Optimal Convergence Rates for the Boltzmann Equation with Potential Forces

Renjun Duan

Department of Mathematics, City University of Hong Kong

This is a joint work with
Professors ***Seiji Ukai, Tong Yang*** *and* ***Huijiang Zhao***

International Conference on Conservation Laws & Kinetic equations
Shanghai Jiao Tong University, December 7-12, 2006

Outline

1. Introduction

2. Main results
    - Problem considered
    - Previous works
    - Main results and ideas of proofs: NS and BE
    - Generalization and further problems

Outline

1. **Introduction**
2. **Main results**
    - **Problem considered**
    - **Previous works**
    - **Main results and ideas of proofs: NS and BE**
    - **Generalization and further problems**

1. Introduction

Basic facts in kinetic theory:

- **A gas in normal conditions is formed of elastic molecules rushing hither and hither at high speed and rebounding according to the *Newton laws of mechanics*:**

$$\dot{\vec{x}}_i = \vec{\xi}_i, \quad \dot{\vec{\xi}}_i = \vec{F}_i, \quad i = 1, 2, \cdots, N;$$

- **There are about $N \approx 2.7 \times 10^{19}$ numbers of molecules in a cubic centimeter of a gas at atmosphere pressure and a temperature of 0°C;**
- **It is *a hopeless task* to attempt to describe the state of the gas by specifying the microscopic state (i.e., position and velocity) for each molecule; we must turn to statistics.**

1. Introduction

Kinetic theory (classical):

"The fundamental assumption of the Kinetic Theory is that all macroscopic observable properties of a substance can be deduced, in principle, from a knowledge of

- *the forces of interaction and*
- *the internal structure of its molecules."*[1]

[1] Ref.: Harold Grad, On the kinetic theory of rarefied gases, CPAM, '49.

Statistical descriptions for gas particles:

- **Introduce a** ***number density*****, or** ***velocity distribution function***

$$f(t, x, \xi),$$

where

$f(t,x,\xi)dxd\xi =$ **"the number of molecules contained in the infinitesimal volume** $dxd\xi$ **centered at the point** (x,ξ) **of the** **single particle phase space** $\mathbb{R}^3_x \times \mathbb{R}^3_\xi$ **at time** t**".**

1. Introduction

- **Gas molecules in an equilibrium state:** $f = \mathbf{M}$ **Maxwellian ditribution—**

$$\mathbf{M} = \mathbf{M}_{[\rho,u,\theta]}(\xi) = \frac{\rho}{(2\pi R\theta)^{n/2}} \exp\left(-\frac{|\xi - u|^2}{2R\theta}\right).$$

(J. C. Maxwell, 1858, 1866)

- **Gas molecules in a non-equilibrium state:**

For the rarefied monoatomic gas, an evolution equation for $f(t,x,\xi)$ was derived by Ludwig Boltzmann in 1872:

$$\partial_t f + \xi \cdot \nabla_x f = Q(f,f).$$

Main assumptions for the Boltzmann gas:

- **Point molecules:** $N \to \infty$, $\sigma \to 0$. (*Perfect gas*)
- **Collision happens:** $N\sigma^2 \to b > 0$. (*Collisions are significant*)
- **Gas is rarefied:** $N\sigma^3 \to 0$. (*Only binary collisions are important*)
- **Gas particles are elastic.** (*Collisions preserve mass, momentum and energy*)
- **Assumption of molecular chaos.** (*States of two molecules that are about to collide are statistically uncorrected*)

1. Introduction

Collision operator:

$$Q(f,g)(\xi) = \frac{1}{2}\int_{\mathbb{R}^n\times S^{n-1}} B(\omega, |\xi-\xi_*|)(f'g'_* + f'_*g' - fg_* - f_*g)d\xi_*d\omega$$

where

$$f = f(\xi), \quad f_* = f(\xi_*), \quad f' = f(\xi'), \quad f'_* = f(\xi'_*)$$

likewise for g,

$$\xi' = \xi - [(\xi-\xi_*)\cdot\omega]\omega, \quad \xi'_* = \xi_* + [(\xi-\xi_*)\cdot\omega]\omega,$$

and

$$B(\omega, |\xi-\xi_*|) = \widetilde{B}(\cos\theta, |\xi-\xi_*|) \geq 0, \text{ collision kernel}$$

