Large-time behavior for fluid and kinetic plasmas with collisions

Renjun Duan

Department of Mathematics, The Chinese University of Hong Kong

HYP2014- XV International Conference on Hyperbolic Problems: Theory, Numerics, Applications
July 28 to August 1, 2014, IMPA, Rio de Janeiro, BRAZIL

Outline

- Introductions
- Darcy's law of two-fluid Euler-Maxwell system with collisions
- Stability of two-fluid Vlasov-Maxwell-Boltzmann system
- Rarefaction wave of Vlasov-Poisson-Boltzmann system

I: Motivations

Consider the two-fluid Vlasov-type system with collisions

$$\partial_t F_\alpha + \xi \cdot \nabla_x F_\alpha + \frac{q_\alpha}{m_\alpha}(E + \frac{\xi}{c} \times B) \cdot \nabla_\xi F_\alpha = \left(\frac{\partial F_\alpha}{\partial t}\right)_{\text{collision}},$$

$$\alpha \in \{i, e\},$$

coupling to the Maxwell system

$$\begin{aligned} &\partial_t E - c\nabla \times B = -4\pi J, \\ &\partial_t B + c\nabla \times E = 0, \\ &\nabla \cdot E = 4\pi\rho, \quad \nabla \cdot B = 0, \end{aligned}$$

with

$$J = \sum_\alpha q_\alpha \int_{\mathbb{R}^3} \xi F_\alpha \, d\xi, \quad \rho = \sum_\alpha q_\alpha \int_{\mathbb{R}^3} F_\alpha \, d\xi.$$

Bilinear collision terms

$$\left(\frac{\partial F_\alpha}{\partial t}\right)_{\text{collision}} = \sum_\beta Q(F_\alpha, F_\beta).$$

- **Conservation laws (⇒macro fluid-type system):**

$$\int_{\mathbb{R}^3} m_\alpha (\frac{\partial F_\alpha}{\partial t})_{\text{collision}}\, d\xi = 0,$$
$$\int_{\mathbb{R}^3} m_\alpha \xi_i (\frac{\partial F_\alpha}{\partial t})_{\text{collision}}\, d\xi = 0, \quad 1 \le i \le 3,$$
$$\int_{\mathbb{R}^3} \frac{1}{2} m_\alpha |\xi|^2 (\frac{\partial F_\alpha}{\partial t})_{\text{collision}}\, d\xi = 0.$$

- **Entropy product (⇒second law of thermodynamics):**

$$\sum_\alpha \int_{\mathbb{R}^3} \ln F_\alpha (\frac{\partial F_\alpha}{\partial t})_{\text{collision}}\, d\xi \le 0,$$

with equality iff F_α for all α are Maxwellians.

The type of binary collisions depends on the physical situation under consideration:

- **Landau collision (Fokker-Planck type): fully ionized plasma, all collisions grazing**
- **Boltzmann collision: fully ionized plasma, collisions grazing at the deflection angle $\theta = 0$ (non-cutoff vs cutoff)**
- **Linear Boltzmann collision: weakly ionized plasma ($\alpha = i, e, n$), $Q_{\alpha\alpha}$ $(\alpha = i, e)$ skipped. In this case, no conservations of momentum and energy.**
- **...**

Boltzmann's H-theorem:

$$\partial_t F = Q(F,F) \Rightarrow \frac{d}{dt} \boxed{\int_{\mathbb{R}^3} d\xi \, \{-F \log F\}} \geq 0.$$

- **(Physical) entropy increasing. This gives a description of the second law of thermodynamics.**
- **Entropy takes the maximization at the Maxwellian**

$$\mathbf{M} = \mathbf{M}_{[\rho,u,T]}(\xi) = \frac{\rho}{(2\pi T)^{3/2}} e^{-\frac{|\xi-u|^2}{2T}}.$$

  ρ : **density,** u : **bulk velocity,** T**: temperature.**

- **Goal: prove stability and convergence rate of solutions around the** *(global) Maxwellian* **or** *a non-trivial profile (wave pattern)* **in the spatially non-homogeneous case.**

Associated with $F_\alpha(t,x,\xi)$, one can introduce the macro moments

$$n_\alpha(t,x) \equiv \int_{\mathbb{R}^3} F_\alpha(t,x,\xi)\,d\xi,$$
$$u_\alpha(t,x) \equiv \frac{1}{n_\alpha(t,x)} \int_{\mathbb{R}^3} \xi F_\alpha(t,x,\xi)\,d\xi,$$
$$\theta_\alpha(t,x) \equiv \frac{1}{3k_\alpha n_\alpha} \int_{\mathbb{R}^3} |\xi - u_\alpha(t,x)|^2 F_\alpha(t,x,\xi)\,d\xi, \quad k_\alpha = \frac{k_B}{m_\alpha},$$

and the high-order moments (thermal quantities)

$$\begin{aligned} P_\alpha(t,x) &\equiv m_\alpha \int_{\mathbb{R}^3} (\xi - u_\alpha) \otimes (\xi - u_\alpha) F_\alpha(t,x,\xi)\,d\xi \\ &= p_\alpha I + \Pi_\alpha, \quad p_\alpha = k_B n_\alpha \theta_\alpha, \\ h_\alpha(t,x) &\equiv \frac{1}{2} m_\alpha \int_{\mathbb{R}^3} |\xi - u_\alpha|^2 (\xi - u_\alpha) F_\alpha(t,x,\xi)\,d\xi, \\ \mathcal{R}_\alpha(t,x) &\equiv \sum_\beta \int_{\mathbb{R}^3} m_\alpha (\xi - u_\alpha) \mathcal{C}_{\alpha\beta}\,d\xi, \\ \mathcal{Q}_\alpha(t,x) &\equiv \sum_\beta \int_{\mathbb{R}^3} \frac{1}{2} m_\alpha |\xi - u_\alpha|^2 \mathcal{C}_{\alpha\beta}\,d\xi. \end{aligned}$$

Macro fluid moment system (Euler-Maxwell, un-closed!!!):

$$(\partial_t + u_\alpha \cdot \nabla_x) n_\alpha + n_\alpha \nabla_x \cdot u_\alpha = 0,$$

$$n_\alpha m_\alpha (\partial_t + u_\alpha \cdot \nabla_x) u_\alpha + \nabla_x (k_B n_\alpha \theta_\alpha) = n_\alpha q_\alpha \left(E + \frac{u_\alpha}{c} \times B\right) - \nabla_x \cdot \Pi_\alpha + \mathcal{R}_\alpha,$$

$$\frac{3}{2} n_\alpha (\partial_t + u_\alpha \cdot \nabla_x) k_B \theta_\alpha + k_B n_\alpha \theta_\alpha \nabla_x \cdot u_\alpha = -\Pi_\alpha : \nabla_x u_\alpha - \nabla_x \cdot h_\alpha + \mathcal{Q}_\alpha,$$

coupled to

$$\partial_t E - c\nabla \times B = -4\pi \sum_\alpha q_\alpha n_\alpha u_\alpha,$$

$$\partial_t B + c\nabla \times E = 0,$$

$$\nabla \cdot E = 4\pi \sum_\alpha q_\alpha n_\alpha, \quad \nabla \cdot B = 0.$$

The mathematical results for the VMB in perturbation framework (Pure BE[1]: Carleman, Grad, UKai,...):

$$Q_{\alpha\beta}(F_\alpha, F_\beta) = \frac{1}{\epsilon}\int_{\mathbb{R}^3}\int_{S^2} q(\xi-\xi_*,\omega)\{F_\alpha(\xi')F_\beta(\xi'_*) - F_\alpha(\xi)F_\beta(\xi_*)\}\, d\xi d\omega,$$

where for $\omega \in S^2$**,**

$$\xi' = \xi - \frac{2m_\beta}{m_\alpha + m_\beta}[(\xi-\xi_*)\cdot\omega]\,\omega,$$
$$\xi'_* = \xi_* + \frac{2m_\alpha}{m_\alpha + m_\beta}[(\xi-\xi_*)\cdot\omega]\,\omega.$$

$$q(\xi-\xi_*,\omega) = |(\xi-\xi_*)\cdot\omega|. \quad \textbf{(Hard-sphere model)}$$

- $\Omega = \mathbb{T}^3$
  - **Global existence: Guo (IM, 2003) (Energy method)**
  - **Large-time behavior of solutions: Jang (ARMA, 2009) (**$t^{-\infty}$**)**

[1] Non-perturbation framework: DiPerna-Lions, Desvillettes-Villani, Gualdani-Mischler-Mouhot (arXiv),...

- $\Omega = \mathbb{R}^3$
  - **Global existence: Strain (CMP, 2006) (Use two-species' cancelation property to control E and pure time derivatives)**
  - **Large-time behavior of solutions: D.-Strain (2010) ($t^{-3/4}$, Linearized analysis + bootstrap to the nonlinear equation)**

Note: Mathematically it is highly nontrivial to generalize all the above results to the case of non hard-sphere model ($\gamma < 1$).

(Why? It is formally due to $L \sim |\xi|^\gamma$ with $\gamma \leq 1$ but $N \sim |\xi|$, for large $|\xi|$! **)**

Remark: **Energy and energy product in the Fourier space at the linearized level for $\Omega = \mathbb{R}^3$ (D.-Strain):**

$$\partial_t \mathcal{E}(t,k) + \lambda \|\nu^{1/2}\{\mathbf{I}-\mathbf{P}\}\hat{u}\|^2_{L^2_\xi} + \frac{\lambda|k|^2}{1+|k|^2}\|\mathbf{P}\hat{u}\|^2_{L^2_\xi} + \lambda|\hat{E}|^2 + \frac{\lambda|k|^2}{(1+|k|^2)^2}|\hat{B}|^2 \leq 0,$$

with

$$\mathcal{E}(t,k) \sim \|\hat{u}\|^2_{L^2_\xi} + |[\hat{E},\hat{B}]|^2.$$

This implies

$$\mathcal{E}(t,k) \leq e^{-\frac{\lambda|k|^2}{(1+|k|^2)^2}t}\mathcal{E}(0,k).$$

- **For** $k \neq 0$, $i\mathbb{R} \cap \sigma(\widehat{\mathcal{B}}(ik)) = \emptyset$ **but one branch of** $\lambda(ik)$ **tends to** 0 **with rate** $1/|k|^2$ **as** $|k| \to \infty$.
- **It also means that** $(i\tau + \mathcal{B})^{-1}$ **is unbounded as** $|\tau| \to \infty$.
- **Abstract spectral theory and applications to the Bresse system for polynomial stability of semigoup:** Rivera-Racke, Liu-Rao, Batty,...

Questions:

- **Is the energy product optimal? (**Li-Yang-Zhong (Spectral analysis, arXiv, 2014)**)**
- **Can the fluid-type system (Euler-Maxwell) with dampings enjoy a similar property?**
  (Ueda-Kawashima(MAA, 2011), D. (JHDE 2011), Ueda-Wang-Kawashima (SIMA 2012), ...**)**
- **What happens to the long-range potentials with/without angular cutoff? (**Guo (VPL, JAMS 2012)$\rightarrow$**VPB, VMB, or VML???)**
- **Abstract theory? (**Kawashima's compensation function$\rightarrow$**coupling system)**
- **Stability of the non-trivial profile (wave patterns)?**

II: Eigenvalue problem on Euler-Maxwell with collisions

Non-damped case:

- One-fluid case for electrons: Germain-Masmoudi (dispersive but still a kind of system of Klein-Gordon equations with different speeds! arXiv 2011)
- Two-fluid case: Guo-Ionescu-Pausader (arXiv 2013)

II-1: Case of one-fluid for electrons

Consider the Euler-Maxwell system with relaxation[2]:

$$\begin{cases} \partial_t n + \nabla \cdot (nu) = 0, \\ \partial_t u + u \cdot \nabla u + \dfrac{1}{n} \nabla p(n) = -(E + u \times B) - \nu u, \\ \partial_t E - \nabla \times B = nu, \\ \partial_t B + \nabla \times E = 0, \\ \nabla \cdot E = n_{\mathrm{b}} - n, \quad \nabla \cdot B = 0. \end{cases}$$

Here, $n = n(t,x) \geq 0$**,** $u = u(t,x) \in \mathbb{R}^3$**,** $E = E(t,x) \in \mathbb{R}^3$ **and** $B = B(t,x) \in \mathbb{R}^3$**, for** $t > 0$**,** $x \in \mathbb{R}^3$**, denote the electron density, electron velocity, electric field and magnetic field, respectively. Initial data is given as**

$$[n, u, E, B]|_{t=0} = [n_0, u_0, E_0, B_0], \quad x \in \mathbb{R}^3.$$

[2] D. Nicholson, Introduction to Plasma Theory, 1992.

Consider the linearized homogeneous system for $U=[\rho,u,E,B]$ around $[\rho=1,u=0,E=0,B=0]$:

$$\begin{cases}\partial_t\rho+\nabla\cdot u=0,\\ \partial_t u+\gamma\nabla\rho+E+u=0,\\ \partial_t E-\nabla\times B-u=0,\\ \partial_t B+\nabla\times E=0,\\ \nabla\cdot E=-\rho,\quad \nabla\cdot B=0,\quad t>0,x\in\mathbb{R}^3,\end{cases}$$

with given initial data

$$U|_{t=0}=U_0:=[\rho_0,u_0,E_0,B_0],\quad x\in\mathbb{R}^3,$$

satisfying the compatible condition

$$\nabla\cdot E_0=-\rho_0,\quad \nabla\cdot B_0=0.$$

For $t \geq 0$ and $k \in \mathbb{R}^3$ with $|k| \neq 0$, define the *decomposition*

$$\begin{bmatrix} \hat{\rho}(t,k) \\ \hat{u}(t,k) \\ \hat{E}(t,k) \\ \hat{B}(t,k) \end{bmatrix} = \begin{bmatrix} \hat{\rho}(t,k) \\ \hat{u}_{\|}(t,k) \\ \hat{E}_{\|}(t,k) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \hat{u}_{\perp}(t,k) \\ \hat{E}_{\perp}(t,k) \\ \hat{B}_{\perp}(t,k) \end{bmatrix},$$

where $\hat{u}_{\|}, \hat{u}_{\perp}$ are defined by

$$\hat{u}_{\|} = \tilde{k}\tilde{k} \cdot \hat{u}, \quad \hat{u}_{\perp} = -\tilde{k} \times (\tilde{k} \times \hat{u}) = (\mathbf{I}_3 - \tilde{k} \otimes \tilde{k})\hat{u},$$

Define

$$U^I = \mathcal{F}^{-1} \begin{bmatrix} \hat{\rho}(t,k) \\ \hat{u}_{\|}(t,k) \\ \hat{E}_{\|}(t,k) \end{bmatrix}, \quad U^{II} = \mathcal{F}^{-1} \begin{bmatrix} \hat{u}_{\perp}(t,k) \\ \hat{E}_{\perp}(t,k) \\ \hat{B}_{\perp}(t,k) \end{bmatrix}.$$

Then,

$$U = U^I + U^{II}.$$

Theorem

U^I, U^{II} satisfies

$$\begin{cases} \partial_t^2 U^I - \gamma \Delta U^I + U^I + \partial_t U^I = 0, \\ \partial_t U^{II} + \begin{pmatrix} \mathbf{I}_3 & \mathbf{I}_3 & 0 \\ -\mathbf{I}_3 & 0 & -\nabla\times \\ 0 & \nabla\times & 0 \end{pmatrix} U^{II} = 0. \end{cases}$$

