2-4\sqrt{N^2+H_N^*}}}{6-N+\sqrt{4+N^2-4\sqrt{N^2+H_N^*}}} & \text{when } 5 \leq N \leq 12, \\ +\infty & \text{when } N \geq 13 \end{cases} \tag{1.10}$$

and for $N = 4$

$$p^1_*(N) = p_*(N), \quad p^2_*(N) = \frac{N + 2 + \sqrt{4 + N^2 - 4\sqrt{N^2 + H^*_N}}}{6 - N - \sqrt{4 + N^2 - 4\sqrt{N^2 + H^*_N}}}. \tag{1.11}$$

Let $\overline{N} \in (4, 5)$ be the unique root of the equation $8(N - 2)(N - 4) = H^*_N$ and $\overline{x} := \frac{1}{2}(\overline{N} - 2) \in (1, 3/2)$. Set

$$\bar{p} = \frac{2 + \bar{x}}{2 - \bar{x}}. \tag{1.12}$$

With all the notations, we now state the following classification results.

Theorem 1.1. *Let $N \geq 3$ and $p > 1$. Then equation* (1.1) *has no classical stable solution $u(x)$ satisfying*

$$u(x) = O(|x|^\alpha), \quad as\ |x| \to \infty, \tag{1.13}$$

if $N \geq 5$ and $p > \max\{\overline{p}, p_*(N)\}$ or $N = 3$ and $p \in \left[\frac{5}{3}, \infty\right) \cap (p^1_0(3), p^2_0(3))$ or $N = 4$ and $p \in [3, \infty) \cap (p^1_*(4), p^2_*(4))$, where $\alpha = 4/(p+1)$ and $\overline{p}, p_*(N), p^1_0(3), p^2_0(3), p^1_*(4), p^2_*(4)$ are given at the above.

Remark 1.2. *Theorem 1.1 is the first result for the stable entire nonradial solutions of* (1.1)*. We know that $p_0(N) = \infty$ for $N \geq 13$. It is easy to see that $p_*(N) > p_0(N)$.*

Remark 1.3. *Note that the condition* (1.13) *is natural since Equation* (1.1) *admits entire radial solutions with growth rate $O(r^2)$. This marks dramatic difference from the analogue fourth order equations with "positive exponents" (*[19, 21, 29]*).*

We also study the corresponding Navier boundary value problem:

$$(P_\lambda) \qquad \begin{cases} \Delta^2 u = \lambda(1 - u)^{-p}, & \text{in } \Omega, \\ 0 < u < 1, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded smooth domain. It is well-known (see [2, 4, 20]) that there exists a critical value $\lambda^* > 0$ depending on p and Ω such that

(a) If $\lambda \in (0, \lambda^*)$, (P_λ) has a minimal and classical solution u_λ which is stable;

(b) If $\lambda = \lambda^*$, $u^* = \lim_{\lambda \to \lambda^*} u_\lambda$ is a weak solution to (P_{λ^*}), u^* is called the *extremal solution*.

(c) No solution of (P_λ) exists whenever $\lambda > \lambda^*$.

An interesting question is the regularity of the extremal solution u^*. If the domain Ω is the unit ball, then one can use the methods of [8, 3] to obtain optimal results for the radial extremal solution in the case of $f(t) = (1 - t)^{-2}$ (see for instance [17, 26]). For general domains, it is known from [2, 20] that u^* is bounded away from 1 provided $N \leq 5$. Theorem 3 of [4] improves this to $N \leq 6$. The expected result that u^* is bounded away from 1 is $N \leq 8$, which holds on the ball, see [26]. In this paper, we obtain the following theorem:

Theorem 1.4. *The extremal solution u^* is smooth, i.e., $\sup_\Omega u^* < 1$, if $5 \leq N \leq 12$ and $p > p_*(N)$; $N = 3$ and $p \in (p^1_0(3), p^2_0(3))$; $N = 4$ and $p \in (p^1_*(4), p^2_*(4))$.*

It is easy to see that $p_*(7) < 2$. Thus we obtain

Corollary 1.5. *The extremal solution u^* to the following MEMS problem*

$$\begin{cases} \Delta^2 u = \frac{\lambda}{(1-u)^2}, & in\ \Omega, \\ 0 < u < 1, & in\ \Omega, \\ u = \Delta u = 0, & on\ \partial\Omega, \end{cases}$$

is smooth up to $N \leq 7$.

This improves the result in Guo-Wei [20] ($N \leq 4$), Cowan-Esposito-Ghoussoub [2] ($N = 5$), Cowan-Ghoussoub [4] ($N = 6$). Moreover, we obtain optimal results for the regularity of u^* for $N = 3$ and general p.

Let us comment on related results. In the second order case, the finite Morse index solutions to the corresponding problem

$$\Delta u + |u|^{q-1}u = 0 \quad \text{in } \mathbb{R}^N, \quad q > 1 \tag{1.14}$$

have been completely classified by Farina [18]. One main result of [18] is that nontrivial finite Morse index solutions to (1.14) exist if and only if $q \geq p_{JL}$ and $N \geq 11$, or $q = \frac{N+2}{N-2}$ and $n \geq 3$. Here p_{JL} is the so-called Joseph-Lundgren exponent. This also yields the optimal regularity for extremal solutions of the corresponding bounded domain problems.

For semilinear equations with negative exponents

$$\Delta u = \frac{1}{u^p},\ u > 0 \quad \text{in } \mathbb{R}^N, \quad p > 1, \tag{1.15}$$

the finite Morse index solutions has also been classified by Esposito [15] and Esposito-Ghoussoub-Guo [16, 17]. See also Du-Guo [14] and Ma-Wei [25].

The finite Morse index solutions for semilinear problems in other contexts have been studied by many authors recently, see, for example, [11, 5, 9, 13, 6].

For the fourth order with positive power case

$$\Delta^2 u = |u|^{p-1}u \quad \text{in } \mathbb{R}^N \tag{1.16}$$

there have been many ractivities recently. It is known that Farina's approach (which ammounts to Moser's iteration) does not work for (1.16). There are several new approaches dealing with (1.16). The first approach is to use the test function $-\Delta u$. To this end, one has to use the following Souplet's inequality ([28]): for $u > 0$ satisfying (1.16) it holds

$$\Delta u + \sqrt{\frac{2}{p+1}}u^{\frac{p+1}{2}} \leq 0. \tag{1.17}$$

This will give an exponent $\frac{N}{N-8} + \epsilon_N$ for some $\epsilon_N > 0$, see [2] and [29]. The second approach, independently obtained by Cowan-Ghoussoub [4] and Dupaigne-Ghergu-Goubet-Warnault [11], is to derive the following interesting intermediate second order stability criterion: for stable positive solutions to (1.16) it holds

$$\sqrt{p}\int_{\mathbb{R}^N} u^{\frac{p-1}{2}}\phi^2 \leq \int_{\mathbb{R}^N} |\nabla\phi|^2, \quad \forall \phi \in C_0^1(\mathbb{R}^N). \tag{1.18}$$

This approaches improves the first upper bound $\frac{N}{N-8}$ but it again fails to obtain the optimal exponent $p_0(N)$ (when $N \geq 13$). By combining these two approaches one can show that stable solutions to (1.16) do not exist when $N \leq 12$ and $p > \frac{N+4}{N-4}$, see [22]. Finally in a recent paper [10], Davila-Dupaigne-Wang-Wei employed a monotonicity formula based approach and gave a complete classification of stable and finite Morse index (positive or sign-changing) solutions to (1.16. A remarkable outcome of this third approach is that it gives the optimal exponent. Unfortunately it seems that the monotonicity formula approach of [10] does not work well with negative exponent. In this paper, we combine the first and second approaches for negative exponent.

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Note added after the completion of the paper. After the paper was completed, we were informed by Prof. L. Dupaigne that he and Prof. P. Esposito [12] have obtained similar results when $p = 2$.