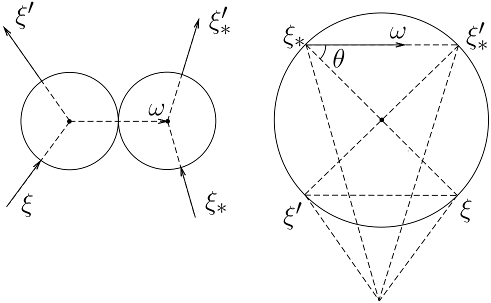
$$\cos\theta = \frac{\xi-\xi_*}{|\xi-\xi_*|}\cdot\omega, \qquad \omega\in S^{n-1}$$

Remarks:

- **Collision mechanism at the microscopic level:**

$$\xi' + \xi'_* = \xi + \xi_*,$$
$$|\xi'|^2 + |\xi'_*|^2 = |\xi|^2 + |\xi_*|^2,$$

Conservations of momentum and energy



1. Introduction

- **An identity:** **for any "good " $\phi = \phi(\xi)$ and $f = f(\xi)$,**

$$\int Q(f,f)\phi d\xi =$$

$$\frac{1}{4}\iiint (f'f'_* - ff_*)(\phi + \phi_* - \phi' - \phi'_*)B(\omega, |\xi - \xi_*|)d\omega d\xi d\xi_*,$$

which gives

Collision invariants; local conservation laws; Boltzmann inequality; Boltzmann's H-theorem (the second law of thermo-dynamics),...

Collision kernel: **Two classical physical models**

— **Hard-sphere gas:**

$$B(\omega, |\xi - \xi_*|) = |(\xi - \xi_*) \cdot \omega|$$

— **Potential of inverse power s :**

$$B(\omega, |\xi - \xi_*|) \sim |\xi - \xi_*|^{\gamma_s} b_{\gamma_s'}(\cos\theta)$$

Grad's cut-off assumption—**(mathematically)**

$$\int_{S^{n-1}} b_{\gamma_s'}(\cos\theta) d\omega = |S^{n-2}| \int_0^\pi b_{\gamma_s'}(\cos\theta) \sin^{n-2}\theta d\theta < +\infty$$

Physically, $= \infty$ **(non cut-off case),...**

1. Introduction

Relations between the BE and other equations:

Regime	Equations	Scale	Unknown
Molecular	Newton	Micro	(x_i, ξ_i)
	$\Downarrow$ *(Boltzamnn-Grad limit)*		
Kinetic	Boltzmann (V/P/M)	Meso: Kn $= O(1)$	$f(t, x, \xi)$
	$\Downarrow$ *(fluid dynamics limit)*		
Fluid	Euler, Navier-Stokes	Macro: Kn $\ll 1$	$(\rho, u, \theta)(t, x)$

Remarks:

- **Validity of the Boltzmann equation: O. Lanford ('75),...**
- **Some equations are beyond the above framework:**
  - Non-classical fluid dynamics equations (Ghost effect, Y. Sone).
  - Models of turbulence (large Reynold number),...

Outline

2. Main results

Consider the Boltzmann equation in $\mathbb{R}^N$ with a potential force

$$\partial_t f + \xi \cdot \nabla_x f - \nabla_x \Phi(x) \cdot \nabla_\xi f = Q(f,f),$$
$$|\Phi(x)| \to 0, \quad \textbf{as} \quad |x| \to +\infty,$$

for the hard-sphere model

$$B(\omega, |\xi - \xi_*|) = |(\xi - \xi_*) \cdot \omega|.$$

Initial data is given by

$$f(0,x,\xi) = f_0(x,\xi).$$

2. Main results

Stationary solution (*local Maxwellian*):

$$\overline{\mathbf{M}} = \frac{\rho_\infty}{(2R\theta_\infty)^{N/2}} \exp\left\{-\frac{1}{R\theta_\infty}\left(\Phi(x) + \frac{1}{2}|\xi|^2\right)\right\}$$
$$= \mathbf{M}_{[\tilde{\rho}(x),0,\theta_\infty]}(\xi) = \tilde{\rho}(x)\mathbf{M}_\infty(\xi),$$

where

$$\tilde{\rho}(x) = \rho_\infty \exp\left(-\frac{\Phi(x)}{R\theta_\infty}\right) \to \rho_\infty.$$

