Stability of the nonrelativistic Vlasov-Maxwell-Boltzmann system for angular non-cutoff potentials

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Consider the Boltzmann equations

$$\partial_t F_+ + \xi \cdot \nabla_x F_+ + (E + \xi \times B) \cdot \nabla_\xi F_+ = Q(F_+, F_+) + Q(F_+, F_-),$$

$$\partial_t F_- + \xi \cdot \nabla_x F_- - (E + \xi \times B) \cdot \nabla_\xi F_- = Q(F_-, F_+) + Q(F_-, F_-),$$

coupling to

$$\partial_t E - \nabla_x \times B = -\int_{\mathbb{R}^3} \xi (F_+ - F_-)\, d\xi,$$

$$\partial_t B + \nabla_x \times E = 0,$$

$$\nabla_x \cdot E = \int_{\mathbb{R}^3} (F_+ - F_-)\, d\xi, \quad \nabla_x \cdot B = 0.$$

Here

$$F_\pm = F_\pm(t, x, \xi) \geq 0,$$

$$x = (x_1, x_2, x_3) \in \mathbb{R}^3, \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3, t \geq 0.$$

Initial data:

$$F_\pm(0, x, \xi) = F_{0,\pm}(x, \xi), \quad E(0, x) = E_0(x), \quad B(0, x) = B_0(x),$$

with

$$\nabla_x \cdot E_0 = \int_{\mathbb{R}^3} (F_{0,+} - F_{0,-})\, d\xi, \quad \nabla_x \cdot B_0 = 0.$$

Boltzmann collision operator:

$$Q(F,G) = \int_{\mathbb{R}^3\times\mathbb{S}^2} q(\xi-\xi_*,\sigma)\left[F(\xi')G(\xi'_*) - F(\xi)G(\xi_*)\right] d\xi_* d\sigma,$$

$$\xi' = \frac{\xi+\xi_*}{2} + \frac{|\xi-\xi_*|}{2}\sigma, \quad \xi'_* = \frac{\xi+\xi_*}{2} - \frac{|\xi-\xi_*|}{2}\sigma.$$

$$q(\xi-\xi_*,\sigma) = C_q|\xi-\xi_*|^\gamma b(\cos\theta),$$

with

$\cos\theta = \sigma\cdot(\xi-\xi_*)/|\xi-\xi_*|$

$C_q > 0$, $\gamma > -3$

$\exists\ C_b > 0$, $0 < s < 1$ **s.t.**

$$\frac{1}{C_b\theta^{1+2s}} \le \sin\theta\, b(\cos\theta) \le \frac{C_b}{\theta^{1+2s}}, \quad \forall\,\theta\in(0,\frac{\pi}{2}].$$

Two physical examples:

- **Hard-sphere model:**

$$q(\xi - \xi_*, \sigma) = C_q|\xi - \xi_*|$$

- **Inverse power law** $U(|x|) = |x|^{-(\ell-1)}$ **with** $2 < \ell < \infty$**:**

$$\gamma = \frac{\ell - 5}{\ell - 1}, \quad s = \frac{1}{\ell - 1}.$$

Remark: For the Coulomb potential $\ell = 2$**, one has** $\gamma = -3$**,** $s = 1$**, for which Boltzmann operator is NOT defined and must be replaced by Landau operator.**

Boltzmann's H-theorem:

$$\partial_t f = Q(f,f) \Rightarrow \frac{d}{dt}\int_{\mathbb{R}^3} d\xi \left\{-f \log f\right\} \geq 0.$$

- (Physical) entropy increasing. This gives a description of the second law of thermodynamics.
- Entropy takes the maximization at the Maxwellian

$$\mathbf{M} = \mathbf{M}_{[\rho,u,T]}(\xi) = \frac{\rho}{(2\pi T)^{3/2}} e^{-\frac{|\xi-u|^2}{2T}}.$$

ρ : density, u : bulk velocity, T: temperature.

- Goal: prove stability and convergence rate of solutions around the (global) Maxwellian in the spatially non-homogeneous case.

Perturbation theory of the Boltzmann equation near Maxwellians:

- **Linearized equation: Grad ('58)**
- **Nonlinear equation based on spectrum analysis: Ukai ('74), Ukai-Asano ('82), Caflish ('80),...**
- **Nonlinear energy method: Guo ('02, '04), Liu-Yang-Yu ('04), Strain-Guo ('06, '08),....**

Recent progress in non-cutoff case:

- **Alexandre-Morimoto-Ukai-Xu-Yang (AMUXY) ('10, '12)**
- **Gressman-Strain ('11)**

Extensions taking in account more physics:

- **Forces occur:**
  - **Self-consistent potential force satisfying Poisson equation: Guo ('02), Yang-Yu-Zhao ('06), Strain-Guo ('08), D.-Strain ('11),...**
  - **Self-consistent electro-magnetic fields satisfying Maxwell equations: Guo ('03), Strain ('06), Duan-Strain ('11),...**
- **Collisions by Landau:**
  - **Linearized operator: Degond-Lemou ('97), Guo ('02), ...**
  - **Energy method: Guo ('02, '12),...**
- **Relativistic effects occur: Dudynski-Ekiel-Jezewska ('88), Glassey-Strauss ('95), Strain-Guo ('04, '12), Strain ('10),...**

Our interest of the talk:

- **Collisions by Boltzmann for angular non-cutoff and for soft potentials**
- **Electric-magnetic fields occur**
- **No relativistic effect**

Reformulation of Cauchy problem:

$$F_{\pm}(t,x,\xi) = \mu + \mu^{1/2} f_{\pm}(t,x,\xi), \quad \mu = \mu(\xi) = (2\pi)^{-3/2} e^{-|\xi|^2/2},$$

satisfies

$$\begin{cases} \partial_t f + \xi \cdot \nabla_x f + q_0 (E + \xi \times B) \cdot \nabla_\xi f - E \cdot \xi \mu^{1/2} q_1 + Lf \\ \qquad\qquad\qquad\qquad\qquad\qquad = \dfrac{q_0}{2} E \cdot \xi u + \Gamma(f,f), \\ \partial_t E - \nabla_x \times B = -\displaystyle\int_{\mathbb{R}^3} \xi \mu^{1/2} (f_+ - f_-)\, d\xi, \\ \partial_t B + \nabla_x \times E = 0, \\ \nabla_x \cdot E = \displaystyle\int_{\mathbb{R}^3} \mu^{1/2} (f_+ - f_-)\, d\xi, \quad \nabla_x \cdot B = 0 \end{cases}$$

Here, $q_0 = \mathrm{diag}(1,-1)$**,** $q_1 = [1,-1]$**,** $f = [f_+, f_-]$**.**

Theorem (D.-Liu-Yang-Zhao, KRM '13)

Assume

$$\max\left\{-3, -\frac{3}{2} - 2s\right\} < \gamma < -2s, \quad \frac{1}{2} \leq s < 1.$$

For initial data $(f_0(x,\xi), E_0(x), B_0(x))$ *regular enough and small enough, Cauchy problem on Vlasov-Maxwell-Boltzmann system admits a unique classical solution*

$$(f(t,x,\xi), E(t,x), B(t,x))$$

satisfying

$$\|f(t)\|_{L^2_{x,\xi}} + \|(E,B)(t)\|_{L^2} \lesssim (1+t)^{-\frac{3}{4}},$$
$$\|\nabla_x f(t)\|_{L^2_{x,\xi}} + \|\nabla_x (E,B)(t)\|_{L^2} \lesssim (1+t)^{-\frac{5}{4}}.$$

Remarks:

- Convergence rates are the same as those obtained by Ukai's spectrum method for the angular cutoff Boltzmann equation without forces ($E = B = 0$).
- Collision kernel includes the inverse power law which can be close to the Coulomb potential, i.e. $\gamma \to -3+$, $s \to 1-$.
- Restriction $s \geq 1/2$ is technical and essentially needed in our proof, since by AMUXY

$$C_1 \left\{ |f|^2_{H^s_{\gamma/2}} + |f|^2_{L^2_{s+\gamma/2}} \right\} \leq |f|^2_{\mathbf{D}} \leq C_2 |f|^2_{H^s_{s+\gamma/2}},$$

for $f \in (\ker L)^\perp$, where $|f|^2_{\mathbf{D}} = \langle -Lf, f \rangle$ is the Dirichlet norm.

Main difficulties and our efforts in the proof

- **Angular non-cutoff:**
  - **Use the commutator estimates by AMUXY**
  - **Extra efforts: Introduce the exponential weight into the non-cutoff framework**
- **Soft potentials:**
  - **Use the weighted energy norm by Guo**
  - **Extra effort: To take care the nonlinear estimates, use the velocity-time-dependent weight (D.-Yang-Zhao, '12):**

$$w_{\tau,\lambda} = w_{\tau,\lambda}(t,\xi) = \langle \xi \rangle^{\gamma\tau} \exp\left\{ \frac{\lambda}{(1+t)^{\vartheta}} \langle \xi \rangle \right\}.$$

Main difficulties and our efforts in the proof (cont.)

- **Regularity-loss of (E, B):**
  - **D.-Strain: The dissipation rate of $\|(E,B)\|_{H^N}^2$ includes only**

    $$\|E\|_{H^{N-1}}^2 + \|\nabla_x B\|_{H^{N-2}}^2.$$

  - **Do the time-weighted estimates with time weight of negative power**

    $$\frac{d}{dt}\left[(1+t)^{-\sigma}\|(E,B)(t)\|_{H^N}^2\right] + \sigma(1+t)^{-\sigma-1}\|(E,B)(t)\|_{H^N}^2 \leq \text{“}h.o.t.\text{”}$$

    Such approach firstly introduced by Hosono-Kawashima (M3AS ’06).

