The Boltzmann Equation with Forcing: Global Existence, Uniform Stability and Optimal Decay Rates

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1. Introduction

1.1 Consider the Boltzmann equation:

$$\partial_t f + \xi \cdot \nabla_x f + \nabla_x \phi(x) \cdot \nabla_\xi f = Q(f,f), \qquad (BE)$$

where

- $t \geq 0$ **(time),** $x \in \mathbb{R}^n$ **(position),** $\xi \in \mathbb{R}^n$ **(velocity),** $n \geq 3$**;**
- $f = f(t,x,\xi) \geq 0$ **(number density), unknown;**
- $\phi(x)$ **(potential of stationary force), bounded & given;**
- Q **is a bilinear collision operator (hard sphere model):**

$$Q(f,g) = \frac{1}{2}\int_{\mathbb{R}^n \times S^{n-1}} (f' g'_* + f'_* g' - f g_* - f_* g)|(\xi - \xi_*) \cdot \omega| d\omega d\xi_*,$$

$$f = f(t,x,\xi), \quad f' = f(t,x,\xi'), \quad f_* = f(t,x,\xi_*), \quad f'_* = f(t,x,\xi'_*),$$

$$\textit{likewise for } g,$$

$$\xi' = \xi - [(\xi - \xi_*) \cdot \omega]\omega, \quad \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \omega]\omega, \quad \omega \in S^{n-1}.$$

1.2 (BE) has a stationary solution:

$$e^{\phi(x)}\mathbf{M},$$

where

$$\mathbf{M} = \frac{1}{(2\pi)^{n/2}} \exp\left(-|\xi|^2/2\right),$$

on the basis of the observations:

- Conservation of energy

$$|\xi'|^2 + |\xi'_*|^2 = |\xi|^2 + |\xi_*|^2$$

$\Longrightarrow$

$$Q(e^{\phi(x)}\mathbf{M}, e^{\phi(x)}\mathbf{M}) = e^{2\phi(x)}Q(\mathbf{M},\mathbf{M}) = 0,$$

- $$\{\partial_t + \xi \cdot \nabla_x + \nabla_x\phi(x) \cdot \nabla_\xi\} \exp\left(\phi(x) - |\xi|^2/2\right) \equiv 0.$$

1.3 Aim (D., PhD thesis, '08):

- **Stability of stationary solution $e^{\phi}\mathbf{M}$:**

$$\left.\begin{array}{c}\|f(0)-e^{\phi}\mathbf{M}\|_X \ll 1\\ |\phi|\ll 1\end{array}\right\} \Rightarrow \sup_t \|f(t)-e^{\phi}\mathbf{M}\|_X \le C\|f(0)-e^{\phi}\mathbf{M}\|_X,$$

$$X = H^N(\mathbb{R}^n_x \times \mathbb{R}^n_\xi; \mathbf{M}^{-1/2}dxd\xi): \text{ no time derivatives.}$$

- **Uniform-in-time stability for two solutions:**

$$\left.\begin{array}{c}\|f(0)-e^{\phi}\mathbf{M}\|_X \ll 1\\ \|g(0)-e^{\phi}\mathbf{M}\|_X \ll 1\\ |\phi|\ll 1\end{array}\right\} \Rightarrow \sup_t \|f(t)-g(t)\|_X \le C\|f(0)-g(0)\|_X.$$

- **Optimal convergence rate in X:**

$$\left.\begin{array}{c}\text{Additional conditions on}\\ f(0)-e^{\phi}\mathbf{M}\ \&\ \phi\end{array}\right\} \Rightarrow \|f(t)-e^{\phi}\mathbf{M}\|_X \le C_{f_0,\phi}(1+t)^{-\frac{n}{4}}.$$

1.4 Energy method

- Liu-Yu (CMP,'02), Liu-Yang-Yu (PD,'02),...
- Guo (IUMJ,'02),...

- **Stability:**

Macro-micro decomposition (micro eqn + macro eqn)

+

macroscopic conservation laws

$\Downarrow$

Energy inequality: $\frac{d}{dt}\mathcal{E}(u(t)) + \mathcal{D}(u(t)) \leq 0$

- **Optimal convergence rate:**

High-order energy estimates

+

Spectral analysis

$\Downarrow$

Optimal decay rate

1.5 Applications:

- **General force (non-potential) [DUYZ, CMP, '08]:**

$$E(t,x), \quad E(t,x) + \xi \times B(t,x).$$

Assumptions: E, B **are small and decay in time (or** $n \geq 5$**).**

- **General intermolecular interaction law:**

$$|\xi - \xi_*|^\gamma b(\theta), \quad 0 \leq \gamma \leq 1.$$

- **Time-periodic solution [DUYZ, CMP, '08]:**

time-periodic force: $E(t+T,x) = E(t,x).$

Assumptions: E **is small and** $n \geq 5$**.**

- **Physical models such as:**

one-species Vlasov-Poisson-Boltzmann system [DY, '08];
one-species Vlasov-Maxwell-Boltzmann system [in progress].

1.6 Back to (BE):

$$\partial_t f + \xi \cdot \nabla_x f + \nabla_x \phi(x) \cdot \nabla_\xi f = Q(f,f).$$

Problems on the stability of $e^{\phi}\mathbf{M}$ in other cases:

(i) $\phi(x)$ can be large:

This situation was studied in the case of the compressible Navier-Stokes equations.

- Matsumura-Padula ('92):
  interior domain, smooth solutions,

$$\phi \in H^4.$$

- Matsumura-Yamagata ('01):
  the whole space $\mathbb{R}^3$, weak solutions,

$$|\phi(x)| \le \frac{C}{1+|x|}, |\nabla_x \phi(x)| \le \frac{C}{(1+|x|)^2}, \cdots$$

C need not be small.

(ii) $e^{\phi}\mathbf{M}$ **is connected to vacuum at infinity:**

$$e^{\phi(x)}\mathbf{M} \to 0 \text{ (or } \phi(x) \to -\infty) \text{ as } |x| \to \infty.$$

Here, ϕ is a confining potential.

