Regularity-loss property of some degenerately dissipative systems with the self-consistent electromagnetic field

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- **Main results: the case of kinetic plasma**

1. Introduction

1.1 Consider a linear symmetric system of the form

$$\partial_t U + \sum_{j=1}^{n} A^j \partial_{x_j} U + LU = 0,$$

where

- $U = U(t,x) \in \mathbb{R}^m$, $t \geq 0$, $x \in \mathbb{R}^n$,
- $A^j,\ L \in \mathbb{R}^{m\times m}$ **with** $(A^j)^T = A^j$, $L^T = L$, $1 \leq j \leq n$,
- $L \geq 0$. **(**$\ker L \neq 0$**) (No spectral gap!)**

Q: Which kind of conditions can guarantee the time-decay of solutions $U(t) = e^{tB}U_0$**?**

1.2 Shizuta-Kawashima condition:

$\forall\, 0 \neq k = (k_1, \cdots, k_n) \in \mathbb{R}^n$, **every eigenvalue of** $\sum_{j=1}^m k_j A^j$ **does not belong to** $\ker L$**.**

Theorem (Shizuta-Kawashima, '85)

Under the SK condition,

$$|\mathcal{F}\{e^{tB}U_0\}| \leq C e^{-\frac{\lambda|k|^2}{1+|k|^2}t} |\hat{U}_0(k)|,$$

and

$$\|\nabla^\ell e^{tB}U_0\| \leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{\ell}{2}}(\|U_0\|_{L^p} + \|\nabla^\ell U_0\|), \quad 1 \leq p \leq 2.$$

Remark: Under the SK condition, one has the normal energy inequality of the form

$$\|U(t)\|_{H^N}^2 + \int_0^t \|\{\mathbf{I} - \mathbf{P}_L\}U(s)\|_{H^N}^2 + \|\nabla \mathbf{P}_L U(s)\|_{H^{N-1}}^2 ds \leq C\|U_0\|_{H^N}^2.$$

1.3 Examples: the linearized versions of the following nonlinear equations near constant states

- **Euler-system with damping**

$$
\begin{aligned}
&\partial_t \rho + \nabla \cdot v = 0, \\
&\partial_t v + v \cdot \nabla v + \frac{1}{\rho} \nabla p(\rho) = -v.
\end{aligned}
$$

- ***p*-system with relaxation**

$$
\begin{aligned}
&\partial_t v - \partial_x u = 0, \\
&\partial_t u + \partial_x p(v) = \frac{1}{\epsilon}(f(v) - u).
\end{aligned}
$$

- **Jin-Xin model**

$$
\begin{aligned}
&\partial_t u + \partial_x v = 0, \\
&\partial_t v + a^2 \partial_x u = \frac{1}{\epsilon}(f(u) - v).
\end{aligned}
$$

1.4 Further progress (not-complete):

- **Under the SK condition:**
  - **Nonlinear stability of solutions for small H^N initial perturbation: Hanouzet-Natalini ('03), Yong ('04)**
  - **Large-time behavior of solutions for the nonlinear system: Bianchini-Hanouzet-Natalini ('06) (Use the Green's function)**
- **Under conditions where the SK condition is NOT satisfied:**
  - **Global nonlinear stability near constant states: Beauchard-Zuazua ('10) (Use Kalman rank condition to extend the SK condition)**
  - **Global nonlinear stability near constant states (the non-dissipative component is degenerate): Mascia-Natalini ('10) (Use the entropy functional)**

1.5 Recently, we have found a new dissipative feature of the regularity-loss type which shows that

(i) **if** $U_0 \in H^N$ **with** N **properly large, it could occur that there is some component** U_i **of the solution** $U = (U_1, \cdots, U_m)$ **such that**

$$\int_0^t \int_{\mathbb{R}^n} \|\nabla^N U_i(s)\|^2 ds = \infty.$$

(ii) **the semigroup** e^{tB} **has the bound of the form**

$$|\mathcal{F}\{e^{tB}U_0\}| \leq Ce^{-p(k)t}|\hat{U}_0|,$$

where the frequency function $p(k)$ **is positive and smooth over** $k \in \mathbb{R}^n$ **with**

$$p(k) \to 0 \ \text{ as } |k| \to 0, \quad p(k) \to 0 \ \text{ as } |k| \to \infty.$$

