How Besov meets Boltzmann for well-posedness

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Outline

I. Introduction

II. Known results on Well-posedness

III. The goal of the talk

IV. The proof of the main result

I. Introduction

Boltzmann Equation

It was derived by L. Boltzmann in 1872 to govern motion of a nonequilibrium gas. Its unknown is a scalar function

$$0 \leq F = F(t, x, v), \quad t \in \mathbb{R}, x \in \mathbb{R}^3, v \in \mathbb{R}^3$$

which stands for the probability (or number, or mass) density function of gas particles having position x and velocity v at time t, and the Boltzmann equation reads

$$\frac{\partial F}{\partial t} + v \cdot \nabla_x F = Q(F, F)$$

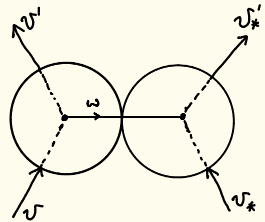
where Q, *the collision operator*, describes the binary collision of molecules and is given by

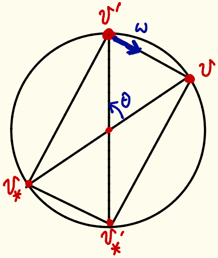
$$Q(F,H)(v) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\omega \underbrace{|(v-v_*)\cdot\omega|}_{\text{collision kernel}} \big(F(v_*')H(v') - F(v_*)H(v)\big),$$

$$v' = v - [(v-v_*)\cdot\omega]\omega, \quad v_*' = v_* + [(v-v_*)\cdot\omega]\omega,$$

Note:

$$v + v_* = v' + v_*',$$
$$|v|^2 + |v_*|^2 = |v'|^2 + |v_*'|^2.$$





The equilibrium state $(Q(F, F) = 0)$ is the Maxwellian distr.:

$$\mu = \mu_{[\rho,u,\theta]}(v) = \frac{\rho}{(2\pi\theta)^{3/2}} e^{-\frac{|v-u|^2}{2\theta}}$$

ρ : **density,** u : **bulk velocity,** θ**: temperature**

which can be built in the BE due to Boltzmann's

H-Theorem

$$H[F] = \int_{\mathbb{R}^3} F \log F \, dv$$

H-function (negative of *physical* **entropy)**

In fact, if $F = F(t, v)$ **is a solution to the BE in the** *spatially homogeneous* **setting, then**

$$\frac{dH[F](t)}{dt} = D[F](t) \leq 0,$$

for all $t \geq 0$**, where** $D[F]$ **is the** *entropy product* **given by**

$$D[F] = \int_{\mathbb{R}^3} Q(F,F) \ln F \, dv$$
$$= -\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} |(v-v_*)\cdot\omega|(F'F'_* - FF_*) \log \frac{F'F'_*}{FF_*} \, dv dv_* d\omega.$$

It holds that

$$Q(F,F) = 0$$
$$\Leftrightarrow D[F] = 0$$
$$\Leftrightarrow F \text{ is the Maxwellian:} \quad F = \mu_{[\rho,u,\theta]}(v)$$

Thus mathematically it shows that the *equilibrium state* **is uniquely described by the Maxwellian, not by any other distribution functions.**

One classical property of Q:

For $\psi(v) = 1, v_1, v_2, v_3, |v|^2$ (five collision invariants),

$$\int_{\mathbb{R}^3} \psi(v) Q(F,F)(v)\, dv = 0.$$

This gives the local macroscopic conservation laws:

$$\partial_t \int_{\mathbb{R}^3} F\, dv + \nabla_x \cdot \int_{\mathbb{R}^3} vF\, dv = 0,$$
$$\partial_t \int_{\mathbb{R}^3} v_i F\, dv + \nabla_x \cdot \int_{\mathbb{R}^3} v v_i F\, dv = 0, i = 1, 2, 3,$$
$$\partial_t \int_{\mathbb{R}^3} |v|^2 F\, dv + \nabla_x \cdot \int_{\mathbb{R}^3} |v|^2 vF\, dv = 0.$$

(Unclosed! They are Euler equations if $F = \mu_{[\rho,u,\theta]}$)

Relation to Fluid Dynamcs:

From

$$\partial_t F + v \cdot \nabla_x F = \frac{1}{\kappa} Q(F, F)$$

where $\kappa > 0$ is the Knudsen number proportional to the mean free path,

- 0^{th}**-order approximation by Hilbert expansion (1912) gives**

  Euler equations

- 1^{st}**-order approximation by Chapman-Enskog expansion (1916/17) gives**

  Navier-Stokes equations

Well-posedness of the Cauchy problem on the Boltzmann equation:

$$\partial_t F + v \cdot \nabla_x F = Q(F, F),$$
$$F|_{t=0} = F_0.$$

Goal: **Find function spaces** X **and** $Y \subset X$ **such that if** $0 \le F_0 \in X$ **with** $\|F_0\|_X < \infty$ **then it admits a (**$unique$**) "solution"** $0 \le F(t) \in Y$ **with**

$$\sup_{0<t<\infty} \|F(t)\|_Y < \infty.$$

Note: It is expected that the global-in-time existence and even large-time behavior of solutions be a consequence of the interplay between the free transport operator and the partially dissipative Boltzmann operator.