Remark: Related results in this situation:

- **Kinetic Fokker-Planck equation:**

$$\partial_t f + \xi \cdot \nabla_x f - \nabla_x \phi(x) \cdot \nabla_\xi f = \Delta_\xi f + \nabla_\xi \cdot (\xi f),$$

Helffer-Nier, Hérau, Hérau-Nier, Desvillettes, Villani,...:

$$\phi \in C^1(\mathbb{R}^3), \quad \inf \phi > -\infty.$$

- **Linearized Boltzmann equation:**

$$\partial_t u + \xi \cdot \nabla_\xi u - \nabla_x \phi(x) \cdot \nabla_\xi u = e^{-\phi}\mathbf{L}u,$$

Tabata (TTSP,'94):

$$\phi = \phi(|x|), \quad \phi''(r) \geq C_1 > 0, \quad \phi'(r) \leq C_2 r + C_3, \ldots$$

2. Well-posedness of the Cauchy problem

2.1 To expose the main idea, suppose

$$n = 3.$$

Set the perturbation u by

$$f = e^{\phi(x)}\mathbf{M} + \sqrt{\mathbf{M}}u.$$

The Cauchy problem for (BE) is reformulated as

$$\begin{cases} \partial_t u + \xi \cdot \nabla_x u + \nabla_x \phi(x) \cdot \nabla_\xi u - \frac{1}{2}\xi \cdot \nabla_x \phi(x) u = e^{\phi(x)}\mathbf{L}u + \Gamma(u,u), \\ u(0,x,\xi) = u_0(x,\xi). \end{cases} \qquad (CP)_1$$

Here

$$\mathbf{L}u = \mathbf{M}^{-1/2}\left[Q(\mathbf{M}, \mathbf{M}^{1/2}u) + Q(\mathbf{M}^{1/2}u, \mathbf{M})\right],$$

$$\Gamma(u,u) = \mathbf{M}^{-1/2}Q(\mathbf{M}^{1/2}u, \mathbf{M}^{1/2}u).$$

2.2 Recall some standard facts on $\mathbf{L}$:

(a) $(\mathbf{L}u)(\xi) = -\nu(\xi)u(\xi) + (Ku)(\xi)$, **where**

- $\nu_0(1+|\xi|) \leq \nu(\xi) \leq \nu_0^{-1}(1+|\xi|), \quad \nu_0 > 0$;
- K **is a self-adjoint compact operator on** $L^2(\mathbb{R}^3_\xi)$;
- $Ker\mathbf{L} = \operatorname{span}\left\{\mathbf{M}^{1/2};\ \xi_i\mathbf{M}^{1/2}, i=1,2,3;\ |\xi|^2\mathbf{M}^{1/2}\right\} := \mathcal{N}$;

(b) $\mathbf{L}$ **is self-adjoint on** $L^2(\mathbb{R}^3_\xi)$ **with the domain**

$$D(\mathbf{L}) = \{u \in L^2(\mathbb{R}^3_\xi) | \nu(\xi)u \in L^2(\mathbb{R}^3_\xi)\},$$

and $-\mathbf{L}$ **is locally coercive:** $\exists\ \lambda > 0$ **s.t.**

$$\begin{aligned} -\int_{\mathbb{R}^3} u\mathbf{L}u\,d\xi &\geq \lambda \int_{\mathbb{R}^3} \nu(\xi)\left(\{\mathbf{I}-\mathbf{P}\}u\right)^2 d\xi,\ \forall\, u \in D(\mathbf{L}) \\ &= \lambda\|\{\mathbf{I}-\mathbf{P}\}u\|_\nu^2, \end{aligned}$$

where $\mathbf{P}$ **is the projector from** $L^2(\mathbb{R}^3_\xi)$ **to** $\mathcal{N}$.

2.3 Define the energy functional

$$[[u(t)]]^2 \equiv \sum_{|\alpha|+|\beta|\le N} \|\partial_x^\alpha \partial_\xi^\beta u(t)\|^2 = \|u(t)\|^2_{H^N_{x,\xi}},$$

and the dissipation rate

$$[[u(t)]]_\nu^2 \equiv \|\{\mathbf{I}-\mathbf{P}\}u(t)\|_\nu^2 + \sum_{0<|\alpha|\le N} \|\partial_x^\alpha u(t)\|_\nu^2$$
$$+ \sum_{|\alpha|+|\beta|\le N,\ |\beta|>0} \|\partial_x^\alpha \partial_\xi^\beta \{\mathbf{I}-\mathbf{P}\}u(t)\|_\nu^2.$$

2.4 Assumptions on the potential $\phi(x)$:

(AP):

$$\phi \in L_x^\infty,$$
$$\delta_\phi := \left\|(1+|x|)^2 \nabla_x \phi\right\|_{L_x^\infty} + \sum_{2\le|\alpha|\le N} \|(1+|x|)\partial_x^\alpha \phi\|_{L_x^\infty} \ll 1.$$

Theorem I (Well-posedness) *Let* $f_0(x,\xi) = \mathbf{M} + \sqrt{\mathbf{M}}u_0(x,\xi) \geq 0$. $\exists\ \delta_0 > 0,\ \lambda_0 > 0$ *and* $C_0 > 0$ *s.t. if*

$$[[u(0)]] + \delta_\phi \leq \delta_0,$$

then $\exists !\ u(t,x,\xi)$ *to* $(CP)_0$ *s.t.* $f(t,x,\xi) = \mathbf{M} + \sqrt{\mathbf{M}}u(t,x,\xi) \geq 0$, *and*

$$[[u(t)]]^2 + \lambda_0 \int_0^t [[u(s)]]_\nu^2 ds \leq C_0 [[u(0)]]^2,\ \forall\ t \geq 0.$$

Theorem II (Uniform stability) *Let*

$$f_0(x,\xi) = \mathbf{M} + \sqrt{\mathbf{M}}u_0(x,\xi) \geq 0, g_0(x,\xi) = \mathbf{M} + \sqrt{\mathbf{M}}v_0(x,\xi) \geq 0.$$

$\exists\ \delta_1 \in (0,\delta_0),\ \lambda_1 > 0$ *and* $C_1 > 0$ *s.t. if*

$$\max\{[[u(0)]], [[v(0)]]\} + \delta_\phi \leq \delta_1,$$

then the solutions $u(t,x,\xi)$, $v(t,x,\xi)$ *obtained in Theorem I satisfy*

$$[[u(t)-v(t)]]^2 + \lambda_1 \int_0^t [[u(s)-v(s)]]_\nu^2 ds \leq C_1 [[u(0)-v(0)]]^2,\ \forall\ t \geq 0.$$

2.5 Related results:

- **Spaces based on the spectral analysis ($\phi \equiv 0$):**
  - **Ukai, Nishida-Imai:** $L^\infty_{\beta_1}(\mathbb{R}^3_\xi; H^k(\mathbb{R}^3_x))$, $\beta_1 > 5/2$, $k \geq 2$, **where** $L^\infty_{\beta_1}(\mathbb{R}^3_\xi) \equiv \{u | (1+|\xi|)^{\beta_1} u \in L^\infty(\mathbb{R}^3_\xi)\}$.
  - **Shizuta (Torus case):** $L^\infty_{\beta_1}(\mathbb{R}^3_\xi; C^k(\mathbb{T}^3_x))$, $\beta_1 > 5/2$, $k = 0, 1, \cdots$.
  - **Ukai-Yang:** $L^2(\mathbb{R}^3_x \times \mathbb{R}^3_\xi) \cap L^\infty_{\beta_2}(\mathbb{R}^3_\xi; L^\infty(\mathbb{R}^3_x))$, $\beta_2 > 3/2$

Remark *Notice that*

$$L^\infty_{\beta_1}(\mathbb{R}^3_\xi) \subsetneq L^\infty_{\beta_2}(\mathbb{R}^3_\xi) \subsetneq L^2(\mathbb{R}^3_\xi),$$

where β_1 and β_2 are sufficiently close to $5/2$ and $3/2$, respectively.

- **Spaces based on the energy method:**
  - **Liu-Yu, Liu-Yang-Yu, Yang-Zhao, D. (JDE, '08) (**Refined energy method, no time derivative)**,...**
  - **Guo, Strain,...**

$$H^{N(n_1,n_2,n_3)}_{t,x,\xi}(\mathbb{R}^3_x \times \mathbb{R}^3_\xi),$$
$$N = N(n_1,n_2,n_3) \equiv n_1 + n_2 + n_3 \geq 4.$$

- **Spaces based on the method of Green's function ($\phi \equiv 0$), Liu-Yu:**

$$L^\infty(\mathbb{R}^3_x, w(t,x)dx; L^\infty_{\beta_3}(\mathbb{R}^3_\xi)), \quad \beta_3 \geq 3,$$

where the pointwise weight function,

$$w(t,x) = e^{-C(|x|+\beta_3 t)} + \frac{e^{-\frac{(|x|-\sqrt{5/3}t)^2}{Ct}}}{(1+t)^2} + \{acoustic\ cone\},$$

exposes the wave structure of convergence. Notice that the initial perturbation u_0 decays exponentially in x.

Remark *The energy method is an effective one in the presence of the external force.*

2.6 Key points of the proof.

(a) Macro-micro decomposition: For fixed (t,x),

$$\begin{cases} u(t,x,\xi) = u_1 + u_2, \\ u_1 \equiv \mathbf{P}u \in \mathcal{N}, \\ u_2 \equiv \{\mathbf{I}-\mathbf{P}\}u \in \mathcal{N}^{\perp}. \end{cases}$$

(b) Our goal is to obtain the dissipation rate $[[u(t)]]_{\nu}^2$, which is equivalent with

$$\underbrace{\sum_{|\alpha|+|\beta|\leq N} \|\partial_x^\alpha \partial_\xi^\beta \{\mathbf{I}-\mathbf{P}\}u(t)\|_\nu^2}_{\text{(I): micro dissipation rate}} + \underbrace{\sum_{0<|\alpha|\leq N} \|\partial_x^\alpha \mathbf{P}u(t)\|^2}_{\text{(II): macro dissipation rate}} .$$

Remark

- *(I)$\Leftarrow$the local coercivity of $-\mathbf{L}$;*
- *(II)$\Leftarrow$macro equations + local macro balance laws.*

(c) Expand $u_1 = \mathbf{P}u$ as

$$u_1 = \left\{ a(t,x) + \sum_{i=1}^{3} b_i(t,x)\xi_i + c(t,x)|\xi|^2 \right\} \mathbf{M}^{1/2}.$$

One can determine the evolution of u_1 and (a,b,c) in terms of u_2:

- **Macroscopic equation on u_1:**

$$\partial_t u_1 + \xi \cdot \nabla_x u_1 + \nabla_x \phi \cdot \nabla_\xi u_1 - \frac{1}{2}\xi \cdot \nabla_x \phi u_1 = r + \ell + n$$

with

$$\begin{aligned} r &= -\partial_t u_2, \\ \ell &= -\xi \cdot \nabla_x u_2 - \nabla_x \phi \cdot \nabla_\xi u_2 + \frac{1}{2}\xi \cdot \nabla_x \phi u_2 + e^{\phi}\mathbf{L}u_2, \\ n &= \Gamma(u,u). \end{aligned}$$

- **Macroscopic equations on coefficients (a, b, c) of u_1:**

$$\partial_t a + b\cdot\nabla_x\phi = -\partial_t\tilde{r}^{(0)} + \ell^{(0)} + n^{(0)} \equiv \gamma^{(0)},$$
$$\partial_t b_i + \partial_i a - (a\partial_i\phi - 2c\partial_i\phi) = -\partial_t\tilde{r}_i^{(1)} + \ell_i^{(1)} + n_i^{(1)} \equiv \gamma_i^{(1)},$$
$$\partial_t c + \partial_i b_i - b_i\partial_i\phi = -\partial_t\tilde{r}_i^{(2)} + \ell_i^{(2)} + n_i^{(2)} \equiv \gamma_i^{(2)},$$
$$\partial_i b_j + \partial_j b_i - (b_j\partial_i\phi + b_i\partial_j\phi) = -\partial_t\tilde{r}_{ij}^{(2)} + \ell_{ij}^{(2)} + n_{ij}^{(2)} \equiv \gamma_{ij}^{(2)}, \quad i\neq j,$$
$$\partial_i c - c\partial_i\phi = -\partial_t\tilde{r}_i^{(3)} + \ell_i^{(3)} + n_i^{(3)} \equiv \gamma_i^{(3)}.$$

Remark *An important observation from Guo is that $b = (b_1, b_2, b_3)$ satisfies an elliptic-type equation:*

$$\begin{aligned}-\Delta_x b_j - \partial_j\partial_j b_j = &\sum_{i\neq j}\partial_j(b_i\partial_i\phi) + \sum_{i\neq j}\partial_j\gamma_i^{(2)} \\ &-\sum_{i\neq j}\partial_i(b_j\partial_i\phi + b_i\partial_j\phi) - \sum_{i\neq j}\partial_i\gamma_{ij}^{(2)} \\ &-2\partial_j(b_j\partial_j\phi) - 2\partial_j\gamma_j^{(2)}.\end{aligned}$$

- **Macroscopic balance laws on coefficients (a,b,c) of u_1:**

$$\begin{aligned}\partial_t a-\frac{1}{2}\nabla_x\cdot\langle|\xi|^2\xi\sqrt{\mathbf{M}},u_2\rangle&=-\frac{1}{2}b\cdot\nabla_x\phi,\\ \partial_t b_i+\partial_i(a+5c)+\nabla_x\cdot\langle\xi\xi_i\sqrt{\mathbf{M}},u_2\rangle&=(a+3c)\partial_i\phi,\\ \partial_t c+\frac{1}{3}\nabla_x\cdot b+\frac{1}{6}\nabla_x\cdot\langle|\xi|^2\xi\sqrt{\mathbf{M}},u_2\rangle&=\frac{1}{6}b\cdot\nabla_x\phi,\end{aligned}$$

where it is noticed that one also has the conservation of mass

$$\partial_t(a+3c)+\nabla_x\cdot b=0.$$

Remark *The time derivatives $\partial_t(a,b,c)$ can be replaced by the spatial derivatives and the nonlinear product terms. For the product terms, the Hardy inequality*

$$\int_{\mathbb{R}^3}\frac{|g(x)|^2}{|x|^2}dx\leq 2\int_{\mathbb{R}^3}|\nabla_x g(x)|^2dx,\ \ \forall\, g\in H^1(\mathbb{R}^3),$$

is used to gain the spatial derivatives.

• **(BE) can be exactly written as the linearized viscous compressible Navier-Stokes equations with remaining terms only related to 13 moments of the micro part u_2 and product terms between (a, b, c) and $\nabla_x \phi$:**

$$
\begin{aligned}
&\partial_t (a + 3c) + \nabla_x \cdot b = 0, \\
&\partial_t b + \nabla_x (a + 3c) + 2\nabla_x c - \Delta_x b - \frac{1}{3}\nabla_x \nabla_x \cdot b = R^b, \\
&\partial_t c + \frac{1}{3}\nabla_x \cdot b - \Delta_x c = R^c,
\end{aligned}
$$

where $R^b = (R^b_1, R^b_2, R^b_3)$ and R^c are defined by

$$
\begin{aligned}
R^b_j = &-\nabla_x \cdot \langle \xi \xi_j \sqrt{\mathbf{M}}, u_2 \rangle - \frac{1}{3}\partial_j \nabla_x \cdot \langle |\xi|^2 \xi \sqrt{\mathbf{M}}, u_2 \rangle \\
&- \sum_{i \neq j} \partial_i \gamma^{(2)}_{ij} - 2\partial_j \gamma^{(2)}_j + \{\textbf{product terms}\}, \\
R^c = &-\frac{1}{6}\nabla_x \cdot \langle |\xi|^2 \xi \sqrt{\mathbf{M}}, u_2 \rangle - \sum_i \partial_i \gamma^{(3)}_i + \{\textbf{product terms}\}.
\end{aligned}
$$

2.7 Proof of Theorem I:

- **Local existence: ...**
- *A prior* **estimates:**

Part I: To obtain the microscopic dissipation

(i) Estimates on zero-order:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \lambda\|u_2\|_\nu^2 \leq C[[u(t)]][[u(t)]]_\nu^2 + C\delta_\phi\|\nabla_x u_1\|^2.$$

(ii) Estimates on pure spatial derivatives:

$$\frac{1}{2}\frac{d}{dt}\sum_{1\leq|\alpha|\leq N}\|\partial_x^\alpha u\|^2 + \lambda\sum_{1\leq|\alpha|\leq N}\|\partial_x^\alpha u_2\|_\nu^2 \leq C[[u(t)]][[u(t)]]_\nu^2$$
$$+C\delta_\phi\sum_{1\leq|\alpha|\leq N}\|\partial_x^\alpha u_1\|^2 + C\delta_\phi\sum_{1\leq|\alpha|\leq N-1}\|\partial_x^\alpha\nabla_\xi u_2\|^2.$$

(iii) Estimates on mixed derivatives:

$$\frac{1}{2}\frac{d}{dt}\sum_{\substack{|\beta|=k\\|\alpha|+|\beta|\le N}}\|\partial_x^\alpha\partial_\xi^\beta u_2\|^2+\lambda\sum_{\substack{|\beta|=k\\|\alpha|+|\beta|\le N}}\|\partial_x^\alpha\partial_\xi^\beta u_2\|_\nu^2$$

$$\le C[[u(t)]][[u(t)]]_\nu^2+C\sum_{|\alpha|\le N-k+1}\|\partial_x^\alpha u_2\|_\nu^2$$

$$+C\chi_{\{k\ge 2\}}\sum_{\substack{1\le|\beta|\le k-1\\|\alpha|+|\beta|\le N}}\|\partial_x^\alpha\partial_\xi^\beta u_2\|_\nu^2+C\sum_{|\alpha|\le N-k}\|\partial_x^\alpha\nabla_x(a,b,c)\|^2,$$

where the integer $1\le k\le N$ and $\chi_{\{k\ge 2\}}$ denotes the characteristic function of the set $\{k\ge 2\}$.

Remark *The above estimate is based on the equation:*

$$\partial_t u_2+\xi\cdot\nabla_x u_2+\nabla_x\phi\cdot\nabla_\xi u_2-1/2\,\xi\cdot\nabla_x\phi u_2+e^\phi\nu(\xi)u_2$$
$$=e^\phi K u_2+\Gamma(u,u)-\partial_t u_1-\xi\cdot\nabla_x u_1-\nabla_x\phi\cdot\nabla_\xi u_1+1/2\,\xi\cdot\nabla_x\phi u_1.$$

Proper linear combinations of (i),(ii),(iii)$_k\Rightarrow$

$$\frac{1}{2}\frac{d}{dt}\left(\sum_{|\alpha|\le N}\|\partial_x^\alpha u\|^2+\sum_{|\alpha|+|\beta|\le N,\ |\beta|\ge 1}\|\partial_x^\alpha\partial_\xi^\beta u_2\|^2\right)$$

$$+\lambda\sum_{|\alpha|+|\beta|\le N}\|\partial_x^\alpha\partial_\xi^\beta u_2\|_\nu^2$$

$$\le C[[u(t)]][[u(t)]]_\nu^2+C\sum_{|\alpha|\le N-1}\|\partial_x^\alpha\nabla_x(a,b,c)\|^2.$$

Part II: To obtain the macroscopic dissipation

$$2\frac{d}{dt}\mathcal{I}(u(t)) + \sum_{|\alpha|\leq N-1} \|\nabla_x \partial_x^\alpha (a,b,c)\|^2 \leq C\left\{\sum_{|\alpha|\leq N} \|\partial_x^\alpha u_2\|^2 + [[u(t)]]^2[[u(t)]]_\nu^2\right\},$$

where $\mathcal{I}(u(t))$ is called the interactive energy functional:

$$\mathcal{I}(u(t)) = \sum_{|\alpha|\leq N-1}\sum_{i=1}^{3}\left[\mathcal{I}^a_{\alpha,i}(u(t)) + \mathcal{I}^b_{\alpha,i}(u(t)) + \mathcal{I}^c_{\alpha,i}(u(t)) + \mathcal{I}^{ab}_{\alpha,i}(u(t))\right],$$

$$\mathcal{I}^a_{\alpha,i}(u(t)) = \left\langle \partial_x^\alpha \tilde{r}_i^{(1)}, \partial_i\partial_x^\alpha a\right\rangle,$$

$$\mathcal{I}^b_{\alpha,i}(u(t)) = -\sum_{j\neq i}\left\langle \partial_x^\alpha \tilde{r}_j^{(2)}, \partial_i\partial_x^\alpha b_i\right\rangle + \sum_{j\neq i}\left\langle \partial_x^\alpha \tilde{r}_{ji}^{(2)}, \partial_j\partial_x^\alpha b_i\right\rangle + 2\left\langle \partial_x^\alpha \tilde{r}_i^{(2)}, \partial_i\partial_x^\alpha b_i\right\rangle,$$

$$\mathcal{I}^c_{\alpha,i}(u(t)) = \left\langle \partial_x^\alpha \tilde{r}_i^{(3)}, \partial_i\partial_x^\alpha c\right\rangle,$$

$$\mathcal{I}^{ab}_{\alpha,i}(u(t)) = \langle \partial_i\partial_x^\alpha a, \partial_x^\alpha b_i\rangle.$$

Idea for Part II:

$$-\Delta_x b_j = -\partial_j \partial_t \tilde{r}^{(2)} + \cdots$$

$\Longrightarrow$

$$\|\nabla_x \partial_x^\alpha b_j\|^2 = -\frac{d}{dt}\langle \partial_j \partial_x^\alpha \tilde{r}^{(2)}, \partial_x^\alpha b_j\rangle + \langle \partial_j \partial_x^\alpha \tilde{r}^{(2)}, \partial_x^\alpha \partial_t b_j\rangle + \cdots$$
$$= -\frac{d}{dt}\underbrace{\langle \partial_x^\alpha \tilde{r}^{(2)}, -\partial_j \partial_x^\alpha b_j\rangle}_{\textbf{(I)}} + \underbrace{\langle \partial_j \partial_x^\alpha \tilde{r}^{(2)}, \partial_x^\alpha \partial_t b_j\rangle}_{\textbf{(II)}} + \cdots .$$

- **(I) is bounded by the temporal energy;**
- **(II) is estimated by the Cauchy-Schwarz inequality and the balance law for** b_j**:**

$$\partial_t b_j = -\partial_j(a+5c) - \nabla_x \cdot \langle \xi \xi_j \sqrt{\mathbf{M}}, u_2\rangle + (a+3c)\partial_j \phi.$$

Further linear combination of Part I and Part II $\Rightarrow$

$$\frac{d}{dt}\mathcal{E}_M(u(t)) + \lambda\mathcal{D}(u(t)) \leq C\sqrt{\mathcal{E}_M(u(t))}\mathcal{D}(u(t)),$$

where the energy functional is in the form

$$\begin{aligned}\mathcal{E}_M(u(t)) \sim & \sum_{|\alpha|\leq N}\|\partial_x^\alpha u\|^2 + \sum_{|\alpha|+|\beta|\leq N \quad |\beta|\geq 1}\|\partial_x^\alpha\partial_\xi^\beta u_2\|^2 \\ & + \frac{M}{2}\sum_{|\alpha|\leq N}\|\partial_x^\alpha u\|^2 + 2\mathcal{I}(u(t)) \\ \sim & \ [[u(t)]]^2,\end{aligned}$$

and the dissipation rate is in the form

$$\begin{aligned}\mathcal{D}(u(t)) \sim & \sum_{|\alpha|+|\beta|\leq N}\|\partial_x^\alpha\partial_\xi^\beta u_2\|_\nu^2 + \sum_{|\alpha|\leq N-1}\|\nabla_x\partial_x^\alpha(a,b,c)\|^2 \\ \sim & \ [[u(t)]]_\nu^2.\end{aligned}$$