Furthermore, $\mathcal{F}U^I = G^I_{7\times 7}(t,k)\mathcal{F}U^I_0$ *with*

$$G^I_{7\times 7} = e^{-\frac{t}{2}} \cos(\sqrt{3/4+\gamma|k|^2}t) \begin{bmatrix} 1 & & \\ & 0_3 & \\ & & 0_3 \end{bmatrix} + e^{-\frac{t}{2}} \frac{\sin(\sqrt{3/4+\gamma|k|^2}t)}{\sqrt{3/4+\gamma|k|^2}} \begin{bmatrix} 1/2 & -ik & 0 \\ -i\gamma k & -1/2\mathbf{I}_3 & -\mathbf{I}_3 \\ 0 & \mathbf{I}_3 & 1/2\mathbf{I}_3 \end{bmatrix}.$$

To solve U^{II}, consider the characteristic equation

$$F(\chi) := \chi^3 + \chi^2 + (1+|k|^2)\chi + |k|^2 = 0.$$

Lemma

Let $|k| \neq 0$. The equation $F(\chi) = 0$, $\chi \in \mathbb{C}$, has a real root $\sigma = \sigma(|k|) \in (-1, 0)$ and two conjugate complex roots $\boxed{\chi_\pm = \beta \pm i\omega}$ with $\beta = \beta(|k|) \in (-1/2, 0)$ and $\omega = \omega(|k|) \in (\sqrt{6}/3, \infty)$ satisfying

$$\beta = -\frac{\sigma + 1}{2}, \quad \omega = \frac{1}{2}\sqrt{3\sigma^2 + 2\sigma + 3 + 4|k|^2}.$$

σ, β, ω are smooth over $|k| > 0$, and $\sigma(|k|)$ is strictly decreasing in $|k| > 0$ with

$$\lim_{|k|\to 0} \sigma(|k|) = 0, \qquad \lim_{|k|\to\infty} \sigma(|k|) = -1.$$

Lemma (cont.)

Mover, the following asymptotic behaviors hold true:

$$\sigma(|k|) = -O(1)|k|^2,$$
$$\beta(|k|) = -\frac{1}{2} + O(1)|k|^2, \quad \omega(|k|) = \frac{\sqrt{3}}{2} + O(1)|k|$$

whenever $|k| \leq 1$ *is small, and*

$$\sigma(|k|) = -1 + O(1)|k|^{-2},$$
$$\beta(|k|) = -O(1)|k|^{-2}, \quad \omega(|k|) = O(1)|k|$$

whenever $|k| \geq 1$ *is large. Here and in the sequel* $O(1)$ *denotes a generic strictly positive constant.*

Remarks:

• Recently Ueda-D.-Kawashima (2012) developed a general theory for characterising the structure of the linear symmetric hyperbolic system with partially relaxation in the case of the regularity-loss type $(p = 1, q = 2)$

$$\mathcal{E}(t,k) \leq e^{-\lambda \frac{|k|^{2p}}{(1+|k|^2)^{2q}} t} \mathcal{E}(0,k).$$

Only Fourier energy method!

The key point is to introduce a symmetric matrix S besides the skew-symmetric matrix K.

• More recently Ueda-D.-Kawashima (2013) also constructed two classes of concrete regularity-loss type $(p < q)$ models for more general values of (p, q) in terms of phase dimensions.

II-2: Case of two-fluid:
Justification of diffusion effect

Consider the two-fluid $(\alpha = i, e)$ Euler-Maxwell system

$$\begin{aligned}
&\partial_t n_\alpha + \nabla \cdot (n_\alpha u_\alpha) = 0,\\
&m_\alpha n_\alpha(\partial_t u_\alpha + u_\alpha \cdot \nabla u_\alpha) + \nabla p_\alpha(n_\alpha)\\
&\qquad\qquad\qquad = q_\alpha n_\alpha\Big(E + \frac{u_\alpha}{c} \times B\Big) - \nu_\alpha m_\alpha n_\alpha u_\alpha,\\
&\partial_t E - c\nabla \times B = -4\pi J,\\
&\partial_t B + c\nabla \times E = 0,\\
&\nabla \cdot E = 4\pi\rho, \quad \nabla \cdot B = 0,
\end{aligned}$$

with

$$J = \sum_\alpha q_\alpha n_\alpha u_\alpha, \quad \rho = \sum_\alpha q_\alpha n_\alpha.$$

$$t \geq 0, x \in \mathbb{R}^3.$$

Our goal is to give a complete analysis of

- eigenvalue problems,
- as well as the optimal large-time behaviour of solutions,

i.e., **tending time-asymptotically linear diffusive waves**.

This can be regarded as an attempt for the mathematical justification of **Darcy's law** in the context of two-fluid plasma with collisions at the linear level.

Heuristic derivation of diffusion waves:

Expected long-term asymptotic profile satisfies the quasi-neutral assumption:

$$n_i = n_e = n(t,x), \qquad u_i = u_e = u(t,x), \qquad B = \text{Const}.$$

Assume: $B = (0, 0, |B|)$ **along** x_3**-direction. Then, the momentum equations**

$$\nabla p_\alpha(n) = q_\alpha n \left(E + \frac{u}{c} \times B \right) - \nu_\alpha m_\alpha n u,$$

can uniquely determine ($p_\alpha(n) = T_\alpha n$ **w.l.g.)**

$$n \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -\frac{T_i+T_e}{m_i\nu_i+m_e\nu_e} & 0 & 0 \\ 0 & -\frac{T_i+T_e}{m_i\nu_i+m_e\nu_e} & 0 \\ 0 & 0 & -\frac{T_i+T_e}{m_i\nu_i+m_e\nu_e} \end{pmatrix} \begin{pmatrix} \partial_1 n \\ \partial_2 n \\ \partial_3 n \end{pmatrix}.$$

and moreover,

$$n\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \frac{1}{e}\begin{pmatrix} \frac{T_i m_e \nu_e - T_e m_i \nu_i}{m_i \nu_i + m_e \nu_e} & \frac{e|B|}{c}\frac{T_i + T_e}{m_i \nu_i + m_e \nu_e} & 0 \\ -\frac{e|B|}{c}\frac{T_i + T_e}{m_i \nu_i + m_e \nu_e} & \frac{T_i m_e \nu_e - T_e m_i \nu_i}{m_i \nu_i + m_e \nu_e} & 0 \\ 0 & 0 & \frac{T_i m_e \nu_e - T_e m_i \nu_i}{m_i \nu_i + m_e \nu_e} \end{pmatrix}\begin{pmatrix} \partial_1 n \\ \partial_2 n \\ \partial_3 n \end{pmatrix}.$$

Notice that n therefore satisfies the diffusion equation

$$\partial_t n - \mu_1 \Delta n = 0, \quad \mu_1 := \frac{T_i + T_e}{m_i \nu_i + m_e \nu_e}.$$

Whenever $p_\alpha(\cdot)$ takes the γ-law, the corresponding diffusion wave is nonlinear with isotropic diffusion coefficient.

From

- R.J. Goldston and P.H. Rutherford, *Introduction to Plasma Physics*, Taylor & Francis (1995).

it is said that

The ambipolar electron and ion fluxes are then obtained from equation (12.22) for electrons:

$$n\mathbf{u}_{e\perp} = n\mathbf{u}_{i\perp} = -D_a \nabla_\perp n \qquad (12.24)$$

where

$$D_a \approx \frac{\nu_{en}(T_e + T_i)}{m\omega_{ce}^2} \approx \nu_{en}\langle r_{Le}^2\rangle \left(1 + \frac{T_i}{T_e}\right) \qquad (12.25)$$

where $\langle r_{Le}^2\rangle = T_e/(m\omega_{ce}^2) = mT_e/(e^2B^2)$ is the mean-square Larmor radius of the electrons. The diffusion coefficient D_a is seen to be inversely proportional to B^2. Clearly, our result agrees—at least in some sense—with the heuristic result given in equation (12.10), except that ambipolar diffusion is at the slower rate similar in order-of-magnitude to that given by equation (12.10) for *electrons*.

Remark: A cancelation effect due to two-fluid could be ignored in the above formal argument of the textbook.

The expected asymptotic equations for the electromagnetic part in the sense of "Darcy's Law":

$$\begin{cases} -e\bar{E}_\perp + m_i\nu_i\bar{u}_{i,\perp} = 0, \\ e\bar{E}_\perp + m_e\nu_e\bar{u}_{e,\perp} = 0, \\ -c\nabla\times\bar{B} + 4\pi n(e\bar{u}_{i,\perp} - e\bar{u}_{e,\perp}) = 0, \\ \partial_t\bar{B} + c\nabla\times\bar{E}_\perp = 0. \end{cases}$$

Letting $n = 1$, this gives

$$\begin{cases} \partial_t\bar{B} - \mu_2\Delta B = 0, \quad \mu_2 = \dfrac{c^2 m_i\nu_i m_e\nu_e}{4\pi e^2(m_i\nu_i + m_e\nu_e)}, \\ \bar{u}_{i,\perp} = \dfrac{e}{m_i\nu_i}\bar{E}_\perp = \dfrac{c}{4\pi e}\dfrac{m_e\nu_e}{m_i\nu_i + m_e\nu_e}\nabla\times\bar{B}, \\ \bar{u}_{e,\perp} = -\dfrac{e}{m_e\nu_e}\bar{E}_\perp = -\dfrac{c}{4\pi e}\dfrac{m_i\nu_i}{m_i\nu_i + m_e\nu_e}\nabla\times\bar{B}, \\ \bar{E}_\perp = \dfrac{c}{4\pi e^2}\dfrac{m_i\nu_i m_e\nu_e}{m_i\nu_i + m_e\nu_e}\nabla\times\bar{B}. \end{cases}$$

Goal: **Mathematical justification!**

Main results: **(D.-Liu-Zhu, arXiv 2014)**

- **Eigenvalue analysis shows that**

$$|\widehat{U}(t,k)-\widehat{\overline{U}}(t,k)| \lesssim \chi_{|k|\leq 1}|k|e^{-\lambda|k|^2 t}|\widehat{U}_0(k)| + \chi_{|k|\geq 1}e^{-\frac{\lambda}{|k|^2}t}|\widehat{U}_0(k)|,$$

where U is the solution to the linearized Cauchy problem with initial data U_0, and $\overline{U}$ is the solution to the asymptotic diffusion equations with the same initial data.

The proof is also combined with the Fourier energy estimate for the high-frequency region.

- **Solutions to the nonlinear Cauchy problem tend time-asymtotically toward the diffusion waves with a faster rate than the one in which solutions themselves decay. Precisely, let $U = [\rho_\alpha, u_\alpha, E, B]$ be the solution the perturbed Cauchy problem on the Euler-Maxwell system with initial data U_0. Define $U^* = U^*(x,t) = [\rho^*, u^*_\alpha, E^*, B^*]$ by**

$$\rho^*(x,t) = G_{\mu_1}(x,t+1)\left(\frac{m_i\nu_i}{m_i\nu_i + m_e\nu_e}\int_{\mathbb{R}^3}\rho_{i0}(x)dx + \frac{m_e\nu_e}{m_i\nu_i + m_e\nu_e}\int_{\mathbb{R}^3}\rho_{e0}(x)dx\right),$$

$$B^*(x,t) = G_{\mu_2}(x,t+1)\int_{\mathbb{R}^3}\bar{B}_0(x)dx = G_{\mu_2}(x,t+1)\int_{\mathbb{R}^3}B_0(x)dx,$$

$$u^*_\alpha(t,x) = -\frac{T_i + T_e}{m_i\nu_i + m_e\nu_e}\nabla n^*(t,x) + \frac{c}{4\pi q_\alpha}\frac{m_e\nu_e}{m_i\nu_i + m_e\nu_e}\nabla\times B^*(t,x),$$

$$E^*(t,x) = \frac{T_i m_e\nu_e - T_e m_i\nu_i}{e(m_i\nu_i + m_e\nu_e)}\nabla n^*(t,x) + \frac{c}{4\pi e^2}\frac{m_i\nu_i m_e\nu_e}{m_i\nu_i + m_e\nu_e}\nabla\times B^*(t,x).$$

Then, one can show:

$$\|(U - U^*)(t)\| \lesssim (1+t)^{-\frac{5}{4}},$$

under additional conditions on initial data:

$$\int_{\mathbb{R}^3} \rho_{\alpha 0}(x)\, dx = 0, \quad \int_{\mathbb{R}^3} B_0(x)\, dx = 0.$$

Remark:

$$\|U^*(t)\| \sim (1+t)^{-\frac{3}{4}}$$

$\Rightarrow U^*$ **is a large-time asymptotic profile in terms of Darcy's law! More precisely,**

$$\sum_{\alpha=i,e} \|\rho_\alpha - \rho^*\| + \|B - B^*\| \leq C(1+t)^{-\frac{5}{4}},$$

and

$$\sum_{\alpha=i,e} \|u_\alpha - u_\alpha^*\| + \|E - E^*\| \leq C(1+t)^{-\frac{7}{4}}.$$

Approach: **Energy estimate combined with the result from the spectral analysis (D.-Ukai-Yang-Zhao 2008)**

$$U - U^* = \underbrace{(U - e^{tL}U_0)}_{I} + \underbrace{(e^{tL}U_0 - \bar{U})}_{II} + \underbrace{(\bar{U} - U^*)}_{III},$$

that is,

$$\begin{aligned}
\rho_\alpha - \rho^* &= \left(\rho_\alpha - \mathbf{P}_{1\alpha}e^{tL}U_0\right) + \left(\mathbf{P}_{1\alpha}e^{tL}U_0 - \bar{\rho}\right) + \left(\bar{\rho} - \rho^*\right), \\
u_\alpha - u^* &= \left(u_\alpha - \mathbf{P}_{2\alpha}e^{tL}U_0\right) + \left(\mathbf{P}_{2\alpha}e^{tL}U_0 - \bar{u}_\alpha\right) + \left(\bar{u}_\alpha - u_\alpha^*\right), \\
E - E^* &= \left(E - \mathbf{P}_3e^{tL}U_0\right) + \left(\mathbf{P}_3e^{tL}U_0 - \bar{E}\right) + \left(\bar{E} - E^*\right), \\
B - B^* &= \left(B - \mathbf{P}_4e^{tL}U_0\right) + \left(\mathbf{P}_4e^{tL}U_0 - \bar{B}\right) + \left(\bar{B} - B^*\right).
\end{aligned}$$

Remark: It seems more interesting to justify the asymptotic equations in the sense of generalised Darcy's laws in the following settings

- nonlinear diffusion connecting different end states along one direction (Hsiao-Liu, CMP 1992);
- appearance of vacuum related to the asymptotic stability of Barenblatt solution (Huang-Marcati-Pan, ARMA 2005).

III: Two-fluid VMB system without angular cutoff

Recent progress in non-cutoff case for the pure Boltzmann:

- Alexandre-Morimoto-Ukai-Xu-Yang (AMUXY) (CMP 2011, JFA 2012)
- Gressman-Strain (JAMS, 2011)

Consider the Boltzmann equations

$$\partial_t F_+ + \xi \cdot \nabla_x F_+ + (E + \xi \times B) \cdot \nabla_\xi F_+ = Q(F_+, F_+) + Q(F_+, F_-),$$

$$\partial_t F_- + \xi \cdot \nabla_x F_- - (E + \xi \times B) \cdot \nabla_\xi F_- = Q(F_-, F_+) + Q(F_-, F_-),$$

coupling to

$$\partial_t E - \nabla_x \times B = -\int_{\mathbb{R}^3} \xi (F_+ - F_-)\, d\xi,$$

$$\partial_t B + \nabla_x \times E = 0,$$

$$\nabla_x \cdot E = \int_{\mathbb{R}^3} (F_+ - F_-)\, d\xi, \quad \nabla_x \cdot B = 0.$$

Here

$$F_{\pm} = F_{\pm}(t, x, \xi) \geq 0,$$

$$x = (x_1, x_2, x_3) \in \mathbb{R}^3, \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3, t \geq 0.$$

Initial data:

$$F_{\pm}(0, x, \xi) = F_{0,\pm}(x, \xi), \quad E(0, x) = E_0(x), \quad B(0, x) = B_0(x),$$

with

$$\nabla_x \cdot E_0 = \int_{\mathbb{R}^3} (F_{0,+} - F_{0,-})\, d\xi, \quad \nabla_x \cdot B_0 = 0.$$

Boltzmann collision operator:

$$Q(F,G) = \int_{\mathbb{R}^3\times\mathbb{S}^2} q(\xi-\xi_*,\sigma)\left[F(\xi')G(\xi'_*) - F(\xi)G(\xi_*)\right] d\xi_* d\sigma,$$

$$\xi' = \frac{\xi+\xi_*}{2} + \frac{|\xi-\xi_*|}{2}\sigma, \quad \xi'_* = \frac{\xi+\xi_*}{2} - \frac{|\xi-\xi_*|}{2}\sigma.$$

$$q(\xi-\xi_*,\sigma) = C_q|\xi-\xi_*|^\gamma b(\cos\theta),$$

with

$\cos\theta = \sigma\cdot(\xi-\xi_*)/|\xi-\xi_*|$

$C_q > 0$, $\gamma > -3$

$\exists\ C_b > 0$, $0 < s < 1$ **s.t.**

$$\frac{1}{C_b\theta^{1+2s}} \le \sin\theta\, b(\cos\theta) \le \frac{C_b}{\theta^{1+2s}}, \quad \forall\,\theta\in(0,\frac{\pi}{2}].$$

Our interest:

- **Collisions by Boltzmann for angular non-cutoff and for soft potentials**
- **Electric-magnetic fields occur**
- **No relativistic effect**

Reformulation of Cauchy problem:

$$F_\pm(t,x,\xi)=\mu+\mu^{1/2}f_\pm(t,x,\xi),\quad \mu=\mu(\xi)=(2\pi)^{-3/2}e^{-|\xi|^2/2},$$

satisfies

$$\begin{cases}\partial_t f+\xi\cdot\nabla_x f+q_0(E+\xi\times B)\cdot\nabla_\xi f-E\cdot\xi\mu^{1/2}q_1+Lf \\ \qquad\qquad =\dfrac{q_0}{2}E\cdot\xi f+\Gamma(f,f), \\ \partial_t E-\nabla_x\times B=-\displaystyle\int_{\mathbb{R}^3}\xi\mu^{1/2}(f_+-f_-)\,d\xi, \\ \partial_t B+\nabla_x\times E=0, \\ \nabla_x\cdot E=\displaystyle\int_{\mathbb{R}^3}\mu^{1/2}(f_+-f_-)\,d\xi,\quad \nabla_x\cdot B=0\end{cases}$$

Here, $q_0=\mathrm{diag}(1,-1)$, $q_1=[1,-1]$, $f=[f_+,f_-]$.

$$L\sim(1+|\xi|)^\gamma\{\mathbf{I}-\mathbf{P}\}!!!$$

Theorem (D.-Liu-Yang-Zhao, KRM '13)

Assume

$$\max\left\{-3, -\frac{3}{2} - 2s\right\} < \gamma < -2s, \quad \frac{1}{2} \leq s < 1.$$

For initial data $(f_0(x,\xi), E_0(x), B_0(x))$ *regular enough and small enough, Cauchy problem on Vlasov-Maxwell-Boltzmann system admits a unique classical solution*

$$(f(t,x,\xi), E(t,x), B(t,x))$$

satisfying

$$\|f(t)\|_{L^2_{x,\xi}} + \|(E,B)(t)\|_{L^2} \lesssim (1+t)^{-\frac{3}{4}},$$

$$\|\nabla_x f(t)\|_{L^2_{x,\xi}} + \|\nabla_x (E,B)(t)\|_{L^2} \lesssim (1+t)^{-\frac{5}{4}}.$$

Remarks:

- Convergence rates are the same as those obtained by Ukai's spectrum method for the angular cutoff Boltzmann equation without forces ($E = B = 0$).
- Collision kernel includes the inverse power law which can be close to the Coulomb potential, i.e. $\gamma \to -3+$, $s \to 1-$.
- Restriction $s \geq 1/2$ is technical and essentially needed in our proof, since by AMUXY

$$C_1 \left\{ |f|^2_{H^s_{\gamma/2}} + |f|^2_{L^2_{s+\gamma/2}} \right\} \leq |f|^2_{\mathbf{D}} \leq C_2 |f|^2_{H^s_{s+\gamma/2}},$$

for $f \in (\ker L)^{\perp}$, where $|f|^2_{\mathbf{D}} = \langle -Lf, f \rangle$ is the Dirichlet norm.

Main difficulties and our efforts in the proof

- **Angular non-cutoff:**
  - Use the commutator estimates by AMUXY
  - Extra effort: Introduce the exponential weight into the non-cutoff framework
- **Soft potentials:**
  - Use the weighted energy norm by Guo
  - Extra effort: To take care the nonlinear estimates, use the velocity-time-dependent weight (D.-Yang-Zhao, '12):

$$w_{\tau,\lambda} = w_{\tau,\lambda}(t,\xi) = \langle\xi\rangle^{\gamma\tau}\exp\left\{\frac{\lambda}{(1+t)^{\vartheta}}\langle\xi\rangle\right\}.$$

Note: Guo's trick (JAMS '12):

$$e^{\mp\phi}f_{\pm}(\mp\frac{1}{2}\nabla_x\phi\cdot\xi f_{\pm}+\xi\cdot\nabla_x f_{\pm})=\frac{1}{2}\xi\cdot\nabla_x(e^{\mp\phi}f_{\pm}^2)$$

fails in the case of non-potential forces!

Main difficulties and our efforts in the proof (cont.)

- **Regularity-loss of (E, B):**
  - **D.-Strain: The dissipation rate of $\|(E,B)\|_{H^N}^2$ includes only**
    $$\|E\|_{H^{N-1}}^2 + \|\nabla_x B\|_{H^{N-2}}^2.$$
  - **Extra effort: Make the time-weighted estimates with time weight of negative power**
    $$\frac{d}{dt}\left[(1+t)^{-\sigma}\|(E,B)(t)\|_{H^N}^2\right] + \sigma(1+t)^{-\sigma-1}\|(E,B)(t)\|_{H^N}^2 \leq \text{“}h.o.t.\text{”}$$

    Such approach firstly introduced by Hosono-Kawashima (M3AS 2006).

IV: Non-trivial large-time behaviour of VPB system

Consider the Vlasov-Poisson-Boltzmann system:

$$0 \le F = F(t,x,\xi),\ t \ge 0, x \in \mathbb{R}, \xi = (\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3\text{:}$$

$$\begin{cases} \partial_t F + \xi_1 \partial_x F - \partial_x \phi \partial_{\xi_1} F = Q(F,F), \\ \qquad -\partial_x^2 \phi = \rho - \rho_e(\phi), \quad \rho = \int_{\mathbb{R}^3} F \, d\xi, \end{cases}$$

with

$$F(0,x,\xi) = F_0(x,\xi) \ge 0.$$

We assume

$$\lim_{x \to \pm\infty} F_0(x,\xi) = \frac{\rho_\pm}{(2\pi\theta_\pm)^{3/2}} e^{-\frac{|\xi - u_\pm|^2}{2\theta_\pm}},\ u_\pm = [u_{1\pm}, 0, 0],$$

$$\lim_{x \to \pm\infty} \phi(t,x) = \phi_\pm,\ \rho_\pm = \rho_e(\phi_\pm).$$

A typical example[3]**:**

$$\rho_e(\phi) = \left[1 + \frac{\gamma_e - 1}{\gamma_e}\frac{\phi}{A_e}\right]^{\frac{1}{\gamma_e - 1}},$$

with

$$\gamma_e > 1, \quad \phi_m = -\frac{\gamma_e}{\gamma_e - 1}A_e, \quad \phi_M = +\infty,$$

motivated by the momentum equation of electrons under the assumption that the electron mass is sufficiently small:

$$m_e\rho_e(\partial_t u_e + u_e\partial_x u_e) + \partial_x\left(A_e\rho_e^{\gamma_e}\right) = \rho_e\partial_x\phi.$$

The limit case $\gamma_e \to 1$ (electron isothermal):

$$\rho_e(\phi) = e^{\frac{\phi}{A_e}}.$$

[3] This has been recently used by D. Han-Kwan (CPDE 2011,...)

In general, we assume

$(\mathcal{A})$ $\rho_e(\phi) : (\phi_m, \phi_M) \to (\rho_m, \rho_M)$ **is a positive smooth function with**

$$\rho_m = \inf_{\phi_m<\phi<\phi_M} \rho_e(\phi), \quad \rho_M = \sup_{\phi_m<\phi<\phi_M} \rho_e(\phi),$$

and

$(\mathcal{A}_1)$ $\rho_e(0) = 1$ **with** $0 \in (\phi_m, \phi_M)$**;**
$(\mathcal{A}_2)$ $\rho_e(\phi) > 0$**,** $\rho_e'(\phi) > 0$ **for each** $\phi \in (\phi_m, \phi_M)$**;**
$(\mathcal{A}_3)$ $\rho_e(\phi)\rho_e''(\phi) \leq [\rho_e'(\phi)]^2$ **for each** $\phi \in (\phi_m, \phi_M)$**.**

Motivation of introducing $(\mathcal{A}_3)$: Define

$$P^{\phi}(\rho) = \int^{\rho} \frac{\varrho}{\rho_e'(\rho_e^{-1}(\varrho))} d\varrho,$$

in terms of

$$\boxed{\partial_x P^{\phi}(\rho) = \rho \partial_x \phi}$$

under the quasi-neutral assumption $\rho = \rho_e(\phi)$. One can check

$$\partial_\rho P^{\phi}(\rho) = \frac{\rho_e(\phi)}{\rho_e'(\phi)}, \quad \partial_\rho^2 P^{\phi}(\rho) = \frac{[\rho_e'(\phi)]^2 - \rho_e(\phi)\rho_e''(\phi)}{[\rho_e'(\phi)]^3},$$

with $\phi = \rho_e^{-1}(\rho)$ on the right. Due to $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$,

$$\partial_\rho P^{\phi}(\rho) > 0, \quad \partial_\rho^2 P^{\phi}(\rho) \geq 0,$$

for each $\rho \in (\rho_m, \rho_M)$.

Previous works on the pure Boltzmann:

$$(\partial_t + \xi \cdot \nabla_x)F = Q(F, F).$$

- **Shock wave: Caflisch-Nicolaenko, Liu-Yu, Yu,...**
- **Rarefaction wave: Liu-Yang-Yu-Zhao,...**
- **Contact discontinuity: Huang-Yang,...**

However,

no result on the Vlasov-Poisson-Boltzmann!

Remark: $\phi(t, x)$ **satisfying the Poisson equation may take the distinct states at both far fields** $x = \pm\infty$**:**

$$\phi(t, -\infty) \neq \phi(t, +\infty), \quad t \geq 0.$$

The goal of this work:

- **Construct the rarefaction wave of VPB system:**

$$\mathbf{M}_{[\rho^R,u^R,\theta^R](x/t)}(\xi), \quad \phi^R(x/t),$$

with

$$\mathbf{M}_{[\rho^R,u^R,\theta^R](z)}(\xi) \to \mathbf{M}_{[\rho_\pm,u_\pm,\theta_\pm]}(\xi) \quad \text{as } z \to \pm\infty$$
$$\phi^R(z) \to \phi_\pm \quad \text{as } z \to \pm\infty;$$

- **Prove that the rarefaction wave is time-asymptotically stable under small perturbation:**

$$f(t,x,\xi) \to \mathbf{M}_{[\rho^R,u^R,\theta^R](x/t)}(\xi), \quad \phi(t,x) \to \phi^R(x/t),$$

as $t \to \infty$, whenever they are sufficiently "*close*" at initial time.

Recall that

$$\psi_0 = 1, \quad \psi_i = \xi_i \ (i = 1, 2, 3), \quad \psi_4 = \frac{1}{2}|\xi|^2,$$

are five collision invariants satisfying

$$\int_{\mathbb{R}^3} \psi_i Q(F, F)\, d\xi = 0 \quad \textbf{for} \quad i = 0, 1, 2, 3, 4.$$

Maro-Micro decomposition (Liu-Yang-Yu):

$$F(t, x, \xi) = \mathbf{M}(t, x, \xi) + \mathbf{G}(t, x, \xi),$$

with

$$\mathbf{M}_{[\rho(t,x), u(t,x), \theta(t,x)]}(\xi) \equiv \frac{\rho(t, x)}{(2\pi R\theta(t, x))^{\frac{3}{2}}} \exp\left(-\frac{|\xi - u(t, x)|^2}{2R\theta(t, x)}\right),$$

through

$$\rho(t, x) \equiv \int_{\mathbb{R}^3} F(t, x, \xi)\, d\xi,$$

$$\rho(t, x) u_i(t, x) \equiv \int_{\mathbb{R}^3} \psi_i(\xi) F(t, x, \xi)\, d\xi, \quad i = 1, 2, 3,$$

$$\left[\rho\left(\frac{3}{2}R\theta(t, x) + \frac{1}{2}|u(t, x)|^2\right)\right] \equiv \int_{\mathbb{R}^3} \psi_4(\xi) F(t, x, \xi)\, d\xi.$$

Euler-Poisson type system (unclosed!): From

$$\int_{\mathbb{R}^3} \psi_i \left(\partial_t F + \xi_1 \partial_x F - \partial_x \phi \partial_{\xi_1} F\right) d\xi = 0, \quad i = 0,1,2,3,4,$$

one can deduce ($P = R\rho\theta$)

$$\begin{cases} \partial_t \rho + \partial_x(\rho u_1) = 0, \\ \partial_t(\rho u_1) + \partial_x(\rho u_1^2) + \partial_x P + \rho \partial_x \phi = -\int_{\mathbb{R}^3} \xi_1^2 \partial_x \mathbf{G}\, d\xi, \\ \partial_t(\rho u_i) + \partial_x(\rho u_1 u_i) = -\int_{\mathbb{R}^3} \xi_i \xi_1 \partial_x \mathbf{G}\, d\xi, \quad i = 2,3, \\ \partial_t \left[\rho\left(\frac{3}{2}R\theta + \frac{1}{2}|u|^2\right)\right] + \partial_x \left[u_1\left(\rho\left(\frac{3}{2}R\theta + \frac{1}{2}|u|^2\right) + P\right)\right] \\ \qquad + \rho u_1 \partial_x \phi = -\frac{1}{2}\int_{\mathbb{R}^3} |\xi|^2 \xi_1 \partial_x \mathbf{G}\, d\xi, \\ -\partial_x^2 \phi = \rho - \rho_e(\phi). \end{cases}$$

To capture viscosity and heat-conductivity, we rewrite

$$(\partial_t + \xi_1\partial_x - \partial_x\phi\partial_{\xi_1})(\mathbf{M}+\mathbf{G}) = Q(\mathbf{M}+\mathbf{G},\mathbf{M}+\mathbf{G}),$$

and apply the projection $\mathrm{P}_1^{\mathrm{M}} : f \mapsto \mathbf{G} = f - \mathrm{M}$ to it, so

$$\partial_t\mathbf{G} + \mathbf{P}_1^{\mathbf{M}}(\xi_1\partial_x\mathbf{M}) + \mathbf{P}_1^{\mathbf{M}}(\xi_1\partial_x\mathbf{G}) - \partial_x\phi\partial_{\xi_1}\mathbf{G} = L_{\mathbf{M}}\mathbf{G} + Q(\mathbf{G},\mathbf{G}).$$

This further implies

$$\mathbf{G} = L_{\mathbf{M}}^{-1}\Big(\mathbf{P}_1^{\mathbf{M}}(\xi_1\partial_x\mathbf{M})\Big) + \Theta,$$

with

$$\Theta = L_{\mathbf{M}}^{-1}\left[\partial_t\mathbf{G} + \mathbf{P}_1^{\mathbf{M}}(\xi_1\partial_x\mathbf{G}) - \partial_x\phi\partial_{\xi_1}\mathbf{G}\right] - L_{\mathbf{M}}^{-1}[Q(\mathbf{G},\mathbf{G})].$$

Plugging to zero-order fluid type system, we obtain

Navier-Stokes-Poisson type system (unclosed!):

$$
\begin{cases}
\partial_t \rho + \partial_x(\rho u_1) = 0, \\
\partial_t u_1 + u_1 \partial_x u_1 + \dfrac{\partial_x P}{\rho} + \partial_x \phi = \dfrac{3}{\rho}\partial_x \left(\mu(\theta)\partial_x u_1\right) - \dfrac{1}{\rho}\displaystyle\int_{\mathbb{R}^3} \xi_1^2 \partial_x \Theta \, d\xi, \\
\partial_t u_i + u_1 \partial_x u_i = \dfrac{1}{\rho}\partial_x \left(\mu(\theta)\partial_x u_i\right) - \dfrac{1}{\rho}\displaystyle\int_{\mathbb{R}^3} \xi_1 \xi_i \partial_x \Theta \, d\xi, \quad i = 2, 3, \\
\partial_t \left(\mathcal{E} + \dfrac{1}{2}|u|^2\right) + u_1 \partial_x \left(\mathcal{E} + \dfrac{1}{2}|u|^2\right) + \dfrac{\partial_x(P u_1)}{\rho} + u_1 \partial_x \phi \\
\qquad = \dfrac{1}{\rho}\partial_x \left(\kappa(\theta)\partial_x \theta\right) + \dfrac{3}{\rho}\partial_x \left(\mu(\theta) u_1 \partial_x u_1\right) + \dfrac{1}{\rho}\displaystyle\sum_{i=2}^{3} \partial_x \left(\mu(\theta) u_i \partial_x u_i\right) \\
\qquad\qquad - \dfrac{1}{2\rho}\displaystyle\int_{\mathbb{R}^3} |\xi|^2 \xi_1 \partial_x \Theta \, d\xi, \\
-\partial_x^2 \phi = \rho - \rho_e(\phi).
\end{cases}
$$

We expect (Why?) that the large-time behaviour is determined by the quasineutral Euler system

$$
\begin{cases}
\partial_t \rho + \rho \partial_x u_1 + u_1 \partial_x \rho = 0, \\
\partial_t u_1 + u_1 \partial_x u_1 + \dfrac{\partial_x P}{\rho} + \partial_x \phi = 0, \\
\partial_t \theta + u_1 \partial_x \theta + \dfrac{P \partial_x u_1}{\rho} = 0, \\
\rho = \rho_e(\phi),
\end{cases}
$$

or equivalently,

$$
\begin{cases}
\partial_t \rho + \rho \partial_x u_1 + u_1 \partial_x \rho = 0, \\
\partial_t u_1 + u_1 \partial_x u_1 + \dfrac{\partial_x [P + P^{\phi}(\rho)]}{\rho} = 0, \\
\partial_t S + u_1 \partial_x S = 0,
\end{cases}
$$

with $P = ke^S \rho^{5/3}$**.**

The Q.E. system has three characteristics

$$\begin{cases} \lambda_1 = \lambda_1(\rho, u_1, S) \equiv u_1 - \sqrt{\partial_\rho P(\rho, S) + \partial_\rho P^\phi(\rho)}, \\ \lambda_2 = \lambda_2(\rho, u_1, S) \equiv u_1, \\ \lambda_3 = \lambda_3(\rho, u_1, S) \equiv u_1 + \sqrt{\partial_\rho P(\rho, S) + \partial_\rho P^\phi(\rho)}. \end{cases}$$

The admissible set of 3-rarefaction wave:

$$R_3(\rho_-, u_{1-}, \theta_-) \equiv \Big\{ [\rho, u_1, \theta] \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \;\Big|\; \frac{\rho^{2/3}}{\theta} = \frac{\rho_-^{2/3}}{\theta_-},$$

$$u_1 - u_{1-} = \int_{\rho_-}^{\rho} \frac{\sqrt{\partial_\rho P(\varrho, S_i) + \partial_\rho P^\phi(\varrho)}}{\varrho} d\varrho,\ \rho > \rho_-,\ u_1 > u_{1-} \Big\}.$$

The 3-rarefaction wave $\left[\rho^R, u_1^R, \theta^R\right](z)$ with $z = x/t \in \mathbb{R}$:

$$
\begin{cases}
\lambda_3\left(\rho^R(z), u_1^R(z), S_i\right) \\
= \begin{cases} \lambda_3(\rho_-, u_{1-}, S_i) & \textbf{for } z < \lambda_3(\rho_-, u_{1-}, S_i), \\ z & \textbf{for } \lambda_3(\rho_-, u_{1-}, S_i) \le z \le \lambda_3(\rho_+, u_{1+}, S_i), \\ \lambda_3(\rho_+, u_{1+}, S_i) & \textbf{for } z > \lambda_3(\rho_+, u_{1+}, S_i), \end{cases} \\
u_1^R(z) - u_{1-} = \displaystyle\int_{\rho_-}^{\rho^R(z)} \sqrt{\frac{5}{3} A_i \varrho^{-\frac{4}{3}} + \varrho^{-1}\left(\frac{d}{d\rho}(\rho_e^{-1})\right)(\varrho)}\, d\varrho, \\
\theta^R(z) = \frac{3}{2} A_i (\rho^R(z))^{2/3}.
\end{cases}
$$

The smooth rarefaction wave $[\rho^r, u^r, \theta^r](t,x)$ and $\phi^r(t,x)$ with $u^r(t,x) = [u_1^r(t,x), 0, 0]$ are defined by

$$\begin{cases} \lambda_3(\rho^r(t,x), u_1^r(t,x), S_i) = w(t,x), \\ u_1^r(t,x) - u_{1-} = \displaystyle\int_{\rho_-}^{\rho^r(t,x)} \sqrt{\frac{5}{3} A_i \varrho^{-\frac{4}{3}} + \varrho^{-1} \left(\frac{d}{d\rho}(\rho_e^{-1}) \right)(\varrho)}\, d\varrho, \\ \theta^r(t,x) = \frac{3}{2} A_i (\rho^r(t,x))^{2/3}, \quad \phi^r(t,x) = \rho_e^{-1}(\rho^r(t,x)), \\ \lim\limits_{x \to \pm\infty} [\rho^r, u_1^r, \theta^r](t,x) = [\rho_\pm, u_{1\pm}, \theta_\pm], \quad [\rho_+, u_{1+}, \theta_+] \in R_3(\rho_-, u_{1-}, \theta_-), \end{cases}$$

with $w = w(t,x)$ being the solution to the Burgers' equation

$$\begin{cases} \partial_t w + w \partial_x w = 0, \\ w(0,x) = w_0(x) \overset{\text{def}}{=} \frac{1}{2}(w_+ + w_-) + \frac{1}{2}(w_+ - w_-) \tanh(\epsilon x), \\ w_\pm \overset{\text{def}}{=} \lambda_3(\rho_\pm, u_{1\pm}, S_i). \end{cases}$$

Here $\epsilon > 0$ is a constant to be chosen later on.

