Asymptotic stability of kinetic plasmas for general collision potentials

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Background

Physical description of a plasma

- Plasma is the 4th state of matter:
  solid→liquid→gas→plasma
- 99.9% of the universe exists in a plasma state
- Plasma is a gas of charged particles, e.g. electrons and ions
- The motion of plasmas strongly responds to the self-consistent electromagnetic field through the Maxwell equations

$$\frac{1}{c}\partial_t E - \nabla \times B = -\frac{4\pi}{c}J, \quad \frac{1}{c}\partial_t B + \nabla \times E = 0,$$
$$\nabla \cdot E = 4\pi\rho, \quad \nabla \cdot B = 0.$$

- Plasma physics involves the physics of classical mechanics, electromagnetism, and non relativistic statistical mechanics
- Challenge lies in the long-range coulomb interaction

Mathematical description of a plasma

- **microscopic particle model for** $[x_i(t), \xi_i(t)]$
- **mesoscopic kinetic model for** $f(t, x, \xi)$
- **macroscopic fluid model for** $[n(t, x), u(t, x)]$

1st type (Klimontovich)

Microscopic motion equations governing $[x_i(t), \xi_i(t)]$ of all plasma particles $1 \leq i \leq N_0$ of s-species at any time t:

$$
\begin{aligned}
& m_s \frac{d\xi_i}{dt} = q_s[E(t,x_i) + \frac{\xi_i}{c} \times B(t,x_i)], \\
& \frac{1}{c}\partial_t E - \nabla \times B = -\frac{4\pi}{c} J, \\
& \frac{1}{c}\partial_t B + \nabla \times E = 0, \\
& \nabla \cdot E = 4\pi\rho, \quad \nabla \cdot B = 0, \\
& \rho = \sum_s q_s \int_{\mathbb{R}^3} N_s(t,x,\xi)\, d\xi, \quad J = \sum_s q_s \int_{\mathbb{R}^3} \xi N_s(t,x,\xi)\, d\xi, \\
& N_s(t,x,\xi) = \sum_{i=1}^{N_0} \delta(x - x_i(t))\delta(\xi - \xi_i(t)).
\end{aligned}
$$

2nd type (kinetic plasma equations)

$f_s = f_s(t, x, \xi)$, $t \geq 0, x \in \mathbb{R}^3, \xi \in \mathbb{R}^3$, $s = i, e$

$$
\begin{aligned}
&\partial_t f_s + \xi \cdot \nabla_x f_s + \frac{q_s}{m_s}\left(E + \frac{\xi}{c} \times B\right) \cdot \nabla_\xi f_s = \left(\frac{\partial f_s}{\partial t}\right)_c, \\
&\frac{1}{c}\partial_t E - \nabla \times B = -\frac{4\pi}{c} J, \\
&\frac{1}{c}\partial_t B + \nabla \times E = 0, \\
&\nabla \cdot E = 4\pi\rho, \quad \nabla \cdot B = 0, \\
&\rho = \sum_s q_s \int_{\mathbb{R}^3} f_s(t, x, \xi)\, d\xi, \quad J = \sum_s q_s \int_{\mathbb{R}^3} \xi f_s(t, x, \xi)\, d\xi.
\end{aligned}
$$

Depending on the collisional feature, the system is called

- **Vlasov-Maxwell-Boltzmann**
- **Vlasov-Maxwell-Landau**

Characterization of collisions

– Boltzmann collision (Boltzmann, 1872)

$$\left(\frac{\partial f_s}{\partial t}\right)_c = \sum_{s'} Q(f_s, f_{s'}),$$

$$Q(f_1, f_2)(\xi) = \iint_{\mathbb{R}^3 \times S^2} B(\xi - \xi_*, \omega)\{f_1(\xi')f_2(\xi'_*) - f_1(\xi)f_2(\xi_*)\}\, d\xi_* d\omega,$$

$$\begin{cases} \xi' = \xi \; - \frac{2m_2}{m_1+m_2}[(\xi - \xi_*)\cdot\omega]\omega, \\ \xi'_* = \xi_* + \frac{2m_1}{m_1+m_2}[(\xi - \xi_*)\cdot\omega]\omega, \end{cases}$$

$$B(\xi - \xi_*, \omega) = \Phi(|\xi - \xi_*|)b(\frac{\xi - \xi_*}{|\xi - \xi_*|}\cdot\omega),$$

$$\Phi(|z|) \sim |z|^\gamma \; (-3 < \gamma \le 1), \quad \sin\theta b(\cos\theta) \sim \frac{1}{\theta^{1+2\nu}} \; (0 < \nu < 1)$$

An example: For the inverse power law $U(r) = r^{-(p-1)}$ $(p > 2)$**,** $\gamma = \frac{p-5}{p-1}$**,** $\nu = \frac{1}{p-1}$.

Grad's angular cutoff assumption:

$$\int_0^{\pi/2} \sin\theta \tilde{b}(\cos\theta)\, d\theta < \infty.$$

– **Landau collision (Landau, 1936):**

$$\left(\frac{\partial f_s}{\partial t}\right)_c = \sum_{s'} Q(f_s, f_{s'}),$$

$$Q(f_1, f_2)$$
$$= \frac{1}{m_1}\nabla_\xi \cdot \int_{\mathbb{R}^3} \Phi(\xi-\xi')\{\frac{1}{m_1} f_1(\xi)\nabla_\xi f_2(\xi') - \frac{1}{m_2} f_2(\xi)\nabla_\xi f_1(\xi')\}\, d\xi',$$

$$\Phi(z) = |z|^{\gamma+2}(\mathbf{I} - \frac{z\otimes z}{|z|^2})\,(\gamma \geq -3),$$
$$\gamma = -3 : \textbf{Coulomb potential}$$

Remark: Grazing limit: Boltzmann⇒Landau

3rd type (fluid plasma equations)

$$
\begin{aligned}
&\partial_t n_s + \nabla \cdot (n_s v_s) = 0, \\
&m_s n_s(\partial_t v_s + v_s \cdot \nabla v_s) + \nabla P_s = q_s n_s\left(E + \frac{v_s}{c} \times B\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \sum_{s'} \nu_{ss'} \frac{m_s m_{s'} n_s n_{s'}}{m_s n_s + m_{s'} n_{s'}} (v_s - v_{s'}), \\
&\frac{1}{c}\partial_t E - \nabla \times B = -\frac{4\pi}{c} J, \\
&\frac{1}{c}\partial_t B + \nabla \times E = 0, \\
&\nabla \cdot E = 4\pi\rho, \qquad \nabla \cdot B = 0, \\
&\rho = \sum_s q_s n_s, \qquad J = \sum_s q_s n_s v_s.
\end{aligned}
$$

Euler-Maxwell system with/without relaxation

The Plasma Stability Problem

- Due to the collision AND particle-field interactive mechanism, a plasma usually relaxes to different kinds of profiles such as equilibrium states, periodic states, and wave patterns.
- Both physically and mathematically, it is an important task to understand the stability of those profiles.
- Stability theory addresses the following three questions:
  - Can the initial (small) perturbation of a given profile imply the global-in-time existence of solutions?
  - Will the solution converge to it? How fast for the rate of convergence?
  - If unstable, how to characterize the growth modes?

Remark: **Problems without collisions are quite different (nonlinear effect and structure) !**

- **Molecule model:** H. Weitzner (CPAM '12)
- **Vlasov-Poisson system:** Lemou-Mehats-Raphael (Inve. '11), Mouhot-Villani (Acta M. '11), ...
- **Euler-Maxwell system:** Germain-Masmoudi (arXiv '11)
- ...

Time-asymptotic stability of
kinetic plasmas for general collision potentials

Boltzmann's celebrated H-theorem

$\partial_t f = Q(f,f) \Rightarrow$ **(Physical) entropy increasing:**

$$\frac{d}{dt}\int_{\mathbb{R}^3} d\xi \left\{-f\log f\right\} \geq 0.$$

This is a manifestation of the second law of thermodynamics.

- **Entropy takes the maximization at the Maxwellian**

$$\mathbf{M} = \mathbf{M}_{[\rho,u,T]}(\xi) = \frac{\rho}{(2\pi T)^{3/2}} e^{-\frac{|\xi-u|^2}{2T}}.$$

  ρ : **density,** u : **bulk velocity,** T: **temperature.**

- **L. Boltzmann himself predicted rapid convergence in large time to the Maxwellian due to the H-theorem. The "proof" was however held back by "analytical difficulties".**

- **Goal: prove convergence and convergence rate around the Maxwellian in the spatially non-homogeneous case.**

Degeneration of H-theorem

$$\{\partial_t + \xi \cdot \nabla_x\} f = Q(f, f) \Rightarrow$$

$$\frac{d}{dt} \int_\Omega dx \int_{\mathbb{R}^3} d\xi \, \{-f \log f\} \geq 0, \quad \Omega = \mathbb{R}^3 \text{ or } \mathbb{T}^3.$$

- **H-theorem fails at the local Maxwellian**

$$\mathbf{M}_{[\rho(t,x), u(t,x), T(t,x)]}(\xi).$$

- **In $\mathbb{T}^3$ case, a key tool to overcome the degeneration is the Poincare inequality:**

$$\|\rho - \frac{1}{|\mathbb{T}^3|} \int_{\mathbb{T}^3} \rho \, dx\|_{L^2_x(\mathbb{T}^3)} \leq C \|\nabla \rho\|_{L^2_x(\mathbb{T}^3)}$$

- **In $\mathbb{R}^3$ case, the Poincare inequality fails.**
- **Idea: seek out the enough dissipative mechanisms for the components of the local Maxwellian**

Degeneration of the electromagnetic field

- For the Maxwell system in vacuum,

  $$\partial_t E - \nabla \times B = 0, \quad \partial_t B + \nabla \times E = 0, \quad \nabla \cdot E = \nabla \cdot B = 0,$$

  the total energy is preserved at all time.

- Can the coupling with the kinetic equation imply a kind of the dissipative mechanism?

- Idea: again, seek out the enough dissipative mechanisms for the electromagnetic field with the understanding of the structure of the system

The Vlasov-Maxwell-Boltzmann/Landau system

$f_\pm = f_\pm(t,x,\xi) \geq 0$ **of two-species:**

$$\partial_t f_+ + \xi \cdot \nabla_x f_+ + (E + \xi \times B) \cdot \nabla_\xi f_+ = Q(f_+, f_+) + Q(f_+, f_-),$$
$$\partial_t f_- + \xi \cdot \nabla_x f_- - (E + \xi \times B) \cdot \nabla_\xi f_- = Q(f_-, f_+) + Q(f_-, f_-).$$

It is coupled with the Maxwell system

$$\begin{aligned}
&\partial_t E - \nabla_x \times B = -\int_{\mathbb{R}^3} \xi (f_+ - f_-) d\xi, \\
&\partial_t B + \nabla_x \times E = 0, \\
&\nabla_x \cdot E = \int_{\mathbb{R}^3} (f_+ - f_-) d\xi, \quad \nabla_x \cdot B = 0.
\end{aligned}$$

The initial data in this system is given as

$$f_\pm(0,x,\xi) = f_{0,\pm}(x,\xi), \quad E(0,x) = E_0(x), \quad B(0,x) = B_0(x).$$

Previous results on VMB

Boltzmann collision term Q takes the hard sphere model:

$$B(\xi - \xi_*, \omega) = |(\xi - \xi_*) \cdot \omega|.$$

- $\Omega = \mathbb{T}^3$
  - Global existence: Guo (IM, '03) (Energy method)
  - Large-time behavior of solutions: Jang (ARMA, '09)
- $\Omega = \mathbb{R}^3$
  - Global existence: Strain (CMP, '06) (Use two-species' cancelation property to control E and pure time derivatives)
  - Large-time behavior of solutions: D.-Strain ('10) (Linearized analysis + bootstrap to the nonlinear equation)

! Unknown for non hard-sphere model !