II. Known results on Well-posedness

Known results (Non-exhausting)**:**

- **Non-perturbation framework:**
  - **DiPerna-Lions (1989): The** *weak stability* **method gives the global existence of** *renormalized* **solutions for general** *large* **initial data with finite mass, energy and entropy. Uniqueness is still unknown.**
  - **Desvillettes-Villani (2005): the $t^{-\infty}$ convergence of a class of large amplitude solutions toward the global Maxwellian with an explicit almost exponential rate in large time,** *conditionally under* **some assumptions on smoothness and polynomial moment bounds of solutions:**

$$\|F(t)-\mu\|_{L^2_v H^N_x(1+|v|^k)} \leq C(\sup_{t\geq 0}\|F(t)\|_{L^2_v H^{N+\ell_s}_x(1+|v|^k)},\dots)t^{-s}$$

  - **Gualdani-Mischler-Mouhot (2010): a sharp exponential time rate by developing an abstract semigroup theory for linear operators which are non-symmetric in some Banach spaces:**

$$\|F(t)-\mu\|_X \leq Ce^{-\lambda t}\|F_0-\mu\|_X, \quad X := L^1_v L^\infty_v(1+|v|^k), k>2$$

Known results (Non-exhausting):

- **Perturbation framework (small data results):**
  - **Close-to-vacuum:**
    - **Illner-Shinbrot (1984): the global existence under smallness assumption on velocity weighted norms.**
  - **Close-to-global-Maxwellians:**
    - **Ukai (1974): existence and uniqueness of *mild* solutions based on the fixed point principle through the spectral analysis of linearized equation and the bootstrap argument.**
    - **Guo (2002, 2010), Liu-Yang-Yu (2004), Liu-Yang-Yu (2004): the existence and uniqueness of *classical or strong* solutions by the nonlinear energy method.**

      Recent progress in non-cutoff case:
      - **Alexandre-Morimoto-Ukai-Xu-Yang (AMUXY) ('10, '12)**
      - **Gressman-Strain ('11)**

Remark: Recall

- $L^2 \cap L^\infty$ **approach for global well-posedness (**even for IBVP**) with small data around global Maxwellians was developed by Guo (2010), and**
- **the only large-amplitude solution (**no uniquesness**) was given by DiPerna-Lions (1989).**

Q.: Is it possible to develop an in-between framework where the unique strong solution exists and is allowed to have large amplitude and thus contain vacuum?

A.: D.-Huang-Wang-Yang ('16, preprint**):** $\exists ! \, F$ **with**

$$\sup_{t\geq 0} \|(F-\mu)\mu^{-1/2}\|_{L^\infty_{x,v}(1+|v|^\beta)} < \infty$$

provided that the above norm is finite initially and

$$\mathcal{E}(F_0) + \|(F-\mu)\mu^{-1/2}\|_{L^1_x L^\infty_v} \ll 1.$$

III. The goal of the talk

Motivation:

- **In the close-to-equilibrium framework, $L^2_{x,v}$ is not enough to close the nonlinear dynamics, and in general**

$$X = L^2_v H^s_x, \quad s > 3/2$$

is needed, for instance Ukai's result.

- **Q.: Is $s = 3/2$ spatially critical for such function spaces to obtain the *global-in-time* well-posedness?**

Notice that the embedding $H^s(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$ is no longer true for $s = 3/2$. A replacement would be

$$B^{3/2}_{2,1}(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3).$$

Recall

$$\partial_t F + v \cdot \nabla_x F = Q(F, F).$$

Set

$$F = \mu + \mu^{1/2} f,$$
$$\mu = \mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}.$$

Then the Boltzmann equation can be reformulated as

$$\boxed{\partial_t f + v \cdot \nabla_x f + Lf = \Gamma(f, f),}$$

with initial data $f(0, x, v) = f_0(x, v)$ **given by** $F_0 = \mu + \mu^{1/2} f_0$**.**
Here

$$Lf = -\mu^{-1/2} [Q(\mu, \mu^{1/2} f) + Q(\mu^{1/2} f, \mu)],$$

$$\Gamma(f, g) = \mu^{-1/2} Q \left[\mu^{1/2} f, \mu^{1/2} g \right].$$

Moreover,

$$L = \nu - K,$$

$$\nu(v) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\omega \, |v - v_*|^\gamma B_0(\theta)\mu(v_*) \sim (1 + |v|)^\gamma,$$

K **is a self-adjoint compact operator on** L^2_v**.**

$$\ker L = \{1, v_1, v_2, v_3, |v|^2\}\sqrt{\mu}, \quad \dim \ker L = 5.$$

Define the macroscopic projection of $f(t, x, v)$ **by**

$$\mathbf{P}f = \left\{a(t,x) + v \cdot b(t,x) + \left(|v|^2 - 3\right) c(t,x)\right\} \sqrt{\mu}.$$