Aim:

- **Stability of the stationary solution.**
- **Optimal convergence rate in some $L^2_\xi(L^p_x)$ spaces.**

Outline

1. Introduction
2. **Main results**
   - Problem considered
   - **Previous works**
   - Main results and ideas of proofs: NS and BE
   - Generalization and further problems

Previous works on the convergence rate of the solution to the steady states for large time:

- **Without forces:**
  - Exponential convergence rate in bounded domain and torus: Ukai ('74), Giraud ('75), Shizuta-Asano ('77),...
  - Algebraic convergence rate in unbounded domain: Ukai ('76), Nishida-Imai ('76), Ukai-Asano ('83),...
  - Almost exponential convergence rate: Strain-Guo ('05), Desvillettes-Villani ('05)
  - Optimal convergence rate: Ukai-Yang ('06)
- **With forces:**
  - Convergence rate in L^∞ framework: Asano ('02),...
  - Convergence rate in L^2 framework: Ukai-Yang-Zhao ('05),...

Remarks.

(I) For the spatially homogeneous BE,

- Exponential convergence rate: Arkeryd ('88), Wennberg ('93)

(II)For the compressible Navier-Stokes system,

- **Without forces:** Matsumura-Nishida ('79), Ponce ('85), Hoff-Zumbrun ('97), Liu-Wang ('98), Kobayashi-Shibata ('99), Kagei-Kobayashi ('05), ...
- **With forces:**
  - Slow rate: Deckelnick ('92), Shibata-Tanaka ('03)(for general forces)
  - Almost optimal rate: Ukai-Yang-Zhao ('06),...
  - Optimal rate: Duan-Ukai-Yang-Zhao ('06), Duan-Ukai-Yang-Liu ('06)

Difficulties:

When the force appears,

- the steady state is dependent of t and x (the stationary potential force produces the **LOCAL** equilibrium);
- the momentum and energy are **NOT** conservative;
- the existence of the steady state is **NOT** known for the general force.

Outline

1. Introduction
2. **Main results**
   - Problem considered
   - Previous works
   - **Main results and ideas of proofs: NS and BE**
   - Generalization and further problems

2. Main results

Compressible Navier-Stokes equations:

$$\begin{cases} \rho_t + \nabla_x \cdot (\rho u) = 0, \\ (\rho u^i)_t + \nabla_x \cdot (\rho u^i u) + P_{x_i} \\ = \sum_j [\mu(\theta)(u^i_{x_j} + u^j_{x_i} - \frac{2}{N}\delta_{ij}\nabla_x \cdot u)]_{x_j} + \rho F^i, \quad i = 1, \cdots, N \\ \left[\rho\left(\frac{1}{2}|u|^2 + \frac{N}{2}R\theta\right)\right]_t + \nabla_x \cdot \left(\left[\rho\left(\frac{1}{2}|u|^2 + \frac{N}{2}R\theta\right) + P\right] u\right) \\ = \sum_j (\kappa(\theta)\theta_{x_j})_{x_j} + \sum_{ij} \left[\mu(\theta)u^i \left(u^i_{x_j} + u^j_{x_i} - \frac{2}{N}\delta_{ij}\nabla_x \cdot u\right)\right]_{x_j} + \rho u \cdot F, \end{cases}$$

with

$$(\rho, u, \theta)(0, x) = (\rho_0, u_0, \theta_0)(x) \to (\rho_\infty, 0, \theta_\infty) \quad \text{as} \quad |x| \to \infty.$$

where $P = P(\rho, \theta)$: pressure, F: external force, μ and κ: viscosity and heat-conduction coefficients.