More details on the a priori estimates:

- **Define the energy functional**

$$\mathcal{E}_{N,\ell,\lambda}(t) \sim \sum_{|\alpha|\leq N} \|\partial_x^\alpha \mathbf{P} f(t)\|^2 + \sum_{|\alpha|+|\beta|\leq N} \left\|\partial_x^\alpha \partial_\xi^\beta \{\mathbf{I}-\mathbf{P}\} f(t)\right\|^2_{|\alpha|+|\beta|-\ell,\lambda} + \|(E,B)(t)\|^2_{H^N},$$

$$\overline{\mathcal{E}}_N(t) \sim \sum_{|\alpha|\leq N} \|\partial_x^\alpha f(t)\|^2 + \|(E,B)(t)\|^2_{H^N},$$

where $\mathbf{P}: L^2_\xi \to \ker L$ is the projection, and the norm $\|\cdot\|_{\tau,\lambda}$ defined by

$$\|f(t)\|^2_{\tau,\lambda} = \int_{\mathbb{R}^3\times\mathbb{R}^3} w^2_{\tau,\lambda}(t,\xi)|f(t,x,\xi)|^2\, dx d\xi.$$

- **Define the dissipation rate:**

$$\mathcal{D}_{N,\ell,\lambda}(t) = \sum_{|\alpha|+|\beta|\le N} \left\| \partial_x^\alpha \partial_\xi^\beta \{\mathbf{I}-\mathbf{P}\} f(t) \right\|^2_{\mathbf{D},|\alpha|+|\beta|-\ell,\lambda}$$

$$+ \sum_{|\alpha|\le N-1} \|\nabla_x \partial_x^\alpha \mathbf{P} f(t)\|^2$$

$$+ \|E(t)\|^2_{H^{N-1}} + \|\nabla_x B(t)\|^2_{H^{N-2}}$$

$$+ \frac{\lambda}{(1+t)^{1+\vartheta}} \sum_{|\alpha|+|\beta|\le N} \left\| \langle \xi \rangle^{1/2} \partial_x^\alpha \partial_\xi^\beta \{\mathbf{I}-\mathbf{P}\} f(t) \right\|^2_{|\alpha|+|\beta|-\ell,\lambda},$$

$$\overline{\mathcal{D}}_N(t) = \sum_{|\alpha|\le N} \left\| \partial_x^\alpha \partial_\xi^\beta \{\mathbf{I}-\mathbf{P}\} f(t) \right\|^2_{\mathbf{D}} + \sum_{|\alpha|\le N-1} \|\nabla_x \partial^\alpha \mathbf{P} f(t)\|^2$$

$$+ \|E(t)\|^2_{H^{N-1}} + \|\nabla_x B(t)\|^2_{H^{N-2}}.$$

- **Define the time-weighted norm:**

$$X(t)=\sup_{0\le s\le t}\left\{\overline{\mathcal{E}}_{N_1}(s)+(1+s)^{\frac{3}{2}}\overline{\mathcal{E}}_{N_1-2}(s)\right\}$$
$$+\sup_{0\le s\le t}\left\{(1+s)^{-\frac{1+\epsilon_0}{2}}\mathcal{E}_{N_1,\ell_1,\lambda_0}(s)+\mathcal{E}_{N_1-1,\ell_1,\lambda_0}(s)\right.$$
$$\left.+(1+s)^{\frac{3}{2}}\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(s)\right\}$$
$$+\sup_{0\le s\le t}\left\{(1+s)^{\frac{5}{2}}\|\nabla_x(E,B)(s)\|_{H^5}^2\right\}.$$

- **Define initial norm:**

$$Y_0=\sum_{|\alpha|+|\beta|\le N_1}\left\|\partial_x^\alpha\partial_\xi^\beta f_0\right\|_{|\alpha|+|\beta|-\ell_1,\lambda_0}$$
$$+\|(E_0,B_0)\|_{H^{N_1}\cap L^1}+\|w_{-\ell_2}f_0\|_{L^2_\xi(L^1_x)}.$$

- **Claim: Take**

$$
\begin{aligned}
&N_1 \geq 14, \\
&\ell_1 \geq 1 + N_1, \\
&\ell_2 > \frac{15(\gamma + 2s)}{4\gamma}, \\
&\lambda_0 > 0, \\
&0 < \epsilon_0 \ll 1.
\end{aligned}
$$

Then,

$$X(t) \lesssim Y_0^2 + X^2(t).$$

Thank you !