2. **Preliminaries.** In this section we collect some properties of the entire solutions of (1.1) which will be useful in the following proofs. We will let $v = \Delta u$ in this paper for notation simplicity. As a very first step, we start with

Lemma 2.1. *For any point $x_0 \in \mathbb{R}^N$ and all $r > 0$,*

$$\overline{u}^{-p}(r) \le \overline{u^{-p}}(r), \quad \overline{u}(r) = \frac{\int_{\partial B_r(x_0)} u d\sigma}{|\partial B_r(x_0)|}. \tag{2.1}$$

Proof. This is an immediate consequence of the Jensen's inequality, as the function $f(u) = u^{-p}$ is convex on the interval $(0, \infty)$. □

Lemma 2.2. *If $u > 0$ is a C^4 solution to* (1.1) *in $\mathbb{R}^N$, then $\Delta u > 0$.*

Proof. The proof of this lemma is quite elementary. First we show that $\Delta u \ge 0$. Assume there were a point x_0 in $\mathbb{R}^N$ such that $\Delta u(x_0) < 0$. Then it follows from Lemma 2.1 that $\overline{u}(r)$ and $\overline{v}(r) := \overline{\Delta u}(r)$ satisfy

$$\begin{cases} \Delta \overline{u} = \overline{v}, \\ \Delta \overline{v} + \overline{u}^{-p} \le 0. \end{cases} \tag{2.2}$$

Since $\overline{v}(0) = v(x_0) < 0$, the inequality in (2.2) dictates that $\overline{v}(r) \le \overline{v}(0)$ for all $r > 0$. Replace the first equation in (2.2) by the differential inequality

$$\Delta \overline{u} \le \overline{v}(0)$$

and integrate it to get

$$\overline{u}(r) \le \overline{u}(0) + \frac{\overline{v}(0)}{2N} r^2. \tag{2.3}$$

If r is large enough, we see that $\overline{u}(r) < 0$. This is clearly impossible since we have assumed that u is positive everywhere.

Next assume $v = 0$ at some x_1. Then $\Delta v(x_1) \ge 0$ since v attains its minimum there. This contradicts Eq. (1.1). This completes the proof of this lemma. □

Note that the definition of $\overline{u}(r)$ or $\overline{v}(r)$ in Lemmas 2.1 and 2.2 depends on x_0. In the following, we still use $\overline{u}(r)$ and $\overline{v}(r)$ to denote the spherical average with $x_0 = 0$.

Lemma 2.3. *We have*

$$\overline{u}'(r) > 0, \quad \overline{u}''(0) > 0, \tag{2.4}$$

$$\overline{v}'(r) < 0 \tag{2.5}$$

for all $r > 0$.

Proof. We multiply the inequality in (2.2) by r^{N-1} and integrate the resulting equation to get

$$r^{N-1}\overline{v}'(r) + \int_0^r t^{N-1}\overline{u}^{-p}(t)dt \le 0,$$

where we have used the condition that $\overline{v}'(0) = 0$. Eq. (2.5) follows immediately. Similarly, from the facts that $v > 0$ and $(r^{N-1}\overline{u}'(r))' = r^{N-1}\overline{v}(r)$, we integrate this last equation to conclude the first inequality in (2.4). Notice that we have used $\overline{u}'(0) = 0$. Since $N\overline{u}''(0) = (\Delta\overline{u})(0)$, we easily see that $\overline{u}''(0) > 0$. □

In this paper, we use C to denote generic positive constants, which may change term by term, but does not depend on the solution u.

Proposition 2.4. *Assume that* $N \ge 3$ *and* $p > 1$, $u > 0$ *is a* C^4 *solution to* (1.1) *satisfying* $u(x) = O(|x|^\alpha)$ *as* $|x| \to \infty$ *with* $\alpha = 4/(p+1)$. *Then for* $R \gg 1$ *sufficiently large,*

$$\int_{B_R} u^{-p}dx \le CR^{N-p\alpha}, \tag{2.6}$$

$$\int_{B_R} u^2 dx \le CR^{N+2\alpha}. \tag{2.7}$$

Proof. It is easily seen from the first equation of (2.2) and the fact $\overline{v}'(r) < 0$ that

$$\overline{v}(r) \le Cr^{-2}\overline{u}(r) \quad \text{for } r > 0, \tag{2.8}$$

where $C = C(N)$. On the other hand, since $u(x) = O(|x|^\alpha)$ as $|x| \to \infty$, it follows that for $R \gg 1$ sufficiently large,

$$\overline{u}(R) \le CR^\alpha \tag{2.9}$$

and hence (2.8) implies

$$\overline{v}(R) \le CR^{\alpha-2}. \tag{2.10}$$

Therefore,

$$\begin{aligned}\int_{B_R} u^{-p}dx &\le CR^{N-2}\int_R^{2R} r^{1-N}\int_{B_r} u^{-p}dx = -CR^{N-2}\int_R^{2R} r^{1-N}\int_{B_r}\Delta v dx\\ &= -CR^{N-2}\int_R^{2R}\overline{v}'(r)dr \le CR^{N-2}\overline{v}(R) \le CR^{N-p\alpha} \ \text{(see (2.10))},\end{aligned}$$

since $N - 4 + \alpha = N - \frac{4p}{p+1} = N - p\alpha$. This shows (2.6).

(2.7) follows from the growth assumption on u. □

Proposition 2.5. *Let* $u > 0$ *be a* C^4 *solution to* (1.1) *with* $p > 1$. *Then the following inequality holds:*

$$v^2 \ge \frac{2}{p-1}u^{1-p} \quad \text{in } \mathbb{R}^N. \tag{2.11}$$

The inequality (2.11) is a generalization of the similar inequality given in [28] for the Lane-Emden system.

Proof. Writing (1.1) as the system of equations

$$\begin{cases} \Delta u = v \\ \Delta v = -u^{-p}, \end{cases} \tag{2.12}$$

and defining $w(x) = \ell u^{\sigma} - v$ with $\ell = \sqrt{\frac{2}{p-1}}$ and $\sigma = \frac{1-p}{2}$, we see that $v > 0$, $\sigma < 0$ and

$$\begin{aligned} \Delta w &= \ell\sigma\Big(u^{\sigma-1}\Delta u + (\sigma-1)u^{\sigma-2}|\nabla u|^2\Big) - \Delta v \\ &\geq \ell\sigma u^{\sigma-1} v + u^{-p} \\ &= u^{\sigma-1}\ell^{-1}(\ell u^{\sigma} - v) \\ &= u^{\sigma-1}\ell^{-1} w. \end{aligned}$$

It follows that

$$\Delta w \geq 0 \text{ in the set } \{w \geq 0\}.$$

Consequently, for any $R > 0$, we have

$$\int_{B_R} |\nabla w_+|^2 = -\int_{B_R} w_+ \Delta w + R^{N-1}\int_{S^{N-1}} w_+(R) w_r(R) \leq \frac{R^{N-1}}{2} f'(R), \quad (2.13)$$

where $f(R) := \int_{S^{N-1}} (w_+)^2(R)$. Let $g(R) := \int_{S^{N-1}} u^{-p}(R)$. We note that $f \leq Cg^{\frac{p-1}{p}}$.

We now show that for $R > 1$ large enough,

$$\int_0^R g(r) r^{N-1} dr = \int_{B_R} u^{-p} dx \leq CR^{N-2}. \quad (2.14)$$

To show (2.14), by the argument

$$\int_{B_R} u^{-p} dx \leq CR^{N-2}\overline{v}(R)$$

in the proof of (2.6) of Proposition 2.4, we only need to show that, for $R > 1$ large enough, $\overline{v}(R) \leq C$, where C is independent of R. Indeed, $\overline{u}$ is a subsolution in the sense that it satisfies $\Delta^2\overline{u} + \overline{u}^{-p} \leq 0$, $\overline{u}'(0) = 0$ and $\overline{u}'''(0) = 0$. Next, define the radially symmetric quadratic function $z(r) = u(0) + \frac{\overline{u}''(0)}{2} r^2$. It is a supersolution as it satisfies $\Delta^2 z + z^{-p} \geq 0$, $z'(0) = 0$ and $z'''(0) = 0$. A comparison principle (Lemma 3.2 in [24]) ensures that $z > \overline{u}$. The radially symmetric solution U of the initial value problem $\Delta^2 U + U^{-p} = 0$ with $U(0) = u(0)$, $U'(0) = 0$, $U''(0) = \overline{u}''(0) > 0$, $U'''(0) = 0$ exists on $(0, \infty)$ and satisfies $\overline{u} \leq U \leq z$ by the same comparison principle. Therefore, for any $R > 1$,

$$\overline{v}(R) \leq CR^{-2}\overline{u}(R) \leq C\Big(u(0)R^{-2} + \frac{\overline{u}''(0)}{2}\Big) \leq C.$$

Therefore, (2.14) holds.

On the other hand, (2.14) implies that $g(R_i) \to 0$ for some sequence $R_i \to \infty$. Consequently, $f(R_i) \to 0$ and there exists a sequence $\tilde{R}_i \to \infty$ such that $f'(\tilde{R}_i) \leq 0$. Letting $i \to \infty$ in (2.13) with $R = \tilde{R}_i$, we conclude that w_+ is constant in $\mathbb{R}^N$. But $w_+ \equiv C > 0$ would imply $w \equiv C$ by continuity, hence $u^{\sigma} \geq C$ which implies that $u^{-(p-1)} \geq C^2$. This contradicts the fact that for $R \gg 1$ sufficiently large,

$$\int_{B_R} u^{-(p-1)} dx \leq \Big(\int_{B_R} u^{-p} dx\Big)^{\frac{(p-1)}{p}} |B_R|^{\frac{1}{p}} \leq CR^{\frac{(N-2)(p-1)}{p} + \frac{N}{p}} = CR^{N-2+\frac{2}{p}}.$$

Thus $w_+ \equiv 0$ and the conclusion follows. □

3. **Some decay estimates for stable solutions of** (1.1)**: Proof of Theorem 1.1.** In this section we obtain some useful decay estimates for stable solutions of (1.1).

Lemma 3.1. *Let $u>0$ be a C^4 solution to* (1.1) *satisfying $u(x)=O(|x|^\alpha)$ as $|x|\to\infty$ and $v=\Delta u$. Then for $R\gg 1$ sufficiently large, there holds*

$$\int_{B_R}(v^2+u^{1-p})dx\le CR^{N-4+\frac{8}{p+1}}. \tag{3.1}$$

Proof. We use an identity given in the proof of Theorem 1.1 of [29] (see also [22]): For any $\xi\in C^4(\mathbb{R}^N)$ and $\eta\in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}\int_{\mathbb{R}^N}(\Delta^2\xi)\xi\eta^2dx= &\int_{\mathbb{R}^N}[\Delta(\xi\eta)]^2dx+\int_{\mathbb{R}^N}[-4(\nabla\xi\cdot\nabla\eta)^2+2\xi\Delta\xi|\nabla\eta|^2]dx\\ &+\int_{\mathbb{R}^N}\xi^2\Big[2\nabla(\Delta\eta)\cdot\nabla\eta+(\Delta\eta)^2\Big]dx.\end{aligned} \tag{3.2}$$

Take $\xi=u$, a solution of (1.1) into (3.2), there holds

$$\begin{aligned}&\int_{\mathbb{R}^N}[\Delta(u\eta)]^2dx+\int_{\mathbb{R}^N}u^{1-p}\eta^2dx\\ &\quad=\int_{\mathbb{R}^N}\Big[4(\nabla u\cdot\nabla\eta)^2-2uv|\nabla\eta|^2-u^2\Big(2\nabla(\Delta\eta)\cdot\nabla\eta+(\Delta\eta)^2\Big)\Big]dx.\end{aligned}$$

Since $\Delta(u\eta)=2\nabla u\cdot\nabla\eta+u\Delta\eta+v\eta$, we obtain that

$$\begin{aligned}&\int_{\mathbb{R}^N}\Big[v^2\eta^2+u^{1-p}\eta^2\Big]dx\\ &\quad\le C\int_{\mathbb{R}^N}\Big[|\nabla u|^2|\nabla\eta|^2+u^2|\nabla(\Delta\eta)\cdot\nabla\eta|+u^2(\Delta\eta)^2\Big]dx+C\int_{\mathbb{R}^N}uv|\nabla\eta|^2dx.\end{aligned}$$

Here and in the following, C denotes generic positive constants independent of u, which could be changed from one line to another. Using

$$2\int_{\mathbb{R}^N}|\nabla u|^2|\nabla\eta|^2dx=\int_{\mathbb{R}^N}u^2\Delta(|\nabla\eta)|^2)dx-2\int_{\mathbb{R}^N}uv|\nabla\eta|^2dx,$$

we can conclude that

$$\begin{aligned}&\int_{\mathbb{R}^N}\Big[v^2\eta^2+u^{1-p}\eta^2\Big]dx\\ &\quad\le C\int_{\mathbb{R}^N}u^2\Big[|\nabla(\Delta\eta)\cdot\nabla\eta|+(\Delta\eta)^2+\Delta(|\nabla\eta)|^2)\Big]dx+C\int_{\mathbb{R}^N}uv|\nabla\eta|^2dx.\end{aligned}$$

Take $\eta=\varphi^m$ with $m>2$ and $\varphi\in C_0^\infty(\mathbb{R}^N)$, $\varphi\ge 0$, it follows that

$$\begin{aligned}\int_{\mathbb{R}^N}uv|\nabla\eta|^2dx&=m^2\int_{\mathbb{R}^N}uv\varphi^{2(m-1)}|\nabla\varphi|^2dx\\ &\le\frac{1}{2C}\int_{\mathbb{R}^N}(v\varphi^m)^2dx+C\int_{\mathbb{R}^N}u^2\varphi^{2(m-2)}|\nabla\varphi|^4dx.\end{aligned}$$

Choose now φ_0 a cut-off function in $C_0^\infty(B_2)$ satisfying $0\le\varphi_0\le 1$, $\varphi_0=1$ in B_1. Input $\eta=\varphi^m$ and $m>2$ with $\varphi=\varphi_0(R^{-1}x)$ for $R\gg 1$ into the above inequalities, we arrive at

$$\begin{aligned}\int_{\mathbb{R}^N}(v^2+u^{1-p})\varphi^{2m}dx &\leq \frac{C}{R^4}\int_{B_{2R}}u^2\varphi^{2m-4}dx\\ &\leq \frac{C}{R^4}\int_{B_{2R}}u^2dx\\ &\leq CR^{N-4+\frac{8}{p+1}} \quad \text{(see (2.7))}.\end{aligned}$$

Since $\varphi^{2m}(x)=1$ for $x\in B_R$, we obtain (3.1). □

For $k\in\mathbb{N}$, let $R_k := 2^kR$ with $R>0$.

Lemma 3.2. *Assume that u is a classical stable solution to (1.1). Then there exists $0<C<\infty$ independent of u such that for any $s>1$*

$$\int_{B_{R_k}}u^{-p}v^{s-1}dx \leq \frac{C}{R^2}\int_{B_{R_{k+1}}}v^s dx, \quad \forall R>0, \tag{3.3}$$

provided that

$$L_1(p,s) := s^4 - \frac{32p}{p-1}s^2 + \frac{32p(p-3)}{(p-1)^2}s + \frac{64p}{(p-1)^2} < 0 \tag{3.4}$$

or for any $1<s\leq \frac{p+1}{\sqrt{2}(p-1)}$, (3.3) holds provided that

$$L_0(p,s) := s^4 - 2\Big(\frac{p+1}{p-1}\Big)^2 s^2 + \Big(\frac{p+1}{p-1}\Big)^4 - \frac{32p}{p-1}s^2 + \frac{32p(p-3)}{(p-1)^2}s + \frac{64p}{(p-1)^2} < 0. \tag{3.5}$$

Remark 3.3. *We see that if $x=\frac{(p-1)}{p+1}s$, then*

$$H_0(p,x) = \Big(\frac{p-1}{p+1}\Big)^4 L_0(p,s),\ H_1(p,x) = \Big(\frac{p-1}{p+1}\Big)^4 L_1(p,s).$$

Since for $x\geq 1$,

$$H_0(p,x)<0 \quad \textit{if and only if } p>p_0(2+2x),$$

then for $s\geq (p+1)/(p-1)$,

$$L_0(p,s)<0 \quad \textit{if and only if } p>p_0\Big(2+\tfrac{2(p-1)}{p+1}s\Big).$$

Similarly, since for $x>\overline{x}$,

$$H_1(p,x)<0 \quad \textit{if and only if } p>p_*(2+2x),$$

then, for $s>\frac{(p+1)}{(p-1)}\overline{x}$,

$$L_1(p,x)<0 \quad \textit{if and only if } p>p_*\Big(2+\tfrac{2(p-1)}{(p+1)}s\Big).$$

For $s\in\Big(\frac{(\sqrt{2}-1)(p+1)}{p-1}, \frac{p+1}{p-1}\overline{x}\Big)$,

$$H_1(p,x)<0 \quad \textit{if and only if } p\in\Big(p_*^1\Big(2+\tfrac{2(p-1)}{p+1}s\Big), p_*^2\Big(2+\tfrac{2(p-1)}{p+1}s\Big)\Big).$$

Moreover, for $s\in\Big(\frac{(\sqrt{2}-1)(p+1)}{p-1}, \frac{p+1}{p-1}\Big)$ and $s_1<s_2$, we have

$$\begin{aligned}1 &< p_0^1\Big(2+\frac{2(p-1)}{p+1}s_2\Big) < p_0^1\Big(2+\frac{2(p-1)}{p+1}s_1\Big)\\ &< p_0^2\Big(2+\frac{2(p-1)}{p+1}s_1\Big) < p_0^2\Big(2+\frac{2(p-1)}{p+1}s_2\Big),\end{aligned}$$

and for $s \in \left(\frac{(p+1)}{p-1}, \frac{p+1}{p-1}\overline{x}\right)$ *and* $s_1 < s_2$*, we have*

$$
\begin{aligned}
1 < p_*^1\Big(2+\frac{2(p-1)}{p+1}s_1\Big) &< p_*^1\Big(2+\frac{2(p-1)}{p+1}s_2\Big)\\
&< p_*^2\Big(2+\frac{2(p-1)}{p+1}s_1\Big) < p_*^2\Big(2+\frac{2(p-1)}{p+1}s_2\Big).
\end{aligned}
$$

Proof of Lemma 3.2. The proof of this lemma is motivated by the proof of Lemma 3.1 of [22]. Let u be a classical stable solution of (1.1). Writing the equation of u as

$$
\begin{cases} -\Delta u = z \\ -\Delta z = -u^{-p} \end{cases} \tag{3.6}
$$

with $z = -v$ and using arguments similar to those in the proof of Lemma 3 in [1], we obtain that

$$
\sqrt{p}\int_{\mathbb{R}^N} u^{-\frac{(p+1)}{2}}\varphi^2 dx \le \int_{\mathbb{R}^N} |\nabla\varphi|^2 dx \tag{3.7}
$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Let $\phi \in C_0^2(\mathbb{R}^N)$ and $\varphi = u^{\frac{(q+1)}{2}}\phi$ with $q < -1$. Take φ into the stability inequality (3.7), we obtain

$$
\sqrt{p}\int_{\mathbb{R}^N} u^{-\frac{p+1}{2}}u^{q+1}\phi^2 \le \int_{\mathbb{R}^N} u^{q+1}|\nabla\phi|^2 + \int_{\mathbb{R}^N} |\nabla u^{\frac{q+1}{2}}|^2\phi^2 + (q+1)\int_{\mathbb{R}^N} u^q\phi\nabla u\cdot\nabla\phi. \tag{3.8}
$$

Integrating by parts, we have

$$
\begin{aligned}
\int_{\mathbb{R}^N} |\nabla u^{\frac{q+1}{2}}|^2\phi^2 dx &= \frac{(q+1)^2}{4}\int_{\mathbb{R}^N} u^{q-1}|\nabla u|^2\phi^2 dx\\
&= \frac{(q+1)^2}{4q}\int_{\mathbb{R}^N} \phi^2\nabla(u^q)\cdot\nabla u dx\\
&= -\frac{(q+1)^2}{4q}\int_{\mathbb{R}^N} u^q v\phi^2 dx - \frac{q+1}{4q}\int_{\mathbb{R}^N} \nabla(u^{q+1})\cdot\nabla(\phi^2)dx\\
&= -\frac{(q+1)^2}{4q}\int_{\mathbb{R}^N} u^q v\phi^2 dx + \frac{q+1}{4q}\int_{\mathbb{R}^N} u^{q+1}\Delta(\phi^2)dx
\end{aligned}
$$

and

$$
(q+1)\int_{\mathbb{R}^N} u^q\phi\nabla u\cdot\nabla\phi dx = \frac{1}{2}\int_{\mathbb{R}^N}\nabla(u^{q+1})\cdot\nabla(\phi^2)dx = -\frac{1}{2}\int_{\mathbb{R}^N} u^{q+1}\Delta(\phi^2)dx.
$$

Combining (3.8) and these two identities, we conclude that

$$
a_1\int_{\mathbb{R}^N} u^{-\frac{(p+1)}{2}}u^{q+1}\phi^2 dx \le \int_{\mathbb{R}^N} u^q v\phi^2 dx + C\int_{\mathbb{R}^N} u^{q+1}(|\Delta(\phi^2)| + |\nabla\phi|^2)dx \tag{3.9}
$$

where $a_1 = -\frac{4q\sqrt{p}}{(q+1)^2} > 0$ (note $q < -1$). Choose now $\phi(x) = h(R_k^{-1}x)$ where $h \in C_0^\infty(B_2)$ such that $h \equiv 1$ in B_1, there holds then

$$
\int_{\mathbb{R}^N} u^{-\frac{(p+1)}{2}}u^{q+1}\phi^2 dx \le \frac{1}{a_1}\int_{\mathbb{R}^N} u^q v\phi^2 dx + \frac{C}{R^2}\int_{B_{R_{k+1}}} u^{q+1}dx. \tag{3.10}
$$

Now, apply the stability inequality (3.7) with $\varphi = v^{\frac{r+1}{2}}\phi$, $r > 0$, there holds

$$
\sqrt{p}\int_{\mathbb{R}^N} u^{-\frac{p+1}{2}}v^{r+1}\phi^2 \le \int_{\mathbb{R}^N} v^{r+1}|\nabla\phi|^2 + \int_{\mathbb{R}^N} |\nabla v^{\frac{r+1}{2}}|^2\phi^2 + (r+1)\int_{\mathbb{R}^N} v^r\phi\nabla v\cdot\nabla\phi. \tag{3.11}
$$

By very similar computation as above (recalling that $-\Delta v = u^{-p}$), we have

$$\int_{\mathbb{R}^N} u^{-\frac{p+1}{2}} v^{r+1}\phi^2 dx \le \frac{1}{a_2}\int_{\mathbb{R}^N} u^{-p}v^r\phi^2 dx + \frac{C}{R^2}\int_{B_{R_{k+1}}} v^{r+1}dx, \tag{3.12}$$

where $a_2 = \frac{4r\sqrt{p}}{(r+1)^2}$.

Using (3.10) and (3.12), there holds

$$\begin{aligned} I_1 + a_2^{r+1}I_2 &:= \int_{\mathbb{R}^N} u^{-\frac{p+1}{2}} u^{q+1}\phi^2 dx + a_2^{r+1}\int_{\mathbb{R}^N} u^{-\frac{p+1}{2}} v^{r+1}\phi^2 dx \\ &\le \frac{1}{a_1}\int_{\mathbb{R}^N} u^q v\phi^2 dx + a_2^r\int_{\mathbb{R}^N} u^{-p}v^r\phi^2 dx + \frac{C}{R^2}\int_{B_{R_{k+1}}} (u^{q+1}+v^{r+1})dx. \end{aligned}$$

Fix now

$$2q = -(p-1)r-(p+1), \quad \text{or equivalently } q+1 = -\tfrac{(p-1)(r+1)}{2}. \tag{3.13}$$

By Young's inequality, we have

$$\begin{aligned} \frac{1}{a_1}\int_{\mathbb{R}^N} u^q v\phi^2 &= \frac{1}{a_1}\int_{\mathbb{R}^N} u^{-\frac{p+1}{2}} u^{-\frac{p-1}{2}r} v\phi^2 dx \\ &= \frac{1}{a_1}\int_{\mathbb{R}^N} u^{-\frac{p+1}{2}} u^{\frac{r}{r+1}(q+1)} v\phi^2 dx \\ &\le \frac{r}{r+1}\int_{\mathbb{R}^N} u^{-\frac{p+1}{2}} u^{q+1}\phi^2 dx + \frac{1}{a_1^{r+1}(r+1)}\int_{\mathbb{R}^N} u^{-\frac{p+1}{2}} v^{r+1}\phi^2 dx \\ &= \frac{r}{r+1}I_1 + \frac{1}{a_1^{r+1}(r+1)}I_2 \end{aligned}$$

and similarly

$$a_2^r\int_{\mathbb{R}^N} u^{-p}v^r\phi^2 dx \le \frac{1}{r+1}I_1 + \frac{a_2^{r+1}r}{r+1}I_2.$$

Using these, we obtain that

$$a_2^{r+1}I_2 \le \Big[\frac{1}{a_1^{r+1}(r+1)} + \frac{a_2^{r+1}r}{r+1}\Big]I_2 + \frac{C}{R^2}\int_{B_{R_{k+1}}} (u^{q+1}+v^{r+1})dx \tag{3.14}$$

hence

$$\frac{(a_1a_2)^{r+1}-1}{r+1}I_2 \le \frac{Ca_1^{r+1}}{R^2}\int_{B_{R_{k+1}}} (u^{q+1}+v^{r+1})dx. \tag{3.15}$$

If we denote $s = r+1$, then for $s \le \frac{p+1}{\sqrt{2(p-1)}}$ and $a_1a_2 > 1 - \frac{(p+1)^2}{(p-1)^2s^4}[2s^2 - (\frac{p+1}{p-1})^2]$, by the choice of ϕ, we see from (3.15) that

$$\int_{B_{R_k}} u^{-\frac{p+1}{2}} v^{r+1}dx \le I_2 \le \frac{C}{R^2}\int_{B_{R_{k+1}}} (u^{q+1}+v^{r+1})dx. \tag{3.16}$$

From (2.11) and (3.13), we get $u^{q+1} \le Cv^s$. Therefore, we obtain that

(i) if $a_1a_2 > 1$ and $s > 1$,

$$\int_{B_{R_k}} u^{-p}v^{s-1} \le C\int_{B_{R_k}} u^{-\frac{p+1}{2}} v^s \le \frac{C}{R^2}\int_{B_{R_{k+1}}} (u^{q+1}+v^{r+1}) \le \frac{C}{R^2}\int_{B_{R_{k+1}}} v^s, \tag{3.17}$$

(ii) if $a_1a_2 > 1 - \frac{(p+1)^2}{(p-1)^2 s^4}[2s^2 - (\frac{p+1}{p-1})^2]$ provided $1 < s \le \frac{p+1}{\sqrt{2(p-1)}}$,

$$\int_{B_{R_k}} u^{-p}v^{s-1} \le C\int_{B_{R_k}} u^{-\frac{p+1}{2}}v^s \le \frac{C}{R^2}\int_{B_{R_{k+1}}}(u^{q+1}+v^{r+1}) \le \frac{C}{R^2}\int_{B_{R_{k+1}}} v^s. \tag{3.18}$$

For case (i),

$$a_1a_2 > 1 \text{ is equivalent to } L_1(p,s) < 0,$$

where

$$L_1(p,s) := s^4 - \frac{32p}{p-1}s^2 + \frac{32p(p-3)}{(p-1)^2}s + \frac{64p}{(p-1)^2}. \tag{3.19}$$

For case (ii),

$$a_1a_2 > 1 - \frac{(p+1)^2}{(p-1)^2s^4}\Big[2s^2 - \Big(\frac{p+1}{p-1}\Big)^2\Big] \text{ is equivalent to } L_0(p,s) < 0,$$

where

$$L_0(p,s) := s^4 - 2\Big(\frac{p+1}{p-1}\Big)^2 s^2 + \Big(\frac{p+1}{p-1}\Big)^4 - \frac{32p}{p-1}s^2 + \frac{32p(p-3)}{(p-1)^2}s + \frac{64p}{(p-1)^2}. \tag{3.20}$$

These complete the proof of this lemma. □

Corollary 3.4. *Assume that u is a classical stable solution to (1.1). For any $\epsilon > 0$ sufficiently small, suppose $5 \le N \le 12$ and any $\frac{(p+1)}{(p-1)}\overline{x} := s_0 < \beta := \frac{N}{N-2}\frac{(p+1)}{p-1}\Big(\frac{N-2}{2}+\epsilon\Big)$. Then there is some integer $n \ge 1$ and $0 < C < \infty$ independent of u and ϵ such that*

$$\Big(\int_{B_R} v^\beta\Big)^{\frac{1}{\beta}} \le CR^{N(\frac{1}{\beta}-\frac{1}{s_0})}\Big(\int_{B_{R_{3n}}} v^{s_0}\Big)^{\frac{1}{s_0}}, \tag{3.21}$$

for all $R \ge 1$, provided that p satisfies

$$H_1\Big(p, \frac{N-2}{2}+\epsilon\Big) < 0.$$

Suppose $N = 3$ and $\frac{1}{\sqrt{2}}\frac{p+1}{p-1} > \frac{p+1}{2(p-1)} := s_0 < \beta := 3\frac{(p+1)}{p-1}\Big(\frac{1}{2}+\epsilon\Big)$ (note that $s_0 > 1$ for $1 < p < 3$). Then there is some integer $n \ge 1$ and $0 < C < \infty$ independent of u and ϵ such that

$$\Big(\int_{B_R} v^\beta\Big)^{\frac{1}{\beta}} \le CR^{3(\frac{1}{\beta}-\frac{1}{s_0})}\Big(\int_{B_{R_{3n}}} v^{s_0}\Big)^{\frac{1}{s_0}}, \tag{3.22}$$

for all $R \ge 1$, provided that p satisfies

$$H_0\Big(p, \frac{1}{2}\Big) < 0.$$

Suppose $N = 4$ and $\frac{p+1}{p-1} := s_0 < \beta := 2\frac{(p+1)}{(p-1)}(1+\epsilon)$. Then there is some integer $n \ge 1$ and $0 < C < \infty$ independent of u and ϵ such that

$$\Big(\int_{B_R} v^\beta\Big)^{\frac{1}{\beta}} \le CR^{4(\frac{1}{\beta}-\frac{1}{s_0})}\Big(\int_{B_{R_{3n}}} v^{s_0}\Big)^{\frac{1}{s_0}}, \tag{3.23}$$

for all $R \ge 1$, provided that p satisfies

$$\text{both } H_1\Big(p,1\Big) < 0 \text{ and } H_1\Big(p, 1+\epsilon\Big) < 0.$$

Proof. To prove this corollary, we can use the iteration argument as in [1]. First we need to claim that for $5 \le N \le 12$, if p satisfies

$$L_1\Big(\frac{(p+1)}{(p-1)}\Big[\frac{N-2}{2}+\epsilon\Big]\Big)<0,$$

then

$$L_1(p,s)<0 \ \text{ for } s\in\Big(s_0,\tfrac{(p+1)}{(p-1)}\Big[\tfrac{N-2}{2}+\epsilon\Big]\Big).$$

To see this, a direct calculation implies that $p_*(N)$ is an increasing function of N for $N\in[5,12]$. Therefore, for any fixed $x_*>\overline{x}$, $H_1(p,x_*)<0$ implies $H_1(p,x)<0$ for $x\in(\overline{x},x_*)$. Note that $p_*(2+2x_*)>p_*(2+2x)$ for $x\in(\overline{x},x_*)$. Our claim can be obtained easily from

$$H_1(p,x):=H_1\Big(p,\frac{(p-1)}{(p+1)}s\Big)=\Big(\frac{p-1}{p+1}\Big)^4L_1(p,s)$$

and $\overline{x}=\frac{(p-1)}{(p+1)}s_0\le x:=\frac{(p-1)}{(p+1)}s<\Big(\frac{N-2}{2}+\epsilon\Big)$.

For $2\sqrt{2}<3<4$, we need to claim that $L_0\Big(p,\frac{1}{2}\frac{p+1}{p-1}\Big)<0$ implies that $L_0(p,s)<0$ for $s\in\Big[s_0,\Big(\frac{1}{2}+\epsilon\Big)\frac{p+1}{p-1}\Big]$. To see this, we need to use the fact that for $\sqrt{2}-1<x_0<x<1$, $H_0(p,x_0)<0$ implies $H_0(p,x)<0$. Note that $(p_0^1(2+2x_0),p_0^2(2+2x_0))\subset(p_0^1(2+2x),p_0^2(2+2x))$. Our claim can be obtained easily from

$$H_0(p,x):=H_0\Big(p,\frac{(p-1)}{(p+1)}s\Big)=\Big(\frac{p-1}{p+1}\Big)^4L_0(p,s)$$

and $\sqrt{2}-1<\frac{1}{2}=\frac{(p-1)}{(p+1)}s_0:=x_0<x:=\frac{(p-1)}{(p+1)}s<\frac{1}{2}+\epsilon<1$. We use $H_0(p,x)$, $L_0(p,s)$, $p_0^1(2+2x)$, $p_0^2(2+2x)$ instead of $H_1(p,x)$, $L_1(p,s)$, $p_*^1(2+2x)$ and $p_*^2(2+2x)$ here, because of $s<\frac{1}{\sqrt{2}}\frac{p+1}{p-1}$ for $s\in\Big[s_0,\Big(\frac{1}{2}+\epsilon\Big)\frac{p+1}{p-1}\Big]$.

The proof of $N=4$ is similar to that of $N=3$. But we should use $H_1(p,x)$, $L_1(p,s)$, $p_*^1(2+2x)$ and $p_*^2(2+2x)$ here, because of $s_0>\frac{1}{\sqrt{2}}\frac{p+1}{p-1}$. Note that for $N_1,N_2\in[4,\overline{N})$ and $N_1<N_2$,

$$1<p_*^1(N_1)<p_*^1(N_2)<p_*^2(N_1)<p_*^2(N_2).$$

Then if p satisfies both $H_1(p,1)<0$ and $H_1(p,1+\epsilon)<0$, we see that $p\in(p_*^1(4),p_*^2(4+2\epsilon))$. Therefore, p satisfies

$$L_1(p,s)<0 \ \text{ for } s\in\Big(\tfrac{(p+1)}{(p-1)},(1+\epsilon)\tfrac{(p+1)}{(p-1)}\Big).$$

These complete the proof of this corollary. □

Proof of Theorem 1.1: We first consider the case of $5\le N\le 12$ and $p>\max\{\overline{p},p_*(N)\}$ with $\overline{p}:=\frac{2+\overline{x}}{2-\overline{x}}$. We see that $\frac{(p+1)}{(p-1)}\overline{x}<2$. Then for s_0 as given in Corollary 3.4 and $R\gg 1$,

$$\begin{aligned}\int_{B_R}v^{s_0}dx &\le C\Big(\int_{B_R}v^2dx\Big)^{\frac{s_0}{2}}R^{\frac{N(2-s_0)}{2}}\\ &\le CR^{\frac{s_0}{2}(N-4+2\alpha)+\frac{N(2-s_0)}{2}}\\ &= CR^{N-(2-\alpha)s_0},\end{aligned}$$

(see Lemma 3.1). It follows from Corollary 3.4 that when p satisfies

$$H_1\Big(p,\frac{N-2}{2}+\epsilon\Big)<0,$$

then

$$\Big(\int_{B_R} v^\beta\Big)^{\frac{1}{\beta}} \leq CR^{N(\frac{1}{\beta}-\frac{1}{s_0})}\Big(\int_{B_{R_{3n}}} v^{s_0}\Big)^{\frac{1}{s_0}}, \tag{3.24}$$

for all $R \geq 1$. It is easily seen that

$$H_1\Big(p, \frac{N-2}{2}+\epsilon\Big) < 0 \iff p > p_*(N+2\epsilon). \tag{3.25}$$

Therefore, for $p > p_*(N+2\epsilon)$,

$$\int_{B_R} v^\beta dx \leq CR^{N-(2-\alpha)\beta}, \quad \forall R \gg 1. \tag{3.26}$$

Note that

$$N-(2-\alpha)\beta < 0 \Longleftrightarrow N < \frac{2(p-1)}{p+1}\beta.$$

Since

$$\frac{2(p-1)}{p+1}\beta = \frac{2N}{N-2}\Big(\frac{N-2}{2}+\epsilon\Big) > N,$$

after sending $R \to \infty$ we get that $\|v\|_{L^\beta(\mathbb{R}^N)} = 0$, which is impossible since v is positive. The arbitrariness of ϵ implies that for $p > \max\Big\{\frac{2+\overline{x}}{2-\overline{x}}, p_*(N)\}$, u does not exist.

Now we consider the case of $N = 3$. It is known from Corollary 3.4 that when p satisfies $H_0(p, \frac{1}{2}) < 0$, then

$$\Big(\int_{B_R} v^\beta\Big)^{\frac{1}{\beta}} \leq CR^{3(\frac{1}{\beta}-\frac{1}{s_0})}\Big(\int_{B_{R_{3n}}} v^{s_0}\Big)^{\frac{1}{s_0}}, \tag{3.27}$$

for all $R \geq 1$. We easily see that

$$H_0\Big(p, \frac{1}{2}\Big) < 0 \iff p \in (p_0^1(3), p_0^2(3)).$$

On the other hand, we see that

$$\frac{(p+1)}{2(p-1)} \leq 2 \quad \text{if } p \geq \tfrac{5}{3}.$$

Then for $p \in [\frac{5}{3}, \infty) \cap (p_0^1(3), p_0^2(3))$, we have

$$\Big(\int_{B_R} v^\beta\Big)^{\frac{1}{\beta}} \leq CR^{3(\frac{1}{\beta}-\frac{1}{s_0})}\Big(\int_{B_{R_{3n}}} v^{s_0}\Big)^{\frac{1}{s_0}} \leq CR^{3(\frac{1}{\beta}-\frac{1}{s_0})+\frac{3}{s_0}-(2-\alpha)}. \tag{3.28}$$

This implies that

$$\int_{B_R} v^\beta dx \leq CR^{3-(2-\alpha)\beta}, \quad \forall R \gg 1. \tag{3.29}$$

Since

$$3-(2-\alpha)\beta = 3 - \frac{2(p-1)}{(p+1)}\beta = 3-(3+6\epsilon) < 0,$$

after sending $R \to \infty$ we get that $\|v\|_{L^\beta(\mathbb{R}^N)} = 0$, which is impossible since v is positive. This implies that for $p \in [\frac{5}{3}, \infty) \cap (p_0^1(3), p_0^2(3))$, u does not exist.

The proof of $N = 4$ is similar to that of $N = 3$. We need to send ϵ to zero.

To conclude, the equation (1.1) has no classical stable solution with $u(x) = O(|x|^\alpha)$ provided that

$$p > \max\{\overline{p}, p_*(N)\} \quad \text{for } 5 \leq N \leq 12,$$

and

$$p \in \Big[\frac{5}{3}, \infty\Big) \cap (p_0^1(3), p_0^2(3)) \quad \text{for } N = 3$$

and

$$p \in [3, \infty) \cap (p_*^1(4), p_*^2(4)) \quad \text{for } N = 4.$$

□

4. **Navier boundary value problem: Proof of Theorem 1.4.** In this section, we consider the regularity of the extremal solution u^* of the problem:

$$(P_\lambda) \qquad \begin{cases} \Delta^2 u = \lambda(1-u)^{-p}, & \text{in } \Omega, \\ 0 < u < 1, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded smooth domain. We give the proof of Theorem 1.4.

Proof of Theorem 1.4: Let u_λ be the minimal solution of (P_λ), it is well known that u_λ is stable. To simplify the presentation, we erase the index λ on u_λ. We easily see from the maximum principle that $\Delta u < 0$. If we define $v = -\Delta u$, then $v > 0$ in Ω.

It is known from [4, 2] that there holds

$$\sqrt{\lambda p}\int_\Omega (1-u)^{-\frac{p+1}{2}}\varphi^2 dx \leq \int_\Omega |\nabla\varphi|^2 dx, \quad \forall \varphi \in H_0^1(\Omega). \tag{4.1}$$

Using $\varphi = (1-u)^{\frac{q+1}{2}} - 1$ with $q < -1$ as test function, we see from (4.1) that

$$\sqrt{\lambda}a_1 \int_\Omega (1-u)^{-\frac{p+1}{2}+q+1}\Big[1-2(1-u)^{-\frac{q+1}{2}}+(1-u)^{-(q+1)}\Big]dx \leq \int_\Omega (1-u)^q v dx \tag{4.2}$$

where $a_1 = -\frac{4q\sqrt{p}}{(q+1)^2}$. Similarly, using $\varphi = v^{\frac{r+1}{2}}$ with $r > 0$ as the test function in (4.1), we obtain

$$\sqrt{\lambda}a_2 \int_\Omega (1-u)^{-\frac{p+1}{2}} v^{r+1} dx \leq \lambda \int_\Omega (1-u)^{-p} v^r dx, \tag{4.3}$$

where $a_2 = \frac{4r\sqrt{p}}{(r+1)^2}$. Take always $2q = -(p-1)r - (p+1)$, i.e., $q+1 = -\frac{(p-1)(r+1)}{2}$. Applying Hölder's inequality, there hold

$$\int_\Omega (1-u)^q v dx \leq \Big(\int_\Omega (1-u)^{-\frac{p+1}{2}} v^{r+1} dx\Big)^{\frac{1}{r+1}} \Big(\int_\Omega (1-u)^{-\frac{p+1}{2}+q+1} dx\Big)^{\frac{r}{r+1}} \tag{4.4}$$

and

$$\int_\Omega (1-u)^{-p} v^r dx \leq \Big(\int_\Omega (1-u)^{-\frac{p+1}{2}} v^{r+1} dx\Big)^{\frac{r}{r+1}} \Big(\int_\Omega (1-u)^{-\frac{p+1}{2}+q+1} dx\Big)^{\frac{1}{r+1}}. \tag{4.5}$$

Multiplying (4.2) with (4.3), using (4.4) and (4.5), we obtain

$$a_1 a_2 \int_\Omega (1-u)^{-\frac{p+1}{2}+q+1}\Big[1-2(1-u)^{-\frac{q+1}{2}}+(1-u)^{-(q+1)}\Big]dx \leq \int_\Omega (1-u)^{-\frac{p+1}{2}+q+1} dx. \tag{4.6}$$

Therefore,

$$\Big(1 - \frac{1}{a_1 a_2}\Big)\int_\Omega (1-u)^{-\frac{p+1}{2}+q+1} dx \leq \int_\Omega (1-u)^{-\frac{p+1}{2}+\frac{q+1}{2}}\Big[2-(1-u)^{-\frac{q+1}{2}}\Big]dx. \tag{4.7}$$

Applying Hölder's inequality, we see that

$$\int_\Omega (1-u)^{-\frac{p+1}{2}+\frac{q+1}{2}}\Big[2-(1-u)^{-\frac{q+1}{2}}\Big]dx \le \Big(\int_\Omega (1-u)^{-\frac{p+1}{2}+q+1}dx\Big)^{\frac{(p+1)-(q+1)}{(p+1)-2(q+1)}}C,$$

(note that $q<-1$). Therefore,

$$\Big(1-\frac{1}{a_1a_2}\Big)\Big(\int_\Omega (1-u)^{-\frac{p+1}{2}+q+1}dx\Big)^{-\frac{(q+1)}{(p+1)-2(q+1)}} \le C. \tag{4.8}$$

Therefore, if $1<s\le \frac{1}{\sqrt{2}}\frac{p+1}{p-1}$ and $L_0(p,s)<0$, we see that

$$a_1a_2 > 1-\frac{(p+1)^2}{(p-1)^2s^4}\Big[2s^2-\Big(\frac{p+1}{p-1}\Big)^2\Big].$$

This and (4.8) imply that for any $1<s_*\le \frac{1}{\sqrt{2}}\frac{p+1}{p-1}$ and $L_0(p,s_*)<0$, then

$$\int_\Omega (1-u)^{-\frac{p+1}{2}+q+1}dx \le C. \tag{4.9}$$

Now, we choose $s_3 := \frac{(p+1)}{2(p-1)}$. Then $1<s_3<\frac{1}{\sqrt{2}}\frac{p+1}{p-1}$ for $N=3$ and $1<p<3$. We also know that $L_0(p,s_3)<0$ is equivalent to $H_0\Big(p,\frac{1}{2}\Big)<0$ and

$$H_0\Big(p,\frac{1}{2}\Big)<0 \quad \text{if and only if } p\in(p_0^1(3),p_0^2(3)).$$

This implies that if $p\in(p_0^1(3),p_0^2(3))$ for $N=3$, we have (4.9).

Similarly, when $N=4$, we also choose $s_4=\frac{(p+1)}{(p-1)}$ and see that $L_1(p,s_4)<0$ is equivalent to $a_1a_2>1$. This implies that if $L_1(p,s_4)<0$, (4.9) holds. But $L_1(p,s_4)<0$ if and only if $H_1(p,1)<0$. But $H_1(p,1)<0$ if and only if $p\in(p_*^1(4),p_*^2(4))$. Therefore, if $p\in(p_*^1(4),p_*^2(4))\subset(p_0^1(4),p_0^2(4))$, (4.9) holds. Note that $p_0^1(4)=1$ and $p_0^2(4)=\infty$.

For $5\le N\le 12$, we choose $s_N=\frac{(N-2)}{2}\frac{(p+1)}{p-1}$ and use the similar arguments to obtain that for $p>p_*(N)$, (4.9) holds.

We see from (4.9) and $u^*=\lim_{\lambda\to\lambda_*}u_\lambda$, that

$$(1-u_*)^{-1}\in L^{\frac{p+1}{2}-(q_N+1)}(\Omega), \tag{4.10}$$

with $q_N+1=-\frac{(p-1)s_N}{2}=-\frac{(N-2)(p+1)}{4}$. Thus,

$$\frac{p+1}{2}-(q_N+1)=\frac{N(p+1)}{4}.$$

This implies that

$$(1-u_*)^{-p}\in L^{\frac{N(p+1)}{4p}}(\Omega).$$

This and Theorem 6.2 of [2] imply $\sup_\Omega u_*<1$. This completes the proof. □

In appendix, we discuus the validicity of the inequality (1.5).

Define the following function

$$g(\alpha,N)=pQ_4(\alpha)=(4-\alpha)(2-\alpha)(\alpha+N-2)(\alpha+N-4),\quad \alpha\in(0,2).$$

We discuss different dimensions. For $N = 3$ and $1 < p < 3$, we see that $\alpha \in (1, 2)$ and $g(1, 3) = g(2, 3) = 0$, $g(\alpha, 3) > 0$ for $\alpha \in (1, 2)$. A simple calculation shows that $g''_\alpha(\alpha, 3) < 0$ for $\alpha \in (1, 2)$. Therefore, there are exactly $1 < \alpha_1 < \alpha_2 < 2$ such that

$$g(\alpha_1, 3) = g(\alpha_2, 3) = \frac{9}{16}, \quad g(\alpha, 3) > \frac{9}{16} \text{ for } \alpha \in (\alpha_1, \alpha_2).$$

Since $g(4/3, 3) > 9/16$, we see that $\alpha_1 < 4/3 < \alpha_2$. These imply that there exist $1 < p_0^1(3) < 2 < p_0^2(3) < 3$ such that $\tilde{g}(p, 3) := g(\alpha, 3)$ satisfies

$$\tilde{g}(p_0^1(3), 3) = \tilde{g}(p_0^2(3), 3) = \frac{9}{16}, \quad \tilde{g}(p, 3) > \frac{9}{16} \text{ for } p \in (p_0^1(3), p_0^2(3)).$$

It is known from [7] that

$$p_0^1(3) = \frac{4 - \alpha_2}{\alpha_2} := \frac{5 - \sqrt{13 - 3\sqrt{17}}}{3 + \sqrt{13 - 3\sqrt{17}}}, \quad p_0^2(3) = \frac{4 - \alpha_1}{\alpha_1} := \frac{5 + \sqrt{13 - 3\sqrt{17}}}{3 - \sqrt{13 - 3\sqrt{17}}}.$$

For $N = 4$, $g(0, 4) = g(2, 4) = 0$ and $g(\alpha, 4) > 0$ for $\alpha \in (0, 2)$. Therefore, $\tilde{g}(p, 4) := g(\alpha, 4) > 0$ for $p > 1$. This implies $p_0(4) = 1$ and when $p > p_0(4)$, $\tilde{g}(p, 4) > \frac{4^2(4-4)^2}{16}$.

For $N \geq 5$, we see that $g(0, N) = 8(N - 2)(N - 4)$, $g(2, N) = 0$ and

$$g'_\alpha(\alpha, N) = -(2\alpha + N - 6)[(2 - \alpha)(\alpha + N - 4) + (4 - \alpha)(\alpha + N - 2)].$$

Therefore, $g'_\alpha(\alpha, N) < 0$ for $\alpha \in (0, 2)$ and $N \geq 6$.

We also know that $g(0, 5) = g(1, 5) = 24 > 25/16$; $g''_\alpha(\alpha, 5) < 0$ for $\alpha \in (0, 2)$. Moreover, a simple calculation implies that

$$g(0, N) > \frac{N^2(N - 4)^2}{16} \quad \text{for } 5 \leq N \leq 12.$$

Therefore, we have the following cases:

(i) $g(\alpha, N) > \frac{N^2(N-4)^2}{16}$ for $N = 4$ and $\alpha \in (0, 2)$,
(ii) $g(\alpha, N) < \frac{N^2(N-4)^2}{16}$ for $N \geq 13$ and $\alpha \in (0, 2)$,
(iii) there exists a unique $\alpha_0 \in (0, 2)$ for $5 \leq N \leq 12$ such that

$$g(\alpha, N) \begin{cases} > \frac{N^2(N-4)^2}{16} & \text{for } \alpha \in (0, \alpha_0), \\ = \frac{N^2(N-4)^2}{16} & \text{for } \alpha = \alpha_0, \\ < \frac{N^2(N-4)^2}{16} & \text{for } \alpha \in (\alpha_0, 2). \end{cases}$$

Therefore, for $5 \leq N \leq 12$, there is a unique

$$p_0(N) := (4 - \alpha_0)/\alpha_0 = \frac{N + 2 - \sqrt{4 + N^2 - 4\sqrt{N^2 + H_N}}}{6 - N + \sqrt{4 + N^2 - 4\sqrt{N^2 + H_N}}} \in (1, \infty)$$

where $H_N = (N(N - 4)/4)^2$, such that

$$\tilde{g}(p, N) := pQ_4(\alpha) \begin{cases} < H_N & \text{for } p \in (1, p_0(N)), \\ = H_N & \text{for } p = p_0(N), \\ > H_N & \text{for } p > p_0(N). \end{cases}$$

Moreover, a simple calculation implies that $p_0(N) \in (1, 2)$ for $5 \leq N \leq 8$ and $p_0(N) \in (2, \infty)$ for $9 \leq N \leq 12$.

If we define $N = 2 + 2x$, a direct calculation shows that (1.5) is equivalent to $H_0(p, x) < 0$, where

$$H_0(p,x) := (x^2-1)^2 - \frac{32p(p-1)}{(p+1)^2}x^2 + \frac{32p(p-1)(p-3)}{(p+1)^3}x + \frac{64p(p-1)^2}{(p+1)^4}.$$

It is clear that $H_0(p, x) = H_N - \tilde{g}(p, N)$. Therefore, for any fixed $x \geq 1$,

$$H_0(p,x) < 0 \quad \text{if and only if } p > p_0(N) := p_0(2+2x). \tag{0.1}$$

If $N = 3$ (i.e. $x = \frac{1}{2}$), then

$$H_0(p, \frac{1}{2}) < 0 \quad \text{if and only if } p \in (p_0^1(3), p_0^2(3)). \tag{0.2}$$

We also know that for $x \in (\sqrt{2}-1, 1)$, there are $1 < p_0^1(2+2x) < p_0^2(2+2x)$ such that

$$H_0(p,x) < 0 \quad \text{if and only if } p \in (p_0^1(2+2x), p_0^2(2+2x)). \tag{0.3}$$

Moreover, for $x_1, x_2 \in (\sqrt{2}-1, 1)$ and $x_1 < x_2$,

$$1 < p_0^1(2+2x_2) < p_0^1(2+2x_1) < p_0^2(2+2x_1) < p_0^2(2+2x_2). \tag{0.4}$$

We now define

$$H_1(p,x) = x^4 - \frac{32p(p-1)}{(p+1)^2}x^2 + \frac{32p(p-1)(p-3)}{(p+1)^3}x + \frac{64p(p-1)^2}{(p+1)^4}.$$

Then

$$H_1(p,x) = H_0(p,x) + 2x^2 - 1 = \Big[H_N + \frac{(N-2)^2}{2} - 1\Big] - \tilde{g}(p,N).$$

We see that $H_N^* := H_N + \frac{(N-2)^2}{2} - 1 < H_N$ for $N = 3$ and $H_N + \frac{(N-2)^2}{2} - 1 \geq H_N$ for $N \geq 2 + \sqrt{2}$. Therefore, for $N \geq \overline{N}$, where $\overline{N} \in (4, 5)$ is the unique root of the equation $8(N-2)(N-4) = H_N^*$ and $x \geq \overline{x} := \frac{1}{2}(\overline{N}-2) \in (1, 3/2)$, there is a unique $p_*(N) := p_*(2+2x) > p_0(2+2x)$ with

$$p_*(N) = \frac{N+2-\sqrt{4+N^2-4\sqrt{N^2+H_N^*}}}{6-N+\sqrt{4+N^2-4\sqrt{N^2+H_N^*}}}$$

such that

$$H_1(p,x) < 0 \quad \text{if and only if } p > p_*(2+2x). \tag{0.5}$$

For $x \in (\sqrt{2}-1, \overline{x})$, i.e., $N \in (2\sqrt{2}, \overline{N})$, there are $p_*^1(N) := p_*^1(2+2x)$ and $p_*^2(N) := p_*^2(2+2x)$ with

$$p_*^1(N) = \frac{N+2-\sqrt{4+N^2-4\sqrt{N^2+H_N^*}}}{6-N+\sqrt{4+N^2-4\sqrt{N^2+H_N^*}}},$$

$$p_*^2(N) = \frac{N+2+\sqrt{4+N^2-4\sqrt{N^2+H_N^*}}}{6-N-\sqrt{4+N^2-4\sqrt{N^2+H_N^*}}},$$

and

$$1 < p_0^1(2+2x) < p_*^1(2+2x) < p_*^2(2+2x) < p_0^2(2+2x)$$

such that

$$H_1(p,x) < 0 \quad \text{if and only if } p_*^1(2+2x) < p < p_*^2(2+2x). \tag{0.6}$$

We also have that for $2\sqrt{2} < N_1 < N_2 < 4$,

$$1 < p_0^1(N_2) < p_0^1(N_1) < p_0^2(N_1) < p_0^2(N_2). \tag{0.7}$$

For $4 \le N_1 < N_2 < \overline{N}$,

$$1 < p_*^1(N_1) < p_*^1(N_2) < p_*^2(N_1) < p_*^2(N_2). \tag{0.8}$$

Moreover, (0.7) implies that for $x \in (\sqrt{2}-1, 1)$ and $x_1 < x_2$,

$$(p_0^1(2+2x_1), p_0^2(2+2x_1)) \subset (p_0^1(2+2x_2), p_0^2(2+2x_2)); \tag{0.9}$$

(0.8) implies that for $x \in (1, \overline{x})$ and $x_1 < x_2$,

$$p_*^1(2+2x_1) < p_*^1(2+2x_2) < p_*^2(2+2x_1) < p_*^2(2+2x_2). \tag{0.10}$$

REFERENCES

[1] C. Cowan, *Liouville theorems for stable Lane-Emden systems and biharmonic problems,* arXiv1207.1081v1, 2012.

[2] C. Cowan, P. Esposito and N. Ghoussoub, *Regularity of extremal solutions in fourth order nonlinear eigevalue problems on general domains,* DCDS-A 28 (2010), 1033-1050.

[3] C. Cowan, P. Esposito, N. Ghoussoub and A. Moradifam, *The critical dimension for a fourth order elliptic problem with singular nonlinearity,* Arch. Ration. Mech. Anal. 198 (2010), 763-787.

[4] C. Cowan and N. Ghoussoub, *Regularity of semi-stable solutions to fourth order nonlinear eigevalue problems on general domains,* arXiv:1206.3471v1, 2012.

[5] A. Capella, J. Dávila, L. Dupaigne and Y. Sire, *Regularity of radial extremal solutions for some non-local semilinear equations,* Comm. PDEs 36 (2011), 1353-1384.

[6] E.N. Dancer, Y. Du and Z.M. Guo, *Finite Morse index solutions of an elliptic equation with supercritical exponent,* J. Differential Equations 250 (2011), 3281-3310.

[7] J. Dávila, I. Flores and I. Guerra, *Multiplicity of solutions for a fourth order equation with power-type nonlinearity,* Math. Ann. 348 (2010), 143-193.

[8] J. Dávila, L. Dupaigne, I. Guerra and M. Montenegro, *Stable solutions for the bilaplacian with exponential nonlinearity,* SIAM J. Math. Anal. 39 (2007), 565-592.

[9] J. Dávila, L. Dupaigne and A. Farina, *Partial regularity of finite Morse index solutions to the Lane-Emden equation,* J. Funct. Anal. 261 (2011), 218-232.

[10] J. Davila, L. Dupaigne, K. Wang and J. Wei, *Monotonicity formula and Liouville theorems for fourth order supercritical problems,* preprint, 2013.

[11] L. Dupaigne, M. Ghergu, O. Goubet and G. Warnault, *Entire large solutions for semilinear elliptic equations,* J. Differential Equations 253 (2012), 2224-2251.

[12] L. Dupaigne and P. Esposito, private communications.

[13] L. Dupaigne and A. Farina, *Stable solutions of $-\Delta u = f(u)$ in $\mathbb{R}^N$,* J. Eur. Math. Soc. 12 (2010), 855-882.

[14] Y. Du and Z.M. Guo, *Positive solutions of an elliptic equation with negative exponent: Stability and critical power,* J. Differential Equations 246 (2009), 2387-2414.

[15] P.Esposito, *Compactness of a nonlinear eigenvalue problem with singular nonlinearity,* Comm. Contemp.Math. 10(2008), no.1, 17-45.

[16] P. Esposito, N. Ghoussoub and Y. Guo, *Compactness along the branch of semi-stable and unstable solutions for an elliptic problem with a singular nonlinearity,* Comm. Pure Appl. Math. 60(2007), no.12, 1731-1768.

[17] P. Esposito, N. Ghoussoub and Y. Guo, *Mathematical Analysis of Partial Differential Equations Modeling Electrostatic MEMS*, Research Monograph, Courant Lecture Notes, Vol. 20 (2010).

[18] A. Farina, *On the classification of solutions of the Lane-Emden equation on unbounded domains of $\mathbb{R}^N$,* J. Math. Pures Appl. 87 (2007), 537-561.

[19] F. Gazzola and H. -Ch. Grunau, *Radial entire solutions for supercritical biharmonic equations,* Math. Ann. 334 (2006), 905-936.

[20] Z.M. Guo and J. Wei, *On a fourth order elliptic problem with negative exponent,* SIAM J. Math. Anal. 40 (2008/09), no.5, 2034-2054.

[21] Z.M. Guo and J. Wei, *Qualitative properties of entire radial solutions for a biharmonic equation with supcritical nonlinearity,* Proc. American Math. Soc. 138 (2010), no.11, 3957-3964.

[22] H. Hajlaoui, A. A. Harrabi and D. Ye, *On stable solutions of biharmonic problem with polynomial growth,* arXiv:1211.2223v2, 2012.

[23] P. Karageorgis, *Stability and intersection properties of solutions to the nonlinear biharmonic equation,* Nonlinearity 22 (2009), 1653-1661.

[24] P.J. McKenna, W. Reichel, *Radial solutions of singular nonlinear biharmonic equations and applications to conformal geometry,* Electron J. Differential Equations no 37 13pp (2003).

[25] L. Ma and J. Wei, *Properties of positive solutions to an Elliptic Equation with negative exponent,* Journal of Functional Analysis 254(2008), no.4, 1058-1087.

[26] A. Moradiam, *On the critical dimension of a fourth order elliptic problem with negative exponent,* J. Differential Equations 248 (2010), 594-616.

[27] F. Rellich, *Perturbation theory of eigenvalue problems,* Gordon and Breach Science Publisher, New York, (1969).

[28] P. Souplet, *The proof of the Lane-Emden conjecture in four space dimensions,* Adv. Math. 221 (2009), 1409-1427.

[29] J. Wei and D. Ye, *Liouville theorems for stable solutions of biharmonic problem* Math. Ann 356(2013), no.4, 1599-1612.