2. Main results

Consider the time-independent potential force

$$F = -\nabla_x \Phi(x).$$

Stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\theta})(x)$:

$$\int_{\rho_\infty}^{\tilde{\rho}(x)} \frac{P_\rho(\eta, \theta_\infty)}{\eta} d\eta + \Phi(x) = 0, \quad \tilde{u}(x) = 0, \quad \tilde{\theta}(x) = \theta_\infty,$$

Remark: $|\Phi(x)| \ll 1 \Rightarrow |\tilde{\rho} - \rho_\infty| \ll 1$:

$$\|\tilde{\rho} - \rho_\infty\|_l \leq C\|\Phi\|_l, \quad 0 \leq l \leq 5.$$

Q: Convergence rates of solutions to the stationary solution under the small perturbation.

2. Main results

For small perturbation, w.l.g., consider ($N = 3$):

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ \rho\left[u_t + (u \cdot \nabla)u\right] + \nabla P(\rho, \theta) = \mu \Delta u + (\mu + \mu')\nabla(\nabla \cdot u) - \rho \nabla_x \Phi(x), \\ \rho c_\nu [\theta_t + (u \cdot \nabla)\theta] + \theta P_\theta(\rho, \theta)\nabla \cdot u = \kappa \Delta \theta + \Psi(u). \end{cases}$$

where $\mu > 0$, μ', $\kappa > 0$ and $c_\nu > 0$ are constants with $\mu' + \frac{2}{3}\mu \geq 0$ and

$$\Psi(u) = \frac{\mu}{2} \sum_{ij} \left(u^j_{x_i} + u^i_{x_j} \right)^2 + \mu' \sum_j \left(u^j_{x_j} \right)^2 .$$

Assumptions:

- $\rho_\infty, \theta_\infty > 0$, $P_\rho(\rho_\infty, \theta_\infty) > 0$ **and** $P_\theta(\rho_\infty, \theta_\infty) > 0$.
- $\|\Phi\|_5 \ll 1$.

Stability Proposition (by Matsumura-Nishida):

There exist constants $C_0 > 0$ and $\epsilon_0 > 0$ such that if

$$\|(\rho_0 - \rho_\infty, u_0, \theta_0 - \theta_\infty)\|_3 + \|\Phi\|_5 \leq \epsilon_0,$$

then the initial value problem for NS has a unique solution (ρ, u, θ) globally in time and a unique stationary state $(\tilde{\rho}, 0, \theta_\infty)$, which satisfy

$$\rho - \tilde{\rho} \in C^0(0, \infty; H^3(\mathbb{R}^3)) \cap C^1(0, \infty; H^2(\mathbb{R}^3)),$$
$$u, \theta - \theta_\infty \in C^0(0, \infty; H^3(\mathbb{R}^3)) \cap C^1(0, \infty; H^1(\mathbb{R}^3)),$$

and

$$\begin{aligned} &\|(\rho - \tilde{\rho}, u, \theta - \theta_\infty)(t)\|_3^2 \\ &+ \int_0^t (\|\nabla(\rho - \tilde{\rho}, u, \theta - \theta_\infty)(s)\|_2^2 + \|\nabla(u, \theta - \theta_\infty)(s)\|_3^2)\, ds \\ &\leq C_0 \|(\rho_0 - \tilde{\rho}, u_0, \theta_0 - \theta_\infty)\|_3^2. \end{aligned}$$

2. Main results

Theorem for Optimal L^p-L^q Convergence Rates:

Let C_0 **and** ϵ_0 **be defined above. Let** $1 \leq p < 6/5$**. Suppose**

$$\|\Phi\|_{L^{\frac{2p}{2-p}} \cap L^\infty} + \sum_{k=1}^{4} \|(1+|x|)\nabla^k \Phi\|_{L^{\frac{2p}{2-p}} \cap L^\infty} \leq \epsilon,$$

for some constant $\epsilon > 0$**, and**

$$\|(\rho_0 - \tilde{\rho}, u_0, \theta_0 - \theta_\infty)\|_{L^p} < +\infty.$$

Then if $\epsilon \in (0, \epsilon_0)$ **is small enough, it holds that**

$$\|\nabla^k(\rho - \tilde{\rho}, u, \theta - \theta_\infty)(t)\|_{L^2} \leq C(1+t)^{-\sigma(p,2;1)}, \quad k = 1, 2, 3,$$
$$\|(\rho - \tilde{\rho}, u, \theta - \theta_\infty)(t)\|_{L^q} \leq C(1+t)^{-\sigma(p,q;0)}, \quad 2 \leq q \leq 6,$$

for any $t \geq 0$**, where** $C > 0$ **is some constant and**

$$\sigma(p, q; k) = \frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{k}{2}.$$

Remark. If we linearize the equations at the constant state $(\rho_\infty, 0, \theta_\infty)$

$$\begin{cases} \rho_t + \gamma \nabla \cdot u = 0, \\ u_t - \mu_1 \Delta u - \mu_2 \nabla \nabla \cdot u + \gamma \nabla \rho + \lambda \nabla \theta = 0, \\ \theta_t - \bar{\kappa} \Delta \theta + \lambda \nabla \cdot u = 0, \end{cases}$$

then the theorem shows that for the bounded L^p initial perturbation with $1 \leq p < 6/5$, the optimal L^q decay rate holds for the solution itself with $2 \leq q \leq 6$, and all first-order derivatives with $q = 2$.

Main Ideas:

- **(I) Use energy estimates to find a Lyapunov type inequality in the form of**

$$\frac{dH(t)}{dt} + CH(t) \leq C\|\nabla U(t)\|^2,$$

(Dissipation: Viscosity + Heat conduction)

where $H(t)$ is an energy functional including all derivatives of at least one order:

$$H(t) \sim \|\nabla U(t)\|_2^2,$$

and U denotes the perturbation:

$$U = (\rho - \tilde{\rho}, u, \theta - \theta_\infty).$$

2. Main results

- (II) Reformulate the system as the linearized version

$$U_t + AU = S[U],$$

where A: differential operator with constant coefficients, $S[U]$: source term (nonlinear+small linear). Mild form:

$$U(t) = E(t)U_0 + \int_0^t E(t-s)S[U](s)ds,$$

where $E(t) = e^{-tA}$. Based on the optimal L^p-L^2 time-decay estimates on the linearized system, the first order derivative $\|\nabla U(t)\|^2$ can be bounded by $H(t)$ in some time-weighted integral form with a small coefficient.

- Combine (I) and (II) and use the Gronwall's inequality to give the decay rate of $H(t)$, i.e. L^p-L^2 rate for derivatives. L^p-L^q rate follows from the interpolation.

Remarks.

- Φ can be taken in the form

$$\Phi(x) = \frac{\epsilon}{(1+|x|)^{1+\delta}},$$

where $\epsilon > 0$ is small enough, and

$$\delta > 3/p - 5/2.$$

- For the optimal convergence rate on the Boltzmann equation, the same idea can be applied, together with more delicate analysis for the macroscopic and microscopic parts of the solution by using the Macro-Micro decomposition.

2. Main results

L^2 energy method:

- Liu-Yu '04, Liu-Yang-Yu '04, Ukai-Yang-Zhao '05, Yang-Zhao '05
- Guo '04, Guo-Strain '06

Macro-Micro decomposition: Given any Maxwellian

$$\mathbf{M} = \mathbf{M}_{[\rho,u,\theta]}(\xi) = \frac{\rho}{(2\pi R\theta)^{N/2}} \exp\left(-\frac{|\xi-u|^2}{2R\theta}\right),$$

define the inner product in $L^2_\xi(\mathbf{M}^{-1}d\xi)$ associated with $\mathbf{M}$:

$$\langle h, g\rangle_{\mathbf{M}} \equiv \int_{\mathbf{R}^N} \mathbf{M}^{-1} h(\xi) g(\xi) d\xi,$$

and the corresponding normal orthogonal basis:

$$\chi_i^{\mathbf{M}}, \quad i = 0, 1, \cdots, N,$$

for the finite dimensional linear subspace:

$$\mathcal{N} = \mathrm{Span}\{\mathbf{M},\ \xi_i\mathbf{M}, i = 1, \cdots, N,\ |\xi|^2\mathbf{M}\},$$

