Global well-posedness for Boltzmann equations with large-amplitude data

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Lectures in AMSS, CAS: Part III
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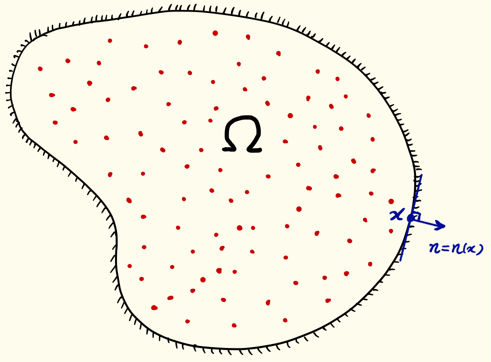
I. Introduction

Consider a rarefied gas contained in a vessel Ω:



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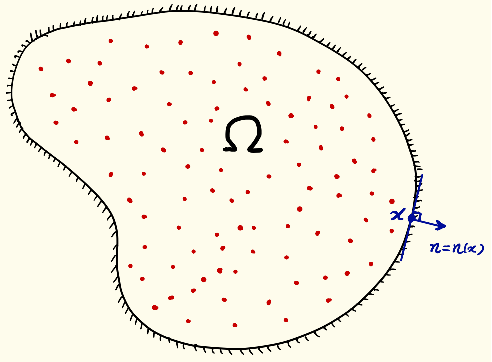

- Ω**: a bounded domain in** $\mathbb{R}^3$**.**
- $\theta_w \equiv \mathrm{cst} > 0$ **(i.e., wall temperature is constant)**
- $u_w = 0$ **(i.e., wall is stationary)**
- $n = n(x)$ $(x \in \partial\Omega)$ **(unit normal vector from gas to wall)**

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- **reverse reflection (**$v \to -v$**)**
- **diffuse reflection (to be considered; clarified later)**
- **specular reflection (**$v \to v - 2n(x) \cdot v$**)**

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Basic problem: Wellposedness on IBVP?

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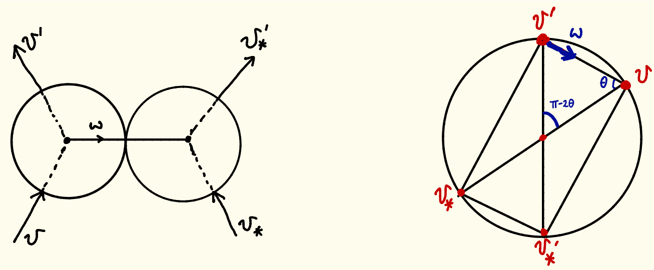

$$v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$$

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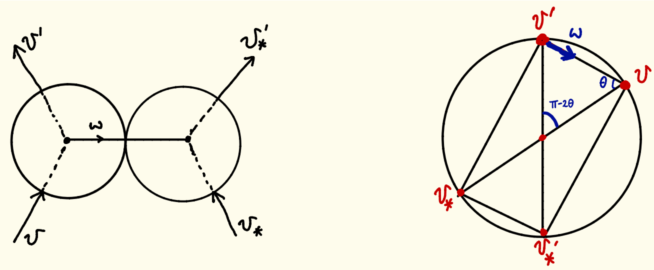

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$$Q(F, H)(v)$$

$$= \int_{\mathbb{R}^3} \mathrm{d}v_* \int_{\mathbb{S}^2} \mathrm{d}\omega \, \underbrace{|v - v_*|^\kappa b_0(\cos\theta)}_{\text{collision kernel (cutoff)}} \big(F(v'_*)H(v') - F(v_*)H(v)\big),$$

$$-3 < \kappa \le 1, \quad 0 \le b_0 \le C|\cos\theta|$$

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  - **Global-in-time existence is a consequence of the interplay between two properties above.**

II. A non-exhausting known results:

Global existence and large-time behavior

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$$\|F(t)-\mu\|_{L^2_v H^N_x(1+|v|^k)} \leq C(\sup_{t\geq 0}\|F(t)\|_{L^2_v H^{N+\ell_s}_x(1+|v|^k)}, \dots)t^{-s}$$

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    - **Gualdani-Mischler-Mouhot (**arXiv:1006.5523, 2010**):**

$$\|F(t)-\mu\|_{L^1_v L^\infty_x(1+|v|^2)} \leq Ce^{-\lambda t},$$

by showing that solutions are time-exponentially stable under small perturbations in $L^1_v L^\infty_x (1+|v|^k)$ $(k>2)$**.**

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Look for $F = \mu + \mu^{1/2} f$**, so** $\{\partial_t + v \cdot \nabla_x + L\} f = \Gamma(f, f)$.

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  - **Solutions in a spatially critical Besov space** $B^{3/2}_{2,1}$**: D.-Liu-Xu** 16**, Morimoto-Sakamoto** 16

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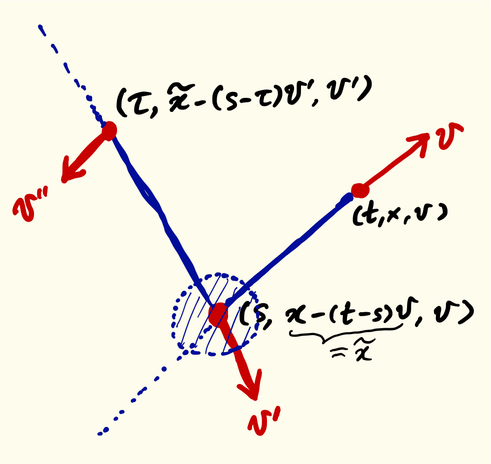
How to estimate L^∞ **of** $h(t) = U(t)h_0 = h_1 + h_2 + h_3$**?**


$(\tau, \tilde{x} - (s-\tau)v', v')$
v
v''
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v'

IDEA of estimate on $h(t) = U(t)h_0 = h_1 + h_2 + h_3$:

- $\Omega = \mathbb{R}^3$: h_3 **includes an integral with** $x_1 = x - v(t-s)$

$$\int_0^t e^{-\nu(v)(t-s)} \mathrm{d}s \int_{|v'|\le N} \mathrm{d}v' \int_{|v''|\le 2N} \mathrm{d}v'' \int_0^{s-\epsilon} e^{-\nu(v')(s-\tau)} \mathrm{d}\tau$$
$$K_w(v,v')K_w(v',v'')h(\tau, x_1 - v'(s-\tau), v''),$$

we take $y = x_1 - v'(s-\tau)$**, so** $\mathrm{d}v' = (s-\tau)^{-3}\mathrm{d}y \le \epsilon^{-3}\mathrm{d}y$**,**
somehow to obtain

$$\|h(t)\|_{L^\infty} \lesssim e^{-\lambda t}\|h_0\|_{L^\infty} + (\epsilon + \frac{C_\epsilon}{N}) \int_0^t e^{-\lambda(t-\tau)}\|h(\tau)\|_{L^\infty}\,\mathrm{d}\tau$$
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  we take $y = x_1 - v'(s-\tau)$**, so** $\mathrm{d}v' = (s-\tau)^{-3}\mathrm{d}y \le \epsilon^{-3}\mathrm{d}y$**, somehow to obtain**

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- Ω **is a bounded domain in** $\mathbb{R}^3$**: Given** (t,x,v)**, we have to treat the case the backward characteristic line hits** $\partial\Omega$ **earlier than** $t=0$**, then to obtain** L^∞ **bound, we need to iterate boundary condition** k **times for** k **large enough.**

- **Further development of Guo's approach:**
  - **Kim** 11**: discontinuity of solutions in non convex domains**
  - **Esposito-Guo-Kim-Marra** 13**:** $\exists\&$**dynamical stability of nontrivial stationary sol. for non-constant** θ_w
  - **Guo-Kim-Tonon-Trescases** 16**: BV-regularity of solutions in non-convex domains**
  - **Guo-Kim-Tonon-Trescases** 16**:** C^1 **regularity of solutions**
  - **Liu-Yang** 16**: soft potentials**

III. Our results

Sum: **In all previous results in near-μ (global Maxwellian) framework ($F(t,x,v)=\mu+\mu^{1/2}f(t,x,v)\geq 0$),**

$$\|f(t)\|_{L^\infty_{x,v}} \ll 1$$

uniformly for all $t\geq 0$, particularly at $t=0$.

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Q.: Is it possible to construct a global-in-time unique *strong* solution allowed to initially have large amplitude (thus also contain vacuum)?

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Q.: Is it possible to construct a global-in-time unique *strong* solution allowed to initially have large amplitude (thus also contain vacuum)?

A.: <u>Yes</u> for a class of initial data when the phase area where F is far from μ is small in a suitable sense!

$\boxed{\text{The 1st result:}}$ **Consider**

$$\Omega = \mathbb{R}^3 \text{ or } \mathbb{T}^3, \ -3 < \kappa \leq 1.$$

Note that any solution $F(t,x,v)$ satisfies with $\mu = \mu_{[1,0,1]}(v)$

$$\int_\Omega \int_{\mathbb{R}^3} (F(t,x,v) - \mu(v)) dv dx = \int_\Omega \int_{\mathbb{R}^3} (F_0(x,v) - \mu(v)) dv dx := M_0,$$

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$$\int_\Omega \int_{\mathbb{R}^3} v(F(t,x,v) - \mu(v))dvdx = \int_\Omega \int_{\mathbb{R}^3} v(F_0(x,v) - \mu(v))dvdx := J_0,$$

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Note that any solution $F(t,x,v)$ **satisfies with** $\mu = \mu_{[1,0,1]}(v)$

$$\int_\Omega \int_{\mathbb{R}^3} (F(t,x,v) - \mu(v))dvdx = \int_\Omega \int_{\mathbb{R}^3} (F_0(x,v) - \mu(v))dvdx := M_0,$$

$$\int_\Omega \int_{\mathbb{R}^3} v(F(t,x,v) - \mu(v))dvdx = \int_\Omega \int_{\mathbb{R}^3} v(F_0(x,v) - \mu(v))dvdx := J_0,$$

$$\int_\Omega \int_{\mathbb{R}^3} |v|^2(F(t,x,v) - \mu(v))dvdx = \int_\Omega \int_{\mathbb{R}^3} |v|^2(F_0(x,v) - \mu(v))dvdx := E_0,$$

for all $t \geq 0$.

Moreover,

$$\int_\Omega \int_{\mathbb{R}^3} \Big\{ F(t,x,v) \ln F(t,x,v) - \mu(v) \ln \mu(v) \Big\} dv dx$$
$$\leq \int_\Omega \int_{\mathbb{R}^3} \Big\{ F_0 \ln F_0 - \mu(v) \ln \mu(v) \Big\} dv dx.$$

Moreover,

$$\int_\Omega \int_{\mathbb{R}^3} \Big\{ F(t,x,v) \ln F(t,x,v) - \mu(v) \ln \mu(v) \Big\} dvdx \\ \leq \int_\Omega \int_{\mathbb{R}^3} \Big\{ F_0 \ln F_0 - \mu(v) \ln \mu(v) \Big\} dvdx.$$

Define

$$\mathcal{E}(F(t)) := \int_\Omega \int_{\mathbb{R}^3} \Big\{ F(t,x,v) \ln F(t,x,v) - \mu \ln \mu \Big\} dvdx \\ + \left[\frac{3}{2} \ln(2\pi) - 1 \right] M_0 + \frac{1}{2} E_0.$$

Moreover,

$$\int_\Omega \int_{\mathbb{R}^3} \Big\{F(t,x,v)\ln F(t,x,v) - \mu(v)\ln\mu(v)\Big\}dvdx$$
$$\leq \int_\Omega \int_{\mathbb{R}^3} \Big\{F_0 \ln F_0 - \mu(v)\ln\mu(v)\Big\}dvdx.$$

Define

$$\mathcal{E}(F(t)) := \int_\Omega \int_{\mathbb{R}^3} \Big\{F(t,x,v)\ln F(t,x,v) - \mu\ln\mu\Big\}dvdx + \left[\frac{3}{2}\ln(2\pi) - 1\right]M_0 + \frac{1}{2}E_0.$$

Then,

$$\mathcal{E}(F(t)) \geq 0,$$

for all $t \geq 0$**.**

Moreover,

$$\int_\Omega \int_{\mathbb{R}^3} \Big\{ F(t,x,v) \ln F(t,x,v) - \mu(v) \ln \mu(v) \Big\} dv dx \\ \leq \int_\Omega \int_{\mathbb{R}^3} \Big\{ F_0 \ln F_0 - \mu(v) \ln \mu(v) \Big\} dv dx.$$

Define

$$\mathcal{E}(F(t)) := \int_\Omega \int_{\mathbb{R}^3} \Big\{ F(t,x,v) \ln F(t,x,v) - \mu \ln \mu \Big\} dv dx \\ + \left[\frac{3}{2} \ln(2\pi) - 1 \right] M_0 + \frac{1}{2} E_0.$$

Then,

$$\mathcal{E}(F(t)) \geq 0,$$

for all $t \geq 0$. Note, in particular, that $\mathcal{E}(F_0) \geq 0$ holds true for any function $F_0(x,v) \geq 0$.

Theorem (D.-Huang-Wang-Yang, 17)

Let $\Omega = \mathbb{T}^3$ or $\mathbb{R}^3$, $-3 < \kappa \leq 1$. Set $w(v) := (1+|v|^2)^{\frac{\beta}{2}}$ with $\beta > \max\{3, 3+\kappa\}$. Let $F_0(x,v) = \mu(v) + \sqrt{\mu(v)} f_0(x,v) \geq 0$. For any $\bar{M} \geq 1$, there is $\varepsilon_0 > 0$ depending on $\gamma, \beta, \bar{M}$ s.t. if

$$\|wf_0\|_{L^\infty} \leq \bar{M},$$
$$\mathcal{E}(F_0) + \|f_0\|_{L^1_x L^\infty_v} \leq \varepsilon_0,$$

then the Cauchy problem on B.E. has a global unique mild solution $F(t,x,v) = \mu(v) + \sqrt{\mu(v)} f(t,x,v) \geq 0$ satisfying

$$\|wf(t)\|_{L^\infty} \leq \tilde{C}_1 \bar{M}^2,$$

where $\tilde{C}_1$ depends only on γ, β.

Remarks:

Remarks:

- **An example for initial data:** $F_0(x,v) = \rho_0(x)\mu$ **with** $\rho_0 \geq 0$, $\rho_0 \in L^\infty_x$, **and**

$$\|\rho_0 \ln \rho_0 - \rho_0 + 1\|_{L^1_x} + \|\rho_0 - 1\|_{L^1_x}$$

is small.

- **It can be shown that for** $\Omega = \mathbb{T}^3$ **and** $(M_0, J_0, E_0) = (0,0,0)$,

$$\|f(t)\|_{L^\infty} \lesssim \begin{cases} e^{-\sigma_0 t} & \textbf{for } 0 \leq \gamma \leq 1, \\ (1+t)^{-1-\frac{2}{|\gamma|}+} & \textbf{for } -3 < \gamma < 0, \end{cases}$$

for all $t \geq 0$, **as long as** $\|w_\beta f_0\|_{L^\infty}$ **is further sufficiently small for** $\beta > 0$ **large enough.**

Key points of the proof:

- **Local-in-time existence: For $\beta > 3$,**

$$\sup_{0\le t\le t_1} \|w_\beta f(t)\|_{L^\infty} \le 2\|w_\beta f_0\|_{L^\infty},$$

$$t_1 := (8\tilde{C}_4[1+\|w_\beta f_0\|_{L^\infty}])^{-1} > 0.$$

- **Global a priori estimates: Let $h = w_\beta f$.**
  - L^∞ **estimate: Let $\beta > 3$, $-3 < \gamma \le 1$, $p > 1$, then**

$$\sup_{0\le s\le t} \|h(s)\|_{L^\infty} \le C_1\Big\{\|h_0\|_{L^\infty} + \|h_0\|_{L^\infty}^2 + \sqrt{\mathcal{E}(F_0)} + \mathcal{E}(F_0)\Big\}$$

$$+ C_1 \sup_{t_1\le s\le t,\ y\in\Omega} \left\{\|h(s)\|_{L^\infty}^{\frac{9p+1}{5p}} \Big(\int_{\mathbb{R}^3} |f(s,y,\eta)|d\eta\Big)^{\frac{p-1}{5p}}\right\}.$$

  - $L^\infty_x L^1_v$ **Estimate: Let $-3 < \gamma \le 1$, $\beta > \max\{3, 3+\gamma\}$, then**

$$\int_{\mathbb{R}^3} |f(t,x,v)|dv$$

$$\le \int_{\mathbb{R}^3} e^{-\nu(v)t}|f_0(x-vt,v)|dv + C_N\lambda^{-\frac{3}{2}}\sqrt{\mathcal{E}(F_0)} + C_N\lambda^{-3}\mathcal{E}(F_0)$$

$$+C\Big(m^{3+\gamma} + C_m[\lambda + \frac{1}{N} + \frac{1}{N^{\beta-3}}]\Big)\cdot \sup_{0\le s\le t}\Big\{\|h(s)\|_{L^\infty} + \|h(s)\|_{L^\infty}^2\Big\}$$

$$+C_N\lambda^{-3}\Big(\sqrt{\mathcal{E}(F_0)} + \mathcal{E}(F_0)\Big)^{1-\frac{1}{p}} \cdot \sup_{0\le s\le t} \|h(s)\|_{L^\infty}^{1+\frac{1}{p}},$$

where $\lambda > 0, m > 0$ are small and $N \ge 1$ is large. □

$\boxed{\text{The 2nd result:}}$ **Consider**

Ω is a bounded domain with diffuse-reflection boundary of constant wall temperature, $0 \leq \kappa \leq 1$.

IBVP under consideration:

$$\begin{cases} \partial_t F + v \cdot \nabla_x F = Q(F, F) \quad \textbf{in } \{t > 0\} \times \Omega \times \mathbb{R}^3 \\ F = F_0 \quad \textbf{on } \{t = 0\} \times \Omega \times \mathbb{R}^3 \\ F|_{\gamma_-} = c_\mu \mu \int_{v \cdot n > 0} F|_{\gamma_+} v \cdot n \, \mathrm{d}v \quad \textbf{on } \{t \geq 0\} \times \gamma_-. \end{cases}$$

Long-time behavior: $F(t, x, v) \to \mu(v) \ (t \to \infty)$**?**

Theorem (D.-Wang, preprint 16)

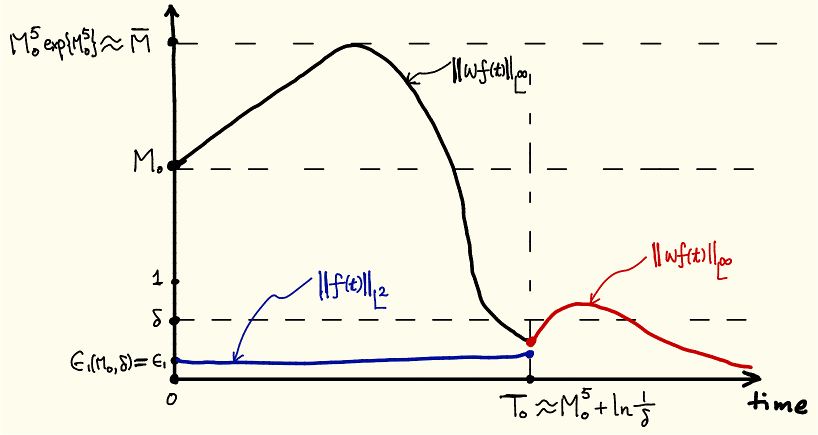
Let $w(v)=(1+\rho^2|v|^2)^\beta e^{\varpi|v|^2}$ *with* $\rho>1$ *large enough,* $\beta\geq 5/2$*, and* $0\leq\varpi\leq 1/64$*. Assume* $F_0(x,v)=\mu+\sqrt{\mu}f_0(x,v)\geq 0$ *with the mass conservation. For any* $M_0\geq 1$*, there is* $\epsilon_0>0$ *depending only on* δ *and* M_0 *such that if*

$$\|wf_0\|_{L^\infty}\leq M_0,\quad \|f_0\|_{L^2}\leq\epsilon_0,$$

then IBVP admits a unique solution $F(t,x,v)=\mu+\sqrt{\mu}f(t,x,v)\geq 0$ *satisfying*

$$\|wf(t)\|_{L^\infty}\leq\widetilde{C}_0M_0^5\exp\left\{\frac{2}{\nu_0}\widetilde{C}_0M_0^5\right\}e^{-\vartheta_1 t},\quad\forall\, t\geq 0,$$

where $\widetilde{C}_0\geq 1$ *is a generic constant,* $\vartheta_1=\min\left\{\vartheta,\frac{\nu_0}{16}\right\}>0$*, and* $\nu_0:=\inf_{v\in\mathbb{R}^3}\nu(v)>0$*. Moreover, if* Ω *is strictly convex, and* $F_0(x,v)$ *is continuous except on* γ_0 *then* $F(t,x,v)$ *is continous in* $[0,\infty)\times\{\overline{\Omega}\times\mathbb{R}^3\backslash\gamma_0\}$*.*


$M_0^5 \exp\{M_0^5\} \approx \bar{M}$
$\|wf(t)\|_{L^\infty}$
M_0
$\|wf(t)\|_{L^\infty}$
1
$\|f(t)\|_{L^2}$
δ
$\epsilon_1(M_0, \delta) = \epsilon_1$
0
$T_0 \approx M_0^5 + \ln \frac{1}{\delta}$
time

The most key ingredients for the proof of Theorem:

- $L^2_{x,v}$–$L^\infty_x L^1_v$–$L^\infty_{x,v}$ **estimates along a bootstrap argument:**

$$\|f(t)\|_{L^2} \le e^{\tilde{C}_1 \bar{M} t} \|f_0\|_{L^2}.$$

The most key ingredients for the proof of Theorem:

- $L^2_{x,v}$–$L^\infty_x L^1_v$–$L^\infty_{x,v}$ **estimates along a bootstrap argument:**

$$\|f(t)\|_{L^2} \le e^{\tilde{C}_1 \bar{M} t} \|f_0\|_{L^2}.$$

- **Pointwise estimates on the upper bound of the gain term by the product of L^∞ norm and L^2 norm:**

$$|w(v)\Gamma_+(f,f)(v)| \le \frac{C_\beta \|wf\|_{L^\infty_v}}{1+|v|} \left(\int_{\mathbb{R}^3} (1+|\eta|)^4 e^{\varpi|\eta|^2} |f(\eta)|^2 \, \mathrm{d}\eta \right)^{\frac{1}{2}}.$$

The most key ingredients for the proof of Theorem:

- $L^2_{x,v}$–$L^\infty_x L^1_v$–$L^\infty_{x,v}$ **estimates along a bootstrap argument:**

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- **An iterative procedure on the nonlinear term:**

$$w(\eta)|f(s,y,\eta)| \leq \cdots$$

Two Key Lemmas:

Lemma

Under the a priori assumption, there exists a generic constant $\tilde{C}_2 \geq 1$ such that given any $T_0 > \tilde{t}$ with

$$\tilde{t} := \frac{2}{\nu_0} \ln\left(\tilde{C}_2 M_0\right) > 0,$$

there is a generally small positive constant $\epsilon_1 = \epsilon_1(\bar{M}, T_0) > 0$, depending only on $\bar{M}$ and T_0, such that if $\|f_0\|_{L^2} \leq \epsilon_1$, then one has

$$R(f)(t,x,v) \geq \frac{1}{2}\nu(v),$$

for all $(t,x,v) \in [\tilde{t}, T_0) \times \Omega \times \mathbb{R}^3$. Here ϵ_1 is decreasing in $\bar{M}$ and T_0.

Two Key Lemmas:

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for all $(t, x, v) \in [\tilde{t}, T_0) \times \Omega \times \mathbb{R}^3$. Here ϵ_1 is decreasing in $\bar{M}$ and T_0.

Proof. Use the mild formulation with k-times reflection + L^2-L^∞ interplay. □

Lemma

Assume $\|f_0\|_{L^2} \leq \epsilon_1 = \epsilon_1(\bar{M}, T_0)$. *There exists a generic constant* $\tilde{C}_3 \geq 1$ *such that*

$$
\begin{aligned}
\|h(t)\|_{L^\infty} \leq{}& \tilde{C}_3 e^{2\nu_0 \tilde{t}} \|h_0\|_{L^\infty} \left[1 + \int_0^t \|h(s)\|_{L^\infty}\,\mathrm{d}s\right] e^{-\frac{\nu_0}{8} t} \\
&+ \tilde{C}_3 e^{2\nu_0 \tilde{t}} \left\{ \left(\varepsilon + \lambda + \frac{C_{\varepsilon, T_0}}{N}\right) \sup_{0 \leq s \leq t} \left[\|h(s)\|_{L^\infty} + \|h(s)\|_{L^\infty}^3\right] \right. \\
&\left. \qquad + C_{\varepsilon, \lambda, N, T_0} \sup_{0 \leq s \leq t} \left[\|f(s)\|_{L^2} + \|f(s)\|_{L^2}^3\right] \right\},
\end{aligned}
$$

holds true for all $0 \leq t \leq T_0$, *where* $\lambda > 0$ *and* $\varepsilon > 0$ *can be arbitrarily small, and* $N > 0$ *can be arbitrarily large.*

The 3rd result (joint with Huang-Wang-Zhang, preprint 17):

Consider

Ω is a bounded domain with diffuse-reflection boundary where the wall temperature can have a small variation around a positive constant, $-3 < \kappa < 0$.

IBVP under consideration:

$$\begin{cases} \partial_t F + v \cdot \nabla_x F = Q(F,F) \quad \textbf{in } \{t>0\} \times \Omega \times \mathbb{R}^3 \\ F = F_0 \quad \textbf{on } \{t=0\} \times \Omega \times \mathbb{R}^3 \\ F|_{\gamma_-} = \mu_\theta \displaystyle\int_{v\cdot n>0} F|_{\gamma_+} v \cdot n \,\mathrm{d}v \quad \textbf{on } \{t \geq 0\} \times \gamma_-, \end{cases}$$

$$\mu_\theta(v) = \frac{1}{2\pi\theta^2(x)} \exp\left[-\frac{|v|^2}{2\theta(x)}\right], \quad \sup_{\partial\Omega} |\theta - 1| \ll 1.$$

Long-time behavior: (Note: $\mu_\theta(v)$ satisfied B.C.: $\int_{v\cdot n>0} \mu_\theta v \cdot n \,\mathrm{d}v = 1$)
$F(t,x,v) \to F_*(x,v)$ which is the stationary solution $(t \to \infty)$**?**

Theorem (Existence of non-Maxwellian stationary solution)

Set $w_0(v) = (1+|v|^2)^{\frac{\beta}{2}} e^{\varpi |v|^2}$. *Let* $-3 < \kappa < 0$, $\beta > 3 + |\kappa|$, $0 \le \varpi \le \frac{1}{64}$. *Let* $M > 0$ *be arbitrary. There are* $\delta_0 > 0$, $C > 0$ *such that if*

$$\delta := |\theta - \theta_0|_{L^\infty(\partial\Omega)} \le \delta_0,$$

then there exists a unique $F_*(x,v) = M\mu + \sqrt{\mu} f_*(x,v) \ge 0$ *to the steady BVP*

$$\begin{cases} v \cdot \nabla_x F = Q(F,F) \quad in\ \Omega \times \mathbb{R}^3 \\ F|_{\gamma_-} = \mu_\theta \displaystyle\int_{v\cdot n>0} F|_{\gamma_+} v \cdot n \, \mathrm{d}v \quad on\ \gamma_- \end{cases}$$

satisfying

$$\|w_0 f_*\|_{L^\infty} \le C\delta.$$

Set

$$w(t,v) = (1+|v|^2)^{\frac{\beta}{2}} \exp\left\{ \frac{\varpi |v|^{\zeta}}{4} + \frac{\varpi |v|^{\zeta}}{4(1+t)^q} \right\}$$

where $0 < q < \frac{\zeta}{|\kappa|}$ **and**

$$0 < \varpi \leq \frac{1}{64} \text{ if } \zeta = 2,$$

$$\text{or } \varpi > 0 \text{ if } 0 < \zeta < 2.$$

Set

$$w(t,v) = (1+|v|^2)^{\frac{\beta}{2}} \exp\left\{\frac{\varpi|v|^\zeta}{4} + \frac{\varpi|v|^\zeta}{4(1+t)^q}\right\}$$

where $0 < q < \frac{\zeta}{|\kappa|}$ **and**

$$0 < \varpi \le \frac{1}{64} \text{ **if** } \zeta = 2,$$

or $\varpi > 0$ **if** $0 < \zeta < 2$**.**

Note: **For the modified collision frequency** $\tilde{\nu}(t,v)$**,**

$$\tilde{\nu}(t,v) \ge C(1+t)^{\frac{(1+q)|\kappa|}{\zeta+|\kappa|}}.$$

Thus, for $s < t$**,**

$$0 < \exp\left[-\int_s^t \tilde{\nu}(\eta, V_{cl}(\eta))\,\mathrm{d}\eta\right] \le e^{-\lambda(t^\alpha - s^\alpha)},$$

$$0 < \alpha = \frac{\zeta - q|\kappa|}{\zeta + |\kappa|} < 1.$$

Theorem (Global dynamics of large-amplitude solutions)

