LIOUVILLE THEOREMS FOR FINITE MORSE INDEX SOLUTIONS OF BIHARMONIC PROBLEM

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Abstract. We prove some Liouville type results for finite Morse index solutions to the biharmonic problem $\Delta^2 u = u^q$, $u > 0$ in $\mathbb{R}^n$ where $1 < q < \infty$. For example, for $n \geq 5$, we show that there are no finite Morse index solutions with $\frac{n+4}{n-4} < q \leq \left(\frac{n-8}{n}\right)_+^{-1}$.

1. Introduction

Consider the following biharmonic equation

$$\Delta^2 u = u^q, \quad u > 0 \text{ in } \mathbb{R}^n \tag{1.1}$$

where $n \geq 5$ and $q > 1$. Define

$$\Lambda_u(\phi) := \int_{\mathbb{R}^n} |\Delta\phi|^2 dx - q\int_{\mathbb{R}^n} u^{q-1}\phi^2 dx, \quad \forall\, \phi \in H^2(\mathbb{R}^n). \tag{1.2}$$

The Morse index of a classical solution to (1.1), $\mathrm{ind}(u)$ is defined as the maximal dimension of all subspaces of $E_{\mathbb{R}^n} := H^2(\mathbb{R}^n)$ such that $\Lambda_u(\phi) < 0$ in $E_{\mathbb{R}^n} \setminus \{0\}$. Similarly, we consider also the solutions $\Delta^2 u = u^q$ on a proper domain $\Omega \neq \mathbb{R}^n$, and define its Morse index with

$$\Lambda_{u,\Omega}(\phi) := \int_{\Omega} |\Delta\phi|^2 dx - q\int_{\Omega} u^{q-1}\phi^2 dx, \quad \forall\, \phi \in E_\Omega := H^2 \cap H_0^1(\Omega). \tag{1.3}$$

A solution u is said stable if $\Lambda_u(\phi) \geq 0$ for any test function $\phi \in E_\Omega$. Clearly, u is stable if and only if its Morse index is equal to zero.

In this paper, we prove the following classification results.

Theorem 1.1. *Let $n \geq 5$.*

(i) *For $\frac{n+4}{n-4} < q \leq \left(\frac{n-8}{n}\right)_+^{-1}$, any solution of (1.1) has infinite Morse index. In particular, for $n \leq 8$ and any $1 < q < \infty$, the equation (1.1) has no stable solution.*
(ii) *For $n \geq 9$, there exists $\epsilon_n > 0$ such that for any $1 < q < \frac{n}{n-8} + \epsilon_n$, the equation (1.1) has no stable solution.*

In the second order case, the finite Morse index solutions to the corresponding nonlinear problem

$$\Delta u + |u|^{q-1}u = 0 \text{ in } \mathbb{R}^n, \quad q > 1 \tag{1.4}$$

have been completely classified by Farina [4]. One main result of [4] is that nontrivial finite Morse index solutions to (1.4) exist if and only if $q \geq p_{JL}$ and $n \geq 11$. Here p_{JL} is the so-called Joeseph-Lundgren exponent, see [8].

In the fourth order case, the nonexistence of positive solutions to (1.1) are showed if $q < \frac{n+4}{n-4}$, and all entire solutions are classified if $q = \frac{n+4}{n-4}$, see [12, 19]. More precisely, when $q = \frac{n+4}{n-4}$ and $n \geq 5$, any classical solution to (1.1) is in the form

$$\widetilde{u}(x) = \frac{c_n \lambda^{\frac{n-4}{2}}}{(1+\lambda^2|x-x_0|^2)^{\frac{n-4}{2}}}, \quad \text{with } x_0 \in \mathbb{R}^n, \lambda > 0.$$

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It was proved by Rozenblum (see [11, 15]) that when $n \geq 5$, the number of negative eigenvalues with multiplicity for the operator $(\Delta^2 - V)$ is bounded by

$$C_n \int_{\mathbb{R}^n} |V(x)|^{\frac{n}{4}} dx.$$

It is easy to check that $\widetilde{u}$ are finite Morse index solutions to (1.1) with the critical exponent.

So our results concern essentially the supercritical case, $n \geq 5$ and $q > \frac{n+4}{n-4}$. As far as we know, there are no results on the classification of entire solutions to (1.1) with finite Morse index and supercritical exponent q. Therefore Theorem 1.1 is a first step towards the understanding of finite Morse index solutions of fourth order problems. We note that only recently the radially symmetric solutions to (1.1) are completely classified in [5, 6, 9]. The radial solutions are shown to be stable if and only if $q \geq p_{JL}^4$ and $n \geq 13$ where p_{JL}^4 stands for the corresponding Joseph-Lundgren exponent (see [5, 6]). Theorem 1.1 classifies finite Morse index solutions in dimensions $n \leq 8$ and for some special cases with $n \geq 9$, there is still a big gap to fill in towards a complete classification.

Our proof borrows crucially an idea from Cowan-Esposito-Ghoussoub [2], who proved the regularity of extremal solutions for fourth order problems in bounded domains. They made a key observation by using a nice result of Souplet [18]. Here we also rely crucially on some results of Souplet [18]. The key argument is to use two different test functions: the first one is u itself, and the other one is $v = -\Delta u$. We believe that further exploration of this idea may help to give the complete classification of stable solutions to (1.1).

At the end, we show some classification results on the half space or compactness results for finite Morse index solutions to $\Delta^2 u = \lambda(u+1)^p$ on bounded domain (see section 3).

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2. Proof of Theorem 1.1

We devide our proof into three steps.

- Step 1. Non existence of stable solution with $1 < q < \left(\frac{n-8}{n}\right)_+^{-1}$.
- Step 2. Non existence of finite Morse index solution with $\frac{n+4}{n-4} < q \leq \left(\frac{n-8}{n}\right)_+^{-1}$.
- Step 3. Non existence of stable solution with q slightly larger than $\frac{n}{n-8}$ with $n \geq 9$.