Macroscopic and Microscopic projections:

$$\mathbf{P}_0^{\mathbf{M}} : L^2(M^{-1}d\xi) \to \mathcal{N}, \quad \mathbf{P}_0^{\mathbf{M}} h \equiv \sum_{\alpha=0}^{N+1} \langle h, \chi_\alpha^{\mathbf{M}} \rangle_{\mathbf{M}} \chi_\alpha^{\mathbf{M}},$$

$$\mathbf{P}_1^{\mathbf{M}} : L^2(M^{-1}d\xi) \to \mathcal{N}^{\perp}, \ \mathbf{P}_1^{\mathbf{M}} = \mathbf{I} - \mathbf{P}_0^{\mathbf{M}}.$$

Let $\mathbf{M}$ be the local Maxwellian determined by the solution to the Boltzmann equation, then

$$f = \mathbf{M} + \mathbf{G}, \quad \mathbf{M} \in \mathcal{N}, \ \mathbf{G} \in \mathcal{N}^{\perp},$$

with

$$\mathbf{M} = \mathbf{P}_0^{\mathbf{M}} f = \mathbf{P}_0^{\mathbf{M}} \mathbf{M}, \quad \mathbf{G} = \mathbf{P}_1^{\mathbf{M}} f = \mathbf{P}_1^{\mathbf{M}} \mathbf{G}.$$

Theorem for Nonlinear Stability:

Assume $f_0(x,\xi) \geq 0$**,** $s \geq N+1$**. Then,** $\exists\, \delta_0 > 0, \epsilon_0 > 0$ **s.t. if**

$$\|\Phi\|_{s+1} + \sum_{|\alpha|+|\beta|\leq s} \left\| \frac{\partial_x^\alpha \partial_\xi^\beta (f_0 - \overline{\mathbf{M}})}{\sqrt{\mathbf{M}_\infty}} \right\|_{L^2_{x,\xi}} \leq \delta_0,$$

then $\exists!$ **global classical solution** $0 \leq f(t,x,\xi) \in H^s(\mathbf{M}_\infty)$ **s.t.**

$$H_1(t) + \int_0^t D_1(\tau) d\tau \leq \epsilon_0^2,$$

where

$$H_1(t) = \|(\rho - \tilde{\rho}, u, \theta - \theta_\infty)(t)\|_s^2 + \sum_{|\gamma|+|\beta|\leq s} \left\| \frac{\partial_\beta^\gamma \mathbf{P}_1^{\mathbf{M}_\infty} f(t)}{\sqrt{\mathbf{M}_\infty}} \right\|_{L^2_{x,\xi}}^2,$$

$$D_1(t) = \|\nabla_x(\rho - \tilde{\rho}, u, \theta - \theta_\infty)(t)\|_s^2 + \sum_{|\gamma|+|\beta|\leq s} \left\| \frac{\sqrt{\nu_{\mathbf{M}_\infty}} \partial_\beta^\gamma \mathbf{P}_1^{\mathbf{M}_\infty} f(t)}{\sqrt{\mathbf{M}_\infty}} \right\|_{L^2_{x,\xi}},$$

2. Main results

and

$$H^s(\mathbf{M}_\infty) = \left\{ f \,\middle|\, \frac{\partial^\gamma_\beta (f - \overline{\mathbf{M}})}{\sqrt{\mathbf{M}_\infty}} \in \mathrm{BC}_t(\mathbb{R}^+, L^2_{x,\xi}), |\gamma| + |\beta| \leq s \right\},$$

$$\partial^\gamma_\beta = \partial^{\gamma_0}_t \partial^{\gamma_1}_x \partial^\beta_\xi, \quad \gamma = (\gamma_0, \gamma_1).$$

Furthermore,

$$\lim_{t\to\infty} \sum_{|\gamma|+|\beta|\leq s-N} \left\| \frac{\partial^\gamma_\beta (f(t) - \overline{\mathbf{M}})}{\sqrt{\mathbf{M}_\infty}} \right\|_{L^\infty_x(L^2_\xi)} = 0.$$

Remark. **Recall that $\mathbf{M}_\infty = \mathbf{M}_\infty(\xi)$: global Maxwellian, $\overline{\mathbf{M}} = \overline{\mathbf{M}}(x,\xi)$: stationary solution with $\overline{\mathbf{M}} = \tilde{\rho}(x)\mathbf{M}_\infty$ (local).**

Theorem for Optimal Convergence Rates:

Under the above theorem, if it further holds that

$$\frac{f_0 - \overline{\mathbf{M}}}{\sqrt{\mathbf{M}_\infty}} \in Z_1 = L^2_\xi(L^1_x), \quad \nabla_x \Phi \in L_x^{\frac{2N}{N+2}},$$

then

$$H_2(t) \leq C(1+t)^{-\frac{N+2}{2}},$$

where

$$H_2(t) = \|\nabla_x(\rho - \tilde{\rho}, u, \theta)(t)\|^2_{s-1} + \left\| \frac{\sqrt{\nu_{\mathbf{M}_\infty}} \nabla_x \mathbf{P}_1^{\mathbf{M}_\infty} f(t)}{\sqrt{\mathbf{M}_\infty}} \right\|^2_{L^2_{x,\xi}} + \sum_{|\gamma|+|\beta|\leq s} \left\| \frac{\partial^\gamma_\beta \mathbf{P}_1^{\mathbf{M}_\infty} f(t)}{\sqrt{\mathbf{M}_\infty}} \right\|^2_{L^2_{x,\xi}}.$$

Moreover,

$$\|(\rho - \tilde{\rho}, u, \theta - \theta_\infty)(t)\|_{L^2_x} \leq C(1+t)^{-\frac{N}{4}},$$

2. Main results

$$\left\|\frac{\nabla_x(f(t)-\overline{\mathbf{M}})}{\sqrt{\mathbf{M}_\infty}}\right\|_{L^2_{x,\xi}}+\left\|\frac{\mathbf{P}_1^{\mathbf{M}_\infty}f(t)}{\sqrt{\mathbf{M}_\infty}}\right\|_{L^2_{x,\xi}}\leq C(1+t)^{-\frac{N+2}{4}}.$$

Finally for $2\leq q\leq \frac{2N}{N-2}$,

$$\left\|\frac{(f(t)-\overline{\mathbf{M}})}{\sqrt{\mathbf{M}_\infty}}\right\|_{L^2_\xi(L^q_x)}\leq C(1+t)^{-\frac{N}{2}\left(1-\frac{1}{q}\right)}.$$

Remarks.

- **The convergence rates given above are optimal up to the first order derivatives, compared with the convergence rates for the linearized Boltzmann equation without forces:**

$$\begin{cases} g_t=Bg, \quad B=-\xi\cdot\nabla_x+L_{\mathbf{M}_\infty} \\ g(0,x,\xi)=g_0(x,\xi), \end{cases}$$

2. Main results

- where

$$\left\| \frac{\nabla_x^m g(t)}{\sqrt{\mathbf{M}_\infty}} \right\|_{Z_2} \leq C(1+t)^{-\sigma(p,2;m)} \left(\left\| \frac{g_0}{\sqrt{\mathbf{M}_\infty}} \right\|_{Z_p} + \left\| \frac{\nabla_x^m g_0}{\sqrt{\mathbf{M}_\infty}} \right\|_{Z_2} \right),$$

$$Z_p = L^2_\xi(L^p_x), \quad 1 \leq p \leq 2.$$

Here for nonnegative integer m, $\sigma(p,q;m)$ is given by

$$\sigma(p,q;m) = \frac{N}{2}\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{m}{2}.$$

- The microscopic component $\mathbf{G}$ decays faster than the macroscopic components by a factor of order $(1+t)^{-\frac{1}{2}}$ in Z_2-norm, since roughly it contains quadratic and differentiation terms.

Main Ideas:

- Based on
  - the analysis of the macroscopic equations: Navier-Stokes + Source (microscopic), and
  - the dissipation of the microscopic component for the linearized collision operator,

find a Lyapunov-type energy inequality

$$\frac{dH_2(t)}{dt} + CH_2(t) \leq C\|\nabla_x(\rho - \tilde{\rho}, u, \theta)(t)\|_{L_x^2}^2,$$

where $H_2(t)$ roughly includes

- Macroscopic variables: at leat first order derivatives;
- Microscopic component: all orders.

2. Main results

- **Control $\|\nabla_x(\rho-\tilde{\rho},u,\theta)(t)\|_{L_x^2}^2$ by $H_2(t)$ in the weighted integral form with a small factor. Observations:**
  - $\nabla_x(\rho-\tilde{\rho},u,\theta)(t,x)\lesssim\left\|\frac{\nabla_x g(t,x)}{\sqrt{\mathbf{M}_\infty}}\right\|_{L_\xi^2}$, $g=f(t,x,\xi)-\overline{\mathbf{M}}$.
  - g **satisfies**

$$g_t=Bg+S[g],$$
$$S[g]=\nabla_x\Phi(x)\cdot\nabla_\xi g+2(\tilde{\rho}(x)-1)Q(\mathbf{M}_0,g)+Q(g,g),$$

    or in the mild form

$$g(t)=e^{tB}g_0+\int_0^t e^{(t-\tau)B}S[g](\tau)d\tau.$$

  - **Time-decay estimates on e^{tB}.**
  - $\nabla_x\Phi(x)$ **and** $\tilde{\rho}(x)-\rho_\infty$ **are small in some space: Sobolev inequality, Poincaré inequality, nonlinearity of** Q**, ...**

Outline

1. Introduction

2. **Main results**
   - Problem considered
   - Previous works
   - Main results and ideas of proofs: NS and BE
   - **Generalization and further problems**

Generalization:

- Z_p **initial perturbation:** $1 \leq p < p_*$**, or Hard potential collision kernel:** $0 \leq \gamma \leq 1$**, (Ukai-Yang, AA, '06);**
- **Time-dependent force:**

$$F = -\nabla_x \Phi(x) + E(t,x),$$

where $E(t,x)$ **is small and tends to zero with some rate as time goes to infinity, (Duan-Ukai-Yang-Zhao, '06). Putting**

$$f = \overline{\mathbf{M}} + \mathbf{M}_\infty^{1/2} u$$

gives

$$\mathcal{T}_F u - \tilde{\rho} L u = \Gamma(u,u) + \tilde{\rho} E \cdot \xi \mathbf{M}_\infty^{1/2},$$

where w.l.g. $\rho_\infty = \theta_\infty = 1$ **is set, and** $\mathcal{T}_F$**,** L **and** Γ **are**

defined by

$$\mathcal{T}_F u = \partial_t u + \xi \cdot \nabla_x u + F \cdot \nabla_\xi u - \frac{1}{2}\xi \cdot F u,$$

$$Lu = \mathbf{M}_\infty^{-1/2}\left[Q(\mathbf{M}_\infty, \mathbf{M}_\infty^{1/2}u) + Q(\mathbf{M}_\infty^{1/2}u, \mathbf{M}_\infty)\right],$$

$$\Gamma(u,v) = \mathbf{M}_\infty^{-1/2} Q(\mathbf{M}_\infty^{1/2}u, \mathbf{M}_\infty^{1/2}v).$$

- **Vlasov-Poisson-Boltzmann system.**

Further problems:

- **General force (non-potential) $F(t,x)$ (NS or BE):**
  - **Existence of the stationary solution?**
  - **In general, the stationary solution has weak regularity.**
- **Gas contained in a torus $\mathbb{T}^3$.**
  - **Poincaré inequality can NOT be applied directly.**

Thanks!