Let $-3 < \kappa < 0$, $\beta > \max\{3 + |\kappa|, 4\}$, *and* $\max\{\frac{3}{2}, \frac{3}{3+\kappa}\} < p < \infty$. *Assume* $F_0(x, v) = \mu + \sqrt{\mu} f_0(x, v) \geq 0$ *has the same mass as* F_* *with* $0 < \delta := |\theta - \theta_0|_{L^\infty(\partial\Omega)} < 1$ *small enough. For any* M_0 *with*

$$1 \leq M_0 \leq \frac{1}{\hat{C} + \frac{5}{2\alpha}} \log \frac{1}{\delta},$$

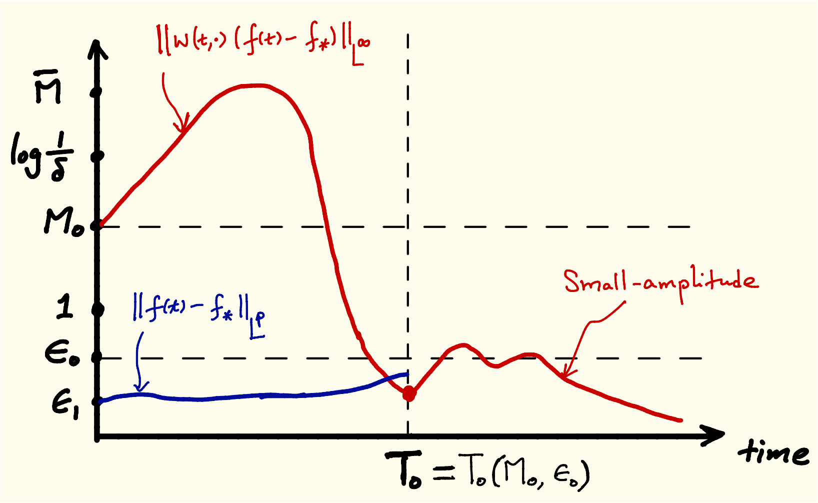
there are $\lambda > 0$, $C_0 > 0$, $\epsilon_1 > 0$ *such that if*

$$\|w(0, \cdot)(f_0 - f_*)\|_{L^\infty} \leq M_0, \quad \|f_0 - f_*\|_{L^p} \leq \epsilon_1,$$

then the IBVP of the Boltzmann equation admits a unique global solution $F(t, x, v) = \mu(v) + \sqrt{\mu(v)} f(t, x, v) \geq 0$ *satisfying*

$$\|w(t, \cdot)[f(t) - f_*]\|_{L^\infty} \leq C_0 e^{C_0 M_0} e^{-\lambda t^\alpha} \|w(0, \cdot)(f_0 - f_*)\|_{L^\infty},$$

for all $t \geq 0$, *where* $0 < \alpha = \frac{\zeta - q|\kappa|}{\zeta + |\kappa|} < 1$.


$\|w(t,\cdot)(f(t)-f_*)\|_{L^\infty}$
$\bar{M}$
$\log\frac{1}{\delta}$
M_0
1
ϵ_0
ϵ_1
$\|f(t)-f_*\|_{L^p}$
Small-amplitude
time
$T_0 = T_0(M_0, \epsilon_0)$

Proof: Existence of stationary solution

$\exists$ **of small-amplitude s.s. is a consequence of construction of iterative solution sequence as well as its L^∞ estimate:**

$$\begin{cases} v\cdot\nabla_x f^{j+1} + Lf^{j+1} = \Gamma(f^j, f^j), \\ f^{j+1}|_{\gamma_-} = P_\gamma f^{j+1} + \frac{\mu_\delta - \mu}{\sqrt{\mu}} \\ \qquad\qquad + \frac{\mu_\delta-\mu}{\sqrt{\mu}} \int_{v'\cdot n(x)>0} f^j(x,v')\sqrt{\mu(v')}\{v'\cdot n(x)\}\mathrm{d}v', \end{cases}$$

for $j = 0, 1, 2\cdots$ **with** $f^0 \equiv 0$**, under the assumption**

$$\mu_\delta := \mu_{\theta(x)},\ \theta(x) = 1 + \delta\theta_0(x),\ 0<\delta \ll 1.$$

Here,

$$P_\gamma f(x,v) := \sqrt{\mu(v)}\int_{v'\cdot n(x)>0} f(x,v')\sqrt{\mu(v')}\{v'\cdot n(x)\}\mathrm{d}v'.$$

Proposition

Assume

$$\iint_{\Omega\times\mathbb{R}^3} g(x,v)\sqrt{\mu(v)}\mathrm{d}v\mathrm{d}x = \int_{\gamma_-} r(x,v)\sqrt{\mu(v)}\mathrm{d}\gamma = 0.$$

Let $\beta > 3 + |\kappa|$, *and assume* $\|\nu^{-1}wg\|_{L^\infty} + |wr|_{L^\infty} < \infty$. *Then there exists a unique solution* $f = f(x,v)$ *to the linearized steady Boltzmann equation*

$$v\cdot\nabla_x f + Lf = g, \quad f(x,v)|_{\gamma_-} = P_\gamma f + r, \tag{P}$$

such that $\int_{\Omega\times\mathbb{R}^3} f\sqrt{\mu}\,\mathrm{d}v\mathrm{d}x = 0$ *and*

$$\|wf\|_{L^\infty} \le C\Big\{|wr|_{L^\infty(\gamma)} + \|\nu^{-1}wg\|_{L^\infty}\Big\}.$$

Proof of Proposition: **The solution $f(x,v)$ to (P) is obtained as a limit (first $n \to \infty$ and then $\epsilon \to 0$) of**

$$\begin{cases} \varepsilon f^{(n,\epsilon)} + v \cdot \nabla_x f^{(n,\epsilon)} + \nu(v) f^{(n,\epsilon)} - K f^{(n,\epsilon)} = g, \\ f^{(n,\epsilon)}(x,v)|_{\gamma_-} = (1 - \frac{1}{n}) P_\gamma f^{(n,\epsilon)} + r, \end{cases} \tag{$P_{n,\epsilon}$}$$

or equivalently, for $h := h^{(n,\epsilon)}(x,v) = w(v) f^{(n,\epsilon)}(x,v)$,

$$\begin{cases} \varepsilon h + v \cdot \nabla_x h + \nu(v) h = K_w h + wg, \\ h(x,v)|_{\gamma_-} = (1 - \frac{1}{n}) \frac{1}{\tilde{w}(v)} \int_{v' \cdot n(x) > 0} h(x,v') \tilde{w}(v') \mathrm{d}\sigma' + wr(x,v), \end{cases}$$

where $\tilde{w}(v) \equiv \frac{1}{w(v)\sqrt{\mu(v)}}$ and $K_w h = wK(\frac{h}{w})$.