1.6 Recall that under the SK condition,

$$p(k) = \frac{\lambda|k|^2}{1+|k|^2}, \quad p(k) \to \lambda > 0 \ \ \textbf{as} \ |k| \to \infty.$$

The new dissipative feature with $p(k) \to 0$ as $|k| \to \infty$ arises from our study of the fluid or kinetic equations with the self-consistent Lorentz force $E + u \times B$ satisfying the Maxwell system

$$\begin{aligned} &\partial_t E - \nabla \times B = -J, \\ &\partial_t B + \nabla \times E = 0, \\ &\nabla \cdot E = \rho, \quad \nabla \cdot B = 0. \end{aligned}$$

Examples:

- **Euler-Maxwell system with relaxation (one-fluid or two-fluid)**
- **Vlasov-Maxwell-Boltzmann system (one-species or two-species) (Boltzmann operator can be extent to the general situation including the Landau operator)**

2. Main results: the case of fluid plasma

2.1 Euler-Maxwell system with relaxation:

$$\begin{cases} \partial_t n + \nabla \cdot (nu) = 0, \\ \partial_t u + u \cdot \nabla u + \dfrac{1}{n} \nabla p(n) = -(E + u \times B) - \nu u, \\ \partial_t E - \nabla \times B = nu, \\ \partial_t B + \nabla \times E = 0, \\ \nabla \cdot E = n_{\mathrm{b}} - n, \quad \nabla \cdot B = 0. \end{cases}$$

Here, $n = n(t,x) \geq 0$**,** $u = u(t,x) \in \mathbb{R}^3$**,** $E = E(t,x) \in \mathbb{R}^3$ **and** $B = B(t,x) \in \mathbb{R}^3$**, for** $t > 0$**,** $x \in \mathbb{R}^3$**, denote the electron density, electron velocity, electric field and magnetic field, respectively. Initial data is given as**

$$[n, u, E, B]|_{t=0} = [n_0, u_0, E_0, B_0], \quad x \in \mathbb{R}^3.$$

2.2 Consider the linearized homogeneous system for $U = [\rho, u, E, B]$**:**

$$\begin{cases} \partial_t \rho + \nabla \cdot u = 0, \\ \partial_t u + \gamma \nabla \rho + E + u = 0, \\ \partial_t E - \nabla \times B - u = 0, \\ \partial_t B + \nabla \times E = 0, \\ \nabla \cdot E = -\rho, \quad \nabla \cdot B = 0, \quad t > 0, x \in \mathbb{R}^3, \end{cases}$$

with given initial data

$$U|_{t=0} = U_0 := [\rho_0, u_0, E_0, B_0], \quad x \in \mathbb{R}^3,$$

satisfying the compatible condition

$$\nabla \cdot E_0 = -\rho_0, \quad \nabla \cdot B_0 = 0.$$

2.3 Estimate on the modulus of $\hat{U}(t,k)$:

Theorem (D., arXiv '10)

Let $U(t,x)$, $t>0, x\in\mathbb{R}^3$ be a well-defined solution to the above linearized system. There is a time-frequency Lyapunov functional $\mathcal{E}(\hat{U}(t,k))$ with

$$\mathcal{E}(\hat{U}) \sim |\hat{U}|^2 := |\hat{\rho}|^2 + |\hat{u}|^2 + |\hat{E}|^2 + |\hat{B}|^2$$

satisfying that there is $\lambda>0$ such that the Lyapunov inequality

$$\frac{d}{dt}\mathcal{E}(\hat{U}(t,k)) + \frac{\lambda|k|^2}{(1+|k|^2)^2}\mathcal{E}(\hat{U}(t,k)) \leq 0$$

holds true for any $t>0$ and $k\in\mathbb{R}^3$.

Proof: Use the energy estimate in the Fourier space. Try to add some L^2 interactive functional into the natually existing one so as to capture the dissipation of all the other degenerate components.

2.4 L^p-L^q time-decay estimate on e^{tB}:

Theorem (D., arXiv '10)

Let $1 \le p, r \le 2 \le q \le \infty$, $\ell \ge 0$ and let $m \ge 0$ be an integer. Then,

$$\|\nabla^m e^{tB} U_0\|_{L^q} \le C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{m}{2}}\|U_0\|_{L^p} + C(1+t)^{-\frac{\ell}{2}}\|\nabla^{m+[\ell+3(\frac{1}{r}-\frac{1}{q})]_+}U_0\|_{L^r}$$

for any $t \ge 0$, where $C = C(p, q, r, \ell, m)$, and

$$[\ell+3(\frac{1}{r}-\frac{1}{q})]_+ = \begin{cases} [\ell+3(\frac{1}{r}-\frac{1}{q})]_- + 1 & \text{when } r \ne 2 \text{ or } q \ne 2 \\ & \text{or } \ell \text{ is not an integer,} \\ \ell & \text{when } r = q = 2 \\ & \text{and } \ell \text{ is an integer,} \end{cases}$$

Remark: Time-decay over the high-frequency domain is gained by putting some extra regularity on initial data.

2.5 Green's function: For $t \geq 0$ and $k \in \mathbb{R}^3$ with $|k| \neq 0$, define the *decomposition*

$$\begin{bmatrix} \hat{\rho}(t,k) \\ \hat{u}(t,k) \\ \hat{E}(t,k) \\ \hat{B}(t,k) \end{bmatrix} = \begin{bmatrix} \hat{\rho}(t,k) \\ \hat{u}_{\|}(t,k) \\ \hat{E}_{\|}(t,k) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \hat{u}_{\perp}(t,k) \\ \hat{E}_{\perp}(t,k) \\ \hat{B}_{\perp}(t,k) \end{bmatrix},$$

where $\hat{u}_{\|}, \hat{u}_{\perp}$ are defined by

$$\hat{u}_{\|} = \tilde{k}\tilde{k}\cdot\hat{u}, \quad \hat{u}_{\perp} = -\tilde{k}\times(\tilde{k}\times\hat{u}) = (\mathbf{I}_3 - \tilde{k}\otimes\tilde{k})\hat{u},$$

Define

$$U^{I} = \mathcal{F}^{-1}\begin{bmatrix} \hat{\rho}(t,k) \\ \hat{u}_{\|}(t,k) \\ \hat{E}_{\|}(t,k) \end{bmatrix}, \quad U^{II} = \mathcal{F}^{-1}\begin{bmatrix} \hat{u}_{\perp}(t,k) \\ \hat{E}_{\perp}(t,k) \\ \hat{B}_{\perp}(t,k) \end{bmatrix}.$$

Then,

$$U = U^{I} + U^{II}.$$

Theorem (D., '10)

U^I, U^{II} satisfies

$$\begin{cases} \partial_t^2 U^I - \gamma\Delta U^I + U^I + \partial_t U^I = 0, \\ \partial_t U^{II} + \begin{pmatrix} \mathbf{I}_3 & \mathbf{I}_3 & 0 \\ -\mathbf{I}_3 & 0 & -\nabla\times \\ 0 & \nabla\times & 0 \end{pmatrix} U^{II} = 0. \end{cases}$$

Furthermore, $\mathcal{F}U^I = G^I_{7\times 7}(t,k)\mathcal{F}U^I_0$ with

$$G^I_{7\times 7} = e^{-\frac{t}{2}}\cos(\sqrt{3/4+\gamma|k|^2}t)\begin{bmatrix} 1 & & \\ & 0_3 & \\ & & 0_3 \end{bmatrix} + e^{-\frac{t}{2}}\frac{\sin(\sqrt{3/4+\gamma|k|^2}t)}{\sqrt{3/4+\gamma|k|^2}}\begin{bmatrix} 1/2 & -ik & 0 \\ -i\gamma k & -1/2\mathbf{I}_3 & -\mathbf{I}_3 \\ 0 & \mathbf{I}_3 & 1/2\mathbf{I}_3 \end{bmatrix}.$$

To solve U^{II}, consider the characteristic equation

$$F(\chi) := \chi^3 + \chi^2 + (1 + |k|^2)\chi + |k|^2 = 0.$$

Lemma

Let $|k| \neq 0$. The equation $F(\chi) = 0$, $\chi \in \mathbb{C}$, has a real root $\sigma = \sigma(|k|) \in (-1, 0)$ and two conjugate complex roots $\chi_\pm = \beta \pm i\omega$ with $\beta = \beta(|k|) \in (-1/2, 0)$ and $\omega = \omega(|k|) \in (\sqrt{6}/3, \infty)$ satisfying

$$\beta = -\frac{\sigma + 1}{2}, \quad \omega = \frac{1}{2}\sqrt{3\sigma^2 + 2\sigma + 3 + 4|k|^2}.$$

σ, β, ω are smooth over $|k| > 0$, and $\sigma(|k|)$ is strictly decreasing in $|k| > 0$ with

$$\lim_{|k| \to 0} \sigma(|k|) = 0, \quad \lim_{|k| \to \infty} \sigma(|k|) = -1.$$

Lemma (cont.)

Mover, the following asymptotic behaviors hold true:

$$\sigma(|k|) = -O(1)|k|^2,$$
$$\beta(|k|) = -\frac{1}{2} + O(1)|k|^2, \quad \omega(|k|) = \frac{\sqrt{3}}{2} + O(1)|k|$$

whenever $|k| \leq 1$ *is small, and*

$$\sigma(|k|) = -1 + O(1)|k|^{-2},$$
$$\beta(|k|) = -O(1)|k|^{-2}, \quad \omega(|k|) = O(1)|k|$$

whenever $|k| \geq 1$ *is large. Here and in the sequel* $O(1)$ *denotes a generic strictly positive constant.*

Theorem (cont.)

Let $M = \mathcal{F}U^{II}$. Then,

$$\begin{aligned} M_1(t,k) = &-\frac{c_1(k)}{1+\sigma}e^{\sigma t} \\ &-\frac{c_2(k)}{(1+\beta)^2+\omega^2}e^{\beta t}\left[(1+\beta)\cos\omega t+\omega\sin\omega t\right] \\ &-\frac{c_3(k)}{(1+\beta)^2+\omega^2}e^{\beta t}\left[(1+\beta)\sin\omega t-\omega\cos\omega t\right], \end{aligned}$$

$$M_2(t,k) = c_1(k)e^{\sigma t} + e^{\beta t}[c_2(k)\cos\omega t + c_3(k)\sin\omega t],$$

$$\begin{aligned} M_3(t,k) = &-ik\times\frac{c_1(k)}{\sigma}e^{\sigma t} \\ &-ik\times\frac{c_2(k)}{\beta^2+\omega^2}e^{\beta t}\left[\beta\cos\omega t+\omega\sin\omega t\right] \\ &-ik\times\frac{c_3(k)}{\beta^2+\omega^2}e^{\beta t}\left[\beta\sin\omega t-\omega\cos\omega t\right], \end{aligned}$$

where $c_i(k)$, $i = 1,2,3$, are determined by initial data $M(0,k)$.

2.6 Asymptotic stability of the nonlinear Cauchy problem:

Theorem (D., arXiv '10)

Let $N \geq 4$. There are $\delta_0 > 0$, C_0 such that if

$$\|[n_0 - 1, u_0, E_0, B_0]\|_N \leq \delta_0,$$

then, the nonlinear Cauchy problem admits a unique global solution with

$$[n - 1, u, E, B] \in C([0, \infty); H^N(\mathbb{R}^3)) \cap Lip([0, \infty); H^{N-1}(\mathbb{R}^3))$$

and

$$\sup_{t \geq 0} \|[n(t) - 1, u(t), E(t), B(t)]\|_N \leq C_0 \|[n_0 - 1, u_0, E_0, B_0]\|_N.$$

Theorem (cont.)

Moreover, there are $\delta_1 > 0$, C_1 *such that if*

$$\|[n_0 - 1, u_0, E_0, B_0]\|_{13} + \|[u_0, E_0, B_0]\|_{L^1} \leq \delta_1,$$

then the solution $[n(t,x), u(t,x), E(t,x), B(t,x)]$ *satisfies that for any* $t \geq 0$,

$$\|n(t) - 1\|_{L^q} \leq C_1(1+t)^{-\frac{11}{4}},$$
$$\|[u(t), E(t)]\|_{L^q} \leq C_1(1+t)^{-2+\frac{3}{2q}},$$
$$\|B(t)\|_{L^q} \leq C_1(1+t)^{-\frac{3}{2}+\frac{3}{2q}},$$

with $2 \leq q \leq \infty$.

2.7 The two-fluid Euler-Maxwell system with relaxation:

$$
\begin{cases}
\partial_t n_\pm + \nabla \cdot (n_\pm u_\pm) = 0, \\
\partial_t u_\pm + u_\pm \cdot \nabla u_\pm + \dfrac{1}{n_\pm} \nabla p_\pm(n_\pm) = \mp (E + u_\pm \times B) - \nu_\pm u_\pm, \\
\partial_t E - \nabla \times B = -(n_+ u_+ - n_- u_-), \\
\partial_t B + \nabla \times E = 0, \\
\nabla \cdot E = n_- - n_+, \quad \nabla \cdot B = 0.
\end{cases}
$$

Q: Is there any dissipative property similar as in the case of the one-fluid EM system?

A: Yes! Two-fluid EM system can be written as the coupling system of the one-fluid EM system and the Euler system without forces.

In fact, suppose $\nu_\pm = \nu > 0$. **Then, the linearized two-fluid EM system near** $(n_\pm = 1, u_\pm = 0, E = B = 0)$ **for**

$$[n_\pm, u_\pm, E, B]$$

$=$ **the one-fluid EM system for** $[\rho_- := \frac{n_+ - n_-}{2}, v_- := \frac{u_+ - u_-}{2}, E, B]$ **satisfying**

$$\begin{cases} \partial_t \rho_- + \nabla \cdot v_- = 0, \\ \partial_t v_- + \nabla \rho_- = -E - \nu v_-, \\ \partial_t E - \nabla \times B = -2 v_-, \\ \partial_t B + \nabla \times E = 0, \\ \nabla \cdot E = -2\rho_-, \quad \nabla \cdot B = 0. \end{cases}$$

$+$**the Euler system without forces for** $[\rho_+ := \frac{n_+ + n_-}{2}, v_+ := \frac{u_+ + u_-}{2}]$ **satisfying**

$$\begin{cases} \partial_t \rho_+ + \nabla \cdot v_+ = 0, \\ \partial_t v_+ + \nabla \rho_+ = -\nu v_+. \end{cases}$$

2.8 Final remark: The linearized Euler-Maxwell system with relaxation is a typical example for the following general system

$$\partial_t U + \sum_{j=1}^{n} A^j \partial_{x_j} U + LU = 0,$$

with

- $U = U(t,x) \in \mathbb{R}^m$**,** $t \geq 0$**,** $x \in \mathbb{R}^n$**,**
- $A^j \in \mathbb{R}^{m \times m}$ **with** $(A^j)^T = A^j$**,** $1 \leq j \leq n$**,**
- $L \in \mathbb{R}^{m \times m}$ **with** $L_1 := \frac{L+L^T}{2} \geq 0$**. (**$\ker L \neq 0$**)**

On-going work with Kawashima-Ueda: Notice that L need not be symmetric and hence could have a non-zero skew-symmetric part. We have found some algebraic condition similar to SK condition under which the solution has the regularity-loss property.

3. Main results: the case of kinetic plasma

3.1 Vlasov-Maxwell-Boltzmann system:

$$
\begin{aligned}
&\partial_t f_+ + \xi \cdot \nabla_x f_+ + (E + \xi \times B) \cdot \nabla_\xi f_+ = Q(f_+, f_+) + Q(f_+, f_-),\\
&\partial_t f_- + \xi \cdot \nabla_x f_- - (E + \xi \times B) \cdot \nabla_\xi f_- = Q(f_-, f_+) + Q(f_-, f_-).
\end{aligned}
$$

It is coupled with the Maxwell system

$$
\begin{aligned}
&\partial_t E - \nabla_x \times B = -\int_{\mathbb{R}^3} \xi (f_+ - f_-) d\xi,\\
&\partial_t B + \nabla_x \times E = 0,\\
&\nabla_x \cdot E = \int_{\mathbb{R}^3} (f_+ - f_-) d\xi, \quad \nabla_x \cdot B = 0.
\end{aligned}
$$

The initial data in this system is given as

$$
f_\pm(0, x, \xi) = f_{0,\pm}(x, \xi), \quad E(0, x) = E_0(x), \quad B(0, x) = B_0(x).
$$

3.2 Known results on the asymptotic stability of near-equilibrium solutions to the two-species VMB system

- $x \in \mathbb{T}^3$
  - Global existence: Guo ('03) (Energy method)
  - Large-time behavior of solutions: Strain-Guo ('06&'08) (Time-velocity splitting)
- $x \in \mathbb{R}^3$
  - Global existence: Strain ('06) (Use two-species' cancelation property to control E)
  - Large-time behavior of solutions: Duan-Strain ('10) (Contruct time-frequency functionals + bootstrap to the nonlinear equation)

Remark: For the case of one-species VMB system, D. ('10) also obtained the global existence. When there is no cancelation, more delicate Lyapunov functionals are desgined to control the dissipation of the electromagnetic field.

3.3 Define the perturbation u as

$$u = \frac{f - \mathbf{M}}{\mathbf{M}^{1/2}}, \quad u = [u_+, u_-], \ f = [f_+, f_-].$$

The linearized homogeneous system takes the form of

$$\begin{cases} \partial_t u + \xi \cdot \nabla_x u - E \cdot \xi \mathbf{M}^{1/2}[1, -1] = \mathbf{L} u, \\ \partial_t E - \nabla_x \times B = -\langle [\xi, -\xi]\mathbf{M}^{1/2}, \{\mathbf{I} - \mathbf{P}\} u \rangle, \\ \partial_t B + \nabla_x \times E = 0, \\ \nabla_x \cdot E = \langle \mathbf{M}^{1/2}, u_+ - u_- \rangle, \quad \nabla_x \cdot B = 0, \\ [u, E, B]|_{t=0} = [u_0, E_0, B_0], \end{cases}$$

Theorem (D.-Strain, '10)

There is a time-frequency functional $\mathcal{E}(t,k)$ such that

$$\mathcal{E}(t,k) \sim \|\hat{u}\|_{L^2_\xi}^2 + |\hat{E}|^2 + |\hat{B}|^2,$$

and

$$\partial_t \mathcal{E}(t,k) + \frac{\lambda |k|^2}{(1+|k|^2)^2} \mathcal{E}(t,k) \le 0, \quad \forall\, t \ge 0, \ k \in \mathbb{R}^3.$$

Remark: The above time-frequency Lyapunov inequality implies the L^p-L^q time-decay property of the solution semigroup similar to the EM system.

3.4 Key idea in the proof: One can use the energy estimate in the Fourier space and construct some L^2 functionals so as to capture the dissipation of all the degenerate components in the solution.

Lemma

There is an interactive time-frequency functional $\mathcal{E}_{\rm int}(t,k)$ such that

$$|\mathcal{E}_{\rm int}(t,k)| \leq C(\|\hat{u}\|_{L^2_\xi}^2 + |\hat{E}|^2 + |\hat{B}|^2),$$

and

$$\partial_t \mathcal{E}_{\rm int}(t,k) + \lambda \|\nu^{1/2}\{\mathbf{I}-\mathbf{P}\}\hat{u}\|_{L^2_\xi}^2 + \frac{\lambda|k|^2}{1+|k|^2}\|\mathbf{P}\hat{u}\|_{L^2_\xi}^2 + \lambda|k\cdot\hat{E}|^2 + \frac{\lambda|k|^2}{(1+|k|^2)^2}(|\hat{E}|^2+|\hat{B}|^2) \leq 0.$$

3.5 Time-decay of the nonlinear VMB system: Define the instant full energy functional $\mathcal{E}_{N,m}(t)$ and the instant high-order energy functional $\mathcal{E}^{\mathrm{h}}_{N,m}(t)$, respectively, as

$$\mathcal{E}_{N,m}(t) \sim \sum_{|\alpha|+|\beta|\leq N} \|\nu^{\frac{m}{2}}\partial^{\alpha}_{\beta}u(t)\|^2 + \sum_{|\alpha|\leq N}\|\partial^{\alpha}[E(t),B(t)]\|^2,$$

$$\mathcal{E}^{\mathrm{h}}_{N,m}(t) \sim \sum_{1\leq|\alpha|\leq N} \|\nu^{\frac{m}{2}}\partial^{\alpha}u(t)\|^2 + \sum_{|\alpha|+|\beta|\leq N}\|\nu^{\frac{m}{2}}\partial^{\alpha}_{\beta}\{\mathbf{I}-\mathbf{P}\}u(t)\|^2 + \sum_{1\leq|\alpha|\leq N}\|\partial^{\alpha}[E(t),B(t)]\|^2,$$

and we define the dissipation rate $\mathcal{D}_{N,m}(t)$ as

$$\mathcal{D}_{N,m}(t) = \sum_{1\leq|\alpha|\leq N} \|\nu^{\frac{m+1}{2}}\partial^{\alpha}u(t)\|^2 + \sum_{|\alpha|+|\beta|\leq N}\|\nu^{\frac{m+1}{2}}\partial^{\alpha}_{\beta}\{\mathbf{I}-\mathbf{P}\}u(t)\|^2 + \sum_{1\leq|\alpha|+|\beta|\leq N-1}\|\partial^{\alpha}[E(t),B(t)]\|^2 + \|E(t)\|^2,$$

for integers N and m. For brevity, we write $\mathcal{E}_N(t)=\mathcal{E}_{N,0}(t)$, $\mathcal{E}^{\mathrm{h}}_N(t)=\mathcal{E}^{\mathrm{h}}_{N,0}(t)$ and $\mathcal{D}_N(t)=\mathcal{D}_{N,0}(t)$ when $m=0$.

Theorem (D.-Strain, '10)

Under some smallness conditions,

$$\frac{d}{dt}\mathcal{E}_{N,m}(t) + \lambda \mathcal{D}_{N,m}(t) \leq 0,$$

$$\frac{d}{dt}\mathcal{E}^{\mathrm{h}}_{N}(t) + \lambda \mathcal{D}_{N}(t) \leq C\|\nabla_x \mathbf{P} u(t)\|^2.$$

Moreover, define $\epsilon_{j,m} = \mathcal{E}_{j,m}(0) + \|u_0\|^2_{Z_1} + \|[E_0, B_0]\|^2_{L^1}$. *Then, under some smallness conditions,*

$$\mathcal{E}_{N,m}(t) \leq C\epsilon_{N+2,m}(1+t)^{-\frac{3}{2}},$$

$$\mathcal{E}^{\mathrm{h}}_{N}(t) \leq C\epsilon_{N+5,1}(1+t)^{-\frac{5}{2}}.$$

Finally, for $1 \leq r \leq 2$,

$$\|u(t)\|_{Z_r} + \|B(t)\|_{L^r_x} \leq C(1+t)^{-\frac{3}{2}+\frac{3}{2r}},$$

$$\|\{\mathbf{I}-\mathbf{P}\}u(t)\|_{Z_r} + \|\langle[1,-1]\mathbf{M}^{1/2}, u(t)\rangle\|_{L^r_x} + \|E(t)\|_{L^r_x} \leq C(1+t)^{-\frac{3}{2}+\frac{1}{2r}}.$$

3.6 One-species VMB:

$$
\begin{cases}
\partial_t f + \xi \cdot \nabla_x f + (E + \xi \times B) \cdot \nabla_\xi f = Q(f, f), \\
\partial_t E - \nabla_x \times B = - \int_{\mathbb{R}^3} \xi f \, d\xi, \\
\partial_t B + \nabla_x \times E = 0, \\
\nabla \cdot E = \int_{\mathbb{R}^3} f \, d\xi - n_{\mathrm{b}}, \quad \nabla_x \cdot B = 0.
\end{cases}
$$

The key difference of one-species case with two-species lies in the fact that the coupling term in the source of the Maxwell system

$$
- \int_{\mathbb{R}^3} \xi f(t, x, \xi) d\xi
$$

corresponds to the momentum component of the macroscopic part of the solution which is degenerate with respect to the linearized operator L.

Theorem (D., '10)

Let $N \geq 4$. Define

$$\mathcal{E}_N(U(t)) \sim \|u(t)\|^2_{H^N_{x,\xi}} + \|[E(t), B(t)]\|^2_{H^N},$$

$$\mathcal{D}_N(U(t)) = \|\nu^{1/2}\{\mathbf{I} - \mathbf{P}\}u(t)\|^2_{H^N_{x,\xi}} + \|\nu^{1/2}\nabla_x u(t)\|^2_{L^2_\xi(H^{N-1}_x)}$$
$$+\|\nabla_x E(t)\|^2_{H^{N-2}} + \|\nabla_x^2 B(t)\|^2_{H^{N-3}}.$$

Suppose $f_0 = \mathbf{M} + \mathbf{M}^{1/2}u_0 \geq 0$. There indeed exists $\mathcal{E}_N(U(t))$ such that if initial data $U_0 = [u_0, E_0, B_0]$ satisfies the compatible condition at $t = 0$ and $\mathcal{E}_N(U_0)$ is sufficiently small, then the nonlinear Cauchy problem admits a global solution $U = [u, E, B]$ satisfying

$$f(t, x, \xi) = \mathbf{M} + \mathbf{M}^{1/2}u(t, x, \xi) \geq 0,$$
$$[u(t), E(t), B(t)] \in C([0, \infty); H^N_{x,\xi} \times H^N \times H^N),$$

and

$$\mathcal{E}_N(U(t)) + \lambda \int_0^t \mathcal{D}_N(U(s))ds \leq \mathcal{E}_N(U_0), \quad \forall\, t \geq 0.$$

3.7 A summary for the dissipation property of kinetic plasma model:

	$\mathcal{E}(t,k)\sim$	$\mathcal{D}(t,k)=$	$\|u(t)\|_{L^2}\leq$
BE	$\|\hat{u}\|^2_{L^2_\xi}$	$\|\nu^{1/2}\{\mathrm{I}-\mathrm{P}\}\hat{u}\|^2_{L^2_\xi} + \frac{\|k\|^2}{1+\|k\|^2}\|[\hat{a},\hat{b},\hat{c}]\|^2$	$C(1+t)^{-\frac{3}{4}}\|u_0\|_{Z_1\cap L^2}$
1-s VPB	$\|\hat{u}\|^2_{L^2_\xi}+\frac{\|\hat{a}\|^2}{\|k\|^2}$	$\|\nu^{1/2}\{\mathrm{I}-\mathrm{P}\}\hat{u}\|^2_{L^2_\xi} + \frac{\|k\|^2}{1+\|k\|^2}\|[\hat{a},\hat{b},\hat{c}]\|^2+\|\hat{a}\|^2$	$C(1+t)^{-\frac{1}{4}}\|u_0\|_{Z_1\cap L^2}$
1-s VMB	$\|\hat{u}\|^2_{L^2_\xi}+\|[\hat{E},\hat{B}]\|^2$	$\|\nu^{1/2}\{\mathrm{I}-\mathrm{P}\}\hat{u}\|_{L^2_\xi} + \frac{\|k\|^2}{1+\|k\|^2}\|[\hat{a},\hat{b},\hat{c}]\|^2+\|k\cdot\hat{E}\|^2 + \frac{\|k\|^2}{(1+\|k\|^2)^2}\|\hat{E}\|^2+\frac{\|k\|^4}{(1+\|k\|^2)^3}\|\hat{B}\|^2$	$C(1+t)^{-\frac{3}{8}} \times(\|U_0\|_{\mathcal{Z}_1}+\|\nabla_x U_0\|_{\mathcal{Z}_2})$
2-s VMB	$\|\hat{u}\|^2_{L^2_\xi}+\|[\hat{E},\hat{B}]\|^2$	$\|\nu^{1/2}\{\mathrm{I}-\mathrm{P}\}\hat{u}\|_{L^2_\xi} + \frac{\|k\|^2}{1+\|k\|^2}\|[\hat{a}_\pm,\hat{b},\hat{c}]\|^2+\|k\cdot\hat{E}\|^2 + \frac{\|k\|^2}{(1+\|k\|^2)^2}(\|\hat{E}\|^2+\|\hat{B}\|^2)$	$C(1+t)^{-\frac{3}{4}} \times(\|U_0\|_{\mathcal{Z}_1}+\|\nabla_x^2 U_0\|_{\mathcal{Z}_2})$

Table: Dissipative and time-decay properties of different models

Remark: The developed approach here is also applicable to other situations such as the Landau operator and the relativistic model.

3.8 Open problem: Green's function of the VMB system.

Thanks for your attention!