2.8 Proof of Theorem II:

Set

$$w(t,x,\xi) = u(t,x,\xi) - v(t,x,\xi).$$

Then w satisfies

$$\partial_t w + \xi\cdot\nabla_x w + \nabla_x\phi(x)\cdot\nabla_\xi w - \frac{1}{2}\xi\cdot\nabla_x\phi(x)w = e^{\phi(x)}\mathbf{L}w + \Gamma(w,u) + \Gamma(v,w).$$

Similar proof yields the Lyapunov-type inequality

$$\frac{d}{dt}\mathcal{E}_M(w(t)) + C\mathcal{D}(w(t)) \leq C\{\mathcal{D}(u(t)) + \mathcal{D}(v(t))\}\mathcal{E}_M(w(t)).$$

By using the time integrability

$$\int_0^\infty \{\mathcal{D}(u(s)) + \mathcal{D}(v(s))\}ds < \infty$$

and the Gronwall's inequality, the uniform-in-time stability estimate holds true.

2.9 Generalized to the general collision kernel:

$$|\xi-\xi_*|^\gamma b(\theta), \quad 0 \le \gamma \le 1.$$

Problem: one of the source terms

$$-1/2\xi \cdot \nabla_x \phi(x) u$$

can not be controlled in terms of the dissipation $\mathbf{L}u$ since

$$-\langle \mathbf{L}u, u\rangle \ge \lambda \int (1+|\xi|)^\gamma (u_2)^2 dx d\xi.$$

Idea: use another kind of perturbation

$$f = e^\phi \mathbf{M} + \sqrt{e^\phi \mathbf{M}} u.$$

Then the reformulated equation reads

$$\partial_t u + \xi \cdot \nabla_x u + \nabla_x \phi(x) \cdot \nabla_\xi u = e^{\phi(x)} \mathbf{L}u + e^{\phi(x)/2} \Gamma(u,u).$$

2.10 Application: Vlasov-Poisson-Boltzmann system

(DY, recent work)

$$\partial_t f + \xi \cdot \nabla_x f + \nabla_x \Phi \cdot \nabla_\xi f = Q(f,f),$$
$$\Delta_x \Phi = \int_{\mathbb{R}^3} f(t,x,\xi)d\xi - \bar{\rho},$$

where $\bar{\rho} > 0$ is (or near) a positive constant.

Energy functional:

$$[[u(t)]]^2 \equiv \sum_{|\alpha|+|\beta|\leq N} \|\partial_x^\alpha \partial_\xi^\beta u\|^2 + \sum_{|\alpha|\leq N} \|\partial_x^\alpha \nabla_x \Phi\|^2$$

Dissipation rate:

$$[[u(t)]]_\nu^2 \equiv \sum_{|\alpha|+|\beta|\leq N, |\beta|>0} \|\partial_x^\alpha \partial_\xi^\beta \{\mathbf{I}-\mathbf{P}\}u\|_\nu^2 + \sum_{0<|\alpha|\leq N} \|\partial_x^\alpha u\|_\nu^2$$
$$+ \sum_{|\alpha|\leq N-1} \|\partial_x^\alpha \nabla_x \Phi\|^2 + \sum_{|\alpha|\leq N-1} \|\partial_x^\alpha b\|^2.$$

A prior estimate:

$$\frac{d}{dt}\mathcal{E}(u(t)) + \lambda\mathcal{D}(u(t)) \le \sqrt{\mathcal{E}(u(t))}\mathcal{D}(u(t)),$$
$$\mathcal{E}(u(t)) \sim [[u(t)]]^2, \quad \mathcal{D}(u(t)) \sim [[u(t)]]_\nu^2.$$

Difficulties:

- **No time-derivatives;**
- **Dissipations include:**

$$\|\nabla_x\Phi\|^2 \text{ (Poisson equation?)}$$

$$\|b\|^2 \text{ (Elliptic equation?)}$$

Remark $\|\nabla_x\Phi\|$ **or** $\|b\|$ **is necessary to be included since the source term contains**

$$\iint \xi\cdot\nabla_x\Phi u_1^2 dxd\xi = \int \nabla_x\Phi\cdot b(a+5c)dx.$$

3. Convergence rate

3.1 The case of $\phi \equiv 0$: the solution semigroup $\{e^{\mathbf{B}t}\}_{t\geq 0}$, where

$$\mathbf{B} = -\xi \cdot \nabla_x + \mathbf{L},$$

decays with an algebraic rate:

$$\|\nabla_x^m e^{\mathbf{B}t} g\|_{L^2_{x,\xi}} \leq C(q,m)(1+t)^{-\sigma_{q,m}}(\|g\|_{Z_q} + \|\nabla_x^m g\|_{L^2_{x,\xi}}),$$
$$m \geq 0, \quad q \in [1,2], \quad Z_q = L^2_\xi(L^q_x),$$
$$\sigma_{q,m} = \frac{3}{2}\left(\frac{1}{q} - \frac{1}{2}\right) + \frac{m}{2}.$$

Remark *The above rate was obtained by the spectral analysis due to Ukai ('74) and Nishida-Imai ('76). Recently, an extra decay was obtained by Ukai-Yang (AA-'06) if g is purely microscopic.*

Theorem III (Optimal convergence rates) *Let all conditions in Theorem I hold. Further assume that $\|u_0\|_{Z_1}$ is bounded and*

$$\||x|\phi\|_{L^\infty_x} + \||x|\nabla_x\phi\|_{L^2_x}$$

is small enough. Then the solution u obtained in Theorem I satisfies

$$[[u(t)]] \le C(1+t)^{-\frac{3}{4}}([[u_0]] + \|u_0\|_{Z_1}), \quad \forall\, t \ge 0.$$

Remarks

(a) By optimal, it means that the decay rate is the same as one of $e^{\mathbf{B}t}$ when $\phi = 0$, at the level of zero order ($\sigma_{1,0} = 3/4$).

(b) The proof is based on

- *the energy estimates of higher order,*
- *the decay-in-time estimates on $e^{\mathbf{B}t}$ (Ukai-Yang),*
- *and the analysis on the damping transport operator*

$$\partial_t + \xi \cdot \nabla_x + \nu(\xi).$$

3.2 **Hypocoercivity:**

degenerate dissipative operator

$$+$$

conservative operator

$$\Downarrow$$

full dissipation and convergence (Villani, etc.)

Models: Boltzmann equation, Fokker-Planck equation, Classical Landau equation, BGK model, etc.

3.3 Some known results on the convergence rates:

- **Without forces:**
  - Exponential convergence rate in bounded domain and torus: Ukai ('74), Giraud ('75), Shizuta-Asano ('77),...
  - Algebraic convergence rate in unbounded domain: Ukai ('76), Nishida-Imai ('76), Ukai-Asano ('83),...
  - Almost exponential convergence rate: Strain-Guo ('05), Desvillettes-Villani ('05)
  - Optimal convergence rate (extra decay): Ukai-Yang ('06)

- **With forces:**
  - Convergence rate in L^∞ framework: Asano ('02),...
  - Convergence rate in L^2 framework: Ukai-Yang-Zhao ('05),...
  - Torus case: Mouhout-Neumann ('06),...

3.4 Sketch of proof of Theorem III.

Step 1. Energy estimates of higher order:

$$\frac{d}{dt}\mathcal{E}_{\mathrm{h.o.}}(u(t)) + \lambda[[u(t)]]_\nu^2 \le C\|\nabla_x(a,b,c)\|^2,$$

where

$$\frac{1}{C}[[u(t)]]_0^2 \le \mathcal{E}_{\mathrm{h.o.}}(u(t)) \le C[[u(t)]]_0^2,$$

$$[[u(t)]]_0^2 \equiv \|\{\mathbf{I}-\mathbf{P}\}u(t)\|^2 + \sum_{0<|\alpha|\le N}\|\partial_x^\alpha u(t)\|^2 + \sum_{|\alpha|+|\beta|\le N,\ |\beta|>0}\|\partial_x^\alpha\partial_\xi^\beta\{\mathbf{I}-\mathbf{P}\}u(t)\|^2.$$

Notice that

$$[[u(t)]]_0 \le C[[u(t)]]_\nu.$$

Then

$$\frac{d}{dt}\mathcal{E}_{\mathrm{h.o.}}(u(t)) + \lambda\mathcal{E}_{\mathrm{h.o.}}(u(t)) \le C\|\nabla_x(a,b,c)\|^2.$$

Step 2. Decay-in-time estimates from the spectral analysis on $\nabla_x u$:

$$u(t) = e^{\mathbf{B}t}u_0 + \int_0^t e^{\mathbf{B}(t-s)}\mathbf{S}[u](s)ds,$$

where

$$\mathbf{S}[u] = -\nabla_x\phi \cdot \nabla_\xi u + \frac{1}{2}\nabla_x\phi \cdot \xi u + (e^{\phi}-1)\mathbf{L}u + \Gamma(u,u).$$

Decomposition of $e^{\mathbf{B}t}$ (Ukai-Yang):

$$e^{\mathbf{B}t} = \underbrace{\mathbf{E}_0(t)}_{\text{damping transport}} + \underbrace{\mathbf{E}_1(t)}_{\text{algebraic decay}} + \underbrace{\mathbf{E}_2(t)}_{\text{exponential decay}},$$

where

$$\mathbf{E}_0(t)u \equiv e^{-\nu(\xi)t}u(x-\xi t,\xi),$$

$$\|\nabla_x^m \mathbf{E}_1(t)\nu u\|_{Z_2} \le C(1+t)^{-\sigma_{q,m}}\|u\|_{Z_q},$$

$$\|\nabla_x^m \mathbf{E}_1(t)\{\mathbf{I}-\mathbf{P}\}\nu u\|_{Z_2} \le C(1+t)^{-\sigma_{q,m+1}}\|u\|_{Z_q} \ : \text{extra decay},$$

$$\|\nabla_x^m \mathbf{E}_2(t)\nu u\|_{Z_2} \le Ce^{-\lambda t}\|\nabla_x^m u\|_{Z_2}.$$

Remark *A technical lemma will be used to deal with the velocity increasing in the source term S. The trouble comes from the transport part in the semigroup* $e^{\mathbf{B}t}$*. For this, define* $\Psi[h](t,x,\xi)$ *as the solution to*

$$\partial_t u + \xi \cdot u + \nu(\xi) u = \nu(\xi) h(t,x,\xi), \quad u|_{t=0} = 0,$$

where $\nu_0 > 0$ *is such that*

$$\nu_0(1+|\xi|) \leq \nu(\xi) \leq \frac{1}{\nu_0}(1+|\xi|).$$

Claim: $\forall\ \lambda \in (0,\nu_0)$, $\exists\ C$ **s.t.**

$$\int_0^t e^{-\lambda(t-s)} \|\Psi[h](s)\|^2_{L^2_{x,\xi}} ds \leq C \int_0^t e^{-\lambda(t-s)} \|h(s)\|^2_{L^2_{x,\xi}} ds.$$

Sketch of proof for Claim : (a) Decompose

$$\|\Psi[h](s)\|^2_{L^2_{x,\xi}} = \sum_{R=0}^{\infty} \|\Psi[h](s)\|^2_{L^2(\mathbb{R}^3_x \times \Omega_\xi(R))},$$

where

$$\Omega_\xi(R) = \{\xi \in \mathbb{R}^3; R \leq |\xi| < R+1\}.$$

(b) Use the pointwise estimates: $\xi \in \Omega_\xi(R)$,

$$\begin{aligned}
&|\Psi[h](s,x,\xi)| \\
&= \left| \int_0^s e^{-\nu(\xi)(s-\theta)} \nu(\xi) h(\theta, x-(s-\theta)\xi, \xi) d\theta \right| \\
&\le \int_0^s e^{-\nu_0(1+R)(s-\theta)} \frac{1}{\nu_0}(2+R) |h(\theta, x-(s-\theta)\xi, \xi)| d\theta.
\end{aligned}$$

(c) Minkowski and Hölder inequalities and Fubini theorem yield

$$\begin{aligned}
&\int_0^t e^{-\lambda(t-s)} \|\Psi[h](s)\|^2_{L^2_{x,\xi}} ds \\
&\le \sum_{R=0}^{\infty} \frac{(2+R)^2}{\nu_0^4 (1+R)^2} \int_0^t e^{-\lambda(t-\theta)} \|h(\theta)\|^2_{L^2(\mathbb{R}^3_x \times \Omega_\xi(R))} d\theta \\
&\le \frac{4}{\nu_0^4} \sum_{R=0}^{\infty} \int_0^t e^{-\lambda(t-\theta)} \|h(\theta)\|^2_{L^2(\mathbb{R}^3_x \times \Omega_\xi(R))} d\theta.
\end{aligned}$$

Step 3. **Time-decay estimates on higher order: From Steps 1 and 2, Gronwall + Hardy + Claim ⇒**

$$\begin{aligned}\mathcal{E}_{\text{h.o.}}(u(t)) &\leq e^{-\lambda t}\mathcal{E}_{\text{h.o.}}(u(0)) + \int_0^t e^{-\lambda(t-s)}\|\nabla_x u(s)\|^2_{L^2_{x,\xi}}\,ds \\ &\leq C(1+t)^{-\frac{5}{2}}[\delta_0^2 + K_0^2 + (\delta_0^2+\delta_\phi^2)\mathcal{E}^\infty_{\text{h.o.}}(t)], \quad \forall\ t\geq 0,\end{aligned}$$

where

$$\mathcal{E}^\infty_{\text{h.o.}}(t) = \sup_{0\leq s\leq t}(1+s)^{\frac{5}{2}}\mathcal{E}_{\text{h.o.}}(u(s)),$$

$$\delta_0 = [[u_0]] = \|u_0\|_{H^N_{x,\xi}},\ K_0 = \|u_0\|_{Z_1},$$

Then

$$\mathcal{E}^\infty_{\text{h.o.}}(t) \leq C(\delta_0^2 + K_0^2).$$

Step 4. Decay-in-time rate for zero order: Add $\|\mathbf{P}u(t)\|^2_{L^2_{x,\xi}}$ to both sides of zero-order energy estimates to obtain

$$\frac{d}{dt}\mathcal{E}_{\mathrm{z.o.}}(u(t)) + \lambda\mathcal{E}_{\mathrm{z.o.}}(u(t)) \leq C\|u(t)\|^2_{L^2_{x,\xi}},$$

where

$$\mathcal{E}_{\mathrm{z.o.}}(u(t)) \sim [[u(t)]]^2 = \|u(t)\|^2_{H^N_{x,\xi}}.$$

Then,

$$\mathcal{E}_{\mathrm{z.o.}}(u(t)) \leq e^{-\lambda t}\mathcal{E}_{\mathrm{z.o.}}(u(0)) + C\int_0^t e^{-\lambda(t-s)}\|u(s)\|^2_{L^2_{x,\xi}}ds.$$

The rest computation is similar as before:

- Use the mild form of $u(t)$ to iterate once;
- Use the decomposition of $e^{\mathbf{B}t}$ and the Claim to find the time-decay rate.

$\Rightarrow$

$$\mathcal{E}_{\mathrm{z.o.}}(u(t)) \lesssim (1+t)^{-\frac{3}{2}}.$$

3.5 Application 1:

The following more general linearized Boltzmann equation with linear and variant-coefficient sources can be considered in the same way:

$$\begin{aligned}\partial_t u + \xi \cdot \nabla_x u - \mathbf{L}u &= A_0 K u + \sum_{|\alpha|+|\beta|\leq 1} A_{\alpha\beta}\partial_x^\alpha \partial_\xi^\beta u,\\ &\equiv \{A_0 K + A_{00}\}u + A_{10}\cdot \nabla_x u + A_{01}\cdot \nabla_\xi u,\end{aligned}$$

where

$$A_0 = A_0(t,x,\xi), \quad A_{\alpha\beta} = A_{\alpha\beta}(t,x,\xi)$$

satisfies some conditions on smallness in (t,x,ξ) and increase in ξ. A physical force inducing the above equation is in the form:

$$F(t,x,\xi) = E(t,x) + \xi \times B(t,x).$$

3.5 Application 2:

$$\partial_t f + \xi \cdot \nabla_\xi f + F(t,x) \cdot \nabla_\xi f = Q(f,f)$$

P: Force $F(t,x)$ is time-periodic $\Rightarrow \exists$ Time-periodic solution?

A: ▶ **Yes if $n \geq 5$, D.-Ukai-Yang-Zhao (CMP, '08);**

Proof:

(i) Optimal time-decay estimates on the linearized equation

(ii) Find the fixed point for certain nonlinear mapping Ψ:

$$\Psi[u](t) = \int_{-\infty}^{t} U(t,s)\mathbf{S}_F[u](s)ds, \quad \forall\, t \in \mathbb{R}.$$

(Well-defined since $U(t,s) \lesssim (1+t-s)^{-\frac{n}{4}}$ and $\frac{n}{4} > 1$)

▶ **Open for $1 \leq n \leq 4$, in particular, $n = 3$ (Physical).**

Thanks!