• **Step 1. A priori L^∞ estimates.**

Definition (Speeded backward bi-characteristics)

Given (t,x,v) with $t>0$,

$$\begin{cases} \frac{d\hat{X}(s)}{ds} = (1+|V(s)|^2)^{\frac{|\gamma|}{2}} V(s) := \hat{V}(s), \\ \frac{dV(s)}{ds} = 0, \\ [X(t), V(t)] = [x, v], \end{cases}$$

has the solution

$$[\hat{X}(s;t,x,v), V(s;t,x,v)] = [x - \hat{v}(t-s), v],$$

with

$$\hat{v} := (1+|v|^2)^{\frac{|\kappa|}{2}} v.$$

Define the speeded back-time cycle:

- **Given** (t, x, v) **with** $t > 0$**,** $x \in \bar{\Omega}$ **and for only outgoing particles if** $x \in \partial\Omega$
- $\hat{t}_{\mathbf{b}}(x, v) = \inf\{\tau \geq 0 : x - \hat{v}\tau \notin \bar{\Omega}\}$.
- $x - \hat{t}_{\mathbf{b}}\hat{v} \in \partial\Omega$**.** $\hat{x}_{\mathbf{b}}(x, v) = \hat{x}(\hat{t}_{\mathbf{b}}) = x - \hat{t}_{\mathbf{b}}\hat{v} \in \partial\Omega$.

Define the speeded back-time cycle:

- **Given** (t, x, v) **with** $t > 0$, $x \in \bar{\Omega}$ **and for only outgoing particles if** $x \in \partial\Omega$
- $\hat{t}_{\mathbf{b}}(x, v) = \inf\{\tau \geq 0 : x - \hat{v}\tau \notin \bar{\Omega}\}$.
- $x - \hat{t}_{\mathbf{b}}\hat{v} \in \partial\Omega$. $\hat{x}_{\mathbf{b}}(x, v) = \hat{x}(\hat{t}_{\mathbf{b}}) = x - \hat{t}_{\mathbf{b}}\hat{v} \in \partial\Omega$.
- **For** $v_{k+1} \in \hat{\mathcal{V}}_{k+1} := \{v_{k+1} \cdot n(\hat{x}_{k+1}) > 0\}$**, inductively define**

$$(\hat{t}_{k+1}, \hat{x}_{k+1}, v_{k+1}) = (\hat{t}_k - \hat{t}_{\mathbf{b}}(\hat{x}_k, v_k), \hat{x}_{\mathbf{b}}(\hat{x}_k, v_k), v_{k+1}).$$

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- $\hat{t}_{\mathbf{b}}(x, v) = \inf\{\tau \geq 0 : x - \hat{v}\tau \notin \bar{\Omega}\}$.
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- **For** $v_{k+1} \in \hat{\mathcal{V}}_{k+1} := \{v_{k+1} \cdot n(\hat{x}_{k+1}) > 0\}$, **inductively define**

$$(\hat{t}_{k+1}, \hat{x}_{k+1}, v_{k+1}) = (\hat{t}_k - \hat{t}_{\mathbf{b}}(\hat{x}_k, v_k), \hat{x}_{\mathbf{b}}(\hat{x}_k, v_k), v_{k+1}).$$

Lemma

For T_0 sufficiently large, there exist generic constants $\hat{C}_1$ and $\hat{C}_2$ independent of T_0 such that for $k = \hat{C}_1 T_0^{\frac{5}{4}}$ and $0 \leq t \leq T_0$, it holds that

$$\int_{\Pi_{j=1}^{k-1}\hat{\mathcal{V}}_j} \mathbf{1}_{\{\hat{t}_k > 0\}} \, \Pi_{j=1}^{k-1} \mathrm{d}\hat{\sigma}_j \leq \left(\frac{1}{2}\right)^{\hat{C}_2 T_0^{\frac{5}{4}}}$$

where $\mathrm{d}\hat{\sigma}_j := \mu(v_j)\{v_j \cdot n(\hat{x}_j)\}\mathrm{d}v_j$.

Consider (Note: 1 can be placed by $1 - \frac{1}{n}$)

$$\begin{cases} \varepsilon h^{i+1} + v \cdot \nabla_x h^{i+1} + \nu(v) h^{i+1} = \lambda K_w^m h^i + \lambda K_w^c h^i + wg, \\ h^{i+1}(x,v)|_{\gamma_-} = \frac{1}{\tilde{w}(v)} \int_{v' \cdot n(x) > 0} h^i(x,v') \tilde{w}(v') \mathrm{d}\sigma' + w(v) r(x,v), \end{cases}$$

for $i = 0, 2, 3, \cdots$ **and** $h^0 = h^0(x,v)$ **is given. Denote**

$$\hat{\nu}(v) := (1 + |v|^2)^{\frac{|\kappa|}{2}} [\varepsilon + \nu(v)].$$

Observe

$$\inf_v \hat{\nu}(v) \geq \inf_v (1 + |v|^2)^{\frac{|\kappa|}{2}} \nu(v) := \hat{\nu}_0 > 0 \text{ **independent of** } \varepsilon.$$

Rewrite the equation as

$$\boxed{\{\hat{v} \cdot \nabla_x + \hat{\nu}(v)\} h^{i+1} = (1 + |v|^2)^{\frac{|\kappa|}{2}} \{\cdots\}.}$$

