INTERFACE FOLIATION OF A POSITIVELY CURVED MANIFOLD NEAR A CLOSED GEODESIC

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Abstract. Let $(\mathcal{M}, g)$ be a two-dimensional compact Riemannian manifold. We consider the singularly perturbed Allen-Cahn equation

$$\varepsilon^2 \Delta_g u + (1-u^2)u = 0 \quad \text{in } \mathcal{M},$$

where ε is a small parameter. Assume that $\Gamma \subset M$ is a simple closed geodesic that separates $\mathcal{M}$ into two disjoint components. Assume that Γ is non-degenerate in the sense that it does not support non-trivial Jacobi fields, and that Gaussian curvature of $\mathcal{M}$ is positive along Γ. Then for any fixed integer $N \geq 2$, we show the existence of a solution u_ϵ with N-transition layers near Γ with mutual distance $O(\varepsilon|\ln \varepsilon|)$, provided that ε stays away from a discrete set of values at which resonance occurs.

1. Introduction

In the gradient theory of phase transitions by Allen-Cahn [2], two phases of a material, +1and −1 coexist in a region $\Omega \subset \mathbb{R}^n$ separated by an $(n-1)$-dimensional interface. The phase is idealized as a smooth ε-regularization of the discrete function, which is selected as a critical point of the energy

$$I_\varepsilon(u) = \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{4\varepsilon}(1-u^2)^2$$

where $\varepsilon > 0$ is a small parameter. While any function with values ± 1 minimize exactly the second term, the presence of the gradient term conveys a balance in which the interface is selected asymptotically as stationary for perimeter. In [19] it is established that a sequence of local minimizers u_ε, with uniformly bounded energy, must converge in L^1_{loc}-sense to a function of the form $\chi_E - \chi_{E^c}$ so that ∂E locally minimizes perimeter. This is the starting point of the Γ-*convergence theory*, in which the constraint of I_ε to a suitable class of separating-phase functions, converges to the perimeter function of the interface. In reality, analogous assertions hold true for general families of critical points, and for stronger notions of interface convergence, see [5, 23, 25]. The principle described above and its variations apply to modeling phase transition phenomena in many contexts: material science, superconductivity, population dynamics and biological pattern formation, see for instance [24] and references therein.

It is natural to consider situations in which pattern formation takes place in a manifold rather than in a subset of Euclidean space. We consider a compact two-dimensional Riemannian manifold $(\mathcal{M}, g)$, and want to investigate critical points in $H^1(\mathcal{M})$ of the functional

$$J_\varepsilon(u) = \int_{\mathcal{M}} \varepsilon|\nabla_g u|^2 + \frac{1}{4\varepsilon}(1-u^2)^2,$$

in connection with geodesic arclength for interfaces. Critical points of J_ε correspond precisely to classical solutions of the Allen-Cahn equation in $\mathcal{M}$,

$$\varepsilon^2 \Delta_g u + (1-u^2)u = 0 \quad \text{in } \mathcal{M}, \tag{1.1}$$

where Δ_g is the Laplace-Beltrami operator on M. In [9] oval surfaces embedded in $\mathbb{R}^3$ are considered and it is shown that interfaces of local minimizers with uniformly bounded energies converge in suitable sense to geodesics. In [22], Pacard and Ritoré considered the n-dimensional case in (1.1)

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and proved that given a minimal $(n-1)$-submanifold Γ which divides $\mathcal{M}$ into two components $\mathcal{M}_\pm$, and is non-degenerate, in the sense that the Jacobi operator associated to Γ is non-singular, there exists a solution u_ε to (1.1) with values close to ± 1 in $\mathcal{M}_\pm$, whose 0-level set lies ε-close to Γ. More precisely, letting w be the unique solution of the ODE Finally, let $\mathrm{w}(x) = \tanh\left(\frac{x}{\sqrt{2}}\right)$ be the unique heteroclinic solution of

$$\mathrm{w}'' + \mathrm{w} - \mathrm{w}^3 = 0 \quad \text{in } \mathbb{R},\ \mathrm{w}(0) = 0,\ \mathrm{w}(\pm\infty) = \pm 1. \tag{1.2}$$

Then the solution u_ε in [22] resembles near Γ the function $w(t/\varepsilon)$, where t is a choice of signed distance to Γ. In particular

$$J_\varepsilon(u_\varepsilon) \;\to\; |\Gamma| \int_{\mathbb{R}} \frac{1}{2}|\mathrm{w}'|^2 + \frac{1}{4}(1-\mathrm{w}^2)^2.$$

In this paper, we shall describe a new phenomenon in the two-dimensional case, induced by the presence of curvature: If in addition to non-degeneracy of the geodesic Γ, it is assumed that *Gauss curvature of* $\mathcal{M}$ *is positive along* Γ, then solutions u_ε with any given number $N \geq 2$ of interfaces foliating Γ exist. This solution has *multiplicity* N in the sense that

$$J_\varepsilon(u_\varepsilon) \;\to\; N\,|\Gamma| \int_{\mathbb{R}} \frac{1}{2}|\mathrm{w}'|^2 + \frac{1}{4}(1-\mathrm{w}^2)^2.$$

Our result is valid under a *nonresonance condition* in ε. In fact, in our construction ε must remain suitably away from a sequence of values where a shift in Morse index occurs. We expect that the solutions we find have a large, ε-dependent Morse index.

To state precisely our main result, let us introduce some notation. In the rest of this paper $\Gamma \subset \mathcal{M}$ will be a fixed closed, embedded geodesic, divides $\mathcal{M}$ into two disjoint components. With no loss of generality we assume that $|\Gamma| = 2\pi$, and consider a natural parameterization $\gamma(\theta)$ of Γ with positive orientation, where θ denotes arclength parameter measured from a fixed point of Γ. Let K denote Gauss curvature on $\mathcal{M}$. We assume that

$$k(\theta) \;:=\; K(\gamma(\theta)) \;>0 \quad \text{for all } \theta \in [0, 2\pi). \tag{1.3}$$

Let $\{\Lambda_i\}$ denote the increasing sequence eigenvalues of the problem:

$$\begin{aligned} \varphi'' + k(\theta)\varphi \;&=\; -\Lambda\, k(\theta)\,\varphi, \\ \varphi(0) \;=\; \varphi(2\pi), &\quad \varphi'(0) \;=\; \varphi'(2\pi). \end{aligned} \tag{1.4}$$

The operator in the left hand side of (1.4) can be interpreted as the linearization of the geodesic curvature operator around the curve Γ. We will assume that this operator is non-degenerate, namely that

$$\Lambda_i \neq 0 \quad \text{for all } i = 1, 2, \ldots. \tag{1.5}$$

In other words, we assume that there is no non-trivial Jacobi field defined along Γ, so that Γ is an isolated closed geodesic in $\mathcal{M}$. We fix in what follows numbers $N \geq 2$, $c > 0$, and assume that ε satisfies that the *gap condition*

$$\min_{j=1,\ldots,N-1} \left| -\Lambda_i + (N-j)j \log\frac{1}{\varepsilon}\right| > c\sqrt{\log\frac{1}{\varepsilon}}, \quad \forall\, i = 1, 2, \ldots. \tag{1.6}$$

As we will see in §2, this condition can be written explicitly for small ε, and amounts precisely to ε staying away from a explicit sequence of values.

Theorem 1.1. *We assume that conditions* (1.3) *and* (1.5) *are satisfied. Then, for any fixed integer* $N \geq 2$ *and any sufficiently small* $\varepsilon > 0$ *satisfying the gap condition* (1.6)*, problem* (1.1) *has a solution* u_ε *with* N *phase transition layers with mutual distance* $O(\varepsilon|\ln\varepsilon|)$*. Near* Γ*,* u_ε *can be approximated by*

$$u_\varepsilon \approx \sum_{k=1}^{N} \mathrm{w}\left(\frac{t-\varepsilon f_j(\theta)}{\varepsilon}\right) \,+\, \frac{1}{2}\left((-1)^{N+1} \,-\, 1\right),$$

where t is a choice of signed distance to Γ along a normal direction ν_0. The functions f_j satisfy

$$\|f_j\|_\infty \;\le\; C|\ln\varepsilon|, \quad f_{j+1}-f_j = O(|\ln\varepsilon|), \quad 1\le j\le N-1, \tag{1.7}$$

and solve the Toda system,

$$\varepsilon^2\big(f_j'' + k(\theta) f_j\big) \;-\; a_0\left[\,e^{-(f_j-f_{j-1})} \;-\; e^{-(f_{j+1}-f_j)}\,\right] \;=\; 0 \quad \text{in } (0,2\pi), \tag{1.8}$$

$$f_j(0) = f_j(2\pi), \qquad f_j'(0) = f_j'(2\pi), \tag{1.9}$$

for $j = 1,\dots,N$, for a universal constant $a_0>0$, with the conventions $f_0=-\infty$, $f_{N+1}=\infty$.

Our result deals with situations in which the geodesic is local but not globally length minimizing. In fact, since condition (1.3) holds, problem (1.4) has at least one negative eigenvalue, and near Γ, $\mathcal{M}$ cannot have parabolic points. In the case of a bounded domain Ω of $\mathbb{R}^2$ under Neumann boundary conditions, a multiple-layer solution near a non-minimizing straight segment orthogonal to the boundary was built in [8]. In ODE cases for the Allen-Cahn equation, clustering interfaces had been previously observed in [6, 20, 21]. No resonance phenomenon is present in those situations, constituting a major qualitative difference with the current setting.

We do not expect that interface foliation occurs only if the limiting interface is a minimizer of the length. On the other hand, negative Gauss curvature seems also prevent interface foliation. This is suggested by a version of De Giorgi-Gibbons conjecture for problem (1.1) with $\mathcal{M}$ the hyperbolic space, established in [3].

The gap condition (1.6) is *unexpected.* It reflects a resonance phenomena caused by the interaction of multiple layers and curvature. Similar resonance has been observed in (simple) concentration phenomena for various problems, see [7, 12, 15, 16]. The phenomenon of clustering of interfaces here discovered has an interesting resemblance with the problem of foliations of a neighborhood of a geodesic by CMC tubes considered in [13, 17].

2. The ansatz

2.1. **Preliminaries.** We will clarify next the meaning of the gap condition (7.18). Given a positive periodic function k, let us consider the eigenvalue problem

$$\begin{aligned} \varphi'' + k(\theta)\varphi \;&=\; -\Lambda\, k(\theta)\,\varphi \quad \text{in } (0,2\pi),\\ \varphi(0) \;=\; \varphi(2\pi), &\quad \varphi'(0) \;=\; \varphi'(2\pi). \end{aligned} \tag{2.1}$$

By the following Liouville transformation

$$\ell_0 = \int_0^{2\pi} \sqrt{k(\theta)}\,\mathrm{d}\,\theta, \qquad t = \frac{\pi}{\ell_0}\int_0^{\theta} \sqrt{k(\theta)}\,\mathrm{d}\,\theta,\ t\in[0,\pi),$$

$$\Psi(\theta) = k(\theta)^{-\frac14}, \qquad e(t) = \Theta(\theta)/\,\Psi, \qquad q(t) = \frac{\ell_0^2\,\Psi''}{\pi^2\,\Psi^2\,k(\theta)},$$

the eigenvalue Λ satisfies the following eigenvalue problem

$$-e'' - q(t)\,e = \frac{\ell_0^2}{\pi^2}\,(1+\Lambda)\,e \quad \text{ in } (0,\pi),\ e(0) \;=\; e(\pi),\ e'(0) = e'(\pi).$$

Now consider the following auxiliary eigenvalue problem

$$-y'' - q(t)y = \xi y \quad \text{ in } (0,\pi),\ y(0) \;=\; y(\pi),\ y'(0) = y'(\pi).$$

The result in [11] shows that, as $i\to\infty$

$$\sqrt{\xi_i} \;=\; 2i \;+\; O(\frac{1}{i^3}).$$

Hence, the eigenvalues of the problem (2.1) have the following asymptotic formula, as $i \to \infty$,

$$\begin{aligned}\Lambda_i &= \frac{\pi^2}{\ell_0^2}\Big[2i + O(\frac{1}{i^3})\Big]^2 - 1\\ &= \frac{4\pi^2 i^2 - \ell_0^2}{\ell_0^2} + O(\frac{1}{i^2}).\end{aligned}$$

The last formula plus trivial computation will imply that we can choose a sequence of small ε which tends to 0 such that the gap condition (1.6) holds.

2.2. **Local coordinate.** Let M be a two dimensional smooth manifold without boundary with Riemannian metric $\tilde{g}$. Assume that Γ is a simple closed geodesic with total length 2π, contained in M in such a way that Γ separates M into two disjoint components. We consider natural parameterization $\gamma(\theta)$ of Γ with positive orientation, where θ denotes unit arclength parameter measured from a fixed point of Γ. Let $\nu(\theta) \in T_{\gamma(\theta)}M$ denote the unit normal to Γ. Given Γ, there exists a small number $\delta_0 > 0$, such that we can define Fermi coordinates $\Phi_0 : [-\delta_0, \delta_0] \times S^1 \to M$, in a neighborhood of Γ:

$$\Phi_0(t,\theta) = \exp_{\gamma(\theta)}\big(t\,\nu(\theta)\big), \quad |t| < \delta_0,\ \theta \in [0, 2\pi), \tag{2.2}$$

where exp is the exponential map on M. Near Γ the metric $\tilde{g}_{t,\theta}$ has form:

$$\tilde{g}_{t,\theta} = \mathrm{d}t^2 + E(t,\theta)\mathrm{d}\theta^2,$$

for some smooth function $E(t,\theta)$ satisfying

$$E(0,\theta) = 1, \quad \forall\,\theta \in S^1. \tag{2.3}$$

Moreover the Gauss curvature assumes the simple expression

$$K(t,\theta) = \frac{-1}{\sqrt{E}}\frac{\partial^2\sqrt{E}}{\partial t^2}. \tag{2.4}$$

The Christoffel symbols of $\tilde{g}$ are given by

$$\tilde{\Gamma}^l_{ij} = \frac{1}{2}\,\tilde{g}^{kl}\Big[\partial_i\tilde{g}_{kj} + \partial_j\tilde{g}_{ki} - \partial_k\tilde{g}_{ij}\Big] \quad \text{where} \quad \partial_1 = \frac{\partial}{\partial t}, \quad \partial_2 = \frac{\partial}{\partial\theta}.$$

Since $\tilde{g}_{ij}(0,\theta) = \delta_{ij}$, then one finds

$$\tilde{\Gamma}^l_{ij}(0,\theta) = \frac{1}{2}\Big[\partial_i\tilde{g}_{jl} + \partial_j\tilde{g}_{li} - \partial_l\tilde{g}_{ij}\Big].$$

Given two vector fields X and Y, Y being possibly defined along an integral curve of X, one has

$$\nabla_X Y = \Big(\frac{\partial Y^i}{\partial X^j} + \tilde{\Gamma}^i_{jk}X^jX^k\Big)\frac{\partial}{\partial_i}.$$

We now turn to the computation of second derivative of E in (t,θ) coordinates. Since Γ is a geodesic, we have

$$\nabla_{(0,1)}(0,1) = 0,$$

where $(0,1) = (0, \frac{\partial}{\partial\theta})$ represents the vector field $\dot{\gamma}$ in the coordinates (t,θ) above. As a consequence, one finds

$$\tilde{\Gamma}^i_{11}(0,\theta) = 0, \quad \forall\,\theta \in S^1 \quad \text{and } i = 1, 2,$$

which implies

$$\frac{\partial E}{\partial t}(0,\theta) = \frac{\partial\tilde{g}_{22}}{\partial t}(0,\theta) = 0, \quad \forall\,\theta \in S^1. \tag{2.5}$$

This equation also follows from the fact that, in Fermi coordinates, the geodesic curvature of the curve $\{t = 0\}$ is given by

$$-\frac{1}{\sqrt{E}}\frac{\partial\sqrt{E}}{\partial t}.$$

In particular, from (2.4) and (2.5) it follows that

$$K(0,\theta) \;=\; -\frac{1}{2}\frac{\partial^2 E}{\partial t^2}\,(0,\theta) \;=\; -\frac{1}{2}\frac{\partial^2 \tilde g_{22}}{\partial t^2}\,(0,\theta), \quad \forall\,\theta\in S^1. \tag{2.6}$$

We also introduce stretched Fermi coordinates defined by

$$\Phi_\varepsilon(x,z)=\frac{1}{\varepsilon}\Phi_0(\varepsilon x,\varepsilon z), \quad (x,z)\in(-\frac{\delta_0}{\varepsilon},\frac{\delta_0}{\varepsilon})\times\frac{1}{\varepsilon}S^1. \tag{2.7}$$

Obviously, the new coefficients g_{ij}'s of the Riemannian metric, after rescaling, can be written as

$$g_{ij}(x,z)=\tilde g_{ij}(\varepsilon x,\varepsilon z), \quad i,j=1,2.$$

Expanding all coefficients g_{ij} up to order ε^2, and taking into account (2.3), (2.5) and (2.6), we find

$$g_{22}=1-\varepsilon^2\,x^2\,K(0,\varepsilon z)+O(\varepsilon^3|x|^3), \quad g_{11}=1, \quad g_{21}=g_{12}=0.$$

The entries of the inverse matrix become

$$g^{22}=1+\varepsilon^2\,x^2\,K(0,\varepsilon z)+O(\varepsilon^3|x|^3), \quad g^{11}=1, \quad g^{21}=g^{12}=0.$$

If we denote by G, the determinant of metric matrix g, then

$$G=1-\varepsilon^2\,x^2\,K(0,\varepsilon z)+O(\varepsilon^3|x|^3).$$

The Laplace-Beltrami operator can be written as

$$\begin{aligned}\Delta_g &= \frac{1}{\sqrt{G}}\,\partial_i\Big(\,g^{ij}\,\sqrt{G}\,\partial_j\,\Big)\\ &= \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+\varepsilon^2x^2K(0,\varepsilon z)\frac{\partial^2}{\partial z^2}-\varepsilon^2x\frac{K(0,\varepsilon z)}{G}\,\frac{\partial}{\partial x}+\varepsilon^3B_2(x,\varepsilon z)\\ &\equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\;+\varepsilon^2B_1(x,\varepsilon z)+\varepsilon^3B_2(x,\varepsilon z),\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}B_1(x,\varepsilon z)&=K(0,\varepsilon z)\Big(x^2\,\frac{\partial^2}{\partial z^2}-x\frac{\partial}{\partial x}\Big),\\ B_2(x,\varepsilon z)&=b_{21}(x,\varepsilon z)\frac{\partial^2}{\partial z^2}+b_{22}(x,\varepsilon z)\frac{\partial}{\partial x}+b_{23}(x,\varepsilon z)\frac{\partial}{\partial z},\end{aligned} \tag{2.9}$$

and functions b_{2n}, $n=1,2,3$ satisfy:

$$|b_{2n}(x,\varepsilon z)|\le C(1+|x|^3).$$

2.3. **The approximate solution.** If we set $u(y)=\tilde u(\varepsilon y)$, then problem (1.1) is thus equivalent to

$$\Delta_g\,u\,+\,F(u)=0 \quad \text{in } M_\varepsilon, \tag{2.10}$$

where $F(u)\,\equiv\,u\,-\,u^3$ and, in the sequel, we will use M_ε and Γ_ε to denote the manifold M and the geodesic Γ after scaling.

To define the approximate solution we recall some basic properties of the heteroclinic solution to (1.2) in the following

$$\begin{aligned}\mathrm{w}(x)\,-\,1 &= -A_0\,e^{-\sqrt{2}\,|x|}\,+\,O\big(e^{-2\sqrt{2}\,|x|}\big), \quad x>1,\\ \mathrm{w}(x)\,+\,1 &= A_0\,e^{-\sqrt{2}\,|x|}\,+\,O\big(e^{-2\sqrt{2}\,|x|}\big), \quad x<-1,\\ \mathrm{w}'(x) &= \sqrt{2}\,A_0\,e^{-\sqrt{2}\,|x|}\,+\,O\big(e^{-2\sqrt{2}\,|x|}\big), \quad |x|>1,\end{aligned} \tag{2.11}$$

where A_0 is a universal constant. For a fixed integer $N \ge 2$, we assume that the location of the N phase transition layers are characterized by periodic functions $x = f_j(\varepsilon z)$, $1 \le j \le N$ in the coordinate (x, z) defined in (2.7). These functions can be defined as follows

$$f_j : (0, 2\pi) \to \mathbb{R}, \tag{2.12}$$

$$||f_j||_{\mathrm{w}^2(0,2\pi)} < C\,|\ln \varepsilon|^2, \tag{2.13}$$

$$f_{j+1}(\zeta) - f_j(\zeta) > \sqrt{2}\,|\ln \varepsilon| - 4\sqrt{2}\,\ln|\ln \varepsilon|. \tag{2.14}$$

For convenience of the notation we will also set

$$f_0(\zeta) = -\infty \ \text{ and } \ f_{N+1}(\zeta) = \infty.$$

Set

$$\mathrm{w}_j(x, z) \equiv (-1)^{j+1}\,\mathrm{w}\big(x - f_j(\varepsilon z)\big),$$

and define the approximate solution of (2.10) by

$$u_0(x, z) \equiv \sum_{j=1}^{N} \mathrm{w}_j(x, z) + \frac{1}{2}\left((-1)^{N+1} - 1\right).$$

Our first goal is to compute the error of approximation in a δ_0/ε neighborhood of Γ_ε, namely the quantity:

$$E_0 \equiv \Delta_g\, u_0 + F(u_0). \tag{2.15}$$

¿From (2.8), it is derived that

$$\Delta_g\, u_0 = \sum_{j=1}^{N} \mathrm{w}_{j,xx} - \varepsilon^2 \sum_{j=1}^{N} \Big((1 + \varepsilon^2 x^2 k) f_j'' + xk\Big) \mathrm{w}_{j,x} + \varepsilon^2 \sum_{j=1}^{N} (1 + \varepsilon^2 x^2 k)(f_j')^2 \mathrm{w}_{j,xx} + O(\varepsilon^3), \tag{2.16}$$

where, from the assumption (1.3) on manifold near the geodesic Γ,

$$k(\varepsilon z) = K(0, \varepsilon z) > 0. \tag{2.17}$$

We now turn to computing other nonlinear terms in E_0. For every fixed n, $1 \le n \le N$, we consider the following set

$$A_n = \left\{ (x, z) \in \left(-\frac{\delta_0}{\varepsilon}, \frac{\delta_0}{\varepsilon}\right) \times \left(0, \frac{2\pi}{\varepsilon}\right) \Big| \ \frac{f_{n-1}(\varepsilon z) + f_n(\varepsilon z)}{2} \le x \le \frac{f_n(\varepsilon z) + f_{n+1}(\varepsilon z)}{2} \right\}. \tag{2.18}$$

For $(x, z) \in A_n$, we write

$$\begin{aligned} F(u_0) &= F(\mathrm{w}_n) + F'(\mathrm{w}_n)(u_0 - \mathrm{w}_n) + \frac{1}{2} F''(\mathrm{w}_n)(u_0 - \mathrm{w}_n)^2 + \max_{j \ne n} O(e^{-3\sqrt{2}\,|f_j - x|}) \\ &= \sum_{j=1}^{N} F(\mathrm{w}_j) + \Big[F'(\mathrm{w}_n)(u_0 - \mathrm{w}_n) - \sum_{j \ne n} F(\mathrm{w}_j) \Big] \\ &\qquad + \frac{1}{2} F''(\mathrm{w}_n)(u_0 - \mathrm{w}_n)^2 + \max_{j \ne n} O(e^{-3\sqrt{2}\,|f_j - x|}). \end{aligned}$$

Following similar computations in [8], we obtain, for $(x, z) \in A_n$, $n = 1, \ldots, N$

$$\begin{aligned} F(u_0) = &\sum_{j=1}^{N} F(\mathrm{w}_j) + \frac{1}{2} F''(\mathrm{w}_n)(u_0 - \mathrm{w}_n)^2 + 3(1 - \mathrm{w}_n^2)(u_0 - \mathrm{w}_n) \\ &- \frac{1}{2} \sum_{j \ne n} F''(\sigma_{nj})(\sigma_{nj} - \mathrm{w}_j)^2 + \max_{j \ne n} O(e^{-3\sqrt{2}\,|f_j - x|}). \end{aligned} \tag{2.19}$$

In the above formula we define σ_{nj} as follows: if n is even, $\sigma_{nj} = (-1)^j$ for $j < n$ and $\sigma_{nj} = (-1)^{j+1}$ for $j > n$; if n is odd, $\sigma_{nj} = (-1)^{j+1}$ for $j < n$ and $\sigma_{nj} = (-1)^j$ for $j > n$.

It follows then for $(x,z)\in A_n$, $n=1,\dots,N$:

$$\begin{aligned} E_0 = & -\varepsilon^2\sum_{j=1}^N f_j''\mathrm{w}_{j,x} + \varepsilon^2\sum_{j=1}^N (1+\varepsilon^2x^2k)(f_j')^2\mathrm{w}_{j,xx} \\ & -\varepsilon^4x^2k(\varepsilon z)\sum_{j=1}^N f_j''\mathrm{w}_{j,xx} - \varepsilon^2 k(\varepsilon z)x\sum_{j=1}^N \mathrm{w}_{j,x} \\ & + 3(1-\mathrm{w}_n^2)(u_0-\mathrm{w}_n) + \frac{1}{2}F''(\mathrm{w}_n)(u_0-\mathrm{w}_n)^2 - \frac{1}{2}F''(\sigma_{nj})(\sigma_{nj}-\mathrm{w}_j)^2 \\ & + \max_{j\neq n} O(e^{-3\sqrt{2}\,|f_j-x|}) + \max_j O(\varepsilon^3)(|x|^4+|f_j'|^2+1)|\mathrm{w}_{j,x}|. \end{aligned} \tag{2.20}$$

Therefore, the following lemma on the accuracy of the first error is readily checked.

Lemma 2.1. *There exists a $q_0>0$ such that we have the following estimate*

$$\big\|E_0\big\|_{L^2((-\frac{\delta_0}{\varepsilon},\frac{\delta_0}{\varepsilon})\times\frac{1}{\varepsilon}S^1)} \le C\,\varepsilon^{3/2}|\ln\varepsilon|^{q_0}. \tag{2.21}$$

We observe that using (2.14) one can derive that $q_0=8$. We will not need such a precision in the sequel and for simplification we will denote by $q\ge q_0$ a generic exponent whose value may change from line to line.

3. The gluing procedure

In this section, we use a gluing technique (as in [8], see also [7]) to reduce the problem in M_ε to the infinite strip $\mathfrak{S}$, where:

$$\mathfrak{S}\equiv\mathbb{R}\times\frac{1}{\varepsilon}S^1. \tag{3.1}$$

Let $\delta<\delta_0/100$ be a fixed number, where δ_0 is a constant defined in (2.2). We consider a smooth cut-off function $\eta_\delta(t)$ where $t\in\mathbb{R}_+$ such that $\eta_\delta(t)=1$ for $0\le t\le\delta$ and $\eta(t)=0$ for $t>2\delta$. Set $\eta_\delta^\varepsilon(x)=\eta_\delta(\varepsilon|x|)$, where x is the normal coordinate to Γ_ε. Let $u_0(x,z)$ denote the approximate solution constructed near the curve Γ_ε in the coordinates (x,z), which was introduced in (2.7). We define our first global approximation by

$$W(y) \;=\; \begin{cases} \eta_{3\delta}^\varepsilon(x)(u_0+1)-1, & \text{for } x<0,\\ \eta_{3\delta}^\varepsilon(x)(u_0-(-1)^{N+1})+(-1)^{N+1}, & \text{for } x>0. \end{cases}$$

Notice that W is defined in the whole manifold M_ε. For $u=W+\hat\phi$ where $\hat\phi$ globally defined in M_ε, denote

$$\Upsilon(u)=\Delta_g u\,+\,u\,(1-u^2),\quad \text{in } M_\varepsilon.$$

Then u satisfies (2.10) if and only if

$$\widetilde{\mathcal{L}}(\hat\phi)\;=\;-\widetilde{E}\,+\,\widetilde{N}(\hat\phi),\quad \text{in } M_\varepsilon. \tag{3.2}$$

where

$$\mathcal{L}(\hat\phi)=\Delta_g\hat\phi\,+\,\big(1-3W^2\big)\hat\phi,\quad \widetilde{N}(\hat\phi)=\hat\phi^3\,+\,3W\hat\phi^2,\quad \widetilde{E}=\Upsilon(W).$$

We will look for $\hat\phi$ in the following form

$$\hat\phi\;=\;\eta_{3\delta}^\varepsilon\phi\,+\,\psi,$$

where, in the coordinates (x,z) in (2.7), we assume that ϕ is defined in the strip $\mathfrak{S}$. Now, let $\widetilde{\mathcal{L}}$ be an extension of the operator $\mathcal{L}$ defined on the whole strip $\mathfrak{S}$. More specifically we set:

$$\widetilde{\mathcal{L}}(\phi)=\eta_{6\delta}^\varepsilon\big[\Delta_g\phi\,+\,\big(1-3W^2\big)\phi\big]+(1-\eta_{6\delta}^\varepsilon)(\Delta\phi-2\phi). \tag{3.3}$$

where $\Delta = \partial_x^2 + \partial_z^2$. With this definition $\hat{\phi}$ is a solution of (3.2) if the pair (ϕ, ψ) satisfies the following coupled system:

$$\begin{aligned} \widetilde{\mathcal{L}}(\phi) &= \eta_\delta^\varepsilon \Big[\widetilde{N}(\eta_{3\delta}^\varepsilon \phi + \psi) - \widetilde{E} - 3(1-W^2)\psi \Big], \qquad (3.4)\\ \Delta_g \psi - 2\psi + 3(1-\eta_\delta^\varepsilon)(1-W^2)\psi &= -(\Delta_g \eta_{3\delta}^\varepsilon)\phi \,-\, 2(\nabla_g \eta_{3\delta}^\varepsilon)\cdot(\nabla_g \phi) \\ &\quad + (1-\eta_\delta^\varepsilon)\widetilde{N}(\eta_{3\delta}^\varepsilon \phi + \psi) \,-\, (1-\eta_\delta^\varepsilon)\widetilde{E}, \qquad (3.5) \end{aligned}$$

where ϕ is defined globally on $\mathfrak{S}$ and ψ is defined in M_ε.

The key observation is that, after solving (3.5), the problem can be transformed to the following nonlinear, nonlocal problem involving $\psi = \psi(\phi)$

$$\widetilde{\mathcal{L}}(\phi) = \eta_\delta^\varepsilon \Big[\widetilde{N}(\eta_{3\delta}^\varepsilon \phi + \psi) - \widetilde{E} - 3(1-W^2)\psi \Big] \qquad (3.6)$$

To solve (3.6) we set up a fixed point argument by first solving (3.5) for a given ϕ. We will assume that ϕ satisfies the following decay property

$$\big|\nabla\phi(y)\big| + \big|\phi(y)\big| \le e^{-\tau/\varepsilon} \ \text{ if } \ |x| > \delta/\varepsilon, \qquad (3.7)$$

for certain constant $\tau > 0$. Let us observe that $1 - W^2$ is exponentially small for $|x| > \delta/\varepsilon$, where x is the normal coordinate to Γ_ε. Then the problem

$$\Delta_g \psi - 2\psi + 3(1-\eta_\delta^\varepsilon)(1-W^2)\psi = h \quad \text{in } M_\varepsilon,$$

has a unique bounded solution ψ whenever $||h||_\infty \le +\infty$. Moreover,

$$||\psi||_\infty \le C||h||_\infty.$$

Since $\widetilde{N}$ is power-like with power greater than one, a direct application of contraction mapping principle yields that (3.5) has a unique (small) solution $\psi = \psi(\phi)$ with

$$||\psi(\phi)||_{L^\infty} \le C\varepsilon\big[\ ||\phi||_{L^\infty(|x|>\delta/\varepsilon)} + ||\nabla\phi||_{L^\infty(|x|>\delta/\varepsilon)} + e^{-\tau/\varepsilon}\ \big], \qquad (3.8)$$

where $|x| > \delta/\varepsilon$ denotes the complement in M_ε of δ/ε-neighborhood of Γ_ε. Moreover, the nonlinear operator ψ satisfies a Lipschitz condition of the form

$$||\psi(\phi_1) - \psi(\phi_2)||_{L^\infty} \le C\varepsilon\big[\ ||\phi_1 - \phi_2||_{L^\infty(|x|>\delta/\varepsilon)} + ||\nabla\phi_1 - \nabla\phi_2||_{L^\infty(|x|>\delta/\varepsilon)}\ \big]. \qquad (3.9)$$

Detailed argument will be postponed until the proof of Proposition 5.1.

¿From the above discussion, the full problem has been reduced to solving (3.6) for $\phi \in H^2(\mathfrak{S})$ satisfying condition (3.7).

Rather than solving problem (3.6), we deal with the following projected problem: given $\mathbf{f} = (f_1, \ldots, f_N)$ satisfying (2.12)-(2.14), finding functions $\phi \in H^2(\mathfrak{S})$, $\mathbf{c} = (c_1, \ldots, c_N)$ with $c_j \in L^2(0, 2\pi/\varepsilon)$ such that

$$\widetilde{\mathcal{L}}(\phi) = \eta_\delta^\varepsilon \Big[\widetilde{N}(\eta_{3\delta}^\varepsilon \phi + \psi) - \widetilde{E} - 3(1-W^2)\psi \Big] + \sum_{j=1}^N \ c_j(z)\chi_j(x,z)\,\mathrm{w}_{j,x}, \qquad (3.10)$$

$$\int_{\mathbb{R}} \phi(x,z)\mathrm{w}_{j,x}(x)\chi_j(x,z)\,\mathrm{d}x = 0, \quad 0 < z < 2\pi/\varepsilon,\ j = 1, \ldots, N. \qquad (3.11)$$

The smooth, positive cut-off functions χ_j are of the form:

$$\chi_j(x,z) = \eta_a^b\Big(\frac{x - f_j(\varepsilon z)}{\log\frac{1}{\varepsilon}}\Big), \ \text{where } \ a = \sqrt{2}\frac{2^6-1}{2^7},\ b = \sqrt{2}\frac{2^7-1}{2^8},\ \eta_a^b(t) = \begin{cases} 1, & |t| < a \\ 0, & |t| > b, \end{cases} \qquad (3.12)$$

We notice that with this choice $\chi_j\chi_i \equiv 0$, for $i \neq j$, provided that ε is taken sufficiently small.

In Proposition 5.1, we will prove that this problem has a unique solution ϕ whose norm is controlled by the L^2-norm of E_2. Moreover, ϕ will satisfies (3.7). After this has been done, our task is to choose suitable parameters f_j's, possessing all properties in (2.12)-(2.14), such that the function $\mathbf{c}$ is identically zero. It is equivalent to solving a nonlocal, nonlinear second order differential equation for the unknown $\mathbf{f}$ under periodic boundary conditions.

4. **Linear Theory**

This section will be devoted to the resolution of the basic linear problem. Given functions $h \in L^2(\mathfrak{S})$, we consider the problem of finding $\phi \in H^2(\mathfrak{S})$ such that for certain functions $c_j \in L^2(0, 2\pi/\varepsilon)$, $j = 1, \dots, N$, we have

$$\widetilde{\mathcal{L}}(\phi) = h + \sum_{j=1}^{N} c_j(z)\chi_j(x,z)\mathrm{w}_{j,x}, \quad \text{in } \mathfrak{S}, \tag{4.1}$$

$$\phi(x,0) = \phi(x, 2\pi/\varepsilon), \quad \phi_z(x,0) = \phi_z(x, 2\pi/\varepsilon), \quad -\infty < x < \infty, \tag{4.2}$$

$$\int_{\mathbb{R}} \phi(x,z)\, \mathrm{w}_{j,x}(x,z)\chi_j(x,z)\, \mathrm{d}x = 0, \quad j = 1, \dots, N, \quad 0 < z < \frac{2\pi}{\varepsilon}. \tag{4.3}$$

Our main result in this section is the following.

Proposition 4.1. *There exists a constant $C > 0$, independent of ε and uniform for the parameters $\mathbf{f}$ in (2.12)-(2.14) such that for all small ε problem (4.1)-(4.3) has a solution $\phi = T_{\mathbf{f}}(h)$, which defines a linear operator of its arguments and satisfies the estimate*

$$\|\phi\|_{H^2(\mathfrak{S})} \le C\, \|h\|_{L^2(\mathfrak{S})}.$$

□

For the proof of Proposition 4.1, we need to show existence result for a simpler problem. Let us define the linear operator

$$L_0(\phi) = \phi_{xx} + \phi_{zz} + (1 - 3\mathrm{w}^2)\phi,$$

and consider the problem: given $h \in L^2(\mathfrak{S})$, finding functions $\phi \in \mathrm{w}^2(\mathfrak{S})$ and $c \in L^2(0, 2\pi/\varepsilon)$ to

$$L_0(\phi) = h + c(z)\chi(x)\mathrm{w}_x \quad \text{in } \mathfrak{S}, \tag{4.4}$$

$$\phi(x,0) = \phi(x, 2\pi/\varepsilon), \quad \phi_z(x,0) = \phi_z(x, 2\pi/\varepsilon), \quad -\infty < x < \infty, \tag{4.5}$$

$$\int_{\mathbb{R}} \phi(x,z)\, \mathrm{w}_x(x)\chi(x)\, \mathrm{d}x = 0, \quad 0 < z < \frac{2\pi}{\varepsilon}, \tag{4.6}$$

where $\chi(x) = \eta_a^b(x)$, and η_a^b is the function in (3.12).

Lemma 4.2. *Problem (4.4)-(4.6) possesses a unique solution, denoted by $(c, \phi) = T_0(h)$. Moreover, we have*

$$\begin{aligned} ||c\,\mathrm{w}_x||_{L^2(\mathfrak{S})} &\le C\|h\|_{L^2(\mathfrak{S})}, \\ \|\phi\|_{H^2(\mathfrak{S})} &\le C\|h\|_{L^2(\mathfrak{S})}. \end{aligned}$$

Proof. We will first prove an apriori estimate for (4.4)-(4.6). To this end let ϕ be a solution of (4.4)-(4.6). We observe that for the purpose of the a priori estimate we can assume that $c \equiv 0$ in (4.4). Let us consider Fourier series decompositions for h and ϕ of the form

$$\phi(x,z) = \sum_{k=0}^{\infty} \phi_k(x)e^{ik\varepsilon z}, \quad h(x,z) = \sum_{k=0}^{\infty} h_k(x)e^{ik\varepsilon z}.$$

Then we have the validity of the equations

$$-\,k^2\varepsilon^2\phi_k + \mathcal{L}_0(\phi_k) = h_k, \quad x \in \mathbb{R}, \tag{4.7}$$

and conditions

$$\int_{\mathbb{R}} \phi_k\, \mathrm{w}_x\chi(x)\, \mathrm{d}x = 0, \tag{4.8}$$

for all k. We have denoted here

$$\mathcal{L}_0(\phi_k) = \phi_{k,xx} + F'(\mathrm{w}(x))\phi_k.$$

Let us consider the bilinear form in $H^1(\mathbb{R})$ associated to the operator $\mathcal{L}_0$, namely

$$B(\psi,\psi)=\int_{\mathbb{R}}\left[|\psi_x|^2-F'(\mathrm{w})|\psi|^2\right]dx\ .$$

Since (4.8) holds uniformly in k we conclude that

$$C[\|\phi_k\|^2_{L^2(\mathbb{R})}+\|\phi_{k,x}\|^2_{L^2(\mathbb{R})}]\ \le\ B(\phi_k,\phi_k) \tag{4.9}$$

for a constant $C>0$ independent of k. Using this fact and equation (4.7) we find the estimate

$$(1+k^4\varepsilon^4)\|\phi_k\|^2_{L^2(\mathbb{R})}+\|\phi_{k,x}\|^2_{L^2(\mathbb{R})}\ \le\ C\|h_k\|^2_{L^2(\mathbb{R})}.$$

In particular, we see from (4.7) that ϕ_k satisfies an equation of the form

$$\phi_{k,xx}-2\phi_k=\tilde h_k,\quad x\in\mathbb{R}.$$

where $\|\tilde h_k\|_{L^2(\mathbb{R})}\le C\|h_k\|_{L^2(\mathbb{R})}$. Hence it follows that additionally we have the estimate

$$\|\phi_{k,xx}\|^2_{L^2(\mathbb{R})}\ \le\ C\|h_k\|^2_{L^2(\mathbb{R})}. \tag{4.10}$$

Adding up estimates (4.9), (4.10) in k we conclude that

$$\|D^2\phi\|^2_{L^2(\mathfrak{S})}+\|D\phi\|^2_{L^2(\mathfrak{S})}+\|\phi\|^2_{L^2(\mathfrak{S})}\ \le\ C\|h\|^2_{L^2(\mathfrak{S})}, \tag{4.11}$$

which ends the proof in the case $c\equiv 0$. To prove the general case we multiply equation (4.4) by $\mathrm{w}_x\chi(x)$ and use (4.6). This yields:

$$\begin{aligned}c(z)\int_{\mathbb{R}}\mathrm{w}_x^2\chi^2(x)\,\mathrm{d}x&=\int_{\mathbb{R}}\mathcal{L}_0(\phi)\mathrm{w}_x\chi\,\mathrm{d}x-\int_{\mathbb{R}}h\mathrm{w}_x\chi\,\mathrm{d}x\\&=\int_{\mathbb{R}}(\mathrm{w}_x\chi_{xx}+2\mathrm{w}_{xx}\chi_x)\phi\,\mathrm{d}x-\int_{\mathbb{R}}h\mathrm{w}_x\chi\,\mathrm{d}x,\end{aligned}$$

hence

$$\|c\mathrm{w}_x\|_{L^2(\mathfrak{S})}\le C\varepsilon^\mu\|\phi\|_{L^2(\mathfrak{S})}+C\|h\|_{L^2(\mathfrak{S})}, \tag{4.12}$$

where $\mu\in(0,1)$. Taking ε sufficiently small and using (4.11) we get the required a priori estimates in the general case.

The existence part of the Lemma follows from standard Fredholm alternative argument. The proof is completed.

□

Now for each fixed j we define:

$$\mathcal{L}^j(\phi)=\eta^\varepsilon_{6\delta}\big[\Delta_g\phi\ +\ \big(1-3\mathrm{w}_j^2\big)\phi\big]+(1-\eta^\varepsilon_{6\delta})(\Delta\phi-2\phi). \tag{4.13}$$

and consider the following problem

$$\begin{aligned}&\mathcal{L}^j(\phi)=h\ +\ c_j(\varepsilon z)\mathrm{w}_{j,x}\chi_j(x,z),\quad \text{in }\mathfrak{S},\\&\phi(x,0)=\phi(x,2\pi/\varepsilon),\quad \phi_z(x,0)\ =\ \phi_z(x,2\pi/\varepsilon),\quad -\infty<x<\infty,\\&\int_{\mathbb{R}}\phi(x,z)\mathrm{w}_{j,x}(x)\chi_j(x,z)\mathrm{d}x=0,\quad 0<z<\frac{2\pi}{\varepsilon}.\end{aligned} \tag{4.14}$$

We have

Lemma 4.3. *Problem (4.14) possesses a unique solution (c_j,ϕ). Moreover,*

$$\begin{aligned}||c_j\,\mathrm{w}_{j,x}||_{L^2(\mathfrak{S})}&\le C\|h\|_{L^2(\mathfrak{S})},\\ \|\phi\|_{H^2(\mathfrak{S})}&\le C\|h\|_{L^2(\mathfrak{S})}.\end{aligned}$$

Proof. We recall that $\mathrm{w}_j = \mathrm{w}\big(x - f_j(\varepsilon z)\big)$ defined in $\mathfrak{S}$ and denote below

$$\tilde{\xi}(x,z) \,=\, \xi\big(x + f_j(\varepsilon z), z\big).$$

Direct computation gives that problem (4.14) is equivalent to

$$\begin{aligned} \tilde\phi_{xx} + \tilde\phi_{zz} + \tilde B_0(\tilde\phi) + \varepsilon^2 \tilde B_1(\tilde\phi) + \varepsilon^3 \tilde B_2(\tilde\phi) + (1-3\mathrm{w}^2)\tilde\phi + 3(1-\tilde\eta^{\varepsilon}_{6\delta})(1-\mathrm{w}^2)\tilde\phi& \\ = \tilde h \,+\, c_j(z)\,\mathrm{w}_x \chi(x) \quad \text{in } \mathfrak{S},& \qquad (4.15)\\ \tilde\phi(x,0) \,=\, \tilde\phi(x, 2\pi/\varepsilon), \quad \tilde\phi_z(x,0) \,=\, \tilde\phi_z(x,2\pi/\varepsilon), \quad -\infty < x < \infty,& \qquad (4.16)\\ \int_{\mathbb{R}} \tilde\phi\, \mathrm{w}_x \chi(x)\,\mathrm{d}x \,=0, \quad 0 < z < \frac{2\pi}{\varepsilon}\,,& \qquad (4.17) \end{aligned}$$

where

$$\tilde B_0(\tilde\phi) \quad = \quad \varepsilon^2 \Big(f_j'(\varepsilon z)\Big)^2 \tilde\phi_{xx} \,-\, \varepsilon^2 f_j''(\varepsilon z)\,\tilde\phi_x \,-\, 2\varepsilon f'(\varepsilon z)\,\tilde\phi_{xz},$$

and $\tilde B_1$, $\tilde B_2$ are second order differential operators of the form:

$$\begin{aligned} \tilde B_1(x,\varepsilon z) &= \eta^{\varepsilon}_{6\delta}\Big[\big(x + f_j(\varepsilon z)\big)^2 K(0,\varepsilon z)\Big(\varepsilon^2[f_j'(\varepsilon z)]^2\partial_{xx} - \varepsilon^2 f_j''(\varepsilon z)\partial_x - 2\varepsilon f'(\varepsilon z)\partial_{xz}\Big) \\ &\quad - \big(x + f_j(\varepsilon z)\big)K(0,\varepsilon z)\partial_x\Big], \\ \tilde B_2(x,\varepsilon z) &= \eta^{\varepsilon}_{6\delta}\Big[\tilde b_{21}(x,\varepsilon z)\Big(\varepsilon^2[f_j'(\varepsilon z)]^2\partial_{xx} - \varepsilon^2 f_j''(\varepsilon z)\partial_x - 2\varepsilon f'(\varepsilon z)\partial_{xz}\Big) \\ &\quad + \tilde b_{22}(x,\varepsilon z)\partial x + \tilde b_{23}(x,\varepsilon z)\Big(\varepsilon f_j'(\varepsilon z)\partial x + \partial z\Big)\Big], \end{aligned} \qquad (4.18)$$

and functions $\tilde b_{2k}$, $k = 1,2,3$ satisfy:

$$|b_{2k}(x,\varepsilon z)| \le C\big(1 + (|x| + |f_j(\varepsilon z)|)^3\big),$$

(see (2.9)). Let us set:

$$\mathtt{B}(\tilde\phi) = \tilde B_0(\tilde\phi) + \varepsilon^2 \tilde B_1(\tilde\phi) + \varepsilon^3 \tilde B_2(\tilde\phi) + 3(1-\tilde\eta^{\varepsilon}_{6\delta})(1-\mathrm{w}^2)\tilde\phi.$$

With these notations (4.15)-(4.17) is equivalent to the fixed point linear problem

$$\tilde\phi \,=\, T_0\big(\tilde h + \mathtt{B}(\tilde\phi)\big),$$

where T_0 is the linear operator defined by Lemma 4.2. The linear operator $\mathtt{B}$ ia small in the sense that

$$\|\mathtt{B}(\tilde\phi)\|_{L^2(\mathfrak{S})} \,\le\, C\delta \|\tilde\phi\|_{\mathrm{w}^2(\mathfrak{S})},$$

with δ is small. From this, unique solvability of the problem and the desired estimate immediately follow. □

Proof of Proposition 4.1. We will first define some cut-off functions that will be important in the sequel. To this end let $\eta^b_a(s)$ be a smooth function with $\eta^b_a(s) = 1$ for $|s| < a$ and $= 0$ for $|s| > b$, where $0 < a < b < 1$. Then, with $R = \log\frac{1}{\varepsilon}$, and $\mathtt{x}_j = x - f_j(\varepsilon z)$ we set

$$\eta_j(x,z) = \eta^b_a\Big(\frac{|\mathtt{x}_j|}{R}\Big), \quad a = \sqrt{2}\frac{2^7-1}{2^8}, \quad b = \sqrt{2}\frac{2^8-1}{2^9}, \qquad (4.19)$$

(c.f. 3.12). We search for a solution of $\phi = T(h)$ to problem (4.1)-(4.3) in the form

$$\phi \,=\, \psi \,+\, \sum_{j=1}^{N} \eta_j\,\bar\phi_j, \qquad (4.20)$$

¿From the definition of the functions η_j, χ_j we have

$$\eta_j\chi_j = \chi_j, \quad \chi_j \nabla\eta_j = \chi_j\Delta\eta_j = 0. \qquad (4.21)$$

We will denote

$$\bar{\chi} = 1 - \sum_{j=1}^{N} \eta_j.$$

It is readily checked that ϕ, given by (4.20), solves problem (4.1)-(4.3) if the functions $\bar{\phi}_j, j = 1, \cdots, N$, and ψ satisfy the following linear system of equations, for $j = 1, \cdots, N$,

$$\begin{aligned}
&\mathcal{L}^j(\bar{\phi}_j) = \eta_j\big(h - \psi + 3W^2\psi\big) + \chi_j\, c_j(\varepsilon z)\, \mathrm{w}_{j,x} + 3\eta_j\, \big(W^2 - \mathrm{w}_j^2\big)\bar{\phi}_j \quad \text{in } \mathfrak{S}, \\
&\bar{\phi}_j(x,0) = \bar{\phi}_j(x, 2\pi/\varepsilon), \quad \bar{\phi}_{j,z}(x,0) = \bar{\phi}_{j,z}(x, 2\pi/\varepsilon), \quad -\infty < x < \infty, \\
&\int_{\mathbb{R}} (\bar{\phi}_j + \chi_j\psi)\, \mathrm{w}_{j,x}\chi_j \, \mathrm{d}x = 0, \quad 0 < z < \frac{2\pi}{\varepsilon},
\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}
\psi_{xx} + \psi_{zz} + \bar{\chi}\eta^{\varepsilon}_{6\delta}\big(1 - 3W^2\big)\psi - 2(1 - \eta^{\varepsilon}_{6\delta})\psi = \bar{\chi}h - \sum_{j=1}^{N} \big(\, 2\nabla\eta_j \cdot \nabla\bar{\phi}_j + \bar{\phi}_j\Delta\eta_j\,\big), \\
\psi(x,0) = \psi(x, 2\pi/\varepsilon), \quad \psi_z(x,0) = \psi_z(x, 2\pi/\varepsilon), \quad -\infty < x < \infty,
\end{aligned} \tag{4.23}$$

(see (3.3). In order to solve this system we will set up a fixed point argument. Observe that the orthogonality condition in (4.22) is satisfied for $\bar{\phi}_j + \chi_j\psi$ rather than $\bar{\phi}_j$, hence it is convenient to introduce new variable $\tilde{\phi}_j = \bar{\phi}_j + \chi_j\psi$. Then combining (4.22) and (4.23) we get the following system for $\tilde{\phi}_j$

$$\begin{aligned}
\mathcal{L}^j(\tilde{\phi}_j) &= \eta_j\big(h - \psi + 3W^2\psi\big) + \chi_j\, c_j(\varepsilon z)\, \mathrm{w}_{j,x} + 3\eta_j\, \big(W^2 - \mathrm{w}_j^2\big)\bar{\phi}_j \\
&\quad + \chi_j\mathcal{L}^j(\psi) + \psi\Delta_g\chi_j + 2\nabla_g\psi \cdot \nabla_g\chi_j, \quad \text{in } \mathfrak{S}, \\
\tilde{\phi}_j(x,0) &= \tilde{\phi}_j(x, 2\pi/\varepsilon), \quad \tilde{\phi}_{j,z}(x,0) = \tilde{\phi}_{j,z}(x, 2\pi/\varepsilon), \quad -\infty < x < \infty, \\
&\int_{\mathbb{R}} \tilde{\phi}_j \mathrm{w}_{j,x}\chi_j \, \mathrm{d}x = 0, \quad 0 < z < \frac{2\pi}{\varepsilon},
\end{aligned} \tag{4.24}$$

To solve (4.24) we assume that functions $\bar{\Phi}_j, j = 1, \cdots, N$, and $\tilde{\Psi}$ are given. First we replace $\bar{\phi}_j$, ψ by $\bar{\Phi}_j$, $\tilde{\Psi}$ on the right hand sides of (4.24) and solve (4.24) for each $\bar{\phi}_j$, $j = 1, \ldots, N$, using Lemma 4.3. We get the following estimates, for all $j = 1, \cdots, N$

$$\|\bar{\phi}_j\|_{\mathrm{w}^2(\mathfrak{S})} \quad \le \quad C\Big[\, \|h\|_{L^2(\mathfrak{S})} + \|\tilde{\Psi}\|_{\mathrm{w}^2(\mathfrak{S})} \Big] + o(1) \sum_{j=1}^{N} \|\bar{\Phi}_j\|_{\mathrm{w}^2(\mathfrak{S})}, \tag{4.25}$$

as $\varepsilon \to 0$. Given $\tilde{\Psi}$ we can now find functions $\bar{\phi}_j = \bar{\phi}_j(\tilde{\Psi})$ which solve (4.24) by a fixed point argument. Next, we can now solve (4.23) for ψ which in addition satisfies

$$\|\psi\|_{H^2(\mathfrak{S})} \le C\|h\|_{L^2(\mathfrak{S})} + \frac{C}{R} \sum_{j=1}^{N} \|\bar{\Phi}_j(\tilde{\Psi})\|_{H^2(\mathfrak{S})}, \quad R = \log\frac{1}{\varepsilon}. \tag{4.26}$$

Combining this with (4.25), taking ε small, and applying a fixed point argument again we get finally a solution to (4.23). This ends the proof. □

5. **Solving the Nonlinear Problem**

For future references we recall that from the estimates in Lemma 2.1, $\tilde{E}_2 \equiv \tilde{E}$ is of order $O(\varepsilon^{3/2}|\ln\varepsilon|^2)$. This fact will be important in for a contraction mapping argument in the proof of the following:

Proposition 5.1. *There exist numbers $D > 0$, $\gamma_0 > 0$, q, such that for all sufficiently small ε and all $\mathbf{f}$ satisfying (2.12)-(2.14) problem (3.10)-(3.11) has a unique solution $\phi = \phi(\mathbf{f})$ which satisfies*

$$\|\phi\|_{H^2(\mathfrak{S})} \leq D\varepsilon^{\frac{3}{2}} |\ln \varepsilon|^q,$$
$$\|\phi\|_{L^\infty(|x|>\delta/\varepsilon)} + \|\nabla\phi\|_{L^\infty(|x|>\delta/\varepsilon)} \leq e^{-\gamma_0\delta/\varepsilon}.$$

Besides ϕ is a Lipschitz function of $\mathbf{f}$, and for given $\mathbf{f}_n : (0, 2\pi) \to \mathbb{R}^N$, $n = 1, 2$ such that:

$$\|\mathbf{f}_1 - \mathbf{f}_2\|_{H^2(0,2\pi)} \leq \frac{|\log \varepsilon|}{2^{12}}, \tag{5.1}$$

it holds

$$\|\phi(\mathbf{f}_1) - \phi(\mathbf{f}_2)\|_{H^2(\mathfrak{S})} \leq C\varepsilon |\log \varepsilon|^q \|\mathbf{f}_1 - \mathbf{f}_2\|_{H^2(0,2\pi)}. \tag{5.2}$$

Proof. Let $T_{\mathbf{f}}$ be the operator defined in Proposition 4.1. Given $\mathbf{f}$ in (2.12)-(2.14), the equation (3.10)-(3.11) is equivalent to the fixed point problem for ϕ:

$$\phi = T_{\mathbf{f}}(h) \tag{5.3}$$

with

$$h = \eta^\varepsilon_\delta \Big[\widetilde{N}(\eta^\varepsilon_{3\delta}\phi + \psi) - \widetilde{E} - 3(1 - W^2)\psi\Big], \tag{5.4}$$

and

$$\begin{aligned} \Delta_g \psi - 2\psi + 3(1 - \eta^\varepsilon_\delta)(1 - W^2)\psi &= -(\Delta_g \eta^\varepsilon_{3\delta})\phi - 2(\nabla_g \eta^\varepsilon_{3\delta}) \cdot (\nabla_g \phi) \\ &\quad + (1 - \eta^\varepsilon_\delta)\widetilde{N}(\eta^\varepsilon_{3\delta}\phi + \psi) - (1 - \eta^\varepsilon_\delta)\widetilde{E} \\ &\equiv A(\phi, \psi) - (1 - \eta^\varepsilon_\delta)\widetilde{E}, \end{aligned} \tag{5.5}$$

where ϕ is defined globally on $\mathfrak{S}$ and ψ is defined in M_ε (see (3.5)).

We will define now the region where contraction mapping principle applies. Let q_0 be the constant in (2.21). We consider the following closed, bounded subset of $H^2(\mathfrak{S})$:

$$\mathcal{B} = \left\{ \phi \in H^2(\mathfrak{S}) \;\middle|\; \begin{array}{l} \|\phi\|_{H^2(\mathfrak{S})} \leq D\varepsilon^{\frac{3}{2}} |\ln \varepsilon|^{q_0}, \\ \|\phi\|_{L^\infty(|x|>\delta/\varepsilon)} + \|\nabla\phi\|_{L^\infty(|x|>\delta/\varepsilon)} \leq e^{-\gamma_0\delta/\varepsilon} \end{array} \right\}$$

and claim that there are constants $D, \gamma_0 > 0$ such that the map $T_{\mathbf{f}}$ defined in (5.3) is a contraction from $\mathcal{B}$ into itself, uniform with respect to $\mathbf{f}$. Given $\tilde\phi \in \mathcal{B}$ we denote $\phi = T_{\mathbf{f}}(\tilde\phi)$. Notice that (5.5) can be solved using a fixed point argument. Indeed, assuming $\tilde\phi \in \mathcal{B}$ we get:

$$\begin{aligned} \|A(\tilde\phi, \psi)\|_{L^\infty(M_\varepsilon)} &= \|A(\tilde\phi, \psi)\|_{L^\infty(M_\varepsilon \cap \{|x|>\delta/\varepsilon\})} \leq C\varepsilon e^{-\gamma_0\delta/\varepsilon} + C\|\psi\|^2_{L^\infty(M_\varepsilon)}, \\ \|(1 - \eta^\varepsilon_\delta)\widetilde{E}\|_{L^\infty(M_\varepsilon)} &= \|\widetilde{E}\|_{L^\infty(M_\varepsilon \cap \{|x|>\delta/\varepsilon\})} \leq Ce^{-\frac{\sqrt{2}}{2}\delta/\varepsilon}. \end{aligned} \tag{5.6}$$

Using these estimates existence of a unique solution of (5.5) such that

$$\|\psi\|_{L^\infty(M_\varepsilon)} \leq C\varepsilon e^{-\gamma_0\delta/\varepsilon}, \tag{5.7}$$

with $\gamma_0 < \frac{\sqrt{2}}{2}$ can be proven by a standard argument. By $\psi(\tilde\phi)$ we denote the solution of (5.5) with ϕ replaced by $\tilde\phi$, and similarly by $\tilde h$ we will denote $h(\tilde\phi)$ (see (5.4)). ¿From this and Lemma 2.1 we get

$$\begin{aligned} \|\phi\|_{H^2(\mathfrak{S})} &= \|T_{\mathbf{f}}(\tilde h)\|_{H^2(\mathfrak{S})} \\ &\leq \|\tilde E\|_{L^2(\mathfrak{S})} + C\|\tilde\phi\|^2_{H^2(\mathfrak{S})} + C\|\psi\|^2_{H^2(\mathfrak{S})} + \|(1 - W^2)\psi\|_{L^2(\mathfrak{S})} \\ &\leq \|\tilde E\|_{L^2(\mathfrak{S})} + C\varepsilon^3 |\log \varepsilon|^{2q_0} + C\varepsilon^{1/2} e^{-\gamma_0\delta/\varepsilon}. \end{aligned} \tag{5.8}$$

Using Lemma 2.1 and Proposition 4.1 we find

$$\|\phi\|_{H^2(\mathfrak{S})} \leq D\varepsilon^{\frac{3}{2}} |\log \varepsilon|^{q_0}, \tag{5.9}$$

with certain constant $D>0$. In addition we have that

$$\|\tilde E\|_{L^\infty(|x|>\delta/\varepsilon)} \le Ce^{-\frac{\sqrt2}{2}\delta/\varepsilon},$$
$$\|(1-W^2)\psi\|_{L^\infty(|x|>\delta/\varepsilon)} \le C\|\psi\|_{L^\infty(M_\varepsilon)}e^{-\frac{\sqrt2}{2}\delta/\varepsilon},$$

hence, using (5.9) and comparison principle we get:

$$\|\phi\|_{L^\infty(|x|>\delta/\varepsilon)}+\|\nabla\phi\|_{L^\infty(|x|>\delta/\varepsilon)} \le Ce^{-2\gamma_0\delta/\varepsilon}+Ce^{-\frac{\sqrt2}{2}\delta/\varepsilon}. \tag{5.10}$$

Combining (5.9) and (5.10) with a straightforward contraction mapping argument for the operator $T_{\mathbf f}$ we conclude the proof. □

We will now analyze the dependence of the solution ϕ found above as a fixed point of the mapping $T_{\mathbf f}$ on the parameter $\mathbf f$. We will denote $\phi=\phi(\mathbf f)$ whenever convenient. We will consider periodic functions $\mathbf f_n:(0,2\pi)\to\mathbb R^N$, $n=1,2$ such that (5.1) holds. A tedious but straightforward analysis of all terms involved in the differential operator and in the error yield that the operator $T_{\mathbf f}(\phi)$ is continuous with respect to $\mathbf f$. Indeed, indicating now the dependence on $\mathbf f$ of the linear operator $\tilde{\mathcal L}$ as well, let us make the following decomposition:

$$\tilde{\mathcal L}_{\mathbf f_1}(\,\phi(\mathbf f_1)\,)-\tilde{\mathcal L}_{\mathbf f_2}(\,\phi(\mathbf f_2)\,) = \tilde{\mathcal L}_{\mathbf f_1}\big(\phi(\mathbf f_1)-\phi(\mathbf f_2)\big) + \big[\,F'(W(\mathbf f_1))-F'(W(\mathbf f_2))\,\big]\phi(\mathbf f_2).$$

Above, and in what follows $\mathbf f_n=\mathbf f_n(\varepsilon z)$. We will denote

$$\bar\phi=\phi(\mathbf f_1)-\phi(\mathbf f_2)+\sum_{j=1}^N\frac{\mathrm w_{j,x}(\mathbf f_1)\chi_j(\mathbf f_1)}{\displaystyle\int_{\mathbb R}\mathrm w_{j,x}(\mathbf f_1)\chi_j(\mathbf f_1)\,\mathrm dx}\int_{\mathbb R}\phi(\mathbf f_2)\mathrm w_{j,x}(\mathbf f_1)\chi_j(\mathbf f_1)\mathrm dx.$$

With these notation we have that $\bar\phi$ satisfies:

$$\begin{aligned}&\tilde{\mathcal L}_{\mathbf f_1}\bar\phi=\mathcal A\big(\mathbf f_1,\mathbf f_2,\phi(\mathbf f_1),\phi(\mathbf f_2)\big)+\sum_{j=1}^N\bar c_j\mathrm w_{j,x}(\mathbf f_1),\\&\int_{\mathbb R}\bar\phi\mathrm w_{j,x}(\mathbf f_1)\chi_j(\mathbf f_1)\,\mathrm dx=0,\end{aligned}\tag{5.11}$$

where

$$\bar c_j=c_j(\mathbf f_1)\chi_j(\mathbf f_1)-c_j(\mathbf f_2)\chi_j(\mathbf f_2),$$

and

$$\begin{aligned}\mathcal A\big(\mathbf f_1,\mathbf f_2,\phi(\mathbf f_1),\phi(\mathbf f_2)\big)=\Bigg\{&\Big[\,F'(W(\mathbf f_1))-F'(W(\mathbf f_2))\Big]\phi(\mathbf f_2)\\&-\tilde{\mathcal L}_{\mathbf f_1}\Big[\sum_{j=1}^N\frac{\mathrm w_{j,x}(\mathbf f_1)\chi_j(\mathbf f_1)}{\displaystyle\int_{\mathbb R}\mathrm w_{j,x}(\mathbf f_1)\chi_j(\mathbf f_1)\,\mathrm dx}\int_{\mathbb R}\phi(\mathbf f_2)[\mathrm w_{j,x}(\mathbf f_1)\chi_j(\mathbf f_1)-\mathrm w_{j,x}(\mathbf f_2)\chi_j(\mathbf f_2)]\mathrm dx\Big]\\&-\sum_{j=1}^N c_j(\mathbf f_2)\chi_j(\mathbf f_2)\Big[\mathrm w_{j,x}(\mathbf f_2)-\mathrm w_{j,x}(\mathbf f_1)\Big]+h(\mathbf f_1)-h(\mathbf f_2)\Bigg\},\end{aligned}\tag{5.12}$$

with $h(\mathbf f_n)$ defined in (5.4). Using these decompositions one can estimate $\|\phi(\mathbf f_1)-\phi(\mathbf f_2)\|_{H^2(\mathfrak S)}$ employing the theory developed in the previous section. Observe that by Proposition 4.1 we only need to estimate $\|\mathcal A\|_{L^2(\mathfrak S)}$. For instance we have:

$$\|\,\big[\,F'(W(\mathbf f_1))-F'(W(\mathbf f_2))\,\big]\phi(\mathbf f_2)\|_{L^2(\mathfrak S)}\le C\varepsilon^{-1/2}\|\mathbf f_1-\mathbf f_2\|_{L^2(\mathfrak S)}\|\phi(\mathbf f_2)\|_{L^2(\mathfrak S)}. \tag{5.13}$$

To estimate the L^2 norm of the second term in (5.13) we fix a j and denote:

$$\mathtt{h} = \frac{\mathrm{w}_{j,x}(\mathbf{f}_1)\chi_j(\mathbf{f}_1)}{\int_{\mathbb{R}} \mathrm{w}_{j,x}(\mathbf{f}_1)\chi_j(\mathbf{f}_1)\,\mathrm{d}x},$$

$$\mathtt{g} = \int_{\mathbb{R}} \phi(\mathbf{f}_2)[\mathrm{w}_{j,x}(\mathbf{f}_1)\chi_j(\mathbf{f}_1) - \mathrm{w}_{j,x}(\mathbf{f}_2)\chi_j(\mathbf{f}_2)]\mathrm{d}x.$$

Then,

$$\begin{aligned}\|\tilde{\mathcal{L}}_{\mathbf{f}_1}(\mathtt{hg})\|_{L^2(\mathfrak{S})} \le\ & C \sup_{z\in(0,2\pi/\varepsilon)} |\mathtt{g}|\|\tilde{\mathcal{L}}_{\mathbf{f}_1}(\mathtt{h})\|_{H^2(\mathfrak{S})} + C \sup_{z\in(0,2\pi/\varepsilon)} |\mathtt{g}_z|\|\nabla \mathtt{h}\|_{H^2(\mathfrak{S})} \\ & + C\|\mathtt{h}\tilde{\mathcal{L}}_{\mathbf{f}_1}(\mathtt{g})\|_{L^2(\mathfrak{S})}. \end{aligned} \tag{5.14}$$

We have:

$$\begin{aligned}|\mathtt{g}| &= \int_{|x|\le C|\log\varepsilon|} |\phi(\mathbf{f}_2)||\mathrm{w}_{j,x}(\mathbf{f}_1)\chi_j(\mathbf{f}_1) - \mathrm{w}_{j,x}(\mathbf{f}_2)\chi_j(\mathbf{f}_2)|\mathrm{d}x \\ &\le C\varepsilon^{-1/2}|\log\varepsilon|^{1/2}\|\phi(\mathbf{f}_2)\|_{H^2(\mathfrak{S})}\|\mathbf{f}_1 - \mathbf{f}_2\|_{H^2(0,2\pi)}.\end{aligned}$$

Using the fact that $\mathrm{w}_{xxx} + F'(\mathrm{w})\mathrm{w}_x = 0$ and that $\operatorname{supp}\chi_j'(\mathbf{f}_1) \subset \{|x - \mathbf{f}_{1j}| > \frac{\sqrt{2}}{4}|\log|\varepsilon|\}$ we can estimate:

$$\|\tilde{\mathcal{L}}_{\mathbf{f}_1}(\mathtt{h})\|_{H^2(\mathfrak{S})} \le C.$$

Furthermore,

$$\begin{aligned}|\mathtt{g}_{zz}|^2 \le\ & C|\log\varepsilon| \int_{|x|\le C|\log\varepsilon|} |\phi_{zz}(\mathbf{f}_2)|^2|\mathrm{w}_{j,x}(\mathbf{f}_1)\chi_j(\mathbf{f}_1) - \mathrm{w}_{j,x}(\mathbf{f}_2)\chi_j(\mathbf{f}_2)|^2\mathrm{d}x \\ & + C|\log\varepsilon| \int_{|x|\le C|\log\varepsilon|} |\phi_z(\mathbf{f}_2)|^2|[\mathrm{w}_{j,x}(\mathbf{f}_1)\chi_j(\mathbf{f}_1) - \mathrm{w}_{j,x}(\mathbf{f}_2)\chi_j(\mathbf{f}_2)]_z|^2\mathrm{d}x \\ & + C|\log\varepsilon| \int_{|x|\le C|\log\varepsilon|} |\phi(\mathbf{f}_2)|^2|[\mathrm{w}_{j,x}(\mathbf{f}_1)\chi_j(\mathbf{f}_1) - \mathrm{w}_{j,x}(\mathbf{f}_2)\chi_j(\mathbf{f}_2)]_{zz}|^2\mathrm{d}x \\ \le\ & C|\log\varepsilon|(I + II + III).\end{aligned}$$

We have:

$$\begin{aligned} I + II \le\ & C\|\mathbf{f}_1 - \mathbf{f}_2\|^2_{H^2(0,2\pi)} \int_{|x|\le C|\log\varepsilon|} (|\phi_{zz}(\mathbf{f}_2)|^2 + |\phi_z(\mathbf{f}_2)|^2)\mathrm{d}x, \\ III \le\ & C(|\mathbf{f}_{1,zz} - \mathbf{f}_{2,zz}|^2 + |\mathbf{f}_{1,z} - \mathbf{f}_{2,z}|^2) \sup_{z\in(0,2\pi/\varepsilon)} \int_{|x|\le C|\log\varepsilon|} |\phi(\mathbf{f}_2)|^2\mathrm{d}x \\ & + C(|\mathbf{f}_{2,zz}|^2 + |\mathbf{f}_{2,z}|^2)\|\mathbf{f}_1 - \mathbf{f}_2\|^2_{H^2(0,2\pi)} \sup_{z\in(0,2\pi/\varepsilon)} \int_{|x|\le C|\log\varepsilon|} |\phi(\mathbf{f}_2)|^2\mathrm{d}x \\ \le\ & C(|\mathbf{f}_{1,zz} - \mathbf{f}_{2,zz}|^2 + (|\mathbf{f}_{1,z} - \mathbf{f}_{2,z}|^2)\|\phi(\mathbf{f}_2)\|^2_{H^2(\mathfrak{S})} \\ & + C(|\mathbf{f}_{2,zz}|^2 + |\mathbf{f}_{2,z}|^2)\|\mathbf{f}_1 - \mathbf{f}_2\|^2_{H^2(0,2\pi)}\|\phi(\mathbf{f}_2)\|^2_{H^2(\mathfrak{S})}.\end{aligned}$$

Similar estimate, but depending only on the first derivatives in z of $\mathbf{f}_n$, $\phi(\mathbf{f}_2)$, holds for $\mathtt{g}_z$. Using these estimates we conclude from (5.14) that

$$\|\tilde{\mathcal{L}}_{\mathbf{f}_1}(\mathtt{hg})\|_{L^2(\mathfrak{S})} \le C\varepsilon^{-1/2}|\log\varepsilon|^q\|\phi(\mathbf{f}_2)\|_{H^2(\mathfrak{S})}\|\mathbf{f}_1 - \mathbf{f}_2\|_{H^2(0,2\pi)}. \tag{5.15}$$

We will now estimate:

$$\begin{aligned}\|h(\mathbf{f}_1) - h(\mathbf{f}_2)\|_{L^2(\mathfrak{S})} \le\ & \|\eta^\varepsilon_\delta(\widetilde{E}(\mathbf{f}_1) - \widetilde{E}(\mathbf{f}_2))\|_{L^2(\mathfrak{S})} \\ & + \|\eta^\varepsilon_\delta[\widetilde{N}(\eta^\varepsilon_{3\delta}\phi(\mathbf{f}_1) + \psi(\mathbf{f}_1)) - \widetilde{N}(\eta^\varepsilon_{3\delta}\phi(\mathbf{f}_2) + \psi(\mathbf{f}_2))\|_{L^2(\mathfrak{S})} \\ & + 3\|\eta^\varepsilon_\delta[(1 - W^2(\mathbf{f}_1))\psi(\mathbf{f}_1) - (1 - W^2(\mathbf{f}_2))\psi(\mathbf{f}_2)]\|_{L^2(\mathfrak{S})}.\end{aligned} \tag{5.16}$$

Using the equation satisfied by $\psi(\mathbf{f}_n)$, $n=1,2$ we find:

$$\|\eta_\delta^\varepsilon(\widetilde{E}(\mathbf{f}_1)-\widetilde{E}(\mathbf{f}_2))\|_{L^2(\mathfrak{S})}\le C\varepsilon^{3/2}|\log\varepsilon|^q\|\mathbf{f}_1-\mathbf{f}_2\|_{H^2(\mathfrak{S})},$$
$$\|\psi(\mathbf{f}_1)-\psi(\mathbf{f}_2)\|_{H^2(\mathfrak{S})}\le C\varepsilon\|\phi(\mathbf{f}_1)-\phi(\mathbf{f}_2)\|_{H^2(\mathfrak{S})}+C\varepsilon^{3/2}|\log\varepsilon|^q\|\mathbf{f}_1-\mathbf{f}_2\|_{H^2(\mathfrak{S})}.$$

Then we get that:

$$\|h(\mathbf{f}_1)-h(\mathbf{f}_2)\|_{L^2(\mathfrak{S})}\le C\varepsilon\|\phi(\mathbf{f}_1)-\phi(\mathbf{f}_2)\|_{H^2(\mathfrak{S})}+C\varepsilon^{3/2}|\log\varepsilon|^q\|\mathbf{f}_1-\mathbf{f}_2\|_{H^2(\mathfrak{S})}. \tag{5.17}$$

Term involving $c_j(\mathbf{f}_2)$ in (5.12) can be estimated in a similar way. In summary we obtain:

$$\begin{aligned}\|\bar\phi\|_{H^2(\mathfrak{S})}&\le C\varepsilon|\log\varepsilon|^q\|\mathbf{f}_1-\mathbf{f}_2\|_{H^2(0,2\pi)}+C\varepsilon\|\phi(\mathbf{f}_1)-\phi(\mathbf{f}_2)\|_{H^2(\mathfrak{S})}\\&\le C\varepsilon|\log\varepsilon|^q\|\mathbf{f}_1-\mathbf{f}_2\|_{H^2(0,2\pi)}+C\varepsilon\|\bar\phi\|_{H^2(\mathfrak{S})}+C\varepsilon\|\bar\phi-\phi(\mathbf{f}_1)+\phi(\mathbf{f}_2)\|_{H^2(\mathfrak{S})}.\end{aligned} \tag{5.18}$$

Since

$$\|\bar\phi-\phi(\mathbf{f}_1)+\phi(\mathbf{f}_2)\|_{H^2(\mathfrak{S})}\le C\varepsilon|\log\varepsilon|^q\|\mathbf{f}_1-\mathbf{f}_2\|_{H^2(0,2\pi)},$$

estimate (5.2) follows from (5.18). This ends the proof. □

6. The Toda System

Clearly Proposition 5.1 and the gluing procedure yield a solution to our original problem (1.1) if we can find $\mathbf{f}$ such that

$$\mathbf{c}(\mathbf{f})\,=\,0. \tag{6.1}$$

As we will see this leads to a Toda type system of N nonlinear ODE's. We carry out this argument and solve the nonlinear system in the next two sections. It is easy to see that the identities (6.1) is equivalent to the following equations

$$\int_{\mathbb{R}}\eta_\delta^\varepsilon\Big[\widetilde{N}(\eta_{3\delta}^\varepsilon\phi+\psi)-\widetilde{E}-3(1-W^2)\psi\Big]\mathrm{w}_{j,x}\chi_j\mathrm{d}x-\int_{\mathbb{R}}\tilde{\mathcal{L}}(\phi)\mathrm{w}_{j,x}\chi_j\,\mathrm{d}x=0,\ \ j=1,...,N. \tag{6.2}$$

(See (3.12) for the definition of χ_j). We will consider for each $j=1,\dots,N$, the following integrals

$$\begin{aligned}\int_{\mathbb{R}}\eta_\delta^\varepsilon\,\tilde{E}(x,z)\,\mathrm{w}_{j,x}\chi_j(x,z)\mathrm{d}x&=\int_{\mathbb{R}}[\Delta_g u_0+u_0(1-u_0^2)]\mathrm{w}_{j,x}\chi_j(x,z)\mathrm{d}x\\&=\int_{\mathbb{R}}[\Delta_g\mathrm{w}_j+\mathrm{w}_j(1-\mathrm{w}_j^2)]\mathrm{w}_{j,x}\chi_j(x,z)\mathrm{d}x\\&\quad+\sum_{n\neq j}\int_{\mathbb{R}}[\Delta_g\mathrm{w}_n+\mathrm{w}_n(1-\mathrm{w}_n^2)]\mathrm{w}_j\chi_j(x,z)\,\mathrm{d}x\\&\quad-\int_{\mathbb{R}}\Big[\sum_{n=1}^N F(\mathrm{w}_n)-F(u_0)\Big]\mathrm{w}_j\chi_j(x,z)\,\mathrm{d}x\\&=I_j+\sum_{n\neq j}^{n}II_{nj}+III_j.\end{aligned}$$

Using (2.16) we get

$$\begin{aligned}I_j=&-\varepsilon^2\int_{\mathbb{R}}\Big(f_j''(1+\varepsilon^2kx^2)+xk\Big)\mathrm{w}_{j,x}^2\chi_j\,\mathrm{d}x+\varepsilon^2\int_{\mathbb{R}}\big[(f_j')^2(1+\varepsilon^2kx^2)\big]\mathrm{w}_{j,xx}\mathrm{w}_{j,x}\chi_j\,\mathrm{d}x\\&+\varepsilon^3\int_{\mathbb{R}}B_2(x,z)[\mathrm{w}_j]\mathrm{w}_{j,x}\chi_j\,\mathrm{d}x,\end{aligned} \tag{6.3}$$

where $B_2(x,z)$ is a second order differential operator, see (2.9) for its definition. Changing variables $\mathrm{x}=x-f_j$ in the leading order terms in the first two integrals we get:

$$I_j=-\varepsilon^2\gamma_0(f_j''+2kf_j)+\varepsilon^3M_{1j}(\mathbf{f},\mathbf{f}',\mathbf{f}''), \tag{6.4}$$

where

$$\gamma_0 = \int_{\mathbb{R}} \mathrm{w}_{\mathtt{x}}^2 \chi(\mathtt{x})\,\mathrm{d}\mathtt{x}, \quad \chi(t) = \eta_a^b(t), \text{with } a, b \text{ as in (3.12)},$$

and

$$M_{1j} = \int_{\mathbb{R}} B_2(x,z)[\mathrm{w}_j]\mathrm{w}_{j,x}\chi_j\,\mathrm{d}x - \varepsilon f_j'' k \int_{\mathbb{R}} x^2 \mathrm{w}_{j,x}^2 \chi_j\,\mathrm{d}x + \varepsilon (f_j')^2 k \int_{\mathbb{R}} x^2 \mathrm{w}_{j,xx}\mathrm{w}_{j,x}\chi_j\,\mathrm{d}x,$$

is a Lipschitz function of $\mathbf{f}, \mathbf{f}'$.

Since in $\operatorname{supp}(\chi_j)$ we have for $n \neq j$

$$|\mathrm{w}_{n,x}| \le C e^{-\frac{\sqrt{2}}{2}|f_j - f_n|} \le C\varepsilon,$$

therefore we get that:

$$II_j = \varepsilon^3 M_{2,j}(\mathbf{f}, \mathbf{f}', \mathbf{f}''), \tag{6.5}$$

where $M_{2,j}$ is a Lipschitz function of its arguments, and in particular it depends linearly on $\mathbf{f}''$.

Finally we will consider III_j. From the computations in section 2.3 (see (2.19)) in, we get the estimates for some terms in III_j as follows

$$\begin{aligned} III_j \;=\; & 3\int_{\mathbb{R}} \big(1 - \mathrm{w}_j^2\big)(u_0 - \mathrm{w}_j)\mathrm{w}_{j,x}\chi_j\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} F''(\mathrm{w}_j)(u_0 - \mathrm{w}_j)^2 \mathrm{w}_{j,x}\chi_j\,\mathrm{d}x \\ & - \frac{1}{2}\sum_{n\neq j}\int_{\mathbb{R}} F''(\sigma_{jn})(\sigma_{jn} - \mathrm{w}_n)^2 \mathrm{w}_j\chi_j\,\mathrm{d}x \\ & + \max_{n\neq j}\int_{\mathbb{R}} O(e^{-3\sqrt{2}\,|f_j - x|})\mathrm{w}_j\chi_j\,\mathrm{d}x \end{aligned} \tag{6.6}$$

It is not hard to calculate that:

$$3\int_{\mathbb{R}} \big(1 - \mathrm{w}_j^2\big)(u_0 - \mathrm{w}_j)\mathrm{w}_{j,x}\chi_j\,\mathrm{d}x = \gamma_1(e^{-\sqrt{2}(f_j - f_{j-1})} - e^{-\sqrt{2}(f_{j+1} - f_j)}) + P_j(\mathbf{f}),$$

where

$$\gamma_1 = 3\int_{\mathbb{R}} (1 - \mathrm{w}^2)\mathrm{w}'\, e^{-\sqrt{2}\,x}\chi\,\mathrm{d}x,$$

and P_j is a Lipschitz function of its arguments, such that:

$$P_j(\mathbf{f}) = \max_{n\neq j} O(e^{-2\sqrt{2}|f_j - f_n|})$$

Notice that the remaining terms in the expression for III_j are of the same type as P_j. Denoting the higher order terms by the same symbols as above we then get:

$$\begin{aligned} \int_{\mathbb{R}} \eta_\delta^\varepsilon\, \tilde{E}(x,z)\, \mathrm{w}_{j,x}\chi_j(x,z)\mathrm{d}x = & -\varepsilon^2\gamma_0(f_j'' + 2kf_j) + \gamma_1(e^{-\sqrt{2}(f_j - f_{j-1})} - e^{-\sqrt{2}(f_{j+1} - f_j)}) \\ & + \varepsilon^3 M_j(\mathbf{f}, \mathbf{f}'\mathbf{f}'') + P_j(\mathbf{f}). \end{aligned} \tag{6.7}$$

Continuing with other terms involved in (6.2), using the quadratic nature of the nonlinear term $\tilde{N}(\phi)$ and Proposition 5.1, we get for

$$Q_{1j} \equiv \int_{\mathbb{R}} [\widetilde{N}(\eta_{3\delta}^\varepsilon\phi + \psi) - 3(1 - W^2)\psi]\mathrm{w}_{j,x}\chi_j\,\mathrm{d}x$$

that

$$\sup_{z\in(0,2\pi/\varepsilon)} |Q_{1j}(z,\mathbf{f})| \le C\varepsilon^3|\log\varepsilon|^q. \tag{6.8}$$

Moreover Q_{1j} is a Lipschitz function, namely we have:

$$\|Q_{1j}(z,\mathbf{f}_1) - Q_{1j}(z,\mathbf{f}_2)\|_{L^2(0,2\pi/\varepsilon)} \le C\varepsilon^{5/2}|\log\varepsilon|^q\|\mathbf{f}_1 - \mathbf{f}_2\|_{\mathrm{w}^2(0,2\pi)}. \tag{6.9}$$

The last term in (6.2) can be written, using orthogonality condition (3.11), as

$$Q_{2j} = \int_{\mathbb{R}} \tilde{\mathcal{L}}(\phi) \mathrm{w}_{j,x}\chi_j \,\mathrm{d}x = \int_R [\Delta_g\phi + (1-3W^2)\phi]\mathrm{w}_{j,x}\chi_j \mathrm{d}x.$$

Using local expression of Δ_g given in (2.8), and the orthogonality condition we get:

$$\begin{aligned} Q_{2j} &= \int_{\mathbb{R}} [\phi_{xx} + (1-3W^2)\phi]\mathrm{w}_{j,x}\chi_j \mathrm{d}x \,+\, \varepsilon^2 \int_{\mathbb{R}} [k(x^2\phi_{zz} - x\phi_x]\mathrm{w}_{j,x}\chi_j \mathrm{d}x \\ &\quad + \varepsilon^3 \int_{\mathbb{R}} B_2(x,z)(\phi)\mathrm{w}_{j,x}\chi_j \mathrm{d}x \\ &= I + II + III. \end{aligned} \tag{6.10}$$

Integrating by parts in the first integral we get:

$$\begin{aligned} I &= \int_{\mathbb{R}} [\mathrm{w}_{j,xxx} + (1-3W^2)\mathrm{w}_{j,x}]\phi\chi_j \mathrm{d}x + \int_{\mathbb{R}} \phi[2\mathrm{w}_{j,xx}\chi_{j,x} + \mathrm{w}_{j,x}\chi_{j,xx}]\mathrm{d}x \\ &= I_1 + I_2. \end{aligned}$$

We have

$$I_1 = 3\int_{\mathbb{R}} (\mathrm{w}_j^2 - W^2)\mathrm{w}_{j,x}\phi\chi_j \mathrm{d}x,$$

hence we can estimate:

$$\|I_1\|_{L^2(0,2\pi/\varepsilon)} \le C\|\phi\|_{\mathrm{w}^2(\mathfrak{S})} \sup_{n\neq j} e^{-\sqrt{2}|f_j - f_n|}.$$

Since

$$\operatorname{supp}\chi_{j,x},\ \operatorname{supp}\chi_{j,xx} \in \Big\{ \frac{\sqrt{2}}{2}(1-2^{-6})|\log\varepsilon| \le |x-f_j| \le \frac{\sqrt{2}}{2}|\log\varepsilon| \Big\},$$

therefore,

$$\|I_2\|_{L^2(0,2\pi/\varepsilon)} \le C\varepsilon^{1-2^{-6}}\|\phi\|_{\mathrm{w}^2(\mathfrak{S})}.$$

A similar estimate holds for II and III above:

$$\|II\|_{L^2(0,2\pi/\varepsilon)} + \|III\|_{L^2(0,2\pi/\varepsilon)} \le C\varepsilon^2\|\phi\|_{\mathrm{w}^2(\mathfrak{S})}.$$

Summarizing we get

$$\|Q_{2j}\|_{L^2(0,2\pi/\varepsilon)} \le C\varepsilon^{1-2^{-6}}\|\phi\|_{\mathrm{w}^2(\mathfrak{S})}. \tag{6.11}$$

Also, using Lipschitz character of the function ϕ, and in particular estimate (5.2) we get

$$\|Q_{2j}(\mathbf{f}_1) - Q_{2j}(\mathbf{f}_2)\|_{L^2(0,2\pi/\varepsilon)} \le C\varepsilon^{2-2^{-6}}|\log\varepsilon|^q\|\mathbf{f}_1 - \mathbf{f}_2\|_{\mathrm{w}^2(0,2\pi)}. \tag{6.12}$$

In the sequel we will denote:

$$Q_j = Q_{1j} + Q_{2j}.$$

We summarize the discussion above in the following proposition:

Proposition 6.1. *Condition (6.1) is equivalent to the following system of equations,*

$$\begin{aligned} 0 &= -\varepsilon^2\gamma_0(f_j'' + 2kf_j) + \gamma_1(e^{-\sqrt{2}(f_j - f_{j-1})} - e^{-\sqrt{2}(f_{j+1}-f_j)}) \\ &\quad + \varepsilon^3 M_j(\mathbf{f}, \mathbf{f}'\mathbf{f}'') + P_j(\mathbf{f}) + Q_j(\mathbf{f}), \quad j = 1,\dots,N. \end{aligned} \tag{6.13}$$

Moreover, functions M_j , P_j and Q_j are Lipschitz functions of their arguments that satisfy estimates

$$\begin{aligned} \|M_j(\mathbf{f}, \mathbf{f}', \mathbf{f}'')\|_{L^2(0,2\pi)} &\le C, \\ \|P_j(\mathbf{f})\|_{L^2(0,2\pi)} &\le C\|\max_{n\neq j} e^{-2\sqrt{2}|f_j - f_n|}\|_{L^2(0,2\pi)}, \\ \|Q_j(\mathbf{f})\|_{L^2(0,2\pi)} &\le C\varepsilon^{3/2-2^{-6}}\|\phi\|_{\mathrm{w}^2(\mathfrak{S})}, \end{aligned} \tag{6.14}$$

and

$$\begin{aligned}\|M_j(\mathbf{f}_1,\mathbf{f}_1',\mathbf{f}_1'')-M_j(\mathbf{f}_2,\mathbf{f}_2',\mathbf{f}_2'')\|_{L^2(0,2\pi)} &\le C\varepsilon^{-1/2}\|\mathbf{f}_1-\mathbf{f}_2\|_{\mathrm{w}^2(0,2\pi)},\\ \|P_j(\mathbf{f}_1)-P_j(\mathbf{f}_2)\|_{L^2(0,2\pi)} &\le C\varepsilon^{5/2}\|\mathbf{f}_1-\mathbf{f}_2\|_{\mathrm{w}^2(0,2\pi)},\\ \|Q_j(\mathbf{f}_1)-Q_j(\mathbf{f}_2)\|_{L^2(0,2\pi)} &\le C\varepsilon^{5/2-2^{-6}}\|\mathbf{f}_1-\mathbf{f}_2\|_{\mathrm{w}^2(0,2\pi)}.\end{aligned} \tag{6.15}$$

7. Location and interaction of clustered Layers

We will define the following Toda type operator $\mathcal{T}(\mathbf{f})=\big(\mathcal{T}_1(\mathbf{f}),\dots,\mathcal{T}_N(\mathbf{f})\big)$ by

$$\mathcal{T}_j(\mathbf{f})\equiv\varepsilon^2\beta\big(f_j''+k(\theta)\,f_j\big)-e^{-\sqrt{2}(f_j-f_{j-1})}+e^{-\sqrt{2}(f_{j+1}-f_j)},\quad \theta\in(0,2\pi),\quad j=1,\dots,N,$$

where $\beta=\gamma_0/\gamma_1$ is a positive constant and $k(\theta)>0$ is defined in (2.17). As a consequence of Proposition 6.1, we have to deal with the following system:

$$\begin{aligned}&\mathcal{T}_j(\mathbf{f})=\varepsilon^3M_j(\mathbf{f},\mathbf{f}',\mathbf{f}'')+P_j(\mathbf{f})+Q_j(\mathbf{f}),\\ &f_j(0)=f_j(2\pi),\quad f_j'(0)=f_j'(2\pi),\end{aligned} \tag{7.1}$$

where $f_0=-\infty$, $f_{N+1}=\infty$.

Before solving the above system, we consider the following Toda system for $j=1,\dots,N$

$$\begin{aligned}&\mathcal{T}_j(\mathbf{f})=\varepsilon^2h_j(\theta),\\ &f_j(0)=f_j(2\pi),\quad f_j'(0)\,=\,f_j'(2\pi),\end{aligned} \tag{7.2}$$

where $f_0=-\infty$, $f_{N+1}=\infty$, and $h_j\in L^2(0,2\pi)$, $j=1,\dots,N$. We will consider the following geometric eigenvalue problem

$$\begin{aligned}\varphi''+k(\theta)\varphi\,&=\,-\Lambda\,k(\theta)\,\varphi,\\ \varphi(0)\,=\,\varphi(2\pi),&\quad\varphi'(0)\,=\,\varphi'(2\pi).\end{aligned} \tag{7.3}$$

Let us recall (see (1.5)) that the operator $\frac{\partial^2}{\partial\theta^2}+k$ is non-degenerate, and under the assumption $k(\theta)>0$, its spectrum necessarily contains at least one negative eigenvalue. By $\{\Lambda_i\},\{\varphi_i\}$ we will denote, respectively, the set of eigenvalues and eigenfunctions of the eigenvalue problem (7.3).

Without loss of generality, we assume that $N=2m$ for $m\ge1$ and give a proof of the main result of this section.

Proposition 7.1. *Consider system (7.2) and assume that its right hand sides satisfy*

$$\|h_j\|_{L^2(0,2\pi)}\le C\varepsilon^{\varrho},\quad j=1,\dots,N, \tag{7.4}$$

with some $C>0$ and $\varrho>0$. We assume that $k(\theta)>0$ and that (1.5) holds. Then for all small ε satisfying the gap condition (1.6), problem (7.2) admits a solution of the following form:

$$f_j(\theta)=\Big(j-m-\frac12\Big)\rho_\varepsilon(\theta)+f_j^0(\theta)+\tilde v_j(\theta)+\tilde f_j(\theta),\quad j=1,\dots,N, \tag{7.5}$$

where $\rho_\varepsilon(\theta)$ satisfies

$$e^{-\sqrt{2}\,\rho_\varepsilon}\,=\,\varepsilon^2\,\beta\,k(\theta)\rho_\varepsilon, \tag{7.6}$$

and in particular

$$\rho_\varepsilon(\theta)\,=\,\sqrt{2}\log\frac1\varepsilon-\frac{1}{\sqrt2}\log\beta-\frac1{\sqrt2}\log k(\theta)+O\Big(\frac{\log\log\frac1\varepsilon}{\log\frac1\varepsilon}\Big).$$

Functions f_j^0 and $\tilde v_j$ do not depend on h_j and satisfy

$$f_j^0(\theta)\,=\,O(1),\quad \tilde v_j(\theta)\,=\,O(1).$$

Finally, for functions $\tilde f_j$ we have

$$\|\tilde f_j\|_{\mathrm{w}^2(0,2\pi)}\,\le\,C|\log\varepsilon|^{3/2}\|\mathbf{h}\|_{L^2(0,2\pi)},\quad \mathbf{h}=(h_1,\dots,h_N).$$

Proof. Let us define the function $\rho_\varepsilon(\theta)$ by

$$e^{-\sqrt{2}\,\rho_\varepsilon} \,=\, \varepsilon^2\beta\, k(\theta)\rho_\varepsilon, \tag{7.7}$$

and set function $\delta(\theta)>0$ by

$$\delta(\theta) \,=\, \varepsilon\sqrt{\beta}\, e^{\rho_\varepsilon(\theta)/\sqrt{2}}.$$

It is straightforward to show

$$\delta^{-2}(\theta) \,=\, k(\theta)\rho_\varepsilon(\theta) \,=\, \sqrt{2}\, k(\theta)\log\frac{1}{\varepsilon}\Big[1+O\Big(\frac{\log\log\frac{1}{\varepsilon}}{\log\frac{1}{\varepsilon}}\Big)\Big]. \tag{7.8}$$

Then multiplying equation (7.2) by $\varepsilon^{-2}\,\beta^{-1}\,\delta^2$ and setting

$$f_j = \Big(j-m-\frac{1}{2}\Big)\rho_\varepsilon + \hat f_j, \quad j=1,\dots,N,$$

we get an equivalent system

$$\begin{aligned}
&\delta^2\Big(\hat f_j'' + k(\theta)\hat f_j\Big) - e^{-\sqrt{2}\big(\hat f_j - \hat f_{j-1}\big)} + e^{-\sqrt{2}\,\big(\hat f_{j+1}-\hat f_j\big)}\\
&\qquad\qquad = \delta^2\, h_j + \delta^2\,\Big(m-j+\frac{1}{2}\Big)\rho_\varepsilon'' + \big(m-j+\frac{1}{2}\big),\\
&\hat f_j(2\pi) = \hat f_j(0), \quad \hat f_j'(2\pi) = \hat f_j'(0),
\end{aligned} \tag{7.9}$$

where $\hat f_0 = -\infty$, $\hat f_{N+1}=\infty$. We look for a solution of (7.9) in the form $\hat f_j = f_j^0 + \bar f_j$, where

$$a_0 \,=\, a_N \,=\, 0, \quad a_j \,=\, e^{-\sqrt{2}\,\big(f^0_{j+1}-f^0_j\big)}, \quad j=1,\cdots,N-1,$$

and $a_j, j=1,\cdots,N-1$, satisfy the following system of equations:

$$\mathbf{M}\cdot\mathbf{A} \,=\, \mathbf{C} \tag{7.10}$$

where we have defined

$$\mathbf{M}_{(N-1)\times(N-1)} \,=\, \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0\\ -1 & 1 & 0 & \cdots & 0 & 0\\ 0 & -1 & 1 & \cdots & 0 & 0\\ \vdots & & \ddots & \ddots & & \vdots\\ 0 & 0 & 0 & \cdots & 1 & 0\\ 0 & 0 & 0 & \cdots & -1 & 1\\ 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}, \ \mathbf{A} \,=\, \begin{pmatrix} a_1\\ a_2\\ \vdots\\ a_{N-2}\\ a_{N-1}\end{pmatrix}, \ \mathbf{C} \,=\, \begin{pmatrix} m-\frac{1}{2}\\ m-\frac{3}{2}\\ \vdots\\ \frac{1}{2}\\ -\frac{1}{2}\\ \vdots\\ -m+\frac{3}{2}\\ -m+\frac{1}{2}\end{pmatrix}.$$

Obviously, system (7.10) can be uniquely solved for unknown variables a_j. In fact, we have that

$$a_m = \sum_{i=1}^{m}(m-i+\frac{1}{2}) = \frac{1}{2}m^2>0, \quad a_j = a_{2m-j} = \sum_{i=1}^{j}(m-i+\frac{1}{2}) = \frac{1}{2}j(2m-j)>0, \tag{7.11}$$

for all $j=1,\dots,m-1$. Hence, f_j^0's are all constants and can be uniquely determined if we set $f_m^0 + f_{m+1}^0 = 0$. By the symmetry and trivial computation, we get

$$\sum_{j=1}^{N} f_j^0 \,=\, 0. \tag{7.12}$$

Then $\bar{f}_j$ satisfy:

$$
\begin{aligned}
&\delta^2\Big(\bar{f}_j'' + k(\theta)\,\bar{f}_j\Big) - a_{j-1}\Big[e^{-\sqrt{2}\big(\bar{f}_j-\bar{f}_{j-1}\big)} - 1\Big] + a_j\Big[e^{-\sqrt{2}\,\big(\bar{f}_{j+1}-\bar{f}_j\big)} - 1\Big] \\
&\qquad\qquad = \delta^2 h_j + \delta^2\Big(m - j + \frac{1}{2}\Big)\rho_\varepsilon'' - \delta^2 k(\theta)\, f_j^0, \qquad (7.13)\\
&\bar{f}_j(2\pi) = \bar{f}_j(0), \quad \bar{f}_j'(2\pi) = \bar{f}_j'(0),
\end{aligned}
$$

where $\bar{f}_0 = -\infty$, $\bar{f}_N = \infty$.

Notice that the right hand of (7.13) is of order $O(\delta^2)$ now. This is not enough to solve our nonlinear problem since there is a term of the same order in front of he linear part of the operator. Thus we need to find one more term in the expansion of $\bar{f}_j$. To this end let $\bar{f}_j = \tilde{f}_j + \delta^2\tilde{v}_j$ where $\tilde{v}_j$, $n = 1, \ldots, N$, solve the following system of equations:

$$\mathbf{B}\cdot\sqrt{2}\,\tilde{\mathbf{v}} \;=\; \tilde{\mathbf{h}}, \qquad (7.14)$$

where

$$
\mathbf{B} \;=\; \begin{pmatrix}
a_1 & -a_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
-a_1 & (a_1+a_2) & -a_2 & 0 & \cdots & 0 & 0 & 0 & 0\\
\vdots & & & & \ddots & & & & \vdots\\
0 & 0 & 0 & 0 & \cdots & 0 & -a_{2m-2} & (a_{2m-2}+a_{2m-1}) & -a_{2m-1}\\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & -a_{2m-1} & a_{2m-1}
\end{pmatrix}, \qquad (7.15)
$$

and

$$
\tilde{\mathbf{v}} \;=\; \begin{pmatrix} \tilde{v}_1\\ \vdots\\ \tilde{v}_{2m}\end{pmatrix}, \qquad
\tilde{\mathbf{h}} \;=\; \begin{pmatrix}
\big(m-\frac{1}{2}\big)\rho_\varepsilon'' - k(\theta)\, f_1^0\\
\big(m-\frac{3}{2}\big)\rho_\varepsilon'' - k(\theta)\, f_2^0\\
\vdots\\
-\big(m-\frac{3}{2}\big)\rho_\varepsilon'' - k(\theta)\, f_{N-1}^0\\
-\big(m-\frac{1}{2}\big)\rho_\varepsilon'' - k(\theta)\, f_N^0
\end{pmatrix}.
$$

Notice that matrix $\mathbf{B}$ is not invertible since its eigenvalues are

$$\mathrm{spect}\,(\mathbf{B}) = \{a_1, \ldots, a_{2m-1}, 0\},$$

however, due to formula (7.12), system (7.14) can be solved uniquely provided that we set $\tilde{v}_m + \tilde{v}_{m+1} = 0$ since the eigenvector corresponding to the eigenvalue 0 is of the form $(1, \ldots, 1)$.

Now, $\tilde{f}_j$ solves the following system of equations, for $j = 1, \ldots, 2m$

$$
\begin{aligned}
&\delta^2\Big(\tilde{f}_j'' + k(\theta)\tilde{f}_j\Big) - \sqrt{2}a_{j-1}\,\tilde{f}_{j-1} + \sqrt{2}\,\big(a_{j-1}+a_j\big)\tilde{f}_j - \sqrt{2}\,a_j\,\tilde{f}_{j+1}\\
&\qquad\qquad = \delta^2\, h_j - \delta^4(\tilde{v}_j)'' + N_j(\tilde{\mathbf{f}}, \tilde{\mathbf{v}}), \qquad (7.16)\\
&\tilde{f}_j(2\pi) = \tilde{f}_j(0), \quad \tilde{f}_j'(2\pi) = \tilde{f}_j'(0),
\end{aligned}
$$

where N_j are given by

$$
\begin{aligned}
N_j(\tilde{\mathbf{f}}, \tilde{\mathbf{v}}) \;=\;& a_{n-1}\Big[\,e^{-\sqrt{2}\,(\tilde{f}_n-\tilde{f}_{n-1}+\tilde{v}_n-\tilde{v}_{n-1})} - 1 + \sqrt{2}\,(\tilde{f}_n - \tilde{f}_{n-1}) + \sqrt{2}\,(\tilde{v}_n - \tilde{v}_{n-1})\,\Big]\\
&- a_n\Big[\,e^{-\sqrt{2}\,(\tilde{f}_{n+1}-\tilde{f}_n+\tilde{v}_{n+1}-\tilde{v}_n)} - 1 + \sqrt{2}\,(\tilde{f}_{n+1} - \tilde{f}_n) + \sqrt{2}\,(\tilde{v}_{n+1} - \tilde{v}_n)\,\Big].
\end{aligned}
$$

In order to use a fixed point argument to solve (7.16) we need the following

Lemma 7.2. *Assume that the function $k(\theta)$ defined in (2.17) is positive and the non-degeneracy condition (1.5) holds. Consider the following problem*

$$\delta^2\big(\mathbf{v}'' + k(\theta)\mathbf{v}\big) + \mathbf{B}\cdot\sqrt{2}\,\mathbf{v} \;=\; \mathbf{g}, \quad \mathbf{v}'(2\pi) \;=\; \mathbf{v}'(0), \quad \mathbf{v}(2\pi) \;=\; \mathbf{v}(0). \qquad (7.17)$$

Then for each ε sufficiently small which satisfy the following gap condition,

$$\min_{j=1,\ldots,N-1}\Big| -\Lambda_i + (2m-j)j\log\frac{1}{\varepsilon}\Big| > c\sqrt{\log\frac{1}{\varepsilon}}, \quad \forall\, i = 1, 2, \ldots, \qquad (7.18)$$

where $c>0$ is a small fixed constant, (7.17) has a unique solution $\mathbf{v}$ *and*

$$\|\mathbf{v}\|_{\mathrm{w}^2(0,2\pi)} \le C|\log\varepsilon|^{3/2}\|\mathbf{g}\|_{L^2(0,2\pi)}. \tag{7.19}$$

Proof. We will denote:

$$\frac{\delta^{-2}(\theta)}{\sqrt{2}\log\frac{1}{\varepsilon}} - k(\theta) = \sigma^\varepsilon(\theta)$$

Notice that by (7.8) we have

$$\sigma^\varepsilon(\theta) = O\Big(\frac{\log\log\frac{1}{\varepsilon}}{\log\frac{1}{\varepsilon}}\Big). \tag{7.20}$$

Let $\{\Lambda_i^\varepsilon\}$, $\{\varphi_i^\varepsilon\}$ be, respectively, the sets eigenvalues and eigenfunctions of the following problem:

$$\begin{aligned} \varphi'' + k(\theta)\varphi &= -\Lambda\big(k(\theta) + \sigma^\varepsilon\big)\varphi,\\ \varphi(0) = \varphi(2\pi), &\quad \varphi'(0) = \varphi'(2\pi). \end{aligned} \tag{7.21}$$

Let us decompose:

$$v_n = \sum_{k=1}^{\infty} a_{ni}\varphi_i^\varepsilon, \quad g_n = \sum_{i=1}^{\infty} g_{ni}\varphi_i^\varepsilon. \tag{7.22}$$

We will use this decomposition to solve

$$\sqrt{2}\delta^2\log\frac{1}{\varepsilon}\left(\mathbf{v}'' + k(\theta)\mathbf{v}\right) + 2\log\frac{1}{\varepsilon}\,\mathbf{B}\cdot\mathbf{v} = \sqrt{2}\log\frac{1}{\varepsilon}\,\mathbf{g}. \tag{7.23}$$

Using (7.22) we get that coefficients a_{ni}, $n = 1,\dots,N$, can be obtained provided that the following systems of linear equations can be solved for each $i \ge 1$

$$\Big(-\Lambda_i^\varepsilon I + 2\log\frac{1}{\varepsilon}\,\mathbf{B}\Big)\cdot\mathbf{a}_i = \log\frac{1}{\varepsilon}\,\mathbf{g}_i \tag{7.24}$$

Notice that because of (7.20) we have

$$\big|\Lambda_i^\varepsilon - \Lambda_i\big| = O\Big(\frac{\log\log\frac{1}{\varepsilon}}{\log\frac{1}{\varepsilon}}\Big), \quad i \ge 1, \tag{7.25}$$

where the O term in (7.25) is uniform for index i. Under the non-degeneracy condition (1.5) we get that matrix

$$-\Lambda_i^\varepsilon I + 2\log\frac{1}{\varepsilon}\,\mathbf{B},$$

is invertible for all small ε satisfying the gap condition (7.18) since its eigenvalues are now:

$$\Big\{-\Lambda_i^\varepsilon + 2a_1\log\frac{1}{\varepsilon},\,\dots,\,-\Lambda_i^\varepsilon + 2a_{2m-1}\log\frac{1}{\varepsilon},\,-\Lambda_i^\varepsilon\Big\}, \quad a_j = \frac{1}{2}j(2m-j)$$

which are bounded away from zero by $\sqrt{\log\frac{1}{\varepsilon}}$. As a consequence, the solution to (7.23) exists and satisfies

$$\|\mathbf{v}\|_{L^2(0,2\pi)} \le C\sqrt{\log\frac{1}{\varepsilon}}\,\|\mathbf{g}\|_{L^2(0,2\pi)}, \tag{7.26}$$

From (7.26) by a standard argument one can show

$$\frac{1}{\log\frac{1}{\varepsilon}}\|\mathbf{v}''\|_{L^2(0,2\pi)} + \frac{1}{\sqrt{\log\frac{1}{\varepsilon}}}\|\mathbf{v}'\|_{L^2(0,2\pi)} + \|\mathbf{v}\|_{L^2(0,2\pi)} \le C\sqrt{\log\frac{1}{\varepsilon}}\,\|\mathbf{g}\|_{L^2(0,2\pi)} \tag{7.27}$$

This ends the proof of the Lemma. □

We claim that using Lemma 7.2, problem (7.16) can be solved by a contraction mapping principle in the set

$$\mathcal{X} \,=\, \left\{ \frac{1}{\sqrt{\log\frac{1}{\varepsilon}}}\,\|\mathbf{v}'\|_{L^2(0,2\pi)} \,+\, \|\mathbf{v}\|_{L^2(0,2\pi)} \,\le\, \frac{1}{\left(\log\frac{1}{\varepsilon}\right)^{1/2+\sigma}} \right\}$$

where $\sigma > \alpha$. In $\mathcal{X}$ we have

$$\begin{aligned}\|\mathbf{N}(\hat{\mathbf{v}}^\varepsilon, \mathbf{v})\|_{L^2(0,2\pi)} \,&\le\, C\Big[\, \|\hat{\mathbf{v}}^\varepsilon\|^2_{L^\infty(0,2\pi)} + \|\mathbf{v}\|_{L^2(0,2\pi)}\big(\|\mathbf{v}\|_{L^2(0,2\pi)} + \|\mathbf{v}'\|_{L^2(0,2\pi)}\big)\Big]\\ &\le\, \frac{C}{|\log\varepsilon|^{1/2+2\sigma}}.\end{aligned}$$

Now, the result follows by a straightforward argument using Lemma 7.2. The proof of Proposition 7.1 is complete. □

In the final part of this section, we solve the system (7.1), which will give a complete proof of Theorem 1.1.

Proof of Theorem 1.1: To finish the proof of Theorem we need to solve problem (7.1). To this end we apply theory developed above combined with a fixed point argument. Thus, keeping the same notations as in the proof of Proposition 7.1 we look for $\mathbf{f}$ solving (7.1) in the form:

$$f_j(\theta) = \Big(j - m - \frac{1}{2}\Big)\rho_\varepsilon(\theta) + f^0_j(\theta) + \tilde{v}_j(\theta) + \tilde{f}_j(\theta), \tag{7.28}$$

where function ρ_ε is given in (7.7), constants f^0_j are given through their relations with the numbers a_j defined in (7.11) and functions $\tilde{v}_j$ solve (7.14). Function $\tilde{f}_j(\theta)$ in turn are determined using Lemma 7.2 where we set the right hand side h_j to be

$$h_j(\tilde{\mathbf{f}}) = \varepsilon\tilde{M}_j(\tilde{\mathbf{f}}, \tilde{\mathbf{f}}', \tilde{\mathbf{f}}'') + \tilde{P}_j(\tilde{\mathbf{f}}) + \tilde{Q}_j(\tilde{\mathbf{f}}).$$

Above we denote

$$\tilde{P}_j(\tilde{\mathbf{f}}) = P_j\big((j - m - \frac{1}{2})\rho_\varepsilon + \mathbf{f}^0 + \tilde{v} + \tilde{\mathbf{f}}\big),$$

with similar rule for $\tilde{Q}_j$ and $\tilde{M}_j$. We observe that using (6.14)-(6.15) and the Lipschitz character of the nonlinear terms involved we can solve (7.1) by a straightforward adaptation of the proof of Lemma 7.2 and Proposition 7.1. We omit the details. This ends the proof of Theorem 1.1. □

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