Previous results on VML

The only existing results concern the case of the absence of the variable magnetic field, i.e.

Vlasov-Poisson-Landau instead of VML

- $\Omega = \mathbb{T}^3$: Guo (JAMS, '12)
- $\Omega = \mathbb{R}^3$:
  - D.-Yang-Zhao (arXiv '11): an application of the exponential weight
  - Strain-Zhu (arXiv '12) and Yu (preprint '12): approach by Guo
  - Wang (arXiv '12): pure energy method without linearized analysis

! Unknown in the case of VML !

Linearization (Carleman, Grad, ...)

- **Define the perturbation u as $u = \mathbf{M}^{-1/2}(f - \mathbf{M})$, $u = [u_+, u_-]$, $f = [f_+, f_-]$, $\mathbf{M} = \mathbf{M}_{[1,0,1]}(\xi)$.**
- **Boltzmann's H-theorem implies: $f_\pm \to \mathbf{M}$, $[E, B] \to 0$.**
- **The linearized system**

$$
\begin{cases}
\partial_t u + \xi \cdot \nabla_x u - E \cdot \xi \mathbf{M}^{1/2}[1, -1] = \mathbf{L}u + g, \\
\partial_t E - \nabla_x \times B = -\langle [\xi, -\xi]\mathbf{M}^{1/2}, \{\mathbf{I} - \mathbf{P}\}u \rangle, \\
\partial_t B + \nabla_x \times E = 0, \\
\nabla_x \cdot E = \langle \mathbf{M}^{1/2}, u_+ - u_- \rangle, \quad \nabla_x \cdot B = 0, \\
[u, E, B]|_{t=0} = [u_0, E_0, B_0],
\end{cases}
$$

$$
\ker \mathbf{L} = \operatorname{span}\left\{ [1,0]\mathbf{M}^{\frac{1}{2}}, [0,1]\mathbf{M}^{\frac{1}{2}}, [\xi_i, \xi_i]\mathbf{M}^{\frac{1}{2}} (1 \le i \le 3), [|\xi|^2, |\xi|^2]\mathbf{M}^{\frac{1}{2}} \right\}.
$$

- **The local Maxwellian**

$$
\mathbf{P}_\pm u = \{a_\pm(t,x) + b(t,x) \cdot \xi + c(t,x)(|\xi|^2 - 3)\}\mathbf{M}^{\frac{1}{2}}.
$$

Dissipation from $\mathbf{L}$:

$$\int_{\mathbb{R}^3} u \cdot \mathbf{L}u \, d\xi \lesssim -\int_{\mathbb{R}^3} \nu(\xi) |\{\mathbf{I} - \mathbf{P}\}u|^2 \, d\xi$$

- **Collision frequency:** $\nu(\xi) = \langle \xi \rangle^\gamma$; $\mathbf{P}$: **projection from** L^2_ξ **to** $\ker \mathbf{L}$
- **A summary of possible difficulties:**
  - **The dissipation of** $\mathbf{P}u$ **is missing: The local Maxwellian is dispersive in the whole space due to the degeneration of** $\mathbf{L}$ **! (Hypocoercivity: Villani)**
  - **If** $\gamma < 1$ **then how to control a nonlinear term which grows in large** $|\xi|$ **at least linearly?**
  - **If** $\gamma < 0$ **then how to control 1st-order velocity derivative of the linear transport term** $\xi \cdot \nabla_x u$**?**
  - **If** $\gamma < 0$ **is much smaller then how to control the nonlinear transport term** $E \cdot \nabla_\xi u$ **provided that the velocity differentiation needs the extra velocity weight?**

Dissipation of $\mathrm{P}u$

Observation (Grad, Kawashima, Liu-Yu, Guo, D.-Strain '10):

- **Find dissipation from the dynamics of the local Maxwellian?**

$$
\begin{aligned}
&\partial_t a_\pm + \nabla_x \cdot b + \nabla_x \cdot \langle \xi \mathbf{M}^{1/2}, \{\mathbf{I}_\pm - \mathbf{P}_\pm\} u \rangle = 0, \\
&\partial_t [b_i + \langle \xi_i \mathbf{M}^{1/2}, \{\mathbf{I}_\pm - \mathbf{P}_\pm\} u \rangle] + \partial_i (a_\pm + 2c) \mp E_i \\
&\qquad + \nabla_x \cdot \langle \xi \xi_i \mathbf{M}^{1/2}, \{\mathbf{I}_\pm - \mathbf{P}_\pm\} u \rangle = 0, \\
&\partial_t \left[c + \frac{1}{6} \langle (|\xi|^2 - 3) \mathbf{M}^{1/2}, \{\mathbf{I}_\pm - \mathbf{P}_\pm\} u \rangle \right] + \frac{1}{3} \nabla_x \cdot b \\
&\qquad + \frac{1}{6} \nabla_x \cdot \langle (|\xi|^2 - 3) \xi \mathbf{M}^{1/2}, \{\mathbf{I}_\pm - \mathbf{P}_\pm\} u \rangle = 0,
\end{aligned}
$$

These equations are NOT closed!

- To close the system of the local Maxwellian, we also need to study the high-order moment equations:

$$\begin{aligned}
&\partial_t[\Theta_{ii}(\{\mathbf{I}_\pm - \mathbf{P}_\pm\}u) + 2c] + 2\partial_i b_i = \Theta_{ii}(l_\pm),\\
&\partial_t\Theta_{ij}(\{\mathbf{I}_\pm - \mathbf{P}_\pm\}u) + \partial_j b_i + \partial_i b_j + \nabla_x \cdot \langle \xi \mathbf{M}^{1/2}, \{\mathbf{I}_\pm - \mathbf{P}_\pm\}u\rangle\\
&\qquad\qquad = \Theta_{ij}(l_\pm), \quad i \neq j,\\
&\partial_t\Lambda_i(\{\mathbf{I}_\pm - \mathbf{P}_\pm\}u) + \partial_i c = \Lambda_i(l_\pm).
\end{aligned}$$

Here, the high-order moment functions are defined by

$$\Theta_{ij}(u_\pm) = \langle(\xi_i\xi_j - 1)\mathbf{M}^{1/2}, u_\pm\rangle, \Lambda_i(u_\pm) = \frac{1}{10}\langle(|\xi|^2 - 5)\xi_i \mathbf{M}^{1/2}, u_\pm\rangle,$$

and $\mathbf{L}_\pm$ are defined in terms of $\{\mathbf{I} - \mathbf{P}\}u$.

- **For $\gamma \geq 0$, there is a time-frequency functional $\mathcal{E}(t,k)$ such that**

$$\mathcal{E}(t,k) \sim \|\hat{u}\|_{L^2_\xi}^2 + |\hat{E}|^2 + |\hat{B}|^2,$$

and

$$\partial_t \mathcal{E}(t,k) + \frac{\lambda |k|^2}{(1+|k|^2)^2} \mathcal{E}(t,k) \leq 0, \quad \forall\, t \geq 0, \ k \in \mathbb{R}^3.$$

Remark:

- **The inequality seems terrible to prove decay rates because the "dissipative term" goes to zero as $|k| \to \infty$.**
- **It is an essential "regularity-loss" feature for the VMB system, not a deficiency of our approach; see D. (**Eigenvalue analysis of damped Euler-Maxwell**, '11), Hosono-Kawashima (M3AS '06), Ueda-D.-Kawashima ('11):**

$$\lambda(ik) \sim -\frac{1}{|k|^2} \pm i|k| \quad (|k| \to \infty).$$

Case when $\gamma < 0$: D. (arXiv '12)

The situation becomes more subtle for $\gamma < 0$; see Strain ('11) and D.-Yang-Zhao ('11)

- **Let $w = w(\xi) = \langle \xi \rangle^{\frac{\gamma+2}{2}}$ for Landau.**
- **Derive**

$$\partial_t M_\ell(t,k) + \kappa D_\ell(t,k) \leq 0,$$

with

$$\begin{aligned}
M_\ell(t,k) &= \|\hat{u}\|_{L^2}^2 + |[\hat{E},\hat{B}]|^2 + \kappa_0 \Re \mathcal{E}^{\rm int}(t,k) \\
&\quad + \kappa_2 \left| w^\ell \{\mathbf{I}-\mathbf{P}\}\hat{u} \right|_{L^2} \chi_{|k|\leq 1} + \frac{\kappa_1}{1+|k|^2} \left| w^\ell \hat{u} \right|_{L^2}^2 \chi_{|k|\geq 1}, \\
D_\ell(t,k) &= |\{\mathbf{I}-\mathbf{P}\}\hat{u}|_{\mathbf{D}}^2 + \frac{1}{1+|k|^2} \left| w^\ell \{\mathbf{I}-\mathbf{P}\}\hat{u} \right|_{\mathbf{D}}^2 \\
&\quad + \frac{|k|^2}{1+|k|^2} (|\widehat{a_+ + a_-}|^2 + |\hat{b}|^2 + |c|^2) + |\widehat{a_+ - a_-}|^2 \\
&\quad + \frac{1}{1+|k|^2} |\hat{E}|^2 + \frac{|k|^2}{(1+|k|^2)^2} |\hat{B}|^2, \quad \rho(k) = \frac{|k|^2}{(1+|k|^2)^2}.
\end{aligned}$$

- **Make the time weighted estimate**

$$M_\ell(t,k) \lesssim [1 + \epsilon \rho(k) t]^{-J} M_{\ell+J+p-1}(0,k).$$

Nonlinear perturbation theory for general collisional potentials

- **Mathematically, when there is an external force, it is highly nontrivial to generalize existing results to the case of non hard-sphere model, which is also of physical importance!**
- **A progress was made by Guo (JAMS, '11):**
  - **the Landau collision with the Coulomb potential** $(\gamma = -3)$
  - **the potential force** $E = -\nabla\phi$ **with vanishing** $B = 0$**:**

$$(\xi \cdot \nabla_x u + \nabla_x \phi \cdot \xi u) e^{\phi} = \xi \cdot \nabla_x (e^{\phi} u).$$

  - $\Omega = \mathbb{T}^3$
- **It is difficult to deal with the non-potential force !!!**

- **A completely different approach was developed by D.-Yang-Zhao (arXiv '11)**
  - **A new dissipative mechanism due to the introduction of the time-velocity dependent weight**

$$\exp\{\lambda\langle\xi\rangle^q/(1+t)^\theta\}$$

$$\Rightarrow \partial_t e^{\frac{\lambda\langle\xi\rangle^q}{(1+t)^\theta}} = -\lambda\theta\frac{\langle\xi\rangle^q}{(1+t)^{1+\theta}}e^{\frac{\lambda\langle\xi\rangle^q}{(1+t)^\theta}}.$$

  - **The approach that we developed can apply to**

    • Landau or Boltzmann
    • $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$
    • For the Boltzmann with most of values of γ: angular cutoff or non-cutoff
    • Maxwell system (non-potential force) can be included!

Main results:

Global classical solutions near a global Maxwellian uniquely exist and time asymptotically tend to the Maxwellian with some rates for the cases of

- D.-Yang-Zhao ('11): Vlasov-Poisson-Boltzmann, angular cutoff with $-2 \leq \gamma \leq 1$
- D.-Liu ('11): Vlasov-Poisson-Boltzmann, angular non cutoff with $-3 < \gamma < -2$ and $1/2 \leq s < 1$
- D. (arXiv '12): Vlasov-Maxwell-Landau, soft potentials $-3 \leq \gamma < -2$ including the Coulomb $\gamma = -3$

Idea in the proofs:

- **Find an energy functional $\mathcal{E}(t)$ and its time-weighted norm $X(t)$ such that**

$$X(t) \lesssim Y_0 + [X(t)]^2.$$

- **To control the term $E \cdot \nabla_\xi u$ and $E \cdot \xi u$, the time-decay of E is needed. Thus, Y_0 generally includes L^1-norm of initial data. Note that Y_0 needs to be small enough to ensure the global-in-time bound by the continuity argument.**
- **To balance an estimate on both $E \cdot \nabla_\xi u$ and $\xi \cdot \nabla_x u$, γ can NOT be too small in the cutoff case. However, in the non cutoff case, since**

$$\int u\mathbf{L}u\, d\xi \lesssim -\int \langle\xi\rangle^{\gamma+2s}|\{\mathbf{I}-\mathbf{P}\}u|^2\, d\xi - \{\cdots\},$$

we may require that $\gamma + 2s$ need not be too small.

Idea in the proofs (cont.):

- **To deal with the degeneration of $\nu(\xi)$ for soft potentials, choose $\mathcal{E}(t)$ in the way that**
  - **higher the differentiation order is, the order of velocity weights is lower, for instance, consider ($\partial_\alpha^\beta = \partial_x^\beta \partial_\xi^\alpha$, $|\alpha| + |\beta| = N$)**

$$\iint \partial_\alpha^\beta Q(u,u) \cdot w_{\alpha,\beta,\ell}^2(t,\xi) \partial_\alpha^\beta u \, dx d\xi.$$

- **To deal with the degeneration of the Maxwell equations, choose $X(t)$ in the way that**
  - **higher the order of $\mathcal{E}_N(t)$ is, the rate of its time weights is lower;**
  - **the highest-order energy norm $\mathcal{E}_N(t)$ may increase in time! For instance, consider**

$$\iint \partial_\alpha^\beta [(B \times \xi) \cdot \nabla_\xi u] \cdot w_{\alpha,\beta,\ell}^2(t,\xi) \partial_\alpha^\beta u \, dx d\xi.$$

Idea in the proofs (cont.):

- **A trouble occurs to the estimate on**

$$\iint \partial_\beta^\alpha (E\cdot\xi \mathbf{M}^{1/2})\cdot w^2_{\alpha,\beta,\ell}(t,\xi)\partial_\beta^\alpha u\,dxd\xi \quad (|\alpha|+|\beta|=N)$$

No dissipation for $\int|\nabla_x^N E|^2$!

Use an idea from Hosono-Kawashima (M3AS '06) to deduce

$$\frac{d}{dt}[(1+t)^{-\epsilon_0}\mathcal{E}_{N_1}(t)]+\kappa(1+t)^{-\epsilon_0}\mathcal{D}_{N_1}(t) + \frac{\epsilon_0}{(1+t)^{1+\epsilon_0}}\mathcal{E}_{N_1}(t)\lesssim (1+t)^{-\epsilon_0}\times\{h.o.t.\}.$$

from the basic energy inequality without any weight

$$\frac{d}{dt}\mathcal{E}_{N_1}(t)+\kappa\mathcal{D}_{N_1}(t)\lesssim h.o.t..$$

Problems for the future:

- **Bounded domain**
- **Eigenvalue analysis and the spectrum**

Thank you !