Mild formulation with parameter $t \in [0, T_0]$ for $T_0 > 0$ sufficiently large:

$$
\begin{aligned}
h^{i+1}(x,v) &= \mathbf{1}_{\{\hat{t}_1 \le 0\}} e^{-\hat{\nu}(v)t} h^{i+1}(x - \hat{v}t) \\
&\quad + \int_{\max\{\hat{t}_1, 0\}}^{t} e^{-\hat{\nu}(v)(t-s)} \nu(v)^{-1} \Big[\lambda K_w^m h^i + \lambda K_w^c h^i + wg\Big](x - \hat{v}(t-s), v) \mathrm{d}s \\
&\quad + e^{-\hat{\nu}(v)(t-\hat{t}_1)} w(v) r(\hat{x}_1, v) \mathbf{1}_{\hat{t}_1 > 0} + \frac{e^{-\hat{\nu}(v)(t-\hat{t}_1)}}{\tilde{w}(v)} \int_{\Pi_{j=1}^{k-1} \hat{\mathcal{V}}_j} \sum_{l=1}^{k-2} \mathbf{1}_{\{\hat{t}_{l+1} > 0\}} \mathrm{d}\Sigma_l^r \\
&\quad + \frac{e^{-\hat{\nu}(v)(t-\hat{t}_1)}}{\tilde{w}(v)} \int_{\Pi_{j=1}^{k-1} \hat{\mathcal{V}}_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{\hat{t}_{l+1} \le 0 < \hat{t}_l\}} h^{i+1-l}(\hat{x}_l - \hat{v}_l \hat{t}_l, v_l) \mathrm{d}\Sigma_l(0) \\
&\quad \frac{e^{-\hat{\nu}(v)(t-\hat{t}_1)}}{\tilde{w}(v)} \int_{\Pi_{j=1}^{k-1} \hat{\mathcal{V}}_j} \sum_{l=1}^{k-1} \int_0^{\hat{t}_l} \mathbf{1}_{\{\hat{t}_{l+1} \le 0 < \hat{t}_l\}} \\
&\qquad \times \nu(v_l)^{-1} \Big[\lambda K_w^m h^{i-l} + \lambda K_w^c h^{i-l} + wg\Big](\hat{x}_l - \hat{v}(\hat{t}_l - s), v_l) \mathrm{d}\Sigma_l(s) \\
&\quad \frac{e^{-\hat{\nu}(v)(t-\hat{t}_1)}}{\tilde{w}(v)} \int_{\Pi_{j=1}^{k-1} \hat{\mathcal{V}}_j} \sum_{l=1}^{k-1} \int_{\hat{t}_{l+1}}^{\hat{t}_l} \mathbf{1}_{\{\hat{t}_{l+1} \le 0 < \hat{t}_l\}} \\
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&\quad + \frac{e^{-\hat{\nu}(v)(t-\hat{t}_1)}}{\tilde{w}(v)} \int_{\Pi_{j=1}^{k-1} \hat{\mathcal{V}}_j} I_{\{\hat{t}_k > 0\}} h^{i+1-k}(\hat{x}_k, v_{k-1}) \mathrm{d}\Sigma_{k-1}(\hat{t}_k)
\end{aligned}
$$

where

$$\mathrm{d}\Sigma_l = \{\Pi_{j=l+1}^{k-1}\mathrm{d}\hat{\sigma}_j\}\cdot\tilde{w}(v_l)\mathrm{d}\hat{\sigma}_l\cdot\{\Pi_{j=1}^{l-1}\mathrm{d}\hat{\sigma}_j\},$$

$$\mathrm{d}\Sigma_l(s) = \{\Pi_{j=l+1}^{k-1}\mathrm{d}\hat{\sigma}_j\}\cdot\{\tilde{w}(v_l)e^{-\hat{\nu}(v_l)(\hat{t}_l-s)}\mathrm{d}\hat{\sigma}_l\}\cdot\{\Pi_{j=1}^{l-1}e^{-\hat{\nu}(v_j)(\hat{t}_j-\hat{t}_{j+1})}\mathrm{d}\hat{\sigma}_j\},$$

$$\mathrm{d}\Sigma_l^r = \{\Pi_{j=l+1}^{k-1}\mathrm{d}\hat{\sigma}_j\}\cdot\{\tilde{w}(v_l)w(v_l)r(\hat{x}_{l+1},v_l)e^{-\hat{\nu}(v_l)(\hat{t}_l-\hat{t}_{l+1})}\mathrm{d}\hat{\sigma}_l\}\cdot\{\Pi_{j=1}^{l-1}e^{-\hat{\nu}(v)(\hat{t}_j-\hat{t}_{j+1})}\mathrm{d}\hat{\sigma}_j\}.$$

Lemma

Let $\beta > 3$. Assume $\|h^i\|_{L^\infty} + |h^i|_{L^\infty(\gamma)} < \infty$ for $i = 0, 1, 2, \cdots$. Then there exist a large positive constant T_0 such that for $k = \hat{C}_2 T_0^{\frac{5}{4}}$, it holds, for $i \geq k$, that

$$\|h^{i+1}\|_{L^\infty} \leq \frac{1}{8}\sup_{0\leq l\leq k}\{\|h^{i-l}\|_{L^\infty}\} + C\Big\{\|\nu^{-1}wg\|_{L^\infty} + |wr|_\infty\Big\} + C\sup_{0\leq l\leq k}\left\{\left\|\frac{\sqrt{\nu}h^{i-l}}{w}\right\|_{L^2}\right\}.$$

Moreover, if $h^i \equiv h$ for $i = 1, 2, \cdots$, that is h is a solution. Then

$$\|h\|_{L^\infty} \leq C\Big\{\|\nu^{-1}wg\|_{L^\infty} + |wr|_\infty\Big\} + C\left\|\frac{\sqrt{\nu}h}{w}\right\|_{L^2}.$$

Here the positive constant $C > 0$ do not depend on $\lambda \in [0, 1]$.

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$$\begin{cases} \mathcal{L}_\lambda f := \varepsilon f + v \cdot \nabla_x f + \nu(v) f - \lambda K f = g, \\ f(x,v)|_{\gamma_-} = (1 - \frac{1}{n}) P_\gamma f + r(x,v). \end{cases}$$

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Claim: $\mathcal{S}_1 = \mathcal{L}_1^{-1}$ is well-defined.

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- **Step 3. Pass the limits $n \to \infty$ and then $\epsilon \to 0$.**

Thank you!