Although Step 1 is almost a special case of Step 2, it is more easier to begin with the stable solution situation, where we introduce the basic ideas and estimates.

2.1. **Step 1.** According to Theorem 3.1 of [19], $v := -\Delta u > 0$ in $\mathbb{R}^n$ since $q > 1$. Rewrite then (1.1) as a system

$$\Delta u + v = 0, \quad \Delta v + u^q = 0, \quad u > 0, \ v > 0 \ \text{ in } \ \mathbb{R}^n. \tag{2.1}$$

We recall several crucial estimates. First, following the idea in [13, 18], we have

Lemma 2.1. *If there exists a stable positive solution to* (1.1) *or* (2.1)*, there exists a bounded stable positive solution u to* (1.1) *such that $v = -\Delta u$ is also bounded in $\mathbb{R}^n$.*

We can prove this lemma by contradiction and proceed exactly as for Theorem 4.3 in [13] (see also Remark 1.1 in [18]). Indeed, if no bounded stable positive solution exists for (2.1), we have the estimate $u(x) \le C_{n,q} d(x, \partial\Omega)^{-\alpha}$ for any stable solution $\Delta^2 u = u^q$ in $\Omega \ne \mathbb{R}^n$, here $C_{n,q}$ depends only on n and $q > 1$. The main reason is that the scaling argument used in [13] does not affect the stability of solutions. Therefore no stable entire solution to (1.1) could exist in $\mathbb{R}^n$, which contradicts the hypothesis.

Let $\alpha = \frac{4}{q-1}$. By Lemma 2.4 of [18], for any solution of (2.1), there exists $C > 0$ such that

$$\int_{B_R} u dx \le CR^{n-\alpha}, \quad \int_{B_R} u^q dx \le CR^{n-q\alpha}, \quad \forall\, R > 0. \tag{2.2}$$

Here and in the following, B_R stands for the ball of radius R centered at the origin. Another important estimate is the following comparison between u and v (see Lemma 2.7 in [18]):

$$\text{As } u \text{ is bounded,} \quad v^2 \ge \frac{2}{q+1} u^{q+1} \ \text{ in } \ \mathbb{R}^n. \tag{2.3}$$

We need also the following identities:

Lemma 2.2. *For any $\xi, \eta \in C^4(\mathbb{R}^n)$, we have*

$$\Delta\xi\Delta(\xi\eta^2) - [\Delta(\xi\eta)]^2 = -4(\nabla\xi\cdot\nabla\eta)^2 - \xi^2(\Delta\eta)^2 + \xi\Delta\xi|\nabla\eta|^2 - 4\xi\Delta\eta\nabla\xi\cdot\nabla\eta.$$

and

Lemma 2.3. *For any $\xi \in C^4(\mathbb{R}^n)$ and $\eta \in C_0^\infty(\mathbb{R}^n)$, there hold*

$$\begin{aligned} \int_{\mathbb{R}^n} (\Delta^2\xi)\xi\eta^2 dx = & \int_{\mathbb{R}^n} [\Delta(\xi\eta)]^2\, dx + \int_{\mathbb{R}^n} \Big[-4(\nabla\xi\cdot\nabla\eta)^2 + 2\xi\Delta\xi|\nabla\eta|^2\Big] dx \\ & + \int_{\mathbb{R}^n} \xi^2 \Big[2\nabla(\Delta\eta)\cdot\nabla\eta + (\Delta\eta)^2\Big] dx, \end{aligned} \tag{2.4}$$

$$\int_{\mathbb{R}^n} |\nabla\xi|^2|\nabla\eta|^2 dx = \int_{\mathbb{R}^n} \Big[\xi(-\Delta\xi)|\nabla\eta|^2 + \frac{1}{2}\xi^2\Delta(|\nabla\eta|^2)\Big]\, dx. \tag{2.5}$$

Proof. The proof of Lemma 2.2 is done by direct verification. The equality (2.5) follows from

$$\frac{1}{2}\Delta(\xi^2) = \xi\Delta\xi + |\nabla\xi|^2.$$

On the other hand, a simple integration by parts yields

$$\begin{aligned} 2\int_{\mathbb{R}^n} \xi\nabla\xi\cdot\nabla\eta\Delta\eta dx &= -\int_{\mathbb{R}^n} \xi^2 \mathrm{div}\,(\Delta\eta\nabla\eta)\, dx \\ &= -\int_{\mathbb{R}^n} \xi^2 \left[(\Delta\eta)^2 + \nabla\eta\cdot\nabla(\Delta\eta)\right] dx. \end{aligned} \tag{2.6}$$

By Lemma 2.2,

$$\begin{aligned} \int_{\mathbb{R}^n} (\Delta^2\xi)\xi\eta^2 dx &= \int_{\mathbb{R}^n} \Delta\xi\Delta\left(\xi\eta^2\right) dx \\ &= \int_{\mathbb{R}^n} [\Delta(\xi\eta)]^2\, dx - 4\int_{\mathbb{R}^n} (\nabla\xi\cdot\nabla\eta)^2 dx - \int_{\mathbb{R}^n} \Big[\xi^2(\Delta\eta)^2 + \xi\Delta\xi|\nabla\eta|^2\Big] dx \\ &\quad - 4\int_{\mathbb{R}^n} \xi\nabla\xi\cdot\nabla\eta\Delta\eta dx \end{aligned}$$

The equality (2.4) is straightforward using (2.6). □

From (2.4) and (1.1), we have

$$\begin{aligned} & \int_{\mathbb{R}^n} [\Delta(u\eta)]^2 dx - \int_{\mathbb{R}^n} u^{q-1}(u\eta)^2 dx \\ = & \ 4\int_{\mathbb{R}^n} (\nabla u\nabla\eta)^2 dx - 2\int_{\mathbb{R}^n} u\Delta u|\nabla\eta|^2 dx - \int_{\mathbb{R}^n} u^2\Big[2\nabla(\Delta\eta)\cdot\nabla\eta + (\Delta\eta)^2\Big] dx. \end{aligned}$$

In the following, we denote C, C' as various generic positive constants which are independent of u. Use stability condition $\Lambda_u(\phi) \geq 0$ with $\phi = u\eta$, we obtain the following estimate.

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left[(\Delta(u\eta))^2 + u^{q+1}\eta^2\right] dx \\
&\leq C \int_{\mathbb{R}^n} \left[|\nabla u|^2|\nabla\eta|^2 + u|\Delta u||\nabla\eta|^2 + u^2|\nabla(\Delta\eta)\cdot\nabla\eta| + u^2(\Delta\eta)^2\right] dx.
\end{aligned} \tag{2.7}$$

Moreover, as

$$\Delta(u\eta) = -v\eta + 2\nabla u\cdot\nabla\eta + u\Delta\eta,$$

by (2.7) and Young's inequality (recalling that $v = -\Delta u > 0$ in $\mathbb{R}^n$),

$$\int_{\mathbb{R}^n} \left[v^2\eta^2 + u^{q+1}\eta^2\right] dx \leq C \int_{\mathbb{R}^n} \left[uv|\nabla\eta|^2 + |\nabla u|^2|\nabla\eta|^2 + u^2|\nabla(\Delta\eta)\cdot\nabla\eta| + u^2(\Delta\eta)^2\right] dx.$$

Applying (2.5) with $\xi = u$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left[(v^2\eta^2 + u^{q+1}\eta^2\right] dx \\
&\leq C \int_{\mathbb{R}^n} uv|\nabla\eta|^2 dx + C \int_{\mathbb{R}^n} u^2\left[|\nabla(\Delta\eta)\cdot\nabla\eta| + \left|\Delta\left(|\nabla\eta|^2\right)\right| + (\Delta\eta)^2\right] dx.
\end{aligned} \tag{2.8}$$

Take $\eta = \varphi^m$ with $m > 2$, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^n} uv|\nabla\eta|^2 dx &= m^2 \int_{\mathbb{R}^n} uv\varphi^{2(m-1)}|\nabla\varphi|^2 dx \\
&\leq \frac{1}{2C}\int_{\mathbb{R}^n} (v\varphi^m)^2 dx + C\int_{\mathbb{R}^n} u^2\varphi^{2(m-2)}|\nabla\varphi|^4 dx.
\end{aligned}$$

Now let us choose φ a cut-off function verifying $0 \leq \varphi \leq 1$, $\varphi = 1$ for $|x| < R$ and $\varphi = 0$ for $|x| > 2R$. Substituting the above inequality into (2.8), we arrive at

$$\int_{\mathbb{R}^n} (v\varphi^m)^2 dx + \int_{\mathbb{R}^n} u^{q+1}\varphi^{2m} dx \leq CR^{-4}\int_{\mathbb{R}^n} u^2\varphi^{2(m-2)} dx. \tag{2.9}$$

We claim:

$$\int_{B_R} u^2 dx \leq CR^{n-2\alpha}, \quad \forall\, R > 0. \tag{2.10}$$

When $q > 2$, the above estimate follows from Hölder's inequality using (2.2) while for $q = 2$, it is just the second estimate in (2.2). If $q \in (1, 2)$, fix $m > \frac{2}{q-1}$, by Hölder's inequality and (2.9), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} u^2\varphi^{2(m-2)} dx &\leq \left(\int_{\mathbb{R}^n} u^q\varphi^{2m-\frac{4}{q-1}} dx\right)^{q-1} \left(\int_{\mathbb{R}^n} u^{q+1}\varphi^{2m} dx\right)^{2-q} \\
&\leq C\left(\int_{B_{2R}} u^q dx\right)^{q-1} \left(R^{-4}\int_{\mathbb{R}^n} u^2\varphi^{2(m-2)} dx\right)^{2-q},
\end{aligned}$$

hence

$$\int_{\mathbb{R}^n} u^2\varphi^{2(m-2)} dx \leq CR^{-\frac{4(2-q)}{q-1}}\int_{B_{2R}} u^q dx.$$

Using (2.2),

$$\int_{B_R} u^2 dx \leq \int_{\mathbb{R}^n} u^2\varphi^{2(m-2)} dx \leq CR^{-\frac{4(2-q)}{q-1}}\int_{B_{2R}} u^q dx \leq C'R^{n-q\alpha}R^{-\frac{4(2-q)}{q-1}} = C'R^{n-2\alpha},$$

so the claim (2.10) is proved. Combining (2.9) and (2.10),

$$\int_{\mathbb{R}^n} \left(v^2 + u^{q+1}\right)\varphi^{2m} dx \leq CR^{n-4-2\alpha}. \tag{2.11}$$

Next we make use of the stability condition again, but this time with the test function $\phi = v\eta$. By equations (2.1), we have

$$\Delta^2 v = -\Delta(u^q) = qu^{q-1}v - q(q-1)u^{q-2}|\nabla u|^2. \tag{2.12}$$

Multiplying (2.12) by $v\eta^2$, similarly as for (2.7), by (2.4) and (2.5),

$$\begin{aligned}
0 \le & \int_{\mathbb{R}^n} \left[(\Delta(v\eta))^2 - qu^{q-1}(v\eta)^2\right] dx \\
\le & -q(q-1)\int_{\mathbb{R}^n} u^{q-2}|\nabla u|^2 v\eta^2 dx + C\int_{\mathbb{R}^n} v|\Delta v||\nabla\eta|^2 dx \\
& + C\int_{\mathbb{R}^n} v^2\left[|\nabla(\Delta\eta)\cdot\nabla\eta| + \left|\Delta\left(|\nabla\eta|^2\right)\right| + |\Delta\eta|^2\right] dx \\
\le & -q(q-1)\int_{\mathbb{R}^n} u^{q-2}|\nabla u|^2 v\eta^2 dx \\
& + C\int_{\mathbb{R}^n} vu^q|\nabla\eta|^2 dx + C\int_{\mathbb{R}^n} v^2\left[|\nabla(\Delta\eta)\cdot\nabla\eta| + \left|\Delta\left(|\nabla\eta|^2\right)\right| + |\Delta\eta|^2\right] dx.
\end{aligned}$$

Hence

$$\begin{aligned}
& \int_{\mathbb{R}^n} u^{q-2}|\nabla u|^2 v\eta^2 dx \\
\le & C\int_{\mathbb{R}^n} vu^q|\nabla\eta|^2 dx + C\int_{\mathbb{R}^n} v^2\left[|\nabla(\Delta\eta)\cdot\nabla\eta| + \left|\Delta\left(|\nabla\eta|^2\right)\right| + |\Delta\eta|^2\right] dx.
\end{aligned} \tag{2.13}$$

Furthermore, for any C^1 function H, integration by parts yields

$$\int_{\mathbb{R}^n} H(u)(-\Delta u)\eta^2 dx = \int_{\mathbb{R}^n} H'(u)|\nabla u|^2\eta^2 dx + \int_{\mathbb{R}^n} H(u)\nabla u\cdot\nabla(\eta^2) dx. \tag{2.14}$$

Following an idea of Cowan-Espositon-Ghoussoub [2], set $H(u) = u^{\frac{3q-1}{2}}$, then

$$\int_{\mathbb{R}^n} u^{\frac{3q-1}{2}} v\eta^2 dx \le \int_{\mathbb{R}^n} u^{\frac{3q-3}{2}}|\nabla u|^2\eta^2 dx + C\int_{\mathbb{R}^n} u^{\frac{3q+1}{2}}\left|\Delta(\eta^2)\right| dx.$$

Recall that $v \ge Cu^{\frac{q+1}{2}}$, we conclude, using (2.13) and (2.14),

$$\begin{aligned}
\int_{\mathbb{R}^n} u^{2q}\eta^2 dx \le & C\int_{\mathbb{R}^n} u^{\frac{3q-1}{2}} v\eta^2 dx \\
\le & C\int_{\mathbb{R}^n} u^{\frac{3q-3}{2}}|\nabla u|^2\eta^2 dx + C\int_{\mathbb{R}^n} u^{\frac{3q+1}{2}}\left|\Delta(\eta^2)\right| dx \\
\le & C\int_{\mathbb{R}^n} u^{q-2}|\nabla u|^2 v\eta^2 dx + C\int_{\mathbb{R}^n} vu^q\left|\Delta(\eta^2)\right| dx \\
\le & C\int_{\mathbb{R}^n} vu^q\left(|\nabla\eta|^2 + \left|\Delta(\eta^2)\right|\right) dx \\
& + C\int_{\mathbb{R}^n} v^2\left[|\nabla(\Delta\eta)\cdot\nabla\eta| + \left|\Delta\left(|\nabla\eta|^2\right)\right| + |\Delta\eta|^2\right] dx.
\end{aligned} \tag{2.15}$$

As before, let $\eta = \varphi^m$ with large m. Similarly to the derivation of inequality (2.9), we get from (2.15) and (2.11),

$$\int_{\mathbb{R}^n} (u^q\varphi^m)^2 dx \le CR^{-4}\int_{B_{2R}\setminus B_R} v^2 dx \le CR^{n-8-2\alpha}. \tag{2.16}$$

If $1 < q < \left(\frac{n-8}{n}\right)_+^{-1}$, $n-8-2\alpha > 0$. So when u is stable, letting $R\to\infty$, we deduce $u \equiv 0$ in $\mathbb{R}^n$ by (2.16). This proves the nonexistence of stable solution to (1.1) for $1 < q < \left(\frac{n-8}{n}\right)_+^{-1}$.

2.2. **Step 2.** Here we show the nonexistence of finite Morse index solutions with $\frac{n+4}{n-4} < q \le \left(\frac{n-8}{n}\right)_+^{-1}$. Our proof is based on the nonexistence of fast decay solutions with supercritical exponent.

Proposition 2.4. *Let $n \ge 5$, $q > \frac{n+4}{n-4}$ and $\alpha = \frac{4}{q-1}$. Then the system* (2.1) *has no classical solution verifying*

$$u(x) = o\left(|x|^{-\alpha}\right), \quad v(x) = o\left(|x|^{-2-\alpha}\right) \quad \text{as } |x| \to \infty. \tag{2.17}$$

Proof. Suppose that such a solution u exists. Let w be the Emden-Fowler transformation of u, i.e. $w(t, \sigma) = r^\alpha u(r\sigma)$ for any $t = \ln r \in \mathbb{R}$ and $\sigma \in S^{n-1}$, direct calculation yields

$$r^{2+\alpha}\Delta u = w_{tt} + (n-2-2\alpha)w_t - \alpha(n-2-\alpha)w + \Delta_{\mathbb{S}^{n-1}}w$$

where $\Delta_{\mathbb{S}^{n-1}}$ denotes the Laplace-Beltrami operator on the standard unit sphere $\mathbb{S}^{n-1}$. Applying again this formula,

$$\begin{aligned} w^q = r^{4+\alpha}u^q = r^{4+\alpha}\Delta^2 u = w_{tttt} + K_3 w_{ttt} + K_2 w_{tt} + K_1 w_t + K_0 w \\ + \Delta^2_{\mathbb{S}^{n-1}}w + 2\Delta_{\mathbb{S}^{n-1}}w_{tt} + K_5\Delta_{\mathbb{S}^{n-1}}w_t + K_6\Delta_{\mathbb{S}^{n-1}}w \end{aligned} \tag{2.18}$$

where K_i are constants depending on α and n, for example

$$K_5 = K_3 = (2n-8-4\alpha), \quad K_6 = -\Big[(\alpha+2)(n-4-\alpha) + \alpha(n-2-\alpha)\Big].$$

In particular, we have (see [6] for K_i, $0 \le i \le 4$)

$$K_1 < 0, \;\; K_3 = K_5 > 0, \quad \forall\, n \ge 5,\; q > \frac{n+4}{n-4}.$$

Set

$$\begin{aligned} E(w) = & \int_{\mathbb{S}^{n-1}} \left(\frac{w^{q+1}}{q+1} - \frac{K_0}{2}w^2 - \frac{K_2}{2}w_t^2 - K_3 w_{tt}w_t + \frac{w_{tt}^2}{2} - w_{ttt}w_t\right) d\sigma \\ & + \int_{\mathbb{S}^{n-1}} \left(\frac{K_6}{2}|\nabla_{\mathbb{S}^{n-1}}w|^2 + |\nabla_{\mathbb{S}^{n-1}}w_t|^2 - \frac{1}{2}|\Delta_{\mathbb{S}^{n-1}}w|^2\right) d\sigma. \end{aligned}$$

Multiplying the equation (2.18) with w_t, we get (as $K_1 > 0$, $K_3 = K_5 < 0$)

$$\frac{\mathrm{d}}{\mathrm{d}t}E(w)(t) = \int_{\mathbb{S}^{n-1}} \left(K_1 w_t^2 - K_5|\nabla_{\mathbb{S}^{n-1}}w_t|^2 - K_3 w_{tt}^2\right) d\sigma \le 0.$$

By the decay conditions (2.17),

$$-\Delta u = v, \quad -\Delta v = u^q = o\left(|x|^{-4-\alpha}\right) \;\text{ as } |x| \to \infty.$$

The standard ellptic estimates imply then

$$\lim_{|x|\to+\infty} |x|^{k+\alpha}|\nabla^k u(x)| = 0, \;\text{ for }\; 1 \le k \le 4 \quad \text{so that} \;\; \lim_{t\to\infty}\|w(t,\cdot)\|_{C^3(\mathbb{S}^{n-1})} = 0. \tag{2.19}$$

Therefore $\lim_{t\to\infty} E(w) = 0$. We have also $\lim_{t\to-\infty} E(w) = 0$ because u is regular at the origin. Finally we conclude

$$\int_{\mathbb{R}}\int_{\mathbb{S}^{n-1}} \left(K_1 w_t^2 - K_5|\nabla_{\mathbb{S}^{n-1}}w_t|^2 - K_3 w_{tt}^2\right) d\sigma dt = 0.$$

So $w_t \equiv 0$, hence $w \equiv 0$ as $\lim_{t\to-\infty} w = 0$, but this contradicts the positivity of u. ☐

Back to Theorem 1.1. Suppose that u is a solution of (1.1) with finite Morse index less than $\ell \in \mathbb{N}^*$. Considering the family of solutions verifying $\mathrm{ind}(u) \le \ell$, with similar argument as for Lemma 2.1, we may assume again u is bounded (see Corollary 3.2). As u is stable outside a large ball B_{R_0} (see for example [4]), all the calculations in Step 1 still hold true by using cut-off functions φ with support in $\mathbb{R}^n \setminus B_{R_0}$, and we replace just the estimate (2.10) by

$$\int_{B_R(y)} u^2 dx \le CR^{n-2\alpha}, \quad \forall\, R > 0 \text{ and } B_R(y) \subset \mathbb{R}^n \setminus B_{R_0}.$$

There hold then the following estimates similar to (2.11) and (2.16).

$$\int_{B_R(y)} v^2 dx \le C R^{n-4-2\alpha}, \quad \int_{B_R(y)} u^{2q} dx \le C R^{n-8-2\alpha}, \quad \text{for all } B_R(y) \subset \mathbb{R}^n \setminus B_{R_0}. \tag{2.20}$$

Applying now the Sobolev embedding of H^2,

$$\|v\|^2_{L^{p_*}(B_R)} \le C\left(\|\Delta v\|^2_{L^2(B_R)} + R^{-4}\|v\|^2_{L^2(B_R)}\right), \quad \text{where } \ p_* = \frac{2n}{n-4}.$$

Combining with (2.20), for any $R > 0$ and $B_R(y) \subset \mathbb{R}^n \setminus B_{R_0}$,

$$\|v\|^2_{L^{p_*}(B_R(y))} \le C R^{n-8-2\alpha}. \tag{2.21}$$

As q is supercritical, we have $n - 4 - 2\alpha < 0$. With the covering argument, it is not difficult to see that

$$\int_{\mathbb{R}^n} v^2 dx + \int_{\mathbb{R}^n} v^{\frac{2n}{n-4}} dx < \infty. \tag{2.22}$$

Now we are ready to prove the decay of u and v. Instead to use the Harnack argument in [16] (see [4]), let us recall a special case of Theorem 4.4 in [10]: For any $p \in [2, \infty)$, there exists $\epsilon(p) > 0$ such that if $\Delta w + \rho w = 0$ in B_1 with $\|\rho\|_{L^{\frac{n}{2}}(B_1)} \le \epsilon(p)$, we have

$$\|w\|_{L^p(B_{\frac{1}{2}})} \le C\|w\|_{L^2(B_1)} \le C'\|w\|_{L^{p_*}(B_1)} \tag{2.23}$$

where the constants C, C' depend only on p and n.

Let $x_0 \in \mathbb{R}^n$ with $|x_0| > 2R_0$ and $R = \frac{|x_0|}{4}$, consider the function $w(y) = v(x_0 + Ry)$. Then w satisfies $\Delta w + \rho w = 0$ where

$$\rho(y) = R^2 \frac{u^q}{v}(x_0 + Ry).$$

Using (2.3), $0 < \rho(y) \le CR^2 u^{\frac{q-1}{2}}(x_0 + Ry) \le C'R^2 v^{\frac{q-1}{q+1}}(x_0 + Ry)$. As $\frac{n+4}{n-4} < q \le \left(\frac{n-8}{n}\right)_+^{-1}$,

$$r = \frac{q-1}{q+1} \times \frac{n}{2} \in (2, p_*].$$

By (2.22) and Hölder's inequality,

$$\int_{\mathbb{R}^n} v^r dx < \infty.$$

Therefore

$$\int_{B_1} |\rho|^{\frac{n}{2}} dx \le C \int_{B_1} R^n v^r(x_0 + Ry) dy = C \int_{B_R(x_0)} v^r dx \to 0, \quad \text{as } \ |x_0| \to \infty.$$

From (2.23) and (2.21), we derive that for any $p \ge 2$,

$$\|v\|_{L^p(B_R(x_0))} = o\left(R^{\frac{n-8-2\alpha}{2} - \frac{n}{p_*} + \frac{n}{p}}\right) = o\left(R^{-2-\alpha+\frac{n}{p}}\right) \quad \text{as } \ |x_0| \to \infty. \tag{2.24}$$

Using classical elliptic estimates (see Theorem 8.17 of [7]), there exists $C > 0$ such that

$$\sup_{B_{\frac{R}{2}}(x_0)} v \le C\left[R^{-\frac{n}{2}}\|v\|_{L^2(B_R(x_0))} + R\|\Delta v\|_{L^n(B_R(x_0))}\right]. \tag{2.25}$$

It is clear that $\widetilde{q} = \frac{2nq}{q+1} \ge 2$ and $|\Delta v|^n = u^{nq} \le C v^{\widetilde{q}}$. Thanks to (2.24), when $|x_0| \to \infty$,

$$R\|\Delta v\|_{L^n(B_R(x_0))} \le CR\|v\|^{\frac{\widetilde{q}}{n}}_{L^{\widetilde{q}}(B_R(x_0))} = o\left(R^{1+\left(-2-\alpha+\frac{n}{\widetilde{q}}\right)\frac{\widetilde{q}}{n}}\right) = o\left(R^{-2-\alpha}\right).$$

Substituting the above estimate into (2.25), applying (2.24) with $p = 2$, we conclude then

$$v(x) = o\left(|x|^{-2-\alpha}\right) \quad \text{as } \ |x| \to \infty.$$

We get also $u(x) = o\left(|x|^{-\alpha}\right)$ at infinity by (2.3), hence the decay estimate (2.17) holds, we reach then a contradiction seeing Proposition 2.4.

2.3. **Step 3.** Here we will prove that no stable solution exists for exponent q slightly higher than $\frac{n}{n-8}$ if $n \geq 9$. The main idea is a blow up argument.

Suppose that the claim (ii) of Theorem 1.1 does not hold, there exist then a sequence $\delta_j > 0$, $\delta_j \to 0$ and a sequence of stable solutions u_j to (1.1) with $q_j = \frac{n}{n-8} + \delta_j$. Lemma 2.1 permits to assume that u_j and $v_j = -\Delta u_j$ are bounded in $\mathbb{R}^n$. Choose $\lambda_j > 0$ such that

$$\frac{1}{\|v_j\|_\infty} = \lambda_j^{\frac{4}{q_j-1}+2}.$$

Let $\widetilde{u}_j(x) = \lambda_j^{\frac{4}{q_j-1}} u_j(\lambda_j x)$, so $\Delta^2 \widetilde{u}_j = \widetilde{u}_j^{q_j}$, $\widetilde{v}_j := -\Delta \widetilde{u}_j$ satisfies $\|\widetilde{v}_j\|_\infty = 1$. Up to a translation, we assume also $\widetilde{v}_j(0) \in (\frac{1}{2}, 1]$. Using (2.3) to $\widetilde{u}_j$, we have also $\|\widetilde{u}_j\|_\infty \leq C$.

By standard elliptic theory, there is a subsequence still denoted by $\widetilde{u}_j$ which tends to a bounded nonnegative function u_* in $C^k_{loc}(\mathbb{R}^n)$ for any $k \in \mathbb{N}$, so $\Delta^2 u_* = u_*^{\frac{n}{n-8}}$ in $\mathbb{R}^n$. As u_j are stable, it is easy to see that u_* is stable (taking the limit in (1.2) with $\widetilde{u}_j$ and q_j). Finally, since $-\Delta u_* \geq 0$ in $\mathbb{R}^n$ and $-\Delta u_*(0) = \lim \widetilde{v}_j(0) > 0$, u_* is nontrivial, hence positive in $\mathbb{R}^n$. This is impossible by the previous step, the claim (ii) is then proved. □

3. Somme applications

As we have mentioned yet, the nonexistence result of entire stable solution yields immediately (with blow-up and scaling argument as in [13, 18])

Corollary 3.1. *Assume that Ω is a proper subdomain of $\mathbb{R}^n$ and u is a classical, positive and stable solution of $\Delta^2 u = u^q$ in Ω where $1 < q < \infty$ if $n \leq 8$; or $1 < q < \frac{n}{n-8} + \epsilon_n$ if $n \geq 9$ with ϵ_n in Theorem 1.1. Then*

$$u(x) \leq C_{n,q} d(x, \partial\Omega)^{-\alpha}, \ |\Delta u(x)| \leq C_{n,q} d(x, \partial\Omega)^{-\alpha-2}, \quad \alpha = \frac{4}{q-1}$$

where the constant C depends only on q and n.

More generally, we have the similar result for solutions with bounded Morse index.

Corollary 3.2. *Assume that Ω is a proper subdomain of $\mathbb{R}^n$ and u is a classical, positive and stable solution of $\Delta^2 u = u^q$ in Ω where $1 < q < \left(\frac{n-8}{n}\right)_+^{-1}$ and the Morse index of $u \leq \ell \in \mathbb{N}$. Then*

$$u(x) \leq C_{n,q,\ell} d(x, \partial\Omega)^{-\alpha}, \ |\Delta u(x)| \leq C_{n,q,\ell} d(x, \partial\Omega)^{-\alpha-2}, \quad \alpha = \frac{4}{q-1}.$$

Here the constant C depends on q, n and ℓ.

Consider now

$$\begin{cases} \Delta^2 u = u^q & \text{in } \mathbb{R}^n_+ = \mathbb{R}_+ \times \mathbb{R}^{n-1}, \quad n \geq 2 \\ u > 0, -\Delta u > 0 & \text{in } \mathbb{R}^n_+ \\ u = -\Delta u = 0 & \text{on } \{0\} \times \mathbb{R}^{n-1}. \end{cases} \tag{3.1}$$

The following result is due to Dancer (Theorem 2 in [3], see also Theorem 10 in [17]).

Lemma 3.3. *Suppose that u is a classical solution of* (3.1) *such that u and $-\Delta u$ are bounded in $\mathbb{R}^n_+$, then $\partial_{x_1} u > 0$ and $-\partial_{x_1} \Delta u > 0$ in $\mathbb{R}^n_+$.*

Therefore, under the condition of this lemma, $w(y) = \lim_{x_1 \to \infty} u(x_1, y)$ exists for all $y \in \mathbb{R}^{n-1}$ and $\Delta^2 w = w^p$ in $\mathbb{R}^{n-1}$. It is not difficult to see that if w is unstable, then $\mathrm{ind}(u)$ is infinite; in other words, if u is of finite Morse index, then w must be stable. This enable us the following classification result.

Theorem 3.4. *Let u be a classical solution of* (3.1) *with $n \geq 2$.*

(i) *The Morse index of u is ∞, $1 < q \leq \left(\frac{n-8}{n}\right)_+^{-1}$, $q \neq \frac{n+4}{n-4}$.*

(ii) *Assume moreover u and $-\Delta u$ are bounded. Then $ind(u) = \infty$, when $q > 1$ and $n \le 9$; or $1 < q < \frac{n-1}{n-9} + \epsilon_{n-1}$ and $n \ge 10$. Here $\epsilon_k > 0$ are given by Theorem 1.1.*

Proof. We prove readily (ii) by the discussion under Lemma 3.3 and Theorem 1.1. For proving (i), we use a doubling argument which is a consequence of Lemma 5.1 in [13] (with $\Sigma = D = \mathbb{R}^n_+$ and $\Gamma = \emptyset$).

Lemma 3.5. *Let $Q : \mathbb{R}^n_+ \to (0, \infty)$ be continuous and fix a real $k > 0$. If $y \in \mathbb{R}^n_+$ verifies $Q(y) > 2k$, then there exists $x \in \mathbb{R}^n_+$ such that*
- *$Q(x) > 2k$, $Q(x) \ge Q(y)$*
- *$Q(z) \le 2Q(x)$, for all $z \in \mathbb{R}^n_+ \cap \overline{B}_{kQ^{-1}(x)}(x)$.*

Define

$$\alpha = \frac{4}{q-1}, \quad \beta = \alpha + 2 = \frac{2(q+1)}{q-1} \quad \text{and} \quad Q(x) = Q_u(x) := |u(x)|^{\frac{1}{\alpha}} + |\Delta u(x)|^{\frac{1}{\beta}} + 1. \tag{3.2}$$

Suppose that there is a positive solution u to (3.1) verifying $\sup_{\mathbb{R}^n_+} Q_u = \infty$ and $\mathrm{ind}(u) < \infty$.

Applying Lemma 3.5, for any $k \in \mathbb{N}^*$, we get $a_k \in \mathbb{R}^n_+$ such that

$$Q(a_k) > 2k, \quad Q(z) \le 2Q(a_k), \;\; \forall\, z \in \mathbb{R}^n_+ \cap \overline{B}_{kQ^{-1}(a_k)}(a_k).$$

Denote $M_k = Q(a_k)$ and consider

$$w_k = M_k^{-\alpha} u\left(a_k + \frac{y}{M_k}\right), \quad \forall\, y \in \Omega_k = B_k \cap \{y_1 \ge -\gamma_k\} \;\text{ where }\; \gamma_k = a_{k,1} M_k. \tag{3.3}$$

We have $\Delta^2 w_k = w_k^q$ in Ω_k, $w_k(y) = \Delta w_k(y) = 0$ on $\{x_1 = -\gamma_k\} \cap \partial\Omega_k$, $\sup_{\Omega_k} Q_{w_k} \le 4$ and $Q_{w_k}(0) \to 2$. Up to a subsequence, we can suppose that $\lim_{k\to\infty} \gamma_k = \gamma \in \mathbb{R}_+ \cup \{\infty\}$ exists.

Case 1: $\gamma = \infty$. Remark that $a_{k,1} = \mathrm{dist}(a_k, \partial\mathbb{R}^n_+)$, hence $M_k \,\mathrm{dist}(a_k, \partial\mathbb{R}^n_+) \to \infty$. By standard elliptic theory, taking the limit of w_k (up to a subsequence), we obtain a solution

$$\Delta^2 w = w^q \;\text{ in } \mathbb{R}^n, \quad Q_w(0) = 2. \tag{3.4}$$

As $\mathrm{ind}(w_k) = \mathrm{ind}(u)$, we have $\mathrm{ind}(w) \le \mathrm{ind}(u) < \infty$, and $Q_w(0) = 2$ implies that w is nontrivial. So we get a solution to (1.1) with finite Morse index, this is impossible seeing the assumption on q and Theorem 1.1.

Case 2: $\gamma \in [0, \infty)$. Let $\widetilde{w}_k = w_k(y + \gamma_k e_1)$, we get a family of bounded solutions for $\Delta^2\varphi = \varphi^q$ in $\widetilde{\Omega}_k := \Omega_k + \gamma_k e_1$ which tends to the half space $\mathbb{R}^n_+$. Fix $R = \gamma + 1$, using classical estimates due to Agmon-Douglis-Nirenberg in [1] successively for $-\Delta\widetilde{w}_k$ and $\widetilde{w}_k$ on $B_{2R,+} := \mathbb{R}^n_+ \cap B_{2R}$, we get

$$\|\widetilde{w}_k\|_{W^{4,p}(B_{R,+})} \le C_{n,p,R} \|\widetilde{w}_k\|_{L^p(B_{2R,+})} \le C.$$

Choosing $p > n$, $\|\widetilde{w}_k\|_{C^3(\overline{B}_{R,+})} \le C$ by Sobolev embedding. As $Q_{\widetilde{w}_k}(0) = 1$ and $Q_{\widetilde{w}_k}(\gamma_k e_1) \to 2$, and $\gamma_k < R$ (for k large enough), there holds $1 \le C\gamma_k$ for large k, so $\gamma > 0$. As $Q_{\widetilde{w}_k} \le 4$ in $\widetilde{\Omega}_k$, passing the limit to a subsequence, we get a solution w on the half space with Navier's boundary condition. Moreover, we have $Q_w \le 4$, $Q_w(\gamma e_1) = 2$ and $\mathrm{ind}(w) \le \mathrm{ind}(u)$. This contradicts (ii) by the assumption on q, so we are done. $\square$

Finally, Consider the bounded domain situation with polynomial growth:

$$(P_\lambda) \qquad \begin{cases} \Delta^2 u = \lambda(u+1)^q & \text{in a bounded smooth domain } \Omega \subset \mathbb{R}^n, \;\; n \ge 1 \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

It is well known that there exists a critical value $\lambda^* > 0$ depending on $q > 1$ and Ω such that

- If $\lambda \in (0, \lambda^*)$, (P_λ) has a minimal and classical solution which is stable;
- If $\lambda = \lambda^*$, a unique weak solution, called the extremal solution u^* exists for (P_{λ^*});

- No weak solution of (P_λ) exists whenever $\lambda > \lambda^*$.

Our objectif is to prove some compactness result for finite Morse index solutions.

Theorem 3.6. *Assume that $1 < q \leq \left(\frac{n-8}{n}\right)_+^{-1}$, $q \neq \frac{n+4}{n-4}$. Let u_k be a sequence of classical solutions of (P_{λ_k}) such that $\lambda_k \to \mu > 0$ and the Morse index of u_k is uniformly bounded, then $\|u_k\|_\infty \leq C < \infty$.*

Proof. The proof is similar to that for Theorem 3.4. Suppose the contrary: There exists $\lambda_k > 0$, u_k solutions of (P_{λ_k}) such that $\lambda_k \to \mu > 0$, $\mathrm{ind}(u_k) \leq \ell < \infty$ and $\|u_k\|_\infty \to \infty$. Define α, β and $Q_u(x)$ as in (3.2). By assumption, there exists $a_k \in \Omega$ such that $M_k = \max_\Omega Q_{u_k} = Q_{u_k}(a_k) \to \infty$. Denote

$$w_k = M_k^{-\alpha} u_k\left(a_k + \frac{y}{M_k}\right), \quad \forall\, y \in \Omega_k = M_k(\Omega - a_k).$$

Clearly,

$$\Delta^2 w_k(y) = \lambda_k\left[w_k + M_k^{-\alpha}\right]^q \ \text{ in } \Omega_k; \quad w_k = -\Delta w_k = 0 \ \text{ on } \partial\Omega_k. \tag{3.5}$$

We have $w_k > 0$, $-\Delta w_k > 0$ in Ω_k and $\max_{\Omega_k} Q_{w_k} = Q_{w_k}(0) \to 2$. Let $\sigma_k = \mathrm{dist}(x_k, \partial\Omega)$, we have two differents situations: Up to a subsequence, either $\sigma_k M_k \to \infty$ or $\sigma_k M_k \to \gamma \in \mathbb{R}_+$.

In the first case, we obtain a nonnegative function w to $\Delta^2 w = \mu w^q$ in $\mathbb{R}^n$ satisfying $\mathrm{ind}(w) \leq \ell < \infty$ and $Q_w(0) = 2$, hence an entire positive solution with finite Morse index to (1.1), which contradicts Theorem 1.1 seeing the hypothesis on q. Assume now $\sigma_k M_k \to \gamma \in \mathbb{R}_+$. Using orthogonal transformation, we obtain $\widetilde{w}_k$ defined on domains $\widetilde{\Omega}_k$ tending to $\mathbb{R}^n_+$. Furthermore, as in [14], we can transform locally the domain $\widetilde{\Omega}_k$ to $B_{2R,+}$ and consider (3.5) as a second order elliptic system with Dirichlet boundary conditions on $\partial\mathbb{R}^n_+ \cap B_{2R,+}$ (see proof of Theorem 1 in [14]). By classical estimates in [1], we obtain again $\gamma > 0$. Taking the limit, there is a solution to (3.1) with $\mathrm{ind}(w) \leq \ell$ and $\max Q_w = 2$. We reach a contradiction with Theorem 3.4 (ii). $\square$

Remark 3.7. We wonder if the results of Theorem 3.4 (i) holds true for $q = \frac{n+4}{n-4}$. In general, the condition $\mu > 0$ in Theorem 3.6 seems to be necessary. For $q \leq \frac{n+4}{n-4}$, a moutain pass solution u^λ to (P_λ) exists always for any $\lambda \in (0, \lambda^*)$ and $\lim_{\lambda\to 0}\|u^\lambda\|_\infty = \infty$, but we wonder if the compactness result holds true when $\lambda_k \to 0$, for general supercritical exponent q.

In the same spirit of Theorem 3.6, we can prove

Theorem 3.8. *There exists $\widetilde{\epsilon}_n > 0$ such that the extremal solution u^*, the unique solution of (P_{λ^*}) is bounded provided that*

$$n \leq 8, \ \ q > 1 \quad \textit{or} \quad n \geq 9, \ \ 1 < q < \frac{n}{n-8} + \widetilde{\epsilon}_n.$$

Here we need just to consider stable solutions u_λ to (P_λ) since $u^* = \lim_{\lambda\to\lambda^*} u_\lambda$, so the conclusion comes from (ii) of Theorem 1.1 or (ii) of Theorem 3.4. The case $1 < q < \left(\frac{n-8}{n}\right)_+^{-1}$ was proved in [2] by different approach.

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