Optimal large-time behavior of the Vlasov-Maxwell-Boltzmann system in the whole space

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1. Introduction

1.1 Consider a linear symmetric hyperbolic system with relaxations

$$U_t + \sum_{j=1}^{n} A^j U_{x_j} + LU = 0$$

where

- $U = U(t,x) \in \mathbb{R}^m$, $t \geq 0$, $x \in \mathbb{R}^n$;
- $A^j \in \mathbb{R}^{m\times m}$ **with** $(A^j)^T = A^j$, $1 \leq j \leq n$;
- $L \in \mathbb{R}^{m\times m}$ **with** $L \geq 0$. **(**$\ker L \neq 0$**).**

Observe:

$$\frac{d}{dt}\|U\|^2 = -2\langle LU, U\rangle.$$

So, no spectral gap in L^2 **because** $\langle LU, U\rangle \geq \lambda\|U\|^2$ **is NOT true.**

Q: Which kind of system structure can guarantee the time-decay of solutions $u(t) = e^{tB}u_0$**?**

1.2 The characteristic equation is

$$\det\left(\lambda I_m + \sum_{j=1}^{n} i k_j A^j + L\right) = 0.$$

Let $\lambda = \lambda(ik)$ be the eigenvalue or the dispersive relation.

Dissipative Structure: Key to the stability for $t \to \infty$.

- **Dissipativity:** $\Re\lambda(ik) \le 0$ **for any** $k \in \mathbb{R}^n$**;**
- **Strict dissipativity:** $\Re\lambda(ik) < 0$ **for any** $0 \ne k \in \mathbb{R}^n$**.**

Definition

The partially dissipative linear system is strictly dissipative of the type (p, q) if

$$\Re\,\lambda(ik) \le -\frac{c|k|^{2p}}{(1+|k|^2)^q}, \quad \forall\, k \in \mathbb{R}^n.$$

1.3 When

$$L = L^T,$$

there are general frameworks to deduce the type $(1,1)$:

- T. Umeda, S. Kawashima & Y. Shizuta (1984): Condition (K)
- Y. Shizuta & S. Kawashima (1985): (SK) stability condition
- K. Beauchard & E. Zuazua (2010): Kalman rank condition

Remark: Feature of the system of type $(1,1)$:

$$\Re\,\lambda(ik) \leq -c|k|^2/(1+|k|^2)$$

implies

$$\Re\,\lambda(ik) \lesssim -c|k|^2 \text{ for } |k| \to 0, \quad \Re\,\lambda(ik) \lesssim -c \text{ for} |k| \to \infty.$$

Moreover,

$$|\mathcal{F}\{e^{tB}U_0\}| \leq Ce^{-\frac{\lambda|k|^2}{1+|k|^2}t}|\hat{U}_0(k)|,$$

$$\|\nabla^\ell e^{tB}U_0\| \leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{\ell}{2}}(\|U_0\|_{L^p} + \|\nabla^\ell U_0\|), \quad 1 \leq p \leq 2.$$

1.4 (SK) condition:

$\forall\, 0 \neq k = (k_1, \cdots, k_n) \in \mathbb{R}^n$, **every eigenvector of**

$$A(k) := \sum_{j=1}^{m} k_j A^j$$

does not belong to $\ker L$**.**

Theorem (Shizuta-Kawashima, 1985)

Let $L = L^T$. The following statements are equivalent:

(a) *(SK) condition holds;*

(b) *The system is strictly dissipative;*

(c) *The system is strictly dissipative of the type $(1,1)$.*

Remark: In such case, one has the normal energy inequality of the form

$$\|U(t)\|_{H^N}^2 + \int_0^t \|\{\mathbf{I} - \mathbf{P}_L\}U(s)\|_{H^N}^2 + \|\nabla \mathbf{P}_L U(s)\|_{H^{N-1}}^2 \, ds \leq C\|U_0\|_{H^N}^2.$$

1.5 There are also concrete models that have properties:

- $L \neq L^T$;
- the system is of the type $(1,2)$.

Some examples:

- Dissipative Timoshenko system:
  J.E.M. Rivera & R. Racke (2003); K. Ide, K. Haramoto & S. Kawashima (2008), K. Ide & S. Kawashima (2008).
- Euler-Maxwell system of one-fluid:
  D (2010); Y. Ueda, S. Wang & S. Kawashima (2010 preprint); Y. Ueda & S. Kawashima (2010 preprint).
- Vlasov-Maxwell-Boltzmann system of two-species:
  D.-Strain (2010): goal of this talk.

Feature of systems of type $(1,2)$: $\Re\lambda(ik) \leq -c|k|^2/(1+|k|^2)^2$ implies

$$\bullet \Re\lambda(ik) \lesssim -c|k|^2 \qquad \text{for } |k| \to 0,$$
$$\bullet \Re\lambda(ik) \lesssim -c/|k|^2 \qquad \text{for } |k| \to \infty.$$

1.6 Systems of type (p, q) with $0 < p < q$ have a new general dissipative feature of the regularity-loss type which shows that

(i) **if $U_0 \in H^N$ with N properly large, it could occur that there is some component U_i of the solution $U = (U_1, \cdots, U_m)$ such that**

$$\int_0^t \int_{\mathbb{R}^n} \|\nabla^N U_i(s)\|^2 ds = \infty.$$

(ii) **the semigroup e^{tB} has the bound of the form**

$$|\mathcal{F}\{e^{tB} U_0\}| \le C e^{-\lambda(ik)t} |\hat{U}_0|,$$

where the frequency function $\lambda(ik)$ is positive and smooth over $k \in \mathbb{R}^n$ with

$$\lambda(ik) \to 0 \ \text{ as } |k| \to 0, \quad \lambda(ik) \to 0 \ \text{ as } |k| \to \infty.$$

2. A motivation

2.1 A special technique due to Kanel (1968) and Matsumura-Nishida (1981) is as follows:

Consider the linearized Navier-Stokes (or damped Euler) system:

$$\begin{cases} \partial_t \rho + \nabla \cdot u = 0, \\ \partial_t u + \nabla \rho = \Delta u. \end{cases} \Longleftrightarrow \begin{cases} \partial_t \widehat{\rho} + ik \cdot \widehat{u} = 0, \\ \partial_t \widehat{u} + ik\widehat{\rho} = -|k|^2 \widehat{u}. \end{cases}$$

$$\Longleftrightarrow \partial_t \widehat{z} = \begin{pmatrix} 0 & -ik \\ -ik & -|k|^2 \end{pmatrix} \widehat{z} = \widehat{\mathbf{L}}(k)\widehat{z}, \quad \widehat{z} = \begin{bmatrix} \widehat{\rho} \\ \widehat{u} \end{bmatrix}.$$

- **"Natural" dissipation from diffusions:**

$$\frac{1}{2}\frac{\partial}{\partial t}(|\widehat{\rho}|^2 + |\widehat{u}|^2) + |k|^2|\widehat{u}|^2 = 0.$$

- **Dissipation along the degenerate component:**

$$|k|^2|\widehat{\rho}|^2 = (ik\widehat{\rho}, ik\widehat{\rho}) = (ik\widehat{\rho}, -|k|^2|\widehat{u}|^2 - \partial_t \widehat{u}) = I + II$$

$$I = (ik\widehat{\rho}, -|k|^2|\widehat{u}|^2) \le \frac{1}{2}|k|^2|\widehat{\rho}|^2 + \frac{1}{2}|k|^4|\widehat{u}|^2.$$

$$II = (ik\widehat{\rho}, -\partial_t \widehat{u}) = -\partial_t (ik\widehat{\rho}, \widehat{u}) + (ik\partial_t \widehat{\rho}, \widehat{u})$$
$$= -\partial_t (ik\widehat{\rho}, \widehat{u}) + |k \cdot \widehat{u}|^2.$$

$$\Rightarrow \partial_t \underbrace{\Re \frac{(ik\widehat{\rho}, \widehat{u})}{1+|k|^2}}_{\mathcal{E}^{\mathrm{int}}(\widehat{z})} + \frac{|k|^2}{2(1+|k|^2)}|\widehat{\rho}|^2 \leq \frac{1}{2}|k|^2|\widehat{u}|^2.$$

- **Combination:**

$$\frac{\partial}{\partial t}\mathcal{E}(\widehat{z}) + \frac{\lambda |k|^2}{1+|k|^2}(|\widehat{\rho}|^2 + |\widehat{u}|^2) \leq 0,$$

where

$$\mathcal{E}(\widehat{z}(t,k)) = \frac{1}{2}(|\widehat{\rho}|^2 + |\widehat{u}|^2) + \kappa_0 \Re \frac{(ik\widehat{\rho}, \widehat{u})}{1+|k|^2}$$

for some $0 < \kappa_0 \ll 1$**.**

2.2 Remarks:

- **An interesting observation by Shizuta-Kawashima:**

$$\Re\,(ik\widehat{\rho},\widehat{u}) = \Re\left(\begin{bmatrix}\hat{\rho}\\ \hat{u}\end{bmatrix}, ik\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{bmatrix}\hat{\rho}\\ \hat{u}\end{bmatrix}\right)$$

which suggests a formulation in terms of a skew-symmetric matrix K**. The general theory was developed by introducing the compensation function related to such skew-symmetric matrix:**

$$\frac{K(\tilde{k})A(\tilde{k}) + [K(\tilde{k})A(\tilde{k})]^T}{2} + L > 0 \quad \textbf{in } \mathbb{R}^n,\ \ \tilde{k} = k/|k|.$$

- **Applications to the kinetic theory:**
  - **Boltzmann equation: Kawashima (1990);**
  - **Vlasov-Poisson-Boltmzann system: Glassey-Strauss (1999);**
  - **Landau equation, VPB system of two-species, VMFP system: Yang-Yu (2010, 2011)**

2.3 Our approach is based on the "explicit step-by-step" **construction of some interactive functionals in terms of the pointwise time-frequency analysis. The key point is to prove a Lyapunov inequality**

$$\partial_t \mathcal{E}(\hat{U}(t,k)) + p(k)\mathcal{E}(\hat{U}(t,k)) \leq 0, \qquad (*)$$

where

$$\mathcal{E}(\hat{U}(t,k)) = |\hat{U}(t,k)|^2 + \kappa \Re \mathcal{E}^{\text{int}}(\hat{U}(t,k)) \sim |\hat{U}(t,k)|^2.$$

Remarks:

- **In general, $(*)$ can yield the linearized time-decay of solutions in L^p with $2 \leq p \leq \infty$. For the case when $1 \leq p < 2$, Green's function need to be used (Boltzmann equation in $\mathbb{R}^3$: Liu-Yu);**
- **For the kinetic equation, $\mathcal{E}(\hat{U}(t,k))$ can be involved in the velocity weight (Boltzmann equation with non-cutoff soft potentials in $\mathbb{R}^3$: Strain, preprint)**

3. Main results

3.1 Vlasov-Maxwell-Boltzmann system:

$$\partial_t f_+ + \xi \cdot \nabla_x f_+ + (E + \xi \times B) \cdot \nabla_\xi f_+ = Q(f_+, f_+) + Q(f_+, f_-),$$
$$\partial_t f_- + \xi \cdot \nabla_x f_- - (E + \xi \times B) \cdot \nabla_\xi f_- = Q(f_-, f_+) + Q(f_-, f_-).$$

It is coupled with the Maxwell system

$$\partial_t E - \nabla_x \times B = -\int_{\mathbb{R}^3} \xi (f_+ - f_-) d\xi,$$
$$\partial_t B + \nabla_x \times E = 0,$$
$$\nabla_x \cdot E = \int_{\mathbb{R}^3} (f_+ - f_-) d\xi, \quad \nabla_x \cdot B = 0.$$

The initial data in this system is given as

$$f_\pm(0, x, \xi) = f_{0,\pm}(x, \xi), \quad E(0, x) = E_0(x), \quad B(0, x) = B_0(x).$$

3.2 Known results on the asymptotic stability of near-equilibrium solutions to the two-species VMB system

- $x \in \mathbb{T}^3$
  - Global existence: Guo ('03) (Energy method)
  - Large-time behavior of solutions: Strain-Guo ('06&'08) (Time-velocity splitting), Jang ('09)
- $x \in \mathbb{R}^3$
  - Global existence: Strain ('06) (Use two-species' cancelation property to control E and pure time derivatives)
  - Large-time behavior of solutions: Duan-Strain ('10) (Contruct time-frequency functionals + bootstrap to the nonlinear equation)

Remark: For the case of one-species VMB system, D. ('10) also obtained the global existence. When there is no cancelation, more delicate Lyapunov functionals are desgined to control the dissipation of the electromagnetic field.

3.3 Define the perturbation u as

$$u = \frac{f - \mathbf{M}}{\mathbf{M}^{1/2}}, \quad u = [u_+, u_-], \ f = [f_+, f_-].$$

The linearized homogeneous system takes the form of

$$\begin{cases} \partial_t u + \xi \cdot \nabla_x u - E \cdot \xi \mathbf{M}^{1/2}[1, -1] = \mathbf{L} u, \\ \partial_t E - \nabla_x \times B = -\langle [\xi, -\xi]\mathbf{M}^{1/2}, \{\mathbf{I} - \mathbf{P}\} u \rangle, \\ \partial_t B + \nabla_x \times E = 0, \\ \nabla_x \cdot E = \langle \mathbf{M}^{1/2}, u_+ - u_- \rangle, \quad \nabla_x \cdot B = 0, \\ [u, E, B]|_{t=0} = [u_0, E_0, B_0], \end{cases}$$

Important observation: Since the null space of $\mathbf{L}$ is given by

$$\ker \mathbf{L} = \operatorname{span}\left\{[1, 0]\mathbf{M}^{1/2}, \ [0, 1]\mathbf{M}^{1/2}, [\xi_i, \xi_i]\mathbf{M}^{1/2}(1 \le i \le 3), [|\xi|^2, |\xi|^2]\mathbf{M}^{1/2}\right\}.$$

due to Guo (2003),

$$\xi \mathbf{M}^{1/2}[1, -1] \notin \ker \mathbf{L}.$$

The case of one-species is different.

Theorem (D.-Strain, 2010)

There is a time-frequency functional $\mathcal{E}(t,k)$ *such that*

$$\mathcal{E}(t,k) \sim \|\hat{u}\|_{L^2_\xi}^2 + |\hat{E}|^2 + |\hat{B}|^2,$$

and

$$\partial_t \mathcal{E}(t,k) + \frac{\lambda |k|^2}{(1+|k|^2)^2}\mathcal{E}(t,k) \leq 0, \quad \forall\, t \geq 0,\ k \in \mathbb{R}^3.$$

Remark: The above time-frequency Lyapunov inequality implies the L^p-L^q time-decay property of the solution semigroup similar to the Euler-Maxwell system (D., 2010):

$$\|\nabla_x^m \mathbb{A}(t) U_0\| \leq C(1+t)^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{2})-\frac{m}{2}} \|U_0\|_{L^r} + C(1+t)^{-\frac{\ell}{2}} \|\nabla_x^{m+\ell} U_0\|.$$

3.4 Key idea in the proof: One can use the energy estimate in the Fourier space and construct some L^2 functionals so as to capture the dissipation of all the degenerate components in the solution.

Lemma

(i)

$$\partial_t \left(\|\hat{u}\|^2_{L^2_\xi} + |\hat{E}|^2 + |\hat{B}|^2 \right) + \lambda \|\nu^{1/2}\{\mathbf{I} - \mathbf{P}\}\hat{u}\|^2_{L^2_\xi} \le 0.$$

(ii) $\exists$ an interactive time-frequency functional $\mathcal{E}_{\rm int}(t,k)$ such that

$$|\mathcal{E}_{\rm int}(t,k)| \le C(\|\hat{u}\|^2_{L^2_\xi} + |\hat{E}|^2 + |\hat{B}|^2),$$

and

$$\partial_t \mathcal{E}_{\rm int}(t,k) + \frac{\lambda |k|^2}{1+|k|^2}\|\mathbf{P}\hat{u}\|^2_{L^2_\xi} + \lambda |k\cdot \hat{E}|^2 + \frac{\lambda |k|^2}{(1+|k|^2)^2}(|\hat{E}|^2 + |\hat{B}|^2) \le C\|\{\mathbf{I}-\mathbf{P}\}\hat{u}\|^2_{L^2_\xi}.$$

3.5 Time-decay of the nonlinear VMB system: Define the instant full energy functional $\mathcal{E}_{N,m}(t)$ and the instant high-order energy functional $\mathcal{E}^{\mathrm{h}}_{N,m}(t)$, respectively, as

$$\mathcal{E}_{N,m}(t) \sim \sum_{|\alpha|+|\beta|\leq N} \|\nu^{\frac{m}{2}}\partial^{\alpha}_{\beta} u(t)\|^2 + \sum_{|\alpha|\leq N} \|\partial^{\alpha}[E(t),B(t)]\|^2,$$

$$\mathcal{E}^{\mathrm{h}}_{N,m}(t) \sim \sum_{1\leq|\alpha|\leq N} \|\nu^{\frac{m}{2}}\partial^{\alpha} u(t)\|^2 + \sum_{|\alpha|+|\beta|\leq N} \|\nu^{\frac{m}{2}}\partial^{\alpha}_{\beta}\{\mathbf{I}-\mathbf{P}\}u(t)\|^2 + \sum_{1\leq|\alpha|\leq N} \|\partial^{\alpha}[E(t),B(t)]\|^2,$$

and we define the dissipation rate $\mathcal{D}_{N,m}(t)$ as

$$\mathcal{D}_{N,m}(t) = \sum_{1\leq|\alpha|\leq N} \|\nu^{\frac{m+1}{2}}\partial^{\alpha} u(t)\|^2 + \sum_{|\alpha|+|\beta|\leq N} \|\nu^{\frac{m+1}{2}}\partial^{\alpha}_{\beta}\{\mathbf{I}-\mathbf{P}\}u(t)\|^2 + \sum_{1\leq|\alpha|+|\beta|\leq N-1} \|\partial^{\alpha}[E(t),B(t)]\|^2 + \|E(t)\|^2,$$

for integers N and m. For brevity, we write $\mathcal{E}_N(t)=\mathcal{E}_{N,0}(t)$, $\mathcal{E}^{\mathrm{h}}_N(t)=\mathcal{E}^{\mathrm{h}}_{N,0}(t)$ and $\mathcal{D}_N(t)=\mathcal{D}_{N,0}(t)$ when $m=0$.

Theorem (D.-Strain, 2010)

Under some smallness conditions,

$$\frac{d}{dt}\mathcal{E}_{N,m}(t) + \lambda\mathcal{D}_{N,m}(t) \leq 0,$$

$$\frac{d}{dt}\mathcal{E}^{\mathrm{h}}_{N}(t) + \lambda\mathcal{D}_{N}(t) \leq C\|\nabla_x \mathbf{P} u(t)\|^2.$$

Moreover, define $\epsilon_{j,m} = \mathcal{E}_{j,m}(0) + \|u_0\|^2_{Z_1} + \|[E_0, B_0]\|^2_{L^1}$. *Then, under some smallness conditions,*

$$\mathcal{E}_{N,m}(t) \leq C\epsilon_{N+2,m}(1+t)^{-\frac{3}{2}},$$

$$\mathcal{E}^{\mathrm{h}}_{N}(t) \leq C\epsilon_{N+5,1}(1+t)^{-\frac{5}{2}}.$$

Finally, for $1 \leq r \leq 2$,

$$\|u(t)\|_{Z_r} + \|B(t)\|_{L^r_x} \leq C(1+t)^{-\frac{3}{2}+\frac{3}{2r}},$$

$$\|\{\mathbf{I}-\mathbf{P}\}u(t)\|_{Z_r} + \|\langle[1,-1]\mathbf{M}^{1/2}, u(t)\rangle\|_{L^r_x} + \|E(t)\|_{L^r_x} \leq C(1+t)^{-\frac{3}{2}+\frac{1}{2r}}.$$

Key points in the proof:

- **Two types of refined energy estimates, compared with Strain (2006):**
  - **velocity weighted;**
  - **energy inequality for the high-order energy functional.**
- **Application of the linearized time-decay property to the nonlinear system by the Duhamel's principle:**
  - **Time weighted:** $1 < \ell < 2$,

$$(1+t)^{\ell}\mathcal{E}_{N,m}(t) + \lambda \int_0^t (1+s)^{\ell}\mathcal{D}_{N,m}(s)ds$$

$$\leq C\mathcal{E}_{N+2,m}(0) + C\ell \int_0^t (1+s)^{\ell-1}(\|\mathbf{P}u(s)\|^2 + \|B(s)\|^2)ds,$$

  - **Define**

$$X_{N,m}(t) = \sup_{0\leq s\leq t} (1+s)^{\frac{3}{2}}\mathcal{E}_{N,m}(s),$$

and prove

$$X_{N,m}(t) \leq C\{\epsilon_{N+2,m\vee 1} + X_{N,m}(t)^2\}.$$

4. Further application

4.1 One-species VMB (D., 2010):

$$\begin{cases} \partial_t f + \xi \cdot \nabla_x f + (E + \xi \times B) \cdot \nabla_\xi f = Q(f,f), \\ \partial_t E - \nabla_x \times B = -\displaystyle\int_{\mathbb{R}^3} \xi f \, d\xi, \\ \partial_t B + \nabla_x \times E = 0, \\ \nabla \cdot E = \displaystyle\int_{\mathbb{R}^3} f \, d\xi - n_{\mathrm{b}}, \quad \nabla_x \cdot B = 0. \end{cases}$$

The key difference of one-species case with two-species lies in the fact that the coupling term in the source of the Maxwell system

$$-\int_{\mathbb{R}^3} \xi f(t,x,\xi) d\xi$$

corresponds to the momentum component of the macroscopic part of the solution which is degenerate with respect to the linearized operator L.

Theorem (D., 2010)

Let $N \geq 4$. *Define*

$$\mathcal{E}_N(U(t)) \sim \|u(t)\|^2_{H^N_{x,\xi}} + \|[E(t), B(t)]\|^2_{H^N},$$

$$\mathcal{D}_N(U(t)) = \|\nu^{1/2}\{\mathbf{I} - \mathbf{P}\}u(t)\|^2_{H^N_{x,\xi}} + \|\nu^{1/2}\nabla_x u(t)\|^2_{L^2_\xi(H^{N-1}_x)}$$

$$+\|\nabla_x E(t)\|^2_{H^{N-2}} + \|\nabla_x^2 B(t)\|^2_{H^{N-3}}.$$

Suppose $f_0 = \mathbf{M} + \mathbf{M}^{1/2}u_0 \geq 0$. *There indeed exists* $\mathcal{E}_N(U(t))$ *such that if initial data* $U_0 = [u_0, E_0, B_0]$ *satisfies the compatible condition at* $t = 0$ *and* $\mathcal{E}_N(U_0)$ *is sufficiently small, then the nonlinear Cauchy problem admits a global solution* $U = [u, E, B]$ *satisfying*

$$f(t, x, \xi) = \mathbf{M} + \mathbf{M}^{1/2}u(t, x, \xi) \geq 0,$$

$$[u(t), E(t), B(t)] \in C([0, \infty); H^N_{x,\xi} \times H^N \times H^N),$$

and

$$\mathcal{E}_N(U(t)) + \lambda \int_0^t \mathcal{D}_N(U(s))ds \leq \mathcal{E}_N(U_0), \quad \forall\, t \geq 0.$$

4.2 A summary for the dissipation property of kinetic plasma model:

	$\mathcal{E}(t,k) \sim$	$\mathcal{D}(t,k) =$	$\Vert u(t)\Vert_{L^2} \leq$
BE **(1,1) type**	$\Vert\hat{u}\Vert^2_{L^2_\xi}$	$\Vert\nu^{1/2}\{\mathrm{I}-\mathrm{P}\}\hat{u}\Vert^2_{L^2_\xi}$ $+\frac{\|k\|^2}{1+\|k\|^2}\|[\hat{a},\hat{b},\hat{c}]\|^2$	$C(1+t)^{-\frac{3}{4}}\Vert u_0\Vert_{Z_1\cap L^2}$
1-s VPB	$\Vert\hat{u}\Vert^2_{L^2_\xi}+\frac{\|\hat{a}\|^2}{\|k\|^2}$	$\Vert\nu^{1/2}\{\mathrm{I}-\mathrm{P}\}\hat{u}\Vert^2_{L^2_\xi}$ $+\frac{\|k\|^2}{1+\|k\|^2}\|[\hat{a},\hat{b},\hat{c}]\|^2+\|\hat{a}\|^2$	$C(1+t)^{-\frac{1}{4}}\Vert u_0\Vert_{Z_1\cap L^2}$
1-s VMB **(2,3) type**	$\Vert\hat{u}\Vert^2_{L^2_\xi}+\|[\hat{E},\hat{B}]\|^2$	$\Vert\nu^{1/2}\{\mathrm{I}-\mathrm{P}\}\hat{u}\Vert_{L^2_\xi}$ $+\frac{\|k\|^2}{1+\|k\|^2}\|[\hat{a},\hat{b},\hat{c}]\|^2+\|k\cdot\hat{E}\|^2$ $+\frac{\|k\|^2}{(1+\|k\|^2)^2}\|\hat{E}\|^2+\frac{\|k\|^4}{(1+\|k\|^2)^3}\|\hat{B}\|^2$	$C(1+t)^{-\frac{3}{8}}$ $\times(\Vert U_0\Vert_{\mathcal{Z}_1}+\Vert\nabla_x U_0\Vert_{\mathcal{Z}_2})$
2-s VMB **(1,2) type**	$\Vert\hat{u}\Vert^2_{L^2_\xi}+\|[\hat{E},\hat{B}]\|^2$	$\Vert\nu^{1/2}\{\mathrm{I}-\mathrm{P}\}\hat{u}\Vert_{L^2_\xi}$ $+\frac{\|k\|^2}{1+\|k\|^2}\|[\hat{a}_\pm,\hat{b},\hat{c}]\|^2+\|k\cdot\hat{E}\|^2$ $+\frac{\|k\|^2}{(1+\|k\|^2)^2}(\|\hat{E}\|^2+\|\hat{B}\|^2)$	$C(1+t)^{-\frac{3}{4}}$ $\times(\Vert U_0\Vert_{\mathcal{Z}_1}+\Vert\nabla_x^2 U_0\Vert_{\mathcal{Z}_2})$

Table: Dissipative and time-decay properties of different models

Remark: The developed approach here is also applicable to other situations such as the Landau operator and the relativistic model in $\mathbb{R}^3$.

Thanks for your attention!