A technical notion for the weighted energy estimates:

Let the *reference weight function* $\mathbf{M}_* = \mathbf{M}_*(\xi) = \mathbf{M}_{[\rho_*, u_*, \theta_*]}(\xi)$ **be a global Maxwellian such that the constant state** $[\rho_*, u_*, \theta_*]$ **with** $u_* = [u_{1*}, 0, 0]$ **satisfies**

$$\begin{cases} \frac{1}{2} \sup_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} \theta^r(t,x) < \theta_* < \inf_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} \theta^r(t,x), \\ \sup_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} \{|\rho^r(t,x) - \rho_*| + |u^r(t,x) - u_*| + |\theta^r(t,x) - \theta_*|\} < \eta_0, \end{cases}$$

for a constant $\eta_0 > 0$ **which is not necessarily small.**

Remark: All inequalities are strict! (Why needed? It is also true for solutions due to small perturbation!!!)

Theorem (D.-Liu, arXiv:1405.2522)

Assume that $[\rho_+, u_{1+}, \theta_+] \in R_3(\rho_-, u_{1-}, \theta_-)$, $\rho_\pm = \rho_e(\phi_\pm)$ *with* $\phi_\pm \in (\phi_m, \phi_M)$, *and* $\rho_e(\cdot)$ *satisfies* $(\mathcal{A})$. *Let*

$$\delta_r = |\rho_+ - \rho_-| + |u_{1+} - u_{1-}| + |\theta_+ - \theta_-|$$

be the wave strength which is not necessarily small. There are $\epsilon_0 > 0$, $0 < \sigma_0 < 1/3$ *and* $C_0 > 0$, *which may depend on* δ_r *and* η_0, *such that if* $F_0(x, \xi) \geq 0$ *and*

$$\sum_{|\alpha|+|\beta|\leq 2} \left\| \partial_x^\alpha \partial_\xi^\beta \left(F_0(x,\xi) - \mathbf{M}_{[\rho^r, u^r, \theta^r](0,x)}(\xi) \right) \right\|^2_{L_x^2\left(L_\xi^2\left(\frac{1}{\sqrt{\mathbf{M}_*(\xi)}}\right)\right)} + \epsilon \leq \epsilon_0^2,$$

then the Cauchy problem on the Vlasov-Poisson-Boltzmann system admits a unique global solution

$$[F(t,x,\xi), \phi(t,x)],$$

satisfying

Theorem (conti.)

$$F(t,x,\xi) \geq 0$$

and

$$\sup_{t\geq 0} \sum_{|\alpha|+|\beta|\leq 2} \left\| \partial_x^\alpha \partial_\xi^\beta \left(F(t,x,\xi) - \mathbf{M}_{[\rho^r,u^r,\theta^r](t,x)}(\xi) \right) \right\|^2_{L^2_x\left(L^2_\xi\left(\frac{1}{\sqrt{\mathbf{M}_*(\xi)}}\right)\right)}$$

$$+ \sup_{t\geq 0} \sum_{|\alpha|\leq 2} \left\| \partial_x^\alpha \left(\phi(t,x) - \rho_e^{-1}(\rho^r(t,x)) \right) \right\|^2_{H^1} \leq C_0 \epsilon_0^{2\sigma_0}.$$

Moreover, it holds that

$$\sup_{t\to+\infty} \sup_{x\in\mathbb{R}} \left\{ \left\| F(t,x,\xi) - \mathbf{M}_{[\rho^R,u^R,\theta^R](x/t)}(\xi) \right\|_{L^2_\xi\left(\frac{1}{\sqrt{\mathbf{M}_*(\xi)}}\right)} \right.$$

$$\left. + \left| \phi(t,x) - \rho_e^{-1}\left(\rho^R(x/t)\right) \right| \right\} = 0.$$

Final Remark: **Two-fluid Navier-Stokes-Poisson system:**

$$\begin{cases} \partial_t n_i + \partial_x(n_i u_i) = 0, \\ m_i n_i(\partial_t u_i + u_i \partial_x u_i) + T_i \partial_x n_i - n_i \partial_x \phi = \mu_i \partial_x^2 u_i, \\ \partial_t n_e + \partial_x(n_e u_e) = 0, \\ m_e n_e(\partial_t u_e + u_e \partial_x u_e) + T_e \partial_x n_e + n_e \partial_x \phi = \mu_e \partial_x^2 u_e, \\ \partial_x^2 \phi = n_i - n_e, \quad t > 0, \; x \in \mathbb{R}. \end{cases}$$

Initial data are given by

$$[n_\alpha, u_\alpha](0, x) = [n_{\alpha 0}(x), u_{\alpha 0}(x)], \quad \alpha = i, e, \quad x \in \mathbb{R},$$

with

$$\lim_{x \to \pm\infty} [n_{\alpha 0}, u_{\alpha 0}](x) = [n_\pm, u_\pm], \quad \alpha = i, e.$$

The boundary values of ϕ at infinity are set by

$$\lim_{x \to \pm\infty} \phi(t, x) = \phi_\pm, \quad t \geq 0.$$

Large-time behavior for rarefaction waves can be determined by the quasineutral Euler system

$$\begin{cases} \partial_t n + \partial_x(nu) = 0, \\ n(\partial_t u + u\partial_x u) + \dfrac{T_i + T_e}{m_i + m_e}\partial_x n = 0, \end{cases}$$

with the potential function ϕ in large time determined by

$$\phi = \frac{T_i m_e - T_e m_i}{m_i + m_e} \ln n.$$

We can also show

$$n_\alpha(t,x) \to n^R(x/t), \quad u_\alpha(t,x) \to u^R(x/t), \quad \alpha = i, e,$$

and

$$\phi(t,x) \to \phi^R(x/t) := \frac{T_i m_e - T_e m_i}{m_i + m_e} \ln n^R(x/t),$$

uniformly for $x \in \mathbb{R}$ as t goes to infinity.

Open problems:

- **Existence and stability of shock wave and contact discontinuity?**
- **Collisional plasma on the half-space? (Related to the kinetic Bohm's criterion, see M. Suzuki's talk for justification at the fluid level)**

Acknowledge my collaborators in the talk:

- `Liu, Shuangqian` (Jinan University)
- `Liu, Qingqing` (Central China Normal University)
- `Kawashima, Shuichi` (Kyushu University)
- `Strain, Robert` (University of Pennsylvania)
- `Ueda, Yoshihiro` (Kobe University)
- `Ukai, Seiji` (Tokyo Institute of Technology)
- `Yang, Tong` (City University of Hong Kong)
- `Zhao, Huijiang` (Wuhan University)
- `Zhu, Changjiang` (Central China Normal University)

Thanks a lot for your attention!