Then, the function $f(t, x, v)$ **can be decomposed as**

$$f = \mathbf{P}f + \{\mathbf{I} - \mathbf{P}\}\mathbf{f}.$$

L **is coercive in the sense that there is** $\lambda_0 > 0$ **such that**

$$\boxed{\int_{\mathbb{R}^3} f \, Lf \, dv \geq \lambda_0 \int_{\mathbb{R}^3} \nu(v) |\{\mathbf{I} - \mathbf{P}\}f|^2.}$$

Spaces and norms: **Define**

$$\mathcal{E}_T(f) \sim \boxed{\|f\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v B^{3/2}_x} = \sum_{q\geq -1} 2^{3q/2} \sup_{0\leq t\leq T} \|\Delta_q f(t,\cdot,\cdot)\|_{L^2_{x,v}}}$$

and

$$\mathcal{D}_T(f) = \|\nabla_x(a,b,c)\|_{\widetilde{L}^2_T B^{1/2}_x} + \|\{\mathbf{I}-\mathbf{P}\}f\|_{\widetilde{L}^2_T \widetilde{L}^2_{v,\nu} B^{3/2}_x},$$

where B^s_x **denotes the Besov space** $B^s_{2,1}(\mathbb{R}^3_x)$ **and**

$$\|f\|_{\widetilde{L}^{\varrho_1}_T \widetilde{L}^{\varrho_2}_v B^s_{p,r}} = \left(\sum_{q\geq -1} 2^{qsr} \left(\int_0^T \left(\int_{\mathbb{R}^3} \|\Delta_q f\|^{\varrho_2}_{L^p_x} dv\right)^{\varrho_1/\varrho_2} dt\right)^{r/\varrho_1}\right)^{\frac{1}{r}},$$

with the usual convention for $\varrho_1, \varrho_2, p, r = \infty$**.**

Main result:

Theorem

There is $\epsilon_0 > 0$ *and* $C > 0$ *such that if*

$$\|f_0\|_{\widetilde{L}^2_v B^{3/2}_x} \leq \epsilon_0,$$

then there is a unique global strong solution $f(t,x,v)$ *to the Boltzmann equation with initial data* $f|_{t=0} = f_0$*, satisfying*

$$\mathcal{E}_T(f) + \mathcal{D}_T(f) \leq C\|f_0\|_{\widetilde{L}^2_v B^{3/2}_x},$$

for any $T > 0$*. Moreover, if* $f_0 \geq 0$ *then* $F(t) \geq 0$ *for all positive time.*

Previous solution spaces for well-posedness in perturbation regime:

- **The first global existence theorem for the mild solution is given by Ukai ('74) in the space**

$$L^\infty\Big(0,\infty; L^\infty_\beta(\mathbb{R}^3_v; H^N(\mathbb{R}^3_x))\Big), \quad \beta > \frac{5}{2},\ N \geq 2,$$

by using the spectrum method as well as the contraction mapping principle, see also Nishida-Imai ('76/'77) and Kawashima ('90). Here $L^\infty_\beta(\mathbb{R}^3_v)$ denotes a space of all functions f with $(1+|v|)^\beta f$ uniformly bounded.

- **Using a similar approach, Shizuta ('83) obtains the global existence of the classical solution** $f(t,x,v) \in C^{1,1,0}((0,\infty)\times\mathbb{T}_x^3\times\mathbb{R}_v^3)$ **on torus, with the uniform bound in the space**

$$L^\infty\Big(0,\infty; L^\infty_\beta(\mathbb{R}^3_v; C^s(\mathbb{T}^3_x))\Big), \quad \beta > \frac{5}{2},\ s > \frac{3}{2}.$$

- **The spectrum method was later improved in Ukai-Yang ('06) for the existence of the mild solution in the space**

$$L^\infty\Big(0,\infty; L^2(\mathbb{R}^3_x\times\mathbb{R}^3_v)\cap L^\infty_\beta(\mathbb{R}^3_v; L^\infty(\mathbb{R}^3_x))\Big), \quad \beta > \frac{3}{2},$$

without any regularity conditions, where some L^∞-L^2 **estimates in terms of the Duhamel's principle are developed.**

- **On the other hand, by means of the robust energy method ('02), for instance, Guo, Liu-Yu and Liu-Yang-Yu, the well-posedness of classical solutions is also established in the space**

$$C\Big(0,\infty; H^N_{t,x,v}(\mathbb{R}^3_x \times \mathbb{R}^3_v)\Big), \quad N \geq 4,$$

where the Sobolev space $H^N_{t,x,v}(\mathbb{R}^3_x \times \mathbb{R}^3_v)$ denotes a set of all functions whose derivatives with respect to all variables t, x and v up to N order are integrable in $L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v)$.

- **If only the strong solution with the uniqueness property is considered then the time differentiation can be disregarded in the above Sobolev space. