Multi-vortex non-radial solutions to the magnetic Ginzburg-Landau equations

F. Ting *J. Wei †

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Abstract

We show that there exists multi-vortex, non-radial, finite energy solutions to the magnetic Ginzburg-Landau equations on all of $\mathbb{R}^2$. We use Lyapunov-Schmidt reduction to construct solutions which are invariant under rotations by $\frac{2\pi}{k}$ (but not by rotations in $O(2)$ in general) and reflections in the $x-$ axis for some $k \geq 7$.

1 Ginzburg-Landau equations

1.1 Introduction

The standard macroscopic theory of superconductivity is due to Ginzburg and Landau [9, 28]. It can be derived from the microscopic theory due to Bardeen, Cooper and Schrieffer [8, 10]. Stationary states of superconductors occupying (for simplicity) the plane $\mathbb{R}^2$, are described by pairs (ψ, A), where $\psi : \mathbb{R}^2 \to \mathbb{C}$ is the *order parameter* and $A : \mathbb{R}^2 \to \mathbb{R}^2$ is the *magnetic potential.* These states satisfy the system of equations

$$-\Delta_A \psi + \lambda(|\psi|^2 - 1)\psi = 0 \tag{1.1}$$

$$\nabla \times \nabla \times A - \operatorname{Im}(\bar{\psi} \nabla_A \psi) = 0 \tag{1.2}$$

called the *Ginzburg-Landau* (GL) equations. Here $\lambda > 0$ is a constant depending on the material in question: when $\lambda < 1/2$ or $> 1/2$, the material is of type I or II superconductor, respectively; $\nabla_A = \nabla - iA$ is the covariant gradient, and $\Delta_A = \nabla_A \cdot \nabla_A$. For a vector field A, $\nabla \times A$ is the scalar $\partial_1 A_2 - \partial_2 A_1$ and for scalar ξ, $\nabla \times \xi$ is the vector $(-\partial_2 \xi, \partial_1 \xi)$. Equation (1.2) is the static Maxwell

*Department of Mathematical Sciences, Lakehead University, Thunder Bay, ON, Canada, P7B 5E1. E-mail: fridolin.ting@lakeheadu.ca.

†Department of Mathematics, Chinese University of Hong Kong, Shatin, N.T., Hong Kong. Email: wei@math.cuhk.edu.hk

equation for the magnetic field $B = curl\, A$ and supercurrent $Im(\bar{\psi}\nabla_A\psi)$. We consider here configurations satisfying the superconducting boundary condition

$$|\psi(x)| \to 1 \qquad as \qquad |x| \to \infty.$$

The Ginzburg-Landau equations on the plane model superconductors in $\mathbb{R}^3$ which are spatially homogeneous in one direction, when neglecting boundary effects [14]. They also describe equilibrium states of the $U(1)$ Higgs model of particle physics [16].

Equations (1.1) and (1.2) are Euler-Lagrange equations for the Ginzburg-Landau energy functional

$$\mathcal{E}(\psi, A) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla_A\psi|^2 + (\nabla \times A)^2 + \frac{\lambda}{2}(|\psi|^2 - 1)^2, \tag{1.3}$$

i.e., solutions of (1.1) and (1.2) are critical points of $\mathcal{E}$: $\mathcal{E}'(\psi, A) = 0$. Here $\mathcal{E}'(u)$ denotes the L^2 gradient of the functional $\mathcal{E}$ at a point $u := (\psi, A)$.

Define the vorticity or the winding number of the vector field $\psi : \mathbb{R}^2 \to \mathbb{C}$ at infinity as $deg\, \psi := deg\left(\frac{\psi}{|\psi|}|_{|x|=R}\right) = \frac{1}{2\pi}\int_{|x|=R} d(arg\psi)$ for R sufficiently large. Assuming a pair (ψ, A) has finite energy, then the degree of the vector field ψ is related to the flux of the magnetic field $B = curl\, A$ as follows:

$$\int_{\mathbb{R}^2} B = 2\pi(deg\, \psi).$$

To date, the only non-trivial, finite energy, rigorously known solutions to equations (1.1)-(1.2) on all of $\mathbb{R}^2$ are the radial and equivariant solutions of the form $u = (\psi^{(n)}, A^{(n)})$, with

$$\psi^{(n)}(x) = f_n(r)e^{in\theta} \qquad and \qquad A^{(n)}(x) = a_n(r)\nabla(n\theta) \tag{1.4}$$

called $n - vortices$. Here (r, θ) are the polar coordinates of the vector $x \in \mathbb{R}^2$ and $n = deg\, \psi_n$ is an integer. Existence of n-vortex solutions of the form (1.4) was proved in [20, 4] using variational methods. The stability and instability properties of n-vortices were established in [12, 11]. More specifically, in [12], they showed that for $\lambda < 1/2$, any integer degree vortex is stable; for $\lambda > 1/2$, only $n = \pm 1$ vortices are stable. When $\lambda = 1/2$, all integer degree vortices are stable [1].

One has the following information on the vortex profiles f_n and a_n (see [20, 4]): $0 < f_n < 1, 0 < a_n < 1$ on $(0, \infty)$; $f_n', a_n' > 0$; and $1 - f_n, 1 - a_n \to 0$ as $r \to \infty$ with exponential rates of decay. In fact,

$$\begin{aligned} f_n(r) &= 1 + O(e^{-m_\lambda r}) \quad \text{and} \\ a_n(r) &= 1 + O(e^{-r}) \quad \text{with} \\ m_\lambda &:= \min(\sqrt{2\lambda}, 2). \end{aligned}$$

At the origin, $f_n \sim cr^n, a_n \sim dr^2$ ($c > 0, d > 0$ are constants) as $r \to 0$.

In addition, we have the asymptotics of the field components as established in [20] as $r = |x| \to \infty$:

$$\begin{array}{rcl} j^{(n)}(r) & = & n\beta_n K_1(r)[1 + o(e^{-m_\lambda |x|})]J\hat{x} \\ B^{(n)}(r) & = & n\beta_n K_1(r)[1 - \dfrac{1}{2r} + O(1/r^2)] \\ |f_n'(r)| & \leq & ce^{-m_\lambda r}. \end{array} \tag{1.5}$$

Here $j^{(n)} = \mathrm{Im}(\overline{\psi^{(n)}}(\nabla_A \psi)^{(n)})$ is the n-vortex supercurrent, $\beta_n > 0$ is a constant, and $K_1(r)$ the order one Bessel's function of the second kind which behaves like $\frac{e^{-r}}{\sqrt{r}}\left(1 + O\left(\frac{1}{r}\right)\right)$ as $r \to \infty$.

Equations (1.1) and (1.2), in addition to being translationally and rotationally invariant, have translational and gauge symmetries: solutions are mapped to solutions under the transformations

$$\psi(x) \mapsto \psi(x - z), \qquad A(x) \mapsto A(x - z)$$

for any $z \in \mathbb{R}^2$, and

$$\psi \mapsto e^{i\gamma}\psi, \qquad A \mapsto A + \nabla\gamma$$

for twice differentiable $\gamma : \mathbb{R}^2 \to \mathbb{R}$. Consequently, solutions (1.4) lead to the following families of solutions

$$\psi_{nz\gamma}(x) = e^{i\gamma(x)}\psi^{(n)}(x - z) \qquad A_{nz\gamma}(x) = A^{(n)}(x - z) + \nabla\gamma(x) \tag{1.6}$$

where n is an integer, $z \in \mathbb{R}^2$ and $\gamma : \mathbb{R}^2 \to \mathbb{R}$.

For reviews and books on the Ginzburg-Landau equations of superconductivity, one can refer to [3, 5, 7, 14, 16, 22, 23], for example.

In the case of the Ginzburg-Landau equation on *bounded* domains, non-radial *non-magnetic* solutions have been found by Bethuel, Brezis, and Helein [2, 3] and non-radial *magnetic* solutions have been found by Sandier and Serfaty (see references in [23]). This is due to the boundary forces which keep repelling vortices within the bounded domain.

In the case of the Ginzburg-Landau equation on *unbounded* domains, Ovchinnikov and Sigal [19] conjectured by numerical evidence that for the non-magnetic Ginzburg-Landau equations on the whole plane, non-radial solutions do exist. In addition, Gustafson, Sigal and Tzaneteas [14] suggest that for magnetic vortices, stationary multi-vortex configurations of degrees ± 1 occur with discrete symmetry group. In this paper, we prove rigorously that this last conjecture is true.

Notation: For the rest of the paper, when we write L^2 and H^s, we mean scalar/vector L^2 spaces and scalar/vector Sobolev spaces or order s. We denote the real inner product on $L^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$ to be

$$\left\langle \begin{pmatrix} \xi \\ \alpha \end{pmatrix}, \begin{pmatrix} \varrho \\ \beta \end{pmatrix} \right\rangle := \int_{\mathbb{R}^2} \{\mathrm{Re}(\bar{\xi}\varrho) + \alpha \cdot \beta\}.$$

We will denote L^p norms as $\|\cdot\|_p = \|\cdot\|_{L^p}$ and H^s norms as $\|\cdot\|_{H^s}$. Finally, we will denote the letter c or C for generic constants that do not depend on any small parameters present.

1.2 Results: Finite-energy non-radial magnetic vortex solutions

In this section, we state the main result of this paper.

We first define a *degree-changing* solution of (1.1) and (1.2) as a multi-vortex configuration containing *both* positive and negative degree vortices.

Theorem 1.1. *Fix $\lambda > 1/2$ and an integer $k \geq 7$. There exists a sequence, $(u_i)_{i\geq 0} := (\psi_i, A_i)_{i\geq 0}$, of non-radial degree-changing solutions to (1.1) and (1.2) containing km_i vortices, $m_i \to \infty$, invariant under rotations by $\frac{2\pi}{k}$ (but not by rotations in $O(2)$ in general) and reflections in the $x_2 = 0$ line. Each u_i has finite-energy of the form*

$$\mathcal{E}(u_i) = km_i\mathcal{E}(v^\bullet) + o(1) \quad as \;\; m_i \to \infty, \tag{1.7}$$

where $v^\bullet$ is the $+1$ or -1 degree vortex.

Recently, Sigal and Tzanateas have found Abrikosov type lattice solutions with *infinite energy* on the whole plane [25] and have proven these lattice solutions are stable under gauge periodic perturbations [26].

We construct these degree-changing, finite-energy, non-radial solutions in the spirit of the construction of *sign*-changing, finite-energy, non-radial solutions to the non-linear Schrödinger equation in $\mathbb{R}^n$ by Musso, Pacard and Wei in [18]. In addition, we use the results of effective interaction of magnetic vortices derived by Gustafson and Sigal in [13].

The solutions u_i will be multi-vortex configurations whose vortices are located on two *equilateral k-gons* (a polygon with k sides of equal length) and on k line segments joining the vertices of the two k-gons. One k-gon, called the *inner* k-gon, will be in the interior of the other one, called the *outer* k-gon. We will show later that there exists two sequences of integers $(p_i)_{i\geq 0}$, $(q_i)_{i\geq 0}$, both tending to infinity, such that $m_i = p_i + 2q_i$ in Theorem 1.1 (see proof of Theorem 2.1). For each multi-vortex configuration u_i, p_i vortices of $+1$ degree will be placed, at approximately equal length l between each other, on the vertices of the inner k-gon and on the line segments joining the vertices of the two k-gons. Also, $2q_i$ vortices of alternating degrees $+1$ and -1 will be placed, at approximately equal length $\bar{l}(> l)$ between each other, on the edges of the outer k-gon. As $p_i, q_i \to \infty$, the number of vortices in the configuration $km_i = k(p_i+2q_i) \to \infty$, and both $l, \bar{l}$ also tend to infinity so that the size of the inner and outer k-gons grows larger and larger. A cartoon example of solution u_i can be found in Figure 2.1 in Section 2 for $k = 7$, and small integers p, q. A more precise description of the solutions u_i can also be found in Section 2.

The rest of the paper is organized as follows. In Section 2, we outline the main ideas and steps and prove Theorem 1.1. We will use Lyapunov-Schmidt reduction and break the problem up into two subproblems: the problem in the "orthogonal" direction and the "tangential" direction, or reduced problem. In Section 3, we solve the problem in the orthogonal direction, and in Section 4, we solve the reduced problem. In the Appendix, we include the outline of proofs of technical results from [13].

2 Main Steps of Proof of Theorem 1.1

In this section, we outline the main steps of and prove Theorem 1.1.

Consider test functions describing $m \geq 2$ number of vortices glued together with centers at $z_1, z_2, ..., z_m$, and degrees $n_1, n_2, ..., n_m$. More specifically, let $m \in \mathbb{Z}^+$ denote the number of vortices with topological degrees $(n_1, n_2, ..., n_m) \in \mathbb{Z}^m$, $n_j \in \mathbb{Z} \setminus \{0\}$; denote the location of the center of each of these m vortices by $\underline{z} = (z_1, z_2, ..., z_m) \in \mathbb{R}^{2m}$, and let $\chi : \mathbb{R}^2 \to \mathbb{R}$ denote a gauge transformation. We associate with each $\underline{z}$ and χ, a function

$$v_{\underline{z}\chi} := (\psi_{\underline{z}\chi}, A_{\underline{z}\chi}) \tag{2.1}$$

where

$$\psi_{\underline{z}\chi} = e^{i\chi(x)} \prod_{j=1}^{m} \psi^{(n_j)}(x - z_j) \tag{2.2}$$

and

$$A_{\underline{z}\chi} = \sum_{j=1}^{m} A^{(n_j)}(x - z_j) + \nabla\chi(x). \tag{2.3}$$

For a given $\underline{z} \in \mathbb{R}^{2m}$, we take our gauge transformations to be of the form

$$\chi(x) = \sum_{j=1}^{m} z_j \cdot A^{(n_j)}(x - z_j) + \tilde{\chi}(x) \tag{2.4}$$

with $\tilde{\chi} \in H^2(\mathbb{R}^2; \mathbb{R})$. Equivalently, our gauge transformations must live in the space

$$H^2_{\underline{z}}(\mathbb{R}^2; \mathbb{R}) := \{\chi \in H^2(\mathbb{R}^2; \mathbb{R}) \mid \chi - \sum_{j=1}^{m} z_j \cdot A^{(n_j)}(x - z_j) \in H^2(\mathbb{R}^2; \mathbb{R})\}$$

to ensure that $v_{\underline{z}\chi} \in X^{(n)}$, where

$$X^{(n)} := \{(\psi, A) : \mathbb{R}^2 \to \mathbb{C} \times \mathbb{R}^2 \mid (\psi, A) - (\psi^{(n)}, A^{(n)}) \in H^1(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)\}$$

is the affine space of degree n configurations (see (A.6) and (A.7) in Appendix A). The pair $(\underline{z}, \chi)$ (or sometimes just $\underline{z}$) is called a *multi-vortex configuration.*

For $\underline{z} = (z_1, ..., z_m) \in \mathbb{R}^{2m}$ denoting the centers of $m \geq 2$ vortices in a vortex configuration, define the *inter-vortex separation* as

$$R(\underline{z}) = \min_{i \neq j} |z_i - z_j|.$$

Define our infinite dimensional manifold of *widely spaced* multi-vortex configurations

$$M_{mv,\epsilon} = \{v_{\underline{z}\chi} \mid (\underline{z}, \chi) \in \Sigma_\epsilon\}$$

parameterized by the set of all centers of vortices and gauge transformations

$$\Sigma_\epsilon = \left\{ (\underline{z}, \chi) \mid \frac{e^{-R(\underline{z})}}{\sqrt{R(\underline{z})}} < \epsilon \ \text{ and } \ \chi \in H^2_{\underline{z}}(\mathbb{R}^2; \mathbb{R}) \right\}.$$

The tangent space to point $v_{\underline{z}\chi} \in M_{mv,\epsilon}$ is

$$T_{v_{\underline{z}\chi}} M_{mv,\epsilon} = \text{span} \left\{ \langle \gamma, \partial_\chi \rangle v_{\underline{z}\chi}, \partial_{z_{jk}}^{A_k^{(j)}} v_{\underline{z}\chi} \ \mid \ j = 1, ..., m; \ k = 1, 2; \ \gamma \in H^2(\mathbb{R}^2; \mathbb{R}) \right\}.$$

consisting of the "almost zero-modes" defined by (2.1) to (2.4) as follows: the gauge-tangent "almost zero-modes" are

$$G_\gamma^{\underline{z}\chi} := \langle \gamma, \partial_\chi \rangle v_{\underline{z}\chi} = \begin{pmatrix} i\gamma\psi_{\underline{z}\chi} \\ \nabla\gamma \end{pmatrix} \tag{2.5}$$

for $\gamma : \mathbb{R}^2 \to \mathbb{R}$. Here, the notation $\langle \gamma, \partial_\chi \rangle v_{\underline{z}\chi}$ denotes the Fréchet derivative of the map $\chi \to (e^{i\chi}\psi, A + \nabla\chi)$ evaluated at χ in the direction of γ. The (gauge-invariant) translational-tangent "almost zero-modes" are

$$\begin{aligned} T_{jk}^{\underline{z}\chi} &:= \partial_{z_{jk}}^{A_k^{(j)}} v_{\underline{z}\chi} = \left(\partial_{z_{jk}} + \langle A_k^{(n_j)}(\cdot - z_j), \partial_\chi \rangle \right) v_{\underline{z}\chi} \qquad (2.6) \\ &= \begin{pmatrix} e^{i\chi(x)} \prod_{l \neq j} \psi^{(n_l)}(x - z_l)[\partial_{x_{jk}} - i(A^{(n_j)}(x - z_j))_k]\psi^{(n_j)}(x - z_j) \\ B^{(n_j)}(x - z_j)e_k^\perp \end{pmatrix} \end{aligned}$$

where $A_k^{(j)} := [A^{(n_j)}(\cdot - z_j)]_k$, $B^{(n)} = \nabla \times A^{(n)}$ and $e_1^\perp = (0, 1)$ and $e_2^\perp = (-1, 0)$. Note that $T_{jk}^{\underline{z}\chi}$ are defined by covariant differentiation to ensure that $\partial_{z_{jk}}^{A_k^{(j)}} v_{\underline{z}\chi} \in H^1 \times L^2$, while $\partial_{z_{jk}} v_{\underline{z}\chi}$ is not. These tangent vectors are called almost zero modes since they "almost solve" $\mathcal{E}''(v_{\underline{z}\chi})\eta = 0$ (see Theorem 3.1(c) below).

Let $u = (\psi, A)$, and denote $F(u) = \mathcal{E}'(u)$, defined as a map from H^2 to L^2. Explicitly,

$$F(u) = \begin{pmatrix} -\Delta_A\psi + \frac{\lambda}{2}(|\psi|^2 - 1)\psi \\ \nabla \times \nabla \times A - Im(\bar{\psi}\nabla_A\psi) \end{pmatrix}. \tag{2.7}$$

Thus, equations (1.1) and (1.2) can be written as $F(u) = 0$.

Define orthogonal projections

$$\begin{aligned} \pi_{\underline{z}\chi} &:= L^2 - \text{orthogonal projection onto} \\ &\qquad \text{span}\{T_{jk}^{\underline{z}\chi}, G_\gamma^{\underline{z}\chi} \mid j = 1, ..., m, \ k = 1, 2, \ \gamma \in H^2(\mathbb{R}^2; \mathbb{R})\} \quad \text{and} \\ \pi_{\underline{z}\chi}^\perp &:= 1 - \pi_{\underline{z}\chi}. \qquad (2.8) \end{aligned}$$

The operator $\pi^{\perp}_{\underline{z}\chi}$ projects onto the L^2 orthogonal complement of $\mathrm{Ran}(\pi_{\underline{z}\chi})$, i.e., $\pi^{\perp}_{\underline{z}\chi} : L^2 \to [\mathrm{Ran}(\pi_{\underline{z}\chi})]^{\perp}$.

It is clear from definitions (2.5), (2.6), and (2.8) that

$$T_{v_{\underline{z}\chi}} M_{mv,\epsilon} = \mathrm{Ran}\pi_{\underline{z}\chi}.$$

The proof of existence of a solution to (1.1) and (1.2) relies on the following two steps:

1. ***Liapunov-Schmidt Reduction and Solution in the Orthogonal Direction.*** We use Liapunov-Schmidt reduction to break the problem up into its tangential and orthogonal components. We will show there exists a solution in the orthogonal direction using an implicit function type argument. More precisely, we will show that for $\epsilon > 0$ sufficiently small and for all widely spaced multi-vortex configurations $v_{\underline{z}\chi}$ with $(\underline{z}, \chi) \in \Sigma_\epsilon$, there exists a unique $w_{\underline{z}\chi\epsilon} \in \mathrm{Ran}(\pi^{\perp}_{\underline{z}\chi})$ such that

$$\pi^{\perp}_{\underline{z}\chi} F(v_{\underline{z}\chi} + w_{\underline{z}\chi\epsilon}) = 0. \tag{2.9}$$

2. ***Reduced Problem and Solution in the Tangential Direction.*** We show that the corresponding problem in the tangential direction,

$$\pi_{\underline{z}\chi} F(v_{\underline{z}\chi} + w_{\underline{z}\chi\epsilon}) = 0, \tag{2.10}$$

can be solved by finding a specific widely spaced multi-vortex configuration $v_{\underline{z}_\sigma\chi}$, built from a perturbation of $v_{\underline{c}\chi}$, an approximate polygonal solution to (1.1) and (1.2). Note that due to gauge equivariance, we do not need to solve for the gauge χ.

Steps 1 and 2 will imply our result. Steps 1 and 2 will be done in Sections 3 and 4, respectively.

Remark: In both Steps 1 and 2 above, we require the solutions that we construct to be invariant under rotations by $\frac{2\pi}{k}$ and reflections along the $x_2 = 0$ line, i.e., solutions u of (1.1) and (1.2) constructed will satisfy

$$u(R_k x) = u(x) \qquad \text{and} \qquad u(\Lambda x) = u(x), \tag{2.11}$$

where R_k is the rotation by $\frac{2\pi}{k}$ and Λ is reflection along the $x_2 = 0$ line. Therefore, we assume for the rest of the paper that our multi-vortex configurations $v_{\underline{z}\chi}$ and our fixed point/perturbation w satisfies symmetry conditions given by (2.11).

We note that we do not have a small perturbation parameter here, as is usually the case in singular perturbation theory techniques such as Lyapunov-Schmidt reduction. Instead, we will perturb a specific multi-vortex configuration which is almost a solution to (1.1) and (1.2) into a genuine solution of

(1.1) and (1.2). We describe this in more detail here.

First, denote $v_0 = \begin{pmatrix} \psi^{(n_0)}(x) \\ A^{(n_0)}(x) \end{pmatrix}$ and $v_{d\mathsf{e}} = \begin{pmatrix} \psi^{(n_{d\mathsf{e}})}(x) \\ A^{(n_{d\mathsf{e}})}(x) \end{pmatrix}$ two vortices with degrees n_0 and $n_{d\mathsf{e}}$ at centers $z_1 = 0$ and $z_2 = d\mathsf{e}$ for e a unit vector and $d > 0$. Define the *interaction energy* of the two vortex configuration $v_{(0,d\mathsf{e})\chi}$ as

$$W(d) = \mathcal{E}(v_{(0,d\mathsf{e})\chi}) - E^0 - E^{d\mathsf{e}}, \tag{2.12}$$

where $E^0 = \mathcal{E}(v_0)$, $E^{d\mathsf{e}} = \mathcal{E}(v_{d\mathsf{e}})$ are the self-energies of vortices v_0 and $v_{d\mathsf{e}}$. By definition, the *magnitude of force* between the two vortices in configuration $v_{(0,d\mathsf{e})\chi}$ is given by $|\partial_d W(d)|$. We have the following crucial lemma, which is essentially proven in Lemma 11 of [13], so we will just summarize the main steps of the proof in Appendix A.

Lemma 2.1. *Fix $\lambda > 1/2$. To leading order as $d \to \infty$, the* force *exerted on vortex v_0 by vortex $v_{d\mathsf{e}}$, in the direction of unit vector e, is given by*

$$\partial_d W(d) = n_0 n_{d\mathsf{e}} \Psi(d) + O\left(e^{-d(1+\delta)}\right) \tag{2.13}$$

for some $\delta > 0$ small. Here, $n_0, n_{d\mathsf{e}}$ are the respective degrees of the vortices and the effective magnitude of the inter-vortex force *is*

$$\Psi(d) = C_{\lambda>1/2} \frac{e^{-d}}{\sqrt{d}} \left(1 + O\left(\frac{1}{d}\right)\right) \tag{2.14}$$

where

$$C_{\lambda>1/2} \quad = \quad \int_{\mathbb{R}^2} e^{x\cdot\mathsf{e}} (-\Delta + 1)\, B\, dx$$

is a positive constant independent of e and $B = \nabla \times A$.

Remark: For our construction of non-radial magnetic vortex solutions to work, it is crucial that the interaction force be exponentially decaying. The construction outlined below does not work for non-magnetic vortices on the whole plane as the interaction energy is of logarithmic order.

Having precise knowledge of the effective interaction force between two vortices, it is natural to ask if there exists a multi-vortex configuration which is almost forceless, static or in equilibrium? In other words, does there exists a $v_{\underline{z}\chi}$ such that

$$\sum_{j=1}^{m} \sum_{k\neq j}^{m} n_k n_j \Psi(|z_k - z_j|) \frac{z_k - z_j}{|z_k - z_j|} = 0 \tag{2.15}$$

or almost zero? The answer is yes. The idea for such the construction of a configuration originated in the work of Kapouleas [17] in finding compact constant mean curvature surfaces. This idea was subsequently used by Wei and Yan [29] to construct infinitely many non-radial positive solutions and by Musso, Pacard and Wei [18] to construct finite energy sign changing solutions with dihedral symmetry for the nonlinear Schrödinger equation in $\mathbb{R}^n$. We describe more precisely this almost forceless, static equilibrium multi-vortex configuration now.

Approximate polygonal solution

First, we construct an approximate polygonal solution of $F(u) = 0$ satisfying (2.11) as in [18]. The approximate polygonal solution will be a multi-vortex configuration whose vertices are placed on an *inner* and *outer equilateral k-gon* (a polygon with k sides of equal length) and on line segments joining the inner and outer k-gons.

We begin by *fixing* an integer $k \geq 7$ and assuming there exists two positive integers, p, q and two positive real numbers $l, \bar{l}$ which satisfy the relation

$$\left(\sin\frac{\pi}{k}\right) pl = (2q-1)\bar{l}. \tag{2.16}$$

Let us define the vertices of the *inner* and *outer equilateral k-gons* to be at points $R_k^i a_1$ and $R_k^i a_{p+1}$ for $i = 0, \ldots, k-1$, where

$$a_1 = \frac{\bar{l}}{2\sin\frac{\pi}{k}}\mathtt{e}_1, \qquad a_{p+1} = a_1 + pl\mathtt{e}_1 \tag{2.17}$$

and R_k is a rotation by $2\pi/k$. In other words, the vertices of the inner and outer k-gons are the orbits of the two points a_1 and a_{p+1} by rotation R_k. Define an *inner spoke* joining the inner and outer k-gons as a line segment connecting the two vertices $R_k^i a_1$ and $R_k^i a_{p+1}$, for a fixed $i = 0, \ldots, k-1$. Note that there are k of these inner spokes or line segments.

Now, we place magnetic vortices on the vertices of the inner and outer k-gons, on the k inner spokes and on the k edges of the outer k-gon. More precisely, the magnetic vortices on one inner spoke are located at points

$$a_r := a_1 + (r-1)l\mathtt{e}_1, \quad for \quad r = 2, ..., p, \tag{2.18}$$

and on the other $k-1$ inner spokes at points $R_k^i a_r$ for $i = 1, ..., k-1$ and $r = 2, ..., p$. The magnetic vortices on one edge of the outer k-gon are located at points

$$b_s := a_{p+1} + s\bar{l}\mathtt{t}, \quad for \quad s = 1, ..., 2q-1, \tag{2.19}$$

and on the other $k-1$ edges of the outer k-gon at points $R_k^i b_s$ for $i = 1, ..., k-1$ and $s = 1, ..., 2q-1$. In (2.19),

$$\mathtt{t} = (-\sin(\pi/k), \cos(\pi/k)). \tag{2.20}$$

Note that the distance from the origin to a_{p+1} is $pl + \frac{\bar{l}}{2\sin(\pi/k)}$, and due to (2.16), the edges of the outer k-gon all have length $2q\bar{l}$. Figure 2.1 gives an

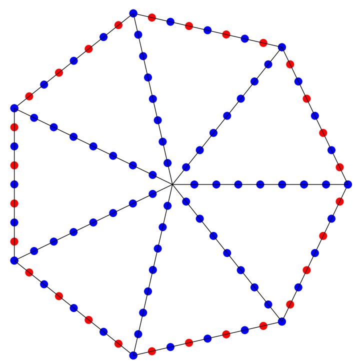

Figure 2.1: The location of the vortices in the approximate polygonal solution $v_{\underline{c}\chi}$ for $k = 7, p = 7$ and $q = 4$. The darker (blue) and lighter (red) dots represent $+1$ and -1 vortices, respectively.

example picture of the approximate polygonal solution described in between equations (2.16) and (2.20) when $k = 7, p = 7$ and $q = 4$.

We collect all the magnetic vortex centers defined above into the vector

$$\underline{c} := (a_1, \ldots, a_{p+1}, b_1, \ldots, b_{2q-1}, ..., R_k^{k-1} a_1, ..., R_k^{k-1} b_{2q-1}) \in \mathbb{R}^{2k(p+2q)} \tag{2.21}$$

and define the *approximate polygonal solution* as

$$v_{\underline{c}\chi} := \begin{pmatrix} e^{i\chi} \prod_{i=0}^{k-1} \left\{ \left[\prod_{r=1}^{p+1} \psi^{(n_r)}(\cdot - R_k^i a_r) \right] \left[\prod_{s=1}^{2q-1} \psi^{(n_s)}(\cdot - R_k^i b_s) \right] \right\} \\ \sum_{i=0}^{k-1} \left[\sum_{r=1}^{p+1} A^{(n_r)}(\cdot - R_k^i a_r) + \sum_{s=1}^{2q-1} A^{(n_s)}(\cdot - R_k^i b_s) \right] + \nabla\chi \end{pmatrix} \tag{2.22}$$

where the degrees of the vortices in $v_{\underline{c}\chi}$ are

$$\begin{aligned} n_r &= 1 \text{ for all } r = 1, \ldots, p+1 \quad \text{ and} \\ n_s &= (-1)^s \text{ for all } s = 1, \ldots, 2q-1. \end{aligned} \tag{2.23}$$

Note that $v_{\underline{c}\chi}$ satisfies symmetry conditions (2.11) and by the arrangement of $+1$ and -1 degree vortices as defined in (2.23), the multi-vortex configuration $v_{\underline{c}\chi}$ is observed to be almost forceless. An example picture of a configuration $v_{\underline{c}\chi}$ can be found in Figure 2.1.

Perturbation of the approximate polygonal solution

Now, we try to perturb the approximate polygonal solution into a "genuine" solution by perturbing the points a_r and b_s: let

$$y_r := a_r + \alpha_r \mathtt{e}_1, \qquad r = 1, \ldots, p+1, \tag{2.24}$$

and

$$z_s := b_s + \beta_s \mathtt{t} + \bar{l}\gamma_s \mathtt{n}, \qquad s = 1, \ldots, 2q-1 \tag{2.25}$$

where

$$\mathtt{n} \;=\; (\cos(\pi/k), \sin(\pi/k)). \tag{2.26}$$

Since we require our solutions to be invariant under rotations by $2\pi/k$ and reflections in the $x_2 = 0$, i.e., we require solutions u to satisfy (2.11), then z_s and z_{2q-s} must be related by the equation

$$z_{2q-s} = R_k \Lambda z_s$$

for all $s = 1, ..., q$. This relation and the fact that $R_k\Lambda \mathtt{t} = -\mathtt{t}$ and $R_k \Lambda \mathtt{n} = -\mathtt{n}$ imply that for $s = 1, ..., q$,

$$\beta_s = -\beta_{2q-s} \quad \text{and} \quad \gamma_s = \gamma_{2q-s}.$$

In particular, $\beta_q = 0$, and therefore, the total number of free parameters is in fact just $2q + p$.

Define the vectors

$$\alpha := (\alpha_1, \ldots, \alpha_{p+1}) \in \mathbb{R}^{p+1}, \;\; \beta := (\beta_1, \ldots, \beta_{q-1}) \in \mathbb{R}^{q-1}, \;\; \gamma := (\gamma_1, \ldots, \gamma_q) \in \mathbb{R}^q$$

and the set of *free* perturbation parameters by

$$\sigma := \left\{ (\alpha, \beta, \gamma) \;:\; \begin{array}{ll} |\alpha_r| \le 1, & r = 1, \ldots, p+1 \\ |\beta_s| \le 1, & s = 1, \ldots, q-1 \\ l|\gamma_s| \le 1, & s = 1, \ldots, q \end{array} \right\} \in \mathbb{R}^{p+2q}. \tag{2.27}$$

We note here that the bound 1 on $\alpha_r, \beta_s, l\gamma_s$ is arbitrary and could be replaced with any finite constant. We collect the perturbed vortex centers into the vector

$$\underline{z}_\sigma := (y_1, ..., y_{p+1}, z_1, ..., z_{2q-1}, \ldots, R_k^{k-1} y_1, \ldots, R_k^{k-1} z_{2q-1}) \in \mathbb{R}^{2k(p+2q)}, \tag{2.28}$$

and denote the *perturbation of the approximate polygonal solution* as

$$v_{\underline{z}_\sigma \chi} \;:=\; \begin{pmatrix} \psi_{\underline{z}_\sigma \chi} \\ A_{\underline{z}_\sigma \chi} \end{pmatrix} \tag{2.29}$$

with $v_{\underline{c}\chi}$ given in (2.22) and $\underline{z}_\sigma$ in (2.28). The *degrees of the vortices in* $v_{\underline{z}_\sigma\chi}$ *are the same as in* $v_{\underline{c}\chi}$ (see (2.23)). Note that for l large enough, $(\underline{z}_\sigma, \chi) \in \Sigma_\epsilon$ since $R(\underline{z}_\sigma) \approx l$.

The main Theorem 1.1 now follows from the following theorem:

Theorem 2.1. *Fix $\lambda > 1/2$ and $k \ge 7$. There exists a positive constant ϵ_0 such that for all ϵ satisfying $0 < \epsilon \log^{1/4}(1/\epsilon) < \epsilon_0$ and for all $l > \frac{1}{\epsilon_0}$, if $\bar{l}$ solves*

(i) Force Balancing Condition:

$$\Psi(l) = 2\sin\frac{\pi}{k}\Psi(\bar{l}) \tag{2.30}$$

and if p, q are positive integers satisfying

(ii) Polygonal Closing Condition:

$$\left(2\sin\frac{\pi}{k}\right)pl = (2q-1)\bar{l}, \tag{2.31}$$

then (1.1) and (1.2) has a solution of the form

$$u = v_{\underline{z}_\sigma\chi} + w$$

where $\underline{z}_\sigma = \underline{c} + O(e^{-\delta l})$ for some $\delta > 0$ and χ is a gauge function with $(\underline{z}_\sigma, \chi) \in \Sigma_\epsilon$; $w = O(\epsilon \log^{1/4}(1/\epsilon))$ in $H^2(\mathbb{R}^2; \mathbb{C}\times\mathbb{R}^2)$ and u has finite energy given by

$$\mathcal{E}(u) = k(p+2q)\mathcal{E}(v^\bullet) + O\left(\frac{e^{-R(\underline{z}_\sigma)}}{\sqrt{R(\underline{z}_\sigma)}}\right)$$

as $R(\underline{z}_\sigma) \approx l \to \infty$. Here, $v^\bullet$ is the $+1$ or -1 degree vortices and $\underline{c}, \underline{z}_\sigma$, and $v_{\underline{z}_\sigma\chi}$ are defined in (2.21), (2.28), and (2.29), respectively.

Remarks:

1. The condition $k \geq 7$ is purely geometrical in nature and is required so that the vortices at y_p and z_1 in $\underline{z}_\sigma$ do not interact too much. More precisely, if $k \leq 6$, one can approximate the distance between vortices at y_p and z_1 to be $2l\sin\left(\frac{\pi}{4} - \frac{\pi}{2k}\right) + O(1) < l + O(1)$ as $l \to \infty$. Therefore, when considering the forces between the nearest neighbors on y_p (see definition (4.20)), one has to also consider z_1 and Λz_1 instead of *only* y_{p-1} and y_{p+1}. This is very important when showing that equation (2.10) can be reduced to an equivalent solvable Toda system for perturbed parameters $\sigma = (\alpha, \beta, \gamma)$ defined in (2.27) (see (4.34) and the proof of Proposition 4.2 below).

2. The "Force Balancing Condition" is required so that the l.h.s. of (2.15) is almost equal to zero. More precisely, (2.30) is required so that the leading order expressions for the inter-vortex forces in the multi-vortex configuration $v_{\underline{z}_\sigma\chi}$ cancel out and have an "almost" forceless equilibrium/static solution (again, see the proof of Proposition 4.2 below).

3. The "Polygonal Closing Condition" is purely geometrical in nature too, and relates the distances between vortices on the "inner spokes" (l) and "outer edges" ($\bar{l}$) of the polygon.

4. For $\lambda < 1/2$, the inter-vortex forces are always *attractive* and do not depend on the degrees of the vortices in the configuration (see Lemma 6.1(b) of [21]). Therefore, we cannot construct almost forceless configurations for $\lambda < 1/2$ as we do here for $\lambda > 1/2$.

Theorem 2.1 will be proven in Section 4.2. Now we are ready to prove our main theorem.

Proof of Theorem 1.1. By (2.14), $\Psi(l)$ is a strictly decreasing function of l for l large enough. Hence, Ψ has an inverse function Ψ^{-1} (defined on a suitable domain) for large enough l, and therefore, (2.30) defines $\bar{l}$ as a function of l via formula $\bar{l} = \Psi^{-1}[(2\sin(\pi/k))^{-1}\Psi(l)]$. Using (2.30) and (2.14), we obtain

$$\bar{l} = l + \ln\left(2\sin\frac{\pi}{k}\right) + O(l^{-1}). \tag{2.32}$$

Substituting (2.32) into (2.31), we obtain

$$\frac{2q-1}{p} = 2\sin\frac{\pi}{k}\left[1 - l^{-1}\ln\left(2\sin\frac{\pi}{k}\right) + O(l^{-2})\right]. \tag{2.33}$$

Now choose positive integers p and q large enough such that

$$0 < \frac{2q-1}{p} - 2\sin\frac{\pi}{k} < \epsilon_0. \tag{2.34}$$

From (2.33) and (2.34), it is clear that there exists a unique l with $l > \epsilon_0^{-1} > \frac{-2\sin\frac{\pi}{k}\ln\left(2\sin\frac{\pi}{k}\right)}{\epsilon_0}$ such that (2.33) is satisfied, i.e., (2.31) is satisfied. Therefore, by Theorem 2.1, there exists two sequences of positive integers $(p_i)_{i\geq 0}, (q_i)_{i\geq 0}$ tending to infinity with $\frac{2q_i-1}{p_i} \to 2\sin\frac{\pi}{k}$, and a sequence, $(u_i)_{i\geq 0}$, of degree-changing non-radial solutions of (1.1) and (1.2) with energy given by (1.7) where $m_i := p_i + 2q_i$. This last statement comes from the fact that as $R(\underline{z}) \to \infty$, the energy of the a multi-vortex configuration $v_{\underline{z}\chi}$ containing m vortices is given by $\mathcal{E}(v_{\underline{z}\chi}) = \sum_{j=1}^m E^{(n_j)} + O\left(\frac{e^{-R(\underline{z})}}{\sqrt{R(\underline{z})}}\right)$ (see Lemma 7 in [13]) where $E^{(n_j)}$ is the self energy of the j^{th} vortex of degree n_j. In this case, our multi-vortex configuration $v_{\underline{z}_\sigma\chi}$ has $k(2q+p) = km$ vortices of degree either $+1$ or -1 with self energy of each vortex $\mathcal{E}(v^\bullet)$. Here, $R(\underline{z}_\sigma) \approx l \to \infty$ since $p, q \to \infty$ in (2.33).

□

3 Solution in the Orthogonal direction

In this section, will show that for $\epsilon > 0$ sufficiently small and for $(\underline{z}, \chi) \in \Sigma_\epsilon$, there exists a unique $w_{\underline{z}\chi\epsilon} \in \mathrm{Ran}(\pi^\perp_{\underline{z}\chi})$ such that (2.9) is satisfied.

We first state the following theorem from [13] which is *crucial* in our analysis. Parts (a) to (e) of the theorem below is proven already in [13] and [27] for $\lambda > 1/2$. We will reprove part (d) in Appendix A, as their is a slight modification to the original proof in [13].

For real numbers g, h, define the error function

$$\Gamma^{g,h}(\epsilon) = \epsilon^g \log^h(1/\epsilon). \tag{3.1}$$

Theorem 3.1. *For $\epsilon > 0$ sufficiently small and for $(\underline{z}, \chi) \in \Sigma_\epsilon$,*

(a) (Almost solution)

$$\|F(v_{\underline{z}\chi})\|_{L^2} = O(\Gamma^{1,\frac{1}{4}}(\epsilon)); \tag{3.2}$$

(b) (Almost Orthogonality)

$$\langle T^{\underline{z}\chi}_{jk}, T^{\underline{z}\chi}_{lm}\rangle = \gamma_{(n_j)}\delta_{jl}\delta_{km} + O(\Gamma^{1,\frac{1}{2}}(\epsilon)), \tag{3.3}$$

where

$$\gamma_{(n_j)} = \frac{1}{2}\|\nabla_{A^{(n_j)}}\psi^{(n_j)}\|_2^2 + \|curl A^{(n_j)}\|_2^2, \tag{3.4}$$

and

$$\langle T^{\underline{z}\chi}_{jk}, G^{\underline{z}\chi}_{\gamma}\rangle = O(\Gamma^{1,\frac{1}{4}}(\epsilon))\|\gamma\|_{L^2}. \tag{3.5}$$

(c) (Almost zero modes) Write

$$L_{\underline{z}\chi} := \mathcal{E}''(v_{\underline{z}\chi}).$$

Then

$$\|L_{\underline{z}\chi}T^{\underline{z}\chi}_{jk}\|_{L^2} = O(\Gamma^{1,\frac{1}{2}}(\epsilon)) \;\; \textit{and} \tag{3.6}$$

$$\|L_{\underline{z}\chi}G^{\underline{z}\chi}_{\gamma}\|_{L^2} \le c\Gamma^{1,\frac{1}{4}}(\epsilon)\|\gamma\|_{L^2}. \tag{3.7}$$

Therefore, from (2.8), (3.6) and (3.7), it follows that

$$L_{\underline{z}\chi}\pi_{\underline{z}\chi} = O(\Gamma^{1,\frac{1}{2}}(\epsilon)) \;\; \textit{in } L^2. \tag{3.8}$$

(d) (Coercivity of Hessian) There exists an $\tilde{\epsilon}_0 > 0$ such that for $0 < \epsilon < \tilde{\epsilon}_0$, $(\underline{z}, \chi) \in \Sigma_\epsilon$ and $w \in Ran(\pi^{\perp}_{\underline{z}\chi})$,

$$\langle w, L_{\underline{z}\chi}w\rangle \ge c_1\|w\|^2_{H^1} \ge c_2\|w\|_2^2 \tag{3.9}$$

for some positive constants c_1 and c_2.

(e) (Invertibility of Hessian) There exists an $\tilde{\epsilon}_0$ such that for all $0 < \epsilon < \tilde{\epsilon}_0$, $(\underline{z}, \chi) \in \Sigma_\epsilon$ and $w \in Ran(\pi^{\perp}_{\underline{z}\chi})$, we have

$$\|L_{\underline{z}\chi}w\|_{L^2} \ge \omega\|w\|_{H^2}$$

for some constant $\omega > 0$.

Remark. If the distance between the two vortices in the two vortex configuration $v_{(0,d\mathbf{e})\chi}$ is $d = l$ and $\Psi(l) < \epsilon$, then by Lemma 2.1, $\frac{e^{-R((0,d\mathbf{e}))}}{\sqrt{R(0,d\mathbf{e})}} < \epsilon$. Therefore, if we choose l large enough in the polygonal multi-vortex configuration $\underline{z}_\sigma$ such that $\Psi(l) < \epsilon$, then since $\bar{l} = l + O(1)$ by (2.32), $\frac{e^{-R(\underline{z}_\sigma)}}{\sqrt{R(\underline{z}_\sigma)}} < \epsilon$, and all the statements of Theorem 3.1 will apply. We will assume this will be the case for the rest of the paper.

Using Theorem (a) and (e) and the nonlinear estimates in Lemma 3.1 below brings us to the following main result of Step 1.

Theorem 3.2. *There exists positive constants δ_0 and ϵ_0 such that for all $0 < \Gamma^{1,\frac{1}{4}}(\epsilon) < \epsilon_0$ and for every $(\underline{z}, \chi) \in \Sigma_\epsilon$, there exists a $w_{\underline{z}\chi\epsilon} \in B_{H^2}(0, \delta_0) \cap Ran(\pi^\perp_{\underline{z}\chi})$ such that (2.9) is true. In addition:*

$$\|w_{\underline{z}\chi\epsilon}\|_{H^2} \leq D\Gamma^{1,\frac{1}{4}}(\epsilon) \tag{3.10}$$

where $D = D(\omega, \kappa)$ and

$$\kappa := \sup_{\epsilon > 0} \frac{1}{\Gamma^{1,\frac{1}{4}}(\epsilon)} \|F(v_{\underline{z}\chi})\|_{L^2}. \tag{3.11}$$

Proof. This is a basic implicit function type argument, which can be found in [24, 21]. We begin with the following definitions. Let $v = v_{\underline{z}\chi} + w$, where $v_{\underline{z}\chi} = \begin{pmatrix} \psi_{\underline{z}\chi} \\ A_{\underline{z}\chi} \end{pmatrix} \in M_{mv,\epsilon}$, and $w = \begin{pmatrix} \xi \\ B \end{pmatrix} \in H^2$ with $w \perp T_{v_{\underline{z}\chi}} M_{mv,\epsilon}$. Using Taylor expansion, we have

$$F(v_{\underline{z}\chi} + w) = F(v_{\underline{z}\chi}) + F'(v_{\underline{z}\chi})w + N(v_{\underline{z}\chi}, w), \tag{3.12}$$

where $F'(v_{\underline{z}\chi})$ and $N(v_{\underline{z}\chi}, w)$ is defined by this relation and explicitly given by

$$F'(v_{\underline{z}\chi})w = \tag{3.13}$$
$$\begin{pmatrix} [-\Delta_{A_{\underline{z}\chi}} + \frac{\lambda}{2}(2|\psi_{\underline{z}\chi}|^2 - 1)]\xi \ + \ \frac{\lambda}{2}\psi_{\underline{z}\chi}^2\bar{\xi} \ + \ i[2\nabla_{A_{\underline{z}\chi}}\psi_{\underline{z}\chi} + \psi_{\underline{z}\chi}\nabla] \cdot B \\ Im([\overline{\nabla_{A_{\underline{z}\chi}}\psi_{\underline{z}\chi}} - \overline{\psi_{\underline{z}\chi}}\nabla_{A_{\underline{z}\chi}}]\xi) \ + \ (-\Delta + \nabla\nabla + |\psi_{\underline{z}\chi}|^2) \cdot B \end{pmatrix}$$

and

$$N(v_{\underline{z}\chi}, w) = \tag{3.14}$$
$$\begin{pmatrix} \lambda(2\psi_{\underline{z}\chi}\bar{\xi} + \bar{\psi}_{\underline{z}\chi}\xi + |\xi|^2)\xi \ + \ ||B||^2(\psi_{\underline{z}\chi} + \xi) \ + \ i[(\nabla \cdot B)\xi + 2B \cdot \nabla_{A_{\underline{z}\chi}}\xi] \\ -Im(\bar{\xi}\nabla_{A_{\underline{z}\chi}}\xi) \ + \ B(2Re(\bar{\psi}_{\underline{z}\chi}\xi) + |\xi|^2) \end{pmatrix}.$$

We need the following nonlinear estimate:

Lemma 3.1. *There exist positive constants C_2, C_3, C_4 independent of $\underline{z}, \chi, \epsilon$ such that for all $w \in H^2$ with $\|w\|_{H^2} \leq C_2$,*

$$\|N(v_{\underline{z}\chi}, w)\|_{L^2} \leq C_3 \|w\|^2_{H^2}, \tag{3.15}$$

and

$$||\partial_w N(v_{\underline{z}\chi}, w)||_{H^2 \to L^2} \leq C_4 ||w||_{H^2}. \tag{3.16}$$

Proof. Lemma 3.1 follows directly from Sobolev embedding theorems and the mean value theorem (see [24]). □

Let ϵ satisfy $0 < \epsilon < \tilde{\epsilon}_0$ so that $L_{\underline{z}\chi}$ is invertible by Theorem 3.1e. Using the Taylor expansion (3.12) and abbreviating $\pi^{\perp}_{\underline{z}\chi} F(v_{\underline{z}\chi})$ to $F^{\perp}_{\underline{z}\chi}$ and $\pi^{\perp}_{\underline{z}\chi} N(v_{\underline{z}\chi}, w)$ to $N^{\perp}_{\underline{z}\chi}(w)$, we rewrite equation (2.9) as a fixed point equation $w = S_{\underline{z}\chi}(w)$ for the map $S_{\underline{z}\chi}$ defined on H^2 by

$$S_{\underline{z}\chi}(w) = -L^{-1}_{\underline{z}\chi} \left[N^{\perp}_{\underline{z}\chi}(w) + F^{\perp}_{\underline{z}\chi} \right]. \tag{3.17}$$

Let ω, C_2, C_3 and C_4 be the constants in Theorem 3.1e and Lemma 3.1. Set $\delta_0 = min(C_2, \frac{\omega}{2C_3}, \frac{\omega}{2C_4})$ and $\epsilon_0 = min(\tilde{\epsilon}_0, \frac{\delta_0}{2\kappa}\omega)$, where κ is defined in (3.11). We will show that for ϵ satisfying $0 < \Gamma^{1,\frac{1}{4}}(\epsilon) < \epsilon_0$, $S_{\underline{z}\chi}$ maps the ball $B^{\perp}_{\delta_0} = B_{H^2}(0, \delta_0) \cap [\mathrm{Ran}(\pi^{\perp}_{\underline{z}\chi})]$ continuously into itself. Let $w \in B^{\perp}_{\delta_0}$. Then for ϵ satisfying $\Gamma^{1,\frac{1}{4}}(\epsilon) < \epsilon_0$ and $||w|| \leq \delta_0 \leq C_2$, we have by Theorem 3.1e and Lemma 3.1

$$\begin{aligned}
\|S_{\underline{z}\chi}(w)\|_{H^2} &\leq \frac{1}{\omega} \left\| N^{\perp}_{\underline{z}\chi}(w) + F^{\perp}_{\underline{z}\chi} \right\|_{L^2} \\
&\leq \frac{1}{\omega} \left(C_3 \|w\|^2_{H^2} + \left\| F^{\perp}_{\underline{z}\chi} \right\|_{L^2} \right) \\
&\leq \frac{1}{\omega} \left(C_3 \delta_0^2 + \kappa \Gamma^{1,\frac{1}{4}}(\epsilon) \right) \leq \delta_0,
\end{aligned}$$

where in the second last inequality, we used (3.11), and in the last inequality, we used the definition of δ_0 and ϵ_0. Therefore $S_{\underline{z}\chi}(w)$ is in $B^{\perp}_{\delta_0}$ too.

In addition, for w and w' in $B^{\perp}_{\delta_0}$, we have from (3.16) and the mean value theorem that

$$\|N(v_{\underline{z}\chi}, w) - N(v_{\underline{z}\chi}, w')\|_{L^2} \leq C_4 \delta_0 \|w - w'\|_{H^2}. \tag{3.18}$$

Hence, (3.18) and our choice of δ_0 imply

$$\begin{aligned}
\|S_{\underline{z}\chi}(w) - S_{\underline{z}\chi}(w')\|_{H^2} &= \left\| L^{-1}_{\underline{z}\chi} (N^{\perp}_{\underline{z}\chi}(w) - N^{\perp}_{\underline{z}\chi}(w')) \right\|_{L^2} \\
&\leq \frac{C_4 \delta_0}{\omega} \|w - w'\|_{H^2} \leq \frac{1}{2} \|w - w'\|_{H^2}.
\end{aligned}$$

Therefore, $S_{\underline{z}\chi}$ is a contraction map and so $S_{\underline{z}\chi}$ has a unique fixed point $w_{\underline{z}\chi\epsilon}$ in $B^{\perp}_{\delta_0}$. By the definition of the map $S_{\underline{z}\chi}$, this fixed point solves (2.9) which proves the first part of Theorem 3.2.

For the second part of Theorem 3.2, we note that

$$\|S_{\underline{z}\chi}(0)\|_{H^2} = \left\|L^{-1}_{\underline{z}\chi} F^{\perp}_{\underline{z}\chi}\right\|_{H^2} \leq \omega^{-1} \|F(v_{\underline{z}\chi})\|_{L^2} .$$

But for the fixed point $w_{\underline{z}\chi\epsilon}$, we have

$$w_{\underline{z}\chi\epsilon} = S_{\underline{z}\chi}(w_{\underline{z}\chi\epsilon}) = S_{\underline{z}\chi}(0) + S_{\underline{z}\chi}(w_{\underline{z}\chi\epsilon}) - S_{\underline{z}\chi}(0).$$

Consequently,

$$\begin{aligned} \|w_{\underline{z}\chi\epsilon}\|_{H^2} &\leq \|S_{\underline{z}\chi}(0)\|_{H^2} + \|S_{\underline{z}\chi}(w_{\underline{z}\chi\epsilon}) - S_{\underline{z}\chi}(0)\|_{H^2} \\ &\leq \omega^{-1} \|F(v_{\underline{z}\chi})\|_{L^2} + \frac{1}{2} \|w_{\underline{z}\chi\epsilon}\|_{H^2} . \end{aligned}$$

Since $||F(v_{\underline{z}\chi})||_{L^2} \leq \kappa\Gamma^{1,\frac{1}{4}}(\epsilon)$ by (3.11), the last inequality implies part a) with $D = 2\omega^{-1}\kappa$:

$$||w_{\underline{z}\chi\epsilon}||_{H^2} \leq 2\omega^{-1}\kappa\Gamma^{1,\frac{1}{4}}(\epsilon). \tag{3.19}$$

□

4 Reduced problem and solution in the tangential direction

In this section, we solve the reduced problem (2.10) for $(\underline{z}, \chi) \in \Sigma_\epsilon$ in Step 2 of section 2 and prove Theorem 2.1.

Denote by $\pi^g_{\underline{z}\chi}$ and $\pi^t_{\underline{z}\chi}$ the L^2-orthogonal projections onto the gauge and translational "almost" tangent vectors, respectively. By definition, we have

$$\pi_{\underline{z}\chi} = \pi^g_{\underline{z}\chi} + \pi^t_{\underline{z}\chi}. \tag{4.1}$$

By (4.1), solving (2.10) for $(\underline{z}, \chi) \in \Sigma_\epsilon$ is equivalent to solving the coupled system of equations

$$\pi^g_{\underline{z}\chi} F(v_{\underline{z}\chi} + w_{\underline{z}\chi\epsilon}) = 0 \tag{4.2}$$

and

$$\pi^t_{\underline{z}\chi} F(v_{\underline{z}\chi} + w_{\underline{z}\chi\epsilon}) = 0 \tag{4.3}$$

for the pair $(\underline{z}, \chi) \in \Sigma_\epsilon$. We will show that if there exists a multi-vortex configuration $\underline{z}$ such that $(\underline{z}, \chi) \in \Sigma_\epsilon$ solves (4.3) for $\epsilon > 0$ sufficiently small and $\chi \in H^2_{\underline{z}}(\mathbb{R}^2, \mathbb{R})$ arbitrary, then due to gauge invariance, one can show that this same $(\underline{z}, \chi) \in \Sigma_\epsilon$ solves (2.10), i.e., this same $(\underline{z}, \chi) \in \Sigma_\epsilon$ solves *both* (4.2) and (4.3). Therefore, to find $(\underline{z}, \chi) \in \Sigma_\epsilon$ which solves (2.10), we are reduced to finding a multi-vortex configuration $\underline{z}$ which solves (4.3).

In subsection 4.1, we will show that if there exists a multi-vortex configuration $\underline{z}$ such that $(\underline{z}, \chi) \in \Sigma_\epsilon$ solves (4.3) for $\epsilon > 0$ sufficiently small and $\chi \in H^2_{\underline{z}}(\mathbb{R}^2, \mathbb{R})$ arbitrary, then this same $(\underline{z}, \chi) \in \Sigma_\epsilon$ solves (2.10). In subsection 4.2, we will prove that (4.3) is satisfied for $\underline{z}_\sigma$ defined in (2.28) of Section 2, and then finish the proof of Theorem 2.1.

4.1 Independence of Gauge and solution to reduced problem

In this subsection, we will prove the following proposition.

Proposition 4.1. *Suppose there exists a multi-vortex configuration $\underline{z}$ such that $(\underline{z}, \chi) \in \Sigma_\epsilon$ solves (4.3) for $\epsilon > 0$ sufficiently small and $\chi \in H^2_{\underline{z}}(\mathbb{R}^2; \mathbb{R})$ arbitrary. Then this same $(\underline{z}, \chi) \in \Sigma_\epsilon$ solves (2.10).*

Before we prove Proposition 4.1, we need some definitions and notation. Define the symmetry action of g_χ on the vectors $v = (\psi, A)$ (solutions of G-L equations) as

$$g_\chi v = \begin{pmatrix} e^{i\chi}\psi \\ A + \nabla\chi \end{pmatrix} \tag{4.4}$$

and the action of $\tilde{g}_\chi$ on vectors $w = (\xi, B)$ (perturbations around solutions) as

$$\tilde{g}_\chi w = \begin{pmatrix} e^{i\chi}\xi \\ B \end{pmatrix}. \tag{4.5}$$

From Theorem 3.2, we know that for $\epsilon > 0$ sufficiently small and for all $(\underline{z}, \chi) \in \Sigma_\epsilon$, $w_{\underline{z}\chi\epsilon}$ is a *unique* fixed point of map $S_{\underline{z}\chi}$ defined in (3.17). Dropping the $\underline{z}$ and ϵ dependance everywhere (equivalently, fixing $\underline{z}$ and ϵ), we have

$$w_\chi = S_\chi(w_\chi), \quad \chi \in H^2_{\underline{z}}(\mathbb{R}^2; \mathbb{R}). \tag{4.6}$$

Now we are ready to state the following required lemma.

Lemma 4.1. *Let w_0 and w_χ be unique fixed points of maps S_0 and S_χ, respectively. Then*

$$w_\chi = \tilde{g}_\chi w_0. \tag{4.7}$$

Proof. Let w_0, w_χ be the unique fixed points of the maps S_0, S_χ, respectively. Denoting $L_\chi := F'(v_{\underline{z}\chi})$, $L^\perp_\chi := \pi^\perp_\chi L_\chi \pi^\perp_\chi$, $v_\chi := g_\chi v_0 = (\psi^\chi, A^\chi)$ and $w_0 = (\xi, B)$, one can show using (4.4) and (4.5) that

$$\pi^\perp_\chi(\tilde{g}_\chi w_0) = \tilde{g}_\chi \pi^\perp_0(w_0); \tag{4.8a}$$

$$F(g_\chi v_0) = \tilde{g}_\chi F(v_0); \tag{4.8b}$$

$$N(v_\chi, w_\chi) = \tilde{g}_\chi N(v_0, w_0); \tag{4.8c}$$

$$L_\chi = \tilde{g}_\chi L_0 \tilde{g}_\chi^{-1} \quad \text{and hence} \quad (L^\perp_\chi)^{-1} = \tilde{g}_\chi (L^\perp_0)^{-1} \tilde{g}_\chi^{-1}. \tag{4.8d}$$

Indeed, (4.8a) comes from (4.1) and the explicit expressions for $\pi^t_\chi = \pi^t_{\underline{z}\chi}$ and $\pi^g_\chi = \pi^g_{\underline{z}\chi}$. More precisely, for $w = (\xi, B)$ and using Dirac notation (see [24]):

$$\pi^t_\chi w = \sum_{jk}\sum_{lm} |T^\chi_{jk}\rangle \left[(\beta^\chi)^{-1}\right]_{(jk)(lm)} \langle T^\chi_{lm}|w\rangle \tag{4.9}$$

$$\pi^g_\chi w = \begin{pmatrix} i\psi^\chi J[Im(\bar{\psi}^\chi\xi) - \nabla\cdot B] \\ \nabla J[Im(\bar{\psi}^\chi\xi) - \nabla\cdot B] \end{pmatrix}$$

where $|T^{\chi}_{jk}\rangle = \begin{pmatrix} e^{i\chi}\prod_{l\neq j}\psi^{(l)}[\nabla_A\psi]^{(j)}_k \\ (\nabla\times A)e^{\perp}_j \end{pmatrix} = \tilde{g}_\chi|T^0_{jk}\rangle$, $[\beta^{\chi}]_{(jk)(lm)} = \langle T^{\chi}_{jk}|T^{\chi}_{lm}\rangle = \langle T^0_{jk}|T^0_{lm}\rangle$ (which is invertible by (3.3)) and $J = (-\Delta + |\psi^{\chi}|^2)^{-1}$. Now, (4.8b) to (4.8d) comes from the explicit expressions for F, L_χ and N from (2.7), (3.13) and (3.14), respectively.

Now, using (4.8) and the fact that $S_\chi(w_\chi) = -(L^{\perp}_\chi)^{-1}(F(v_\chi) + N(v_\chi, w_\chi))$ (see (3.17)), we have

$$S_\chi(w_\chi) = \tilde{g}_\chi S_0(w_0). \tag{4.10}$$

By assumption that w_0 and w_χ are unique fixed points of the maps S_0 and S_χ, respectively (see (4.6)), (4.10) implies (4.7) and are done with the proof of Lemma 4.1.

□

From the above lemma, we have the following important

Lemma 4.2. *For $\epsilon > 0$ sufficiently small and $(\underline{z}, \chi) \in \Sigma_\epsilon$, define the* reduced energy *by*

$$\Phi(\underline{z}, \chi) := \mathcal{E}(v_{\underline{z}\chi} + w_{\underline{z}\chi\epsilon})$$

for $w_{\underline{z}\chi\epsilon}$ defined in Theorem 3.2. Then Φ is independent of gauge $\chi \in H^2_{\underline{z}}(\mathbb{R}^2;\mathbb{R})$.

Proof. Let $v_{\underline{z}\chi} := g_\chi v_{\underline{z}0}$ and $w_{\underline{z}\chi\epsilon} := \tilde{g}_\chi w_{\underline{z}0\epsilon}$. Then for any $\chi \in H^2_{\underline{z}}(\mathbb{R}^2;\mathbb{R})$, we have

$$v_{\underline{z}\chi} + w_{\underline{z}\chi\epsilon} = g_\chi(v_{\underline{z}0} + w_{\underline{z}0\epsilon}) \tag{4.11}$$

by (4.4), (4.5), and (4.7) in Lemma 4.1. Therefore, by (4.11) and the gauge invariance of the G-L energy functional (1.3), we obtain

$$\Phi(\underline{z}, \chi) = \mathcal{E}(v_{\underline{z}\chi} + w_{\underline{z}\chi\epsilon}) = \mathcal{E}(v_{\underline{z}0} + w_{\underline{z}0\epsilon}) = \Phi(\underline{z}, 0).$$

□

We are ready to prove Proposition 4.1. First, let's denote

$$v_{\underline{z}\chi\epsilon} := v_{\underline{z}\chi} + w_{\underline{z}\chi\epsilon} =: (\psi_{\underline{z}\chi\epsilon}, A_{\underline{z}\chi\epsilon}). \tag{4.12}$$

Proof of Proposition 4.1. Suppose there exists a multi-vortex configuration $\underline{z}$ such that $(\underline{z}, \chi) \in \Sigma_\epsilon$ solves (4.3) for $\epsilon > 0$ sufficiently small and $\chi \in H^2_{\underline{z}}(\mathbb{R}^2;\mathbb{R})$ arbitrary. We will show that the equation

$$F(v_{\underline{z}\chi} + w_{\underline{z}\chi\epsilon}) = \sum_{jk} c_{jk} T^{\underline{z}\chi}_{jk} + G^{\underline{z}\chi}_{\gamma_0}, \tag{4.13}$$

where $G^{\underline{z}\chi}_{\gamma}$ and $T^{\underline{z}\chi}_{jk}$ is the almost gauge and translational zero mode defined in (2.5) and (2.6), respectively, only has solutions $c_{jk} = 0$ for all vortices j, $k = 1, 2$ and $\gamma_0(x) = 0$, i.e., $(\underline{z}, \chi) \in \Sigma_\epsilon$ solves (2.10).

To begin, by (4.12), Lemma 4.2 and chain rule, we have for any $\gamma_0 \in H^2(\mathbb{R}^2;\mathbb{R})$,

$$0 = \langle \gamma_0, \partial_\chi\rangle\Phi(\underline{z}, \chi) = \langle \gamma_0, \partial_\chi\rangle\mathcal{E}(v_{\underline{z}\chi\epsilon}) = \langle \mathcal{E}'(v_{\underline{z}\chi\epsilon}), \langle\gamma_0, \partial_\chi\rangle v_{\underline{z}\chi\epsilon}\rangle. \tag{4.14}$$

Using the fact that $\langle\gamma_0, \partial_\chi\rangle g_\chi v = \begin{pmatrix} i\gamma_0\psi \\ \nabla\gamma_0 \end{pmatrix}$ by (4.4), then by (4.11) and the notation in (2.5) and (4.12), we obtain

$$\langle\gamma_0, \partial_\chi\rangle v_{\underline{z}\chi\epsilon} = \begin{pmatrix} i\gamma_0\psi_{\underline{z}\chi\epsilon} \\ \nabla\gamma_0 \end{pmatrix}. \tag{4.15}$$

Since $\mathcal{E}'(u) = F(u)$, then by (4.14) and (4.15),

$$0 = \left\langle F(v_{\underline{z}\chi} + w_{\underline{z}\chi\epsilon}), \begin{pmatrix} i\gamma_0\psi_{\underline{z}\chi\epsilon} \\ \nabla\gamma_0 \end{pmatrix}\right\rangle. \tag{4.16}$$

Now, taking the inner product of (4.13) with $G_{\gamma_0}^{\underline{z}\chi\epsilon} = (i\gamma_0\psi_{\underline{z}\chi\epsilon}, \nabla\gamma_0)$, we obtain using (4.16) and denoting $\psi_{\underline{z}\chi\epsilon} = \psi_{\underline{z}\chi} + \xi_{\underline{z}\chi\epsilon}$,

$$0 = \sum_{jk} c_{jk}\langle T_{jk}^{\underline{z}\chi}, G_{\gamma_0}^{\underline{z}\chi\epsilon}\rangle + \langle\gamma_0, [-\Delta + |\psi_{\underline{z}\chi}|^2 + Re(\psi_{\underline{z}\chi}\overline{\xi}_{\underline{z}\chi\epsilon})]\gamma_0\rangle. \tag{4.17}$$

Similarly, taking the inner product of (4.13) with $T_{j'k'}^{\underline{z}\chi}$ and using the fact that $(\underline{z}, \chi) \in \Sigma_\epsilon$ solves (4.3), we have

$$0 = \sum_{jk} c_{jk}\langle T_{jk}^{\underline{z}\chi}, T_{j'k'}^{\underline{z}\chi}\rangle + \langle T_{j'k'}^{\underline{z}\chi}, G_{\gamma_0}^{\underline{z}\chi}\rangle. \tag{4.18}$$

Note that (4.17) and (4.18) is a coupled system for real coefficients c_{jk} and gauge function $\gamma_0 \in H^2(\mathbb{R}^2; \mathbb{R})$. Using Theorem 3.1(b) and the fact that $[-\Delta + |\psi_{\underline{z}\chi}|^2 + Re(\psi_{\underline{z}\chi}\overline{\xi}_{\underline{z}\chi\epsilon})]$ is a positive operator for ϵ sufficiently small (by (3.10) in Theorem 3.2 and using Lemma 5.1 in [15]), the only solution to (4.17) and (4.18) is $c_{jk} \equiv 0$ for all vortices j, $k = 1, 2$ and $\gamma_0(x) \equiv 0$. This concludes the proof of Proposition 4.1. $\square$

4.2 Solution along translational modes

In the last subsection, we found that if there exists a multi-vortex configuration $\underline{z}$ such that $(\underline{z}, \chi) \in \Sigma_\epsilon$ solves (4.3) for $\epsilon > 0$ sufficiently small and $\chi \in H^2_{\underline{z}}(\mathbb{R}^2; \mathbb{R})$ arbitrary, then this same $(\underline{z}, \chi) \in \Sigma_\epsilon$ solves (2.10). In this section, we find a $\underline{z}$ with $(\underline{z}, \chi) \in \Sigma_\epsilon$ which solves (4.3) and complete the proof of Theorem 2.1. In fact, we will show that $\underline{z}_\sigma$ defined in (2.28) will solve (4.3) for a specific perturbation parameter $\sigma \in \mathbb{R}^{p+2q}$, defined in (2.27).

To begin with, let's Taylor expand (4.3) to obtain

$$\pi_{\underline{z}\chi}^t \left[F(v_{\underline{z}\chi}) + \overbrace{F'(v_{\underline{z}\chi})w_{\underline{z}\chi\epsilon}}^{O(\Gamma^{1,\frac{1}{4}}(\epsilon))^2} + \overbrace{N_{\underline{z}\chi}(v_{\underline{z}\chi}, w_{\underline{z}\chi\epsilon})}^{O(\Gamma^{1,\frac{1}{4}}(\epsilon))^2} \right] = 0. \tag{4.19}$$

The last two terms can be estimated to be of $O(\Gamma^{1,\frac{1}{4}}(\epsilon))^2$ in L^2. Indeed, the last term is $O(\Gamma^{1,\frac{1}{4}}(\epsilon))^2$ due to (3.15) and (3.10) in Theorem 3.2. Also, from the expression for $\pi^t_{\underline{z}\chi} = \pi^t_\chi$ in (4.9), we can estimate $\|F'(v_{\underline{z}\chi})w_{\underline{z}\chi\epsilon}\|_{L^2}$ by

$$\begin{aligned}\langle T^{\underline{z}\chi}_{jk}, F'(v_{\underline{z}\chi})w_{\underline{z}\chi\epsilon}\rangle &= \langle F'(v_{\underline{z}\chi})T^{\underline{z}\chi}_{jk}, w_{\underline{z}\chi\epsilon}\rangle \\ &\leq \|F'(v_{\underline{z}\chi})T^{\underline{z}\chi}_{jk}\|_{L^2}\|w_{\underline{z}\chi\epsilon}\|_{L^2} = O(\Gamma^{1,\frac{1}{4}}(\epsilon))^2\end{aligned}$$

where we used self-adjointness of $F'(v_{\underline{z}\chi})$ in the first equality and (3.6) and (3.10) in the last estimate. Note that by Theorem 3.1(a) and (4.19), $\pi^t_{\underline{z}\chi}F(v_{\underline{z}\chi})$ is the leading order term in (4.3). Therefore, to solve for $\underline{z}$ in (4.3), we need to investigate the solutions of

$$\pi^t_{\underline{z}\chi}F(v_{\underline{z}\chi}) = 0$$

for a configuration $\underline{z}$ more closely (and not for χ due to gauge invariance). In fact, we will find that the perturbed approximate polygonal solution $v_{\underline{z}_\sigma\chi}$ defined in (2.29) is the right candidate to solve this equation for a specific perturbation parameter $\sigma \in \mathbb{R}^{p+2q}$ defined in (2.27). We need some definitions and notation first, followed by a crucial lemma.

For the perturbation of the approximate polygonal solution $\underline{z}_\sigma$ in (2.28), we define two vortices located at z_k and $z_j \in \underline{z}_\sigma$ to be *nearest neighbors* to each other, and write "z_k n.n. $z_j \in \underline{z}_\sigma$", as follows:

$$z_k \text{ n.n. } z_j \in \underline{z}_\sigma \iff |z_k - z_j| \leq l + O(1) \quad \text{as } l \to \infty. \tag{4.20}$$

We write "z_k ~~n.n.~~ $z_j \in \underline{z}_\sigma$" if z_k and $z_j \in \underline{z}_\sigma$ are *not nearest neighbors* to each other, i.e., if (4.20) is false. In addition, it will be useful to write the multi-vortex interaction energy in terms of the sum of interaction energy between two vortices W (defined in (2.12)) plus a remainder. Hence, let's define the remainder term as

$$\Upsilon_{\underline{z}\chi} := \mathcal{E}(v_{\underline{z}\chi}) - \sum_j E^{(n_j)} - \sum_{j\neq k} W(|z_j - z_k|). \tag{4.21}$$

Finally, for compactness in notation in the statements below, denote the *effective interaction force* between two vortices of degrees n_k, n_j, at positions z_k, z_j, to be

$$\vec{\Psi}(z_j, z_k) := n_k n_j \Psi(|z_k - z_j|)\frac{z_k - z_j}{|z_k - z_j|}, \tag{4.22}$$

where Ψ is defined in (2.14).

Now, we are ready to state the following crucial lemma.

Lemma 4.3. *For the perturbation of the approximate polygonal solution $v_{\underline{z}_\sigma\chi}$ defined in (2.29), we have, as $l \to \infty$:*

$$\pi^t_{\underline{z}_\sigma\chi}F(v_{\underline{z}_\sigma\chi}) = 0 \iff \sum_{z_k \text{ n.n. } z_j \in \underline{z}_\sigma} \vec{\Psi}(z_j, z_k) + Rem_j(\underline{z}_\sigma) = 0$$
$$\textit{for all vortices } z_j \in \underline{z}_\sigma \tag{4.23}$$

where $\vec{\Psi}(z_j, z_k)$ *is defined in (4.22) and*

$$Rem_j(\underline{z}_\sigma) := \sum_{z_k \, \cancel{n.n.} \, z_j \in \underline{z}_\sigma} \vec{\Psi}(z_j, z_k) + \sum_{k \neq j} \left[\nabla_{z_j} W(|z_k - z_j|) - \vec{\Psi}(z_j, z_k) \right] + \nabla_{z_j} \Upsilon_{\underline{z}_\sigma \chi}. \tag{4.24}$$

In addition, we have

$$Rem_j(\underline{z}_\sigma) = O(e^{-l(1+\delta)}) \quad as \ l \to \infty \tag{4.25}$$

for all vortices $z_j \in \underline{z}_\sigma$.

Proof. By (4.9) and the fact the $(\beta^\chi)^{-1}$ is non-degenerate, we have

$$\begin{aligned} \pi^t_{\underline{z}_\sigma \chi} F(v_{\underline{z}_\sigma \chi}) = 0 \iff & \langle F(v_{\underline{z}_\sigma \chi}), T^{\underline{z}_\sigma \chi}_{j'k'} \rangle = 0 \quad \text{for all vortices } j', \ k' = 1, 2; \\ \iff & \nabla_{z_{j'}} \left[\mathcal{E}(v_{\underline{z}_\sigma \chi}) - \sum_j E^{(n_j)} \right] = 0 \quad \text{for all vortices } j'; \\ \iff & \nabla_{z_{j'}} \left[\sum_{j \neq k} W(|z_k - z_j|) + \Upsilon_{\underline{z}_\sigma \chi} \right] = 0 \quad \text{for all vortices } j'; \\ \iff & \sum_{z_k \text{ n.n. } z_{j'} \in \underline{z}_\sigma} \vec{\Psi}(z_{j'}, z_k) + \text{Rem}_{j'}(\underline{z}_\sigma) = 0 \quad \text{for all vortices } j'. \end{aligned} \tag{4.26}$$

In the second equivalence, we used the chain rule and the fact that our gauge transformations are of form (2.4); in the third equivalence, we used (4.21); and in the last equivalence, we used (4.24) and the fact that $\sum_{k \neq j'} = \sum_{z_k \text{n.n.} z_{j'}} + \sum_{z_k \cancel{\text{n.n.}} z_{j'}}$.

To show that $\text{Rem}_j(\underline{z}_\sigma) = O(e^{-l(1+\delta)})$, we note that the first term in (4.24) is of $O(e^{-l(1+\delta)})$ since the sum is over *non*-nearest neighbors and by definition (4.20) and form of Ψ in (2.14). The second term in (4.24) is of $O(e^{-l(1+\delta)})$ since $\nabla_{z_j} W(|z_k - z_j|) - \vec{\Psi}(z_j, z_k) = O(e^{-l(1+\delta)})$ by (2.13) and (4.22). For the last term in (4.24), we note that by (4.21), it is straightforward to show

$$\begin{aligned} \Upsilon_{\underline{z}\chi} = & \sum_{j \neq k \neq l; 1 \leq p,q,r \leq 2}^{m} \int (f_j^2 - 1)^p (f_k^2 - 1)^q (f_l^2 - 1)^r + \cdots \\ & + \sum_{j_1 \neq j_2 \neq \cdots \neq j_m; 1 \leq p_1, p_2, \ldots, p_m \leq 2}^{m} \int (f_{j_1}^2 - 1)^{p_1} (f_{j_2}^2 - 1)^{p_2} \cdots (f_{j_m}^2 - 1)^{p_m} \\ & + \sum_{j=1; j \neq k \neq l}^{m} \int (f_k^2 - 1)(f_l^2 - 1) |(\nabla_A \psi)_j|^2 \\ & + 2 \sum_{j < l; k \neq j, l}^{m} \int (f_k^2 - 1) \text{Re} \left[\overline{(\bar{\psi} \nabla_A \psi)_j} (\bar{\psi} \nabla_A \psi)_k \right]. \end{aligned} \tag{4.27}$$

Therefore, $\nabla_{z_j}\Upsilon_{\underline{z}\chi} = O(e^{-l(1+\delta)})$ for some $\delta > 0$ by (4.27), (1.5) and Lemma 12 of [13].

$\square$

With Lemma 4.3 in place, we are now in a position to write down more explicitly the form of the reduced translational projection problem (4.3)

$$\pi^t_{\underline{z}_\sigma\chi} F(v_{\underline{z}_\sigma\chi} + w_{\underline{z}_\sigma\chi^\epsilon}) = 0 \tag{4.28}$$

for perturbed approximate polygonal solution $v_{\underline{z}_\sigma\chi}$. As we will show below, due to the fact that our solution must satisfy symmetry relations in (2.11), solving (4.28) for unknown vortex positions $z_j \in \underline{z}_\sigma$ is equivalent to solving a non-linear system of $p+2q$ equations for $p+2q$ unknown perturbation parameters $\sigma = (\alpha, \beta, \gamma) \in \mathbb{R}^{p+2q}$ of $v_{\underline{z}_\sigma\chi}$ defined in (2.27) (see (4.34) below). To describe this system of $p+2q$ equations in $p+2q$ unknowns, we require a discussion first, followed by some definitions.

We first note that by (4.9) and the fact the $(\beta^\chi)^{-1}$ is non-degenerate, solving (4.28) for $\underline{z}_\sigma$ is equivalent to solving the following system of $k(p+2q)$ equations

$$\langle F(v_{\underline{z}_\sigma\chi} + w_{\underline{z}_\sigma\chi^\epsilon}), \vec{T}_j^{\underline{z}_\sigma\chi}\rangle = 0 \tag{4.29}$$

for unknown vortex positions $z_j \in \underline{z}_\sigma$ defined in (2.28). Here, we used the notation $\vec{T}_j^{\underline{z}_\sigma\chi} = \{T_{jk}^{\underline{z}_\sigma\chi}\}_{k=1,2}$. Now, due to the fact that our configuration of vortices $\underline{z}_\sigma$ must satisfy symmetries in (2.11), we only need to solve (4.29) for z_j in a "base configuration of $\underline{z}_\sigma$" defined by the set

$$\Pi := \{y_r, z_s \in \underline{z}_\sigma \mid r = 1, ..., p+1,\ s = 1, ..., q\} \tag{4.30}$$

(see discussion in between (2.26) and (2.27)). Recall that y_r and z_s are defined in (2.24) and (2.25), respectively. Therefore, solving for $\underline{z}_\sigma$ in (4.28) is equivalent to solving the $p+q+1$ equations

$$\langle F(v_{\underline{z}_\sigma\chi} + w_{\underline{z}_\sigma\chi^\epsilon}), \vec{T}_j^{\underline{z}_\sigma\chi}\rangle = 0 \quad \text{for } z_j \in \Pi. \tag{4.31}$$

To solve (4.31) for $z_j \in \Pi$, Taylor expand F around $v_{\underline{z}_\sigma\chi}$ and denoting

$$\overline{\mathrm{Rem}}_j(\underline{z}_\sigma) := \langle F'(v_{\underline{z}_\sigma\chi}) w_{\underline{z}_\sigma\chi^\epsilon} + N(v_{\underline{z}_\sigma\chi}, w_{\underline{z}_\sigma\chi^\epsilon}), \vec{T}_j^{\underline{z}_\sigma\chi}\rangle, \tag{4.32}$$

we obtain

$$0 = \sum_{z_k \text{n.n.} z_j \in \Pi} \vec{\Psi}(z_j, z_k) + \mathrm{Rem}_j(\underline{z}_\sigma) + \overline{\mathrm{Rem}}_j(\underline{z}_\sigma), \quad z_j \in \Pi, \tag{4.33}$$

where we used Lemma 4.3. Note that (4.33) is a non-linear system of $p+q+1$ equations for $p+2q$ unknown perturbation parameters $\sigma = (\alpha, \beta, \gamma) \in \mathbb{R}^{p+2q}$.

To summarize this discussion above, we have shown that solving for $z_j \in \underline{z}_\sigma$ in (4.28) is equivalent to solving for $\sigma \in \mathbb{R}^{p+2q}$ in (4.33). In Proposition 4.2 below, we will show that solving for $\sigma \in \mathbb{R}^{p+2q}$ in (4.33) is equivalent to solving for $\sigma \in \mathbb{R}^{p+2q}$ in (4.35) below. We will require some definitions before we state Proposition 4.2.

Let $T_{\bar{n}}$ denote the $\bar{n} \times \bar{n}$ invertible Toda matrix

$$T_{\bar{n}} = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix} \in \mathbb{R}^{\bar{n} \times \bar{n}}$$

with inverse given by

$$(T_{\bar{n}}^{-1})_{ij} = min(i,j) - \frac{ij}{\bar{n}+1}.$$

Define vectors

$$\alpha' = \begin{bmatrix} -\lambda_1 \alpha_1 \\ 0 \\ \vdots \\ \frac{\bar{\kappa}}{\kappa}\beta_1 - \frac{1}{\kappa}\cot\left(\frac{\pi}{k}\right)\gamma_1 - \lambda_2 \alpha_{p+1} \end{bmatrix} \in \mathbb{R}^{p+1}, \qquad \beta' = \begin{bmatrix} \sin(\pi/k)\alpha_{p+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{q-1},$$

and

$$\gamma' = \begin{bmatrix} -\frac{1}{l}\cos(\pi/k)\alpha_{p+1} \\ 0 \\ \vdots \\ -\gamma_q \end{bmatrix} \in \mathbb{R}^{q}$$

where

$$\lambda_1 = -2\frac{\bar{\kappa}}{\kappa}\sin\left(\frac{\pi}{k}\right), \qquad \lambda_2 = 1 + \frac{\bar{\kappa}}{\kappa}\sin\left(\frac{\pi}{k}\right) - \frac{1}{\kappa\bar{l}}\cot\left(\frac{\pi}{k}\right)\cos\left(\frac{\pi}{k}\right)$$

and $\kappa, \bar{\kappa} = -(log\Psi)'(l), -(log\Psi)'(\bar{l})$. Note that by (2.14), $\kappa = 1 + \frac{1}{2l} + O(l^{-2})$ as $l \to \infty$.

The explicit form of (4.28) can now be described in the following

Proposition 4.2. *Suppose $\epsilon > 0$ satisfies $0 < \Gamma^{1,\frac{1}{4}}(\epsilon) < \epsilon_0$ (defined in Theorem 3.2), $\Psi(l) < \epsilon$ and l and $\bar{l}$ satisfy the force balancing condition (2.30). Then there exists real valued functions $A_\sigma = (A_\alpha, A_\beta, A_\gamma)$ and $Q_\sigma = (Q_\alpha, Q_\beta, Q_\gamma)$ of vector $(l,p,q,\alpha,\beta,\gamma)$, uniformly bounded as $l \to \infty$, such that solving the reduced problem in (4.28) for $\underline{z}_\sigma$ is equivalent to solving the non-linear, finite dimensional system*

$$\begin{aligned} T_{p+1}\alpha + \alpha' + Q_\alpha + A_\alpha &= 0 \\ T_{q-1}\beta + \beta' + Q_\beta + A_\beta &= 0 \\ T_q\gamma + \gamma' + Q_\gamma + A_\gamma &= 0 \end{aligned} \tag{4.34}$$

for unknown free perturbation parameters $\sigma = (\alpha, \beta, \gamma) \in \mathbb{R}^{p+2q}$ with $\alpha \in \mathbb{R}^{p+1}$, $\beta \in \mathbb{R}^{q-1}$ and $\gamma \in \mathbb{R}^q$. In addition, $A_\sigma = O(e^{-\delta_2 l})$ for some $\delta_2 > 0$ and Q_σ is of at least quadratic order in α, β and γ.

Proof. Proposition 4.2 has already been proven rigourously in Proposition 5.1 of [18]. The reduced translational equation (5.4) in Proposition 5.1 of [18] is of the same form (and has the same properties) as (4.34) above, except here, we have labeled $Q_\sigma = D_\sigma$ and $A_\sigma = e^{-\delta_2 l} B_\sigma$. Therefore, we will just outline the proof of Proposition 4.2 here. At the end of the outline of proof, we will also give an idea of why conditions $k \geq 7$ and (2.30) are required in Theorem 2.1.

The basic outline of the proof of Proposition 4.2 has already been discussed after equation (4.28), and summarized in the paragraph after (4.33). We will now complete the missing steps and refer readers to the proofs of statements in [18].

Recall that that solving for $z_j \in \underline{z}_\sigma$ in (4.28) is equivalent to solving for $\sigma \in \mathbb{R}^{p+2q}$ in (4.33). Now, it was proven in Lemmas 5.2 and 5.3 in [18] that solving for $\sigma \in \mathbb{R}^{p+2q}$ in (4.33) is equivalent to solving for $\sigma \in \mathbb{R}^{p+2q}$ in (4.34). More precisely, in Lemma 5.2 of [18], they show that

$$\sum_{z_k \text{n.n.} z_j \in \Pi} \vec{\Psi}(z_j, z_k) = 0 \quad \forall z_j \in \Pi \iff \begin{array}{ll} T_{p+1}\alpha + \alpha' + Q_\alpha & = 0 \\ T_{q-1}\beta + \beta' + Q_\beta & = 0 \\ T_q\gamma + \gamma' + Q_\gamma & = 0 \end{array} \tag{4.35}$$

for some $Q_\sigma = (Q_\alpha, Q_\beta, Q_\gamma)$ of at least quadratic order in α, β, γ. In other words, they show that computing the $p+q+1$ equations for the nearest neighbor forces for each $z_j \in \Pi$ in the first term on the right hand side of (4.33) is equivalent to the $p+2q$ equations in first 3 sets of terms in system (4.34).

To show (4.35), Musso, Pacard and Wei in Lemma 5.2 of [18] consider the nearest neighbor forces on each spike/soliton in the same multi-spike/soliton configuration as our multi-vortex configuration $v_{\underline{z}_\sigma \chi}$ in (2.29). In addition, the interaction function between spikes/solitons (see equation (5.3) and Lemma 5.1 of [18]) is of the same exponential order as our effective interaction force between magnetic vortices (see (2.14)). The fact that the configuration and the interaction function of spikes/solitons in [18] are the same as the configuration and effective interaction force of magnetic vortices in this work leads to the important observation that the proof of Lemma 5.2 in [18] is the same as the proof of (4.35) here. The only difference between the proof in Lemma 5.2 in [18] and the proof of (4.35) here is that to balance the forces, we use the natural degree of the vortex to determine direction of the force instead of manually putting in the "sign" of the force for the spike or soliton. One can find some example computations that go into the proof of (4.35) at the end of this outline of proof of Proposition 4.2.

To complete the outline of proof of Proposition 4.2, we define for each $z_j \in \Pi$,

$$(\tilde{A}_\sigma)_j := \frac{1}{\Psi(l)}[\mathrm{Rem}_j(\underline{z}_\sigma) + \overline{\mathrm{Rem}}_j(\underline{z}_\sigma)], \tag{4.36}$$

where $\mathrm{Rem}_j(\underline{z}_\sigma)$ and $\overline{\mathrm{Rem}}_j(\underline{z}_\sigma)$ are defined in (4.24) and (4.32), respectively. One can show that there exists smooth functions $A_\sigma = (A_\alpha, A_\beta, A_\gamma) : \mathbb{R}^{p+1} \times$

$\mathbb{R}^{q-1} \times \mathbb{R}^q \to \mathbb{R}^{p+1} \times \mathbb{R}^{q-1} \times \mathbb{R}^q$ of vector $(l, p, q, \alpha, \beta, \gamma)$ such that

$$\begin{array}{r} \sum_{z_k \text{n.n.} z_j \in \Pi} \vec{\Psi}(z_j, z_k) \\ +\Psi(l)(\tilde{A}_\sigma)_j = 0 \\ \forall z_j \in \Pi \end{array} \iff \begin{array}{rl} T_{p+1}\alpha + \alpha' + Q_\alpha + A_\alpha & = 0 \\ T_{q-1}\beta + \beta' + Q_\beta + A_\beta & = 0 \\ T_q\gamma + \gamma' + Q_\gamma + A_\gamma & = 0. \end{array} \qquad (4.37)$$

Indeed, (4.37) was shown in Lemma 5.2 and 5.3 of [18]. Finally, by (4.25), (4.19), and since $\frac{(\Gamma^{1,\frac{1}{4}}(\epsilon))^2}{\Psi(l)} = O(e^{-\delta_2 l})$ for some $\delta_2 > 0$ (since $\Psi(l) < \epsilon$), then $(\tilde{A}_\sigma)_j = O(e^{-\delta_2 l})$ in (4.36), and hence $A_\sigma = O(e^{-\delta_2 l})$ also, which proves the last statement in Proposition 4.2.

To conclude this outline of the proof of Proposition 4.2, we will compute the nearest neighbor forces on y_r for $r = 2, \ldots, p$ and y_{p+1} on the l.h.s. of the equivalence in (4.35) to give readers an idea why (a) (4.35) is true and (b) why conditions $k \geq 7$ and (2.30) are required in Theorem 2.1. For further details, please see proof of Proposition 5.1 and Lemma 5.2 in [18].

To begin with, we Taylor expand the interaction/inter-vortex force in (2.14) as in [18]:

$$\Psi(|\tilde{l}\mathtt{e} + \mathtt{a}|)\frac{\tilde{l}\mathtt{e} + \mathtt{a}}{|\tilde{l}\mathtt{e} + \mathtt{a}|} = \Psi(\tilde{l})(\mathtt{e} - \tilde{\kappa}\mathtt{a}^{\parallel} + \tilde{l}^{-1}\mathtt{a}^{\perp} + O(|\mathtt{a}|^2)) \qquad (4.38)$$

as $\tilde{l} \to \infty$. Here, $|\mathtt{e}| = 1$ and $\mathtt{a} = \mathtt{a}^{\parallel} + \mathtt{a}^{\perp}$ where $\mathtt{a}^{\parallel}$ is parallel to $\mathtt{e}$ and $\mathtt{a}^{\perp}$ is perpendicular to $\mathtt{e}$, and $\tilde{\kappa} = -(log\Psi)'(\tilde{l})$.

For fixed $r = 2, \ldots, p$, the nearest neighbors to y_r is y_{r-1} and y_{r+1}. Since $y_{r+1} - y_r = l\mathtt{e}_1 + (\alpha_{r+1} - \alpha_r)\mathtt{e}_1$ by (2.24), then using (4.38) and Lemma 2.1, the total force on the vortex at y_r is

$$\begin{aligned} \Psi(|y_{r+1} - y_r|)(\widehat{y_{r+1} - y_r}) + \Psi(|y_{r-1} - y_r|)(\widehat{y_{r-1} - y_r}) \\ = \Psi(l)[\kappa(-\alpha_{r-1} + 2\alpha_r - \alpha_{r+1})\mathtt{e}_1 + Q_\alpha] \end{aligned} \qquad (4.39)$$

where Q_α is of quadratic order in α and $\hat{x} = \frac{x}{|x|}$ is the unit vector notation. Note that since the degrees of the vortices are all $+1$ for y_r, $r = 1, ..., p+1$ (see by (2.23)), then the sign in front of both terms in (4.39) must be positive by (2.13). Also note that if $k \leq 6$ and $r = p$, then we have to consider the force of z_1 and Λz_1 on y_p too (since they are a distance $< l + O(1)$ from y_p and so are nearest neighbors by (4.20) - see Remark 1 under Theorem 2.1). But since $k \geq 7$, then the distance between z_1 (and also Λz_1) to y_p is greater than l, and so they are not nearest neighbors and therefore we can group the contribution of these forces into the $\tilde{A}_\alpha := \Psi(l)A_\alpha = O(\Psi(l)e^{-\delta_2 l})$ term.

For y_{p+1}, the nearest neighbor forces are y_p, z_1 and Λz_1. Using (2.24) and (2.25), we have

$$\begin{aligned} y_p - y_{p+1} &= -l\mathtt{e}_1 + (\alpha_p - \alpha_{p+1})\mathtt{e}_1 \\ z_1 - y_{p+1} &= \bar{l}\mathtt{t} + (\beta_1 + \sin(\pi/k)\alpha_{p+1})\mathtt{t} + (\bar{l}\gamma_1 - \cos(\pi/k)\alpha_{p+1})\mathtt{n}, \text{ and} \\ \Lambda z_1 - y_{p+1} &= \bar{l}\Lambda\mathtt{t} + (\beta_1 + \sin(\pi/k)\alpha_{p+1})\Lambda\mathtt{t} + (\bar{l}\gamma_1 - \cos(\pi/k)\alpha_{p+1})\Lambda\mathtt{n}. \end{aligned}$$

Therefore by (4.38), the total force on y_{p+1} is

$$
\begin{aligned}
&\Psi(|y_p - y_{p+1}|)(\widehat{y_p - y_{p+1}}) - \Psi(|z_1 - y_{p+1}|)(\widehat{z_1 - y_{p+1}}) \\
&-\Psi(|\Lambda z_1 - y_{p+1}|)(\widehat{\Lambda z_1 - y_{p+1}}) \qquad (4.40)\\
&\quad = \left[\Psi(l) - 2\sin\frac{\pi}{k}\Psi(\bar{l})\right] \mathsf{e}_1 - \Psi(l)\kappa(\alpha_{p+1} - \alpha_p)\mathsf{e}_1 \\
&\quad +\Psi(\bar{l}) \left[\bar{\kappa}(\beta_1 + \sin\frac{\pi}{k}\alpha_{p+1}) \left(2\sin\frac{\pi}{k}\right) + \frac{1}{\bar{l}} \left(\bar{l}\gamma_1 - \cos\frac{\pi}{k}\alpha_{p+1}\right) \left(\cos\frac{\pi}{k}\right)\right] \mathsf{e}_1 \\
&\quad +\Psi(l)Q_\alpha. \qquad (4.41)
\end{aligned}
$$

Note that the middle two terms in (4.40) have negative signs since the vortices at z_1 and Λz_1 have degree -1 by (2.23). Now, using the force balancing condition (2.30), the leading order first term in (4.41) vanishes. This is precisely what we meant by Remark 2 under Theorem 2.1, i.e., for the forces to balance, it is crucial that the relation between l and $\bar{l}$ satisfies (2.30). □

To proceed, we need the following lemma from [18] (see Lemma 5.4 of [18] and the discussion right before this lemma).

Lemma 4.4. *Suppose all the conditions of Proposition 4.2 are satisfied. In addition, if $l, \bar{l}$ and positive integers p, q satisfy polygonal closing condition (2.31), then system (4.34) has a solution $\sigma = (\alpha, \beta, \gamma) \in \mathbb{R}^{2q+p}$ with $|\sigma| = O(e^{-\delta l})$ for some $\delta > 0$.*

Now, we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. From Theorem 3.2, there exists an $\epsilon_0 > 0$ such that for all $\epsilon > 0$ satisfying $0 < \Gamma^{1,\frac{1}{4}}(\epsilon) < \epsilon_0$ and $(\underline{z}, \chi) \in \Sigma_\epsilon$, there exists a $w_{\underline{z}\chi\epsilon} \in Ran(\pi^\perp_{\underline{z}\chi})$ such that (2.9) is true. Hence, all that remains to solve is the reduced problem (2.10) for the pair $(\underline{z}, \chi) \in \Sigma_\epsilon$. By Proposition 4.1, if there exists a multi-vortex configuration $\underline{z}$ such that $(\underline{z}, \chi) \in \Sigma_\epsilon$ solves (4.3) for $\epsilon > 0$ sufficiently small and $\chi \in H^2_{\underline{z}}(\mathbb{R}^2; \mathbb{R})$ arbitrary, then this same $(\underline{z}, \chi) \in \Sigma_\epsilon$ solves (2.10). Therefore, we are reduced to solving (4.3) for a multi-vortex configuration $\underline{z}$.

By Proposition 4.2, if $l, \bar{l}$ satisfy force balancing condition (2.30), then finding a multi-vortex configuration $\underline{z} = \underline{z}_\sigma$ to solve (4.3) is equivalent to finding perturbation parameters $\sigma = (\alpha, \beta, \gamma) \in \mathbb{R}^{p+2q}$ to solve system (4.34). By Lemma 4.4, if $l, \bar{l}$ and positve integers p, q satisfy polygonal closing condition (2.31), then system (4.34) has a solution $\sigma = (\alpha, \beta, \gamma) \in \mathbb{R}^{p+2q}$ satisfying $|\sigma| = O(e^{-\delta l})$ for some $\delta > 0$.

Therefore, by Theorem 3.2, Propositions 4.1 and 4.2 and Lemma 4.4, we have shown that there exists a $\sigma = (\alpha, \beta, \gamma) \in \mathbb{R}^{p+2q}$ satisfying $|\sigma| = O(e^{-\delta l})$ such that

$$\underline{z}_\sigma := \underline{c} + O(e^{-\delta l}) \quad \text{solves} \quad \pi_{\underline{z}_\sigma\chi} F(v_{\underline{z}_\sigma\chi} + w_{\underline{z}_\sigma\chi\epsilon}) = 0. \qquad (4.42)$$

Therefore,

$$u = v_{\underline{z}_\sigma\chi} + w_{\underline{z}_\sigma\chi\epsilon} \quad \text{solves} \quad (1.1) \text{ and } (1.2) \tag{4.43}$$

with $w = O(\epsilon \log^{1/4}(1/\epsilon))$ by (3.10) in Theorem 3.2. Also, note that solution (4.43) satisfies symmetry condition (2.11) by construction.

To end the proof of Theorem 2.1, we note a couple of technicalities. Firstly, $0 < \Gamma^{1,\frac{1}{4}}(\epsilon) < \epsilon_0$ and $(\underline{z},\chi) \in \Sigma_\epsilon$ implies that $0 < \epsilon < \epsilon_0$ and $\frac{e^{-R(\underline{z}_\sigma)}}{\sqrt{R(\underline{z}_\sigma)}} < \epsilon$. Therefore, $R(\underline{z}_\sigma) > \log\left(\frac{1}{\epsilon_0}\right) - \frac{1}{2}\log(R(\underline{z}_\sigma))$, and the condition $R(\underline{z}_\sigma) \approx l > \frac{1}{\epsilon_0} > \log\left(\frac{1}{\epsilon_0}\right) - \frac{1}{2}\log(R(\underline{z}_\sigma))$ in Theorem 2.1 guarantees that all the assumptions and conclusions of Theorem 3.2 hold. Secondly, $\underline{z}_\sigma - \underline{c} \in \mathbb{R}^{2k(p+2q)}$ is of $O(e^{-\delta l})$ in (4.42) since $\underline{z}_\sigma - \underline{c}$ contains elements of the form $\alpha_r R_k^i \mathtt{e}_1$, $\beta_s R_k^i \mathtt{t} + \bar{l}\gamma_s R_k^i \mathtt{n}$ for $r = 1, ..., p+1$, $s = 1, \ldots, 2q-1$, $i = 0, \ldots, k-1$ (see (2.21), (2.24), (2.25) and (2.28)) and $|\sigma| = O(e^{-\delta l})$. $\square$

A Proof of Lemma 2.1 and Theorem 3.1(d)

In this Appendix, we prove Lemma 2.1 and Theorem 3.1 (d). Denote $[u]_{\psi,A}$ as the complex and vector components of $u \in L^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$.

Proof of Lemma 2.1. This Lemma is proven in Lemma 11 of [13] so we will just summarize the proof.

For

$$W(\underline{z}) := \mathcal{E}(v_{\underline{z}\chi}) - \sum_{j=1}^{m} E^{(n_j)},$$

[13] showed that for *fixed* l, and since the integral of of over $\mathbb{R}^2$ of three nonoverlapping widely spaced vortices is of $O(e^{-R(\underline{z})(1+\delta)})$ for some $\delta > 0$, we have

$$\partial_{z_{lm}} W(\underline{z}) = \langle \mathcal{E}'(v_{\underline{z}\chi}), T_{lm}^{\underline{z}\chi} \rangle = \langle E^{\underline{z}\chi}, T_{lm}^{\underline{z}\chi} \rangle + O(e^{-R(\underline{z})(1+\delta)}) \tag{A.1}$$

where

$$E^{\underline{z}\chi} = \begin{pmatrix} -e^{-i\chi} \sum_{j\neq k} \left(\prod_{l\neq j,k} \psi_l\right) (\nabla_A \psi)_j (\nabla_A \psi)_k \\ \sum_{j=1}^m \left[\sum_{k\neq j}^m (1 - f_k^2)\right] j^{(n_j)} \end{pmatrix}$$

and $T_{lm}^{\underline{z}\chi}$ is defined in (2.6). Therefore,

$$\begin{aligned} \langle [E^{\underline{z}\chi}]_\psi, [T_{lm}^{\underline{z}\chi}]_\psi \rangle &= n_l \sum_{k\neq l} n_k \int_{\mathbb{R}^2} \left(\frac{(1-a)f^2}{r}\right)(|x - z_k|) \left(\frac{2(1-a)ff'}{r}\right)(|x - z_l|) \\ &\quad \times \frac{x - z_k}{|x - z_k|} + O\left(e^{-R(\underline{z})(1+\delta)}\right) \end{aligned} \tag{A.2}$$

and

$$\begin{aligned}\langle [E^{\underline{z}\chi}]_A, [T^{\underline{z}\chi}_{lm}]_A\rangle &= n_l \sum_{k\neq l} n_k \int_{\mathbb{R}^2} \left(\frac{(1-a)f^2}{r}\right)(|x-z_k|)\left(\frac{a'(1-f^2)}{r}\right)(|x-z_l|) \\ &\quad \times \frac{x-z_k}{|x-z_k|} + O\left(e^{-R(\underline{z})(1+\delta)}\right). \end{aligned} \tag{A.3}$$

Note that in the expression for $\langle [E^{\underline{z}\chi}]_\psi, [T^{\underline{z}\chi}_{lm}]_\psi\rangle$ differs from the one in [13] by 2 since the sum $\sum_{j\neq k}$ is symmetric in j and k in the expression for $[E^{\underline{z}\chi}]_\psi$.

Now, using the fact that $|\nabla \times B| = (1-a)f^2/r$ by (1.2) and $(-\Delta+1)B = (2(1-a)ff' + a'(1-f^2))/r$ by straight forward computation (see also Lemma 7 in [13]), we see that by (A.1) to (A.3),

$$\begin{aligned}\partial_{z_l} W(\underline{z}) &= n_l \sum_{k\neq l} n_k \int_{\mathbb{R}^2} |\nabla\times B|(|x-z_k|)[(-\Delta+1)B](|x-z_l|)\frac{x-z_k}{|x-z_k|} \\ &\quad + O\left(e^{-R(\underline{z})(1+\delta)}\right). \end{aligned}$$

If we define

$$\Psi(|z_k - z_l|) := \int_{\mathbb{R}^2} |\nabla \times B|(|x-z_k|)[(-\Delta+1)B](|x-z_l|)\,dx,$$

then using (1.2), the first equation in (1.5), $|(-\Delta+1)B| \leq e^{-m_\lambda r}$ and Lemma 13 of [13], we have our result (specializing to just a two vortex configuration case). □

Now we prove Theorem 3.1 (d). This is proven already for $\lambda > 1/2$ in Lemma 3 of [13], however, there is a slight modification in it, and therefore, we give the main ideas of how to reprove it here.

Proof of Theorem 3.1 *(d).* It is straight forward to check that the decomposition of $L_{\underline{z}\chi}$ is

$$L_{\underline{z}\chi} = L_j + L^{1/2}_{(j)} + V_{(j)} \tag{A.4}$$

with

$$L_j := \mathcal{E}''(g_{\chi_{(j)}} u^{(n_j)}(x-z_j)),$$

$L^{1/2}_{(j)}$ is a first order differential operator at $g_{\chi_{(j)}} u^{(n_j)}(x-z_j)$ given by

$$L^{1/2}_{(j)}(g_{\chi_{(j)}} u^{(n_j)}(x-z_j)) \begin{pmatrix} \xi \\ B \end{pmatrix} = \begin{pmatrix} -2i\left[\Theta_{(j)}\right]_A \cdot \nabla\xi + \left[\Theta_{(j)}\right]_\psi \nabla\cdot B \\ \overline{\left[\Theta_{(j)}\right]_\psi}\nabla\xi \end{pmatrix} \tag{A.5}$$

and

$$\Theta_j(x) := v_{\underline{z}\chi} - g_{\chi_{(j)}} u^{(n_j)}(x-z_j) \tag{A.6}$$

with $\chi_{(j)} := \chi + \sum_{k\neq j}\theta(\cdot - z_k)$ and $V_{(j)}$, $\Theta_{(j)}$ are multiplication operators satisfying

$$\|V_{(j)}\|_\infty, \|\Theta_{(j)}\|_\infty \leq \sum_{k\neq j} e^{-\min(1,m_\lambda)|x-z_k|}. \tag{A.7}$$

You can see that (A.4) is the correctly modified decomposition of $L_{\underline{z}\chi} = F'(v_{\underline{z}\chi})$ by looking at (3.13). For example, for the decomposition of the covariant Laplacian $\Delta_{A_{\underline{z}\chi}}$, we write $A_{\underline{z}\chi} = C + D$ where $C = A^{(n_j)}(x - z_j) + \nabla\chi_{(j)}$ and $D = A_{\underline{z}\chi} - (A^{(n_j)}(x - z_j) + \nabla\chi_{(j)})$ and note

$$\Delta_{C+D} = \underbrace{\Delta_C}_{L_{(j)}} - \underbrace{2iD \cdot \nabla}_{L_{(j)}^{1/2}} - i\underbrace{[(\nabla \cdot D) - 2iC \cdot D - i\|D\|^2]}_{V_{(j)}}.$$

The rest of the proof of Theorem 3.1 (d) then follows in a similar manner as that of the proof of Lemma 3 of [13] using a the IMS formula from [6] (note this formula still holds for operators of the form $L = \Delta + \nabla + V$ for V a multiplication operator). Note also that with the modified decomposition in (A.4), the rest of the proof of Lemma 3 in [13] and the estimate in (3.6) still holds due to the fact that $L_{(j)}^{1/2}$ in (A.5) has the form $\Theta_{(j)}\nabla$ and $\Theta_{(j)}$ decays like (A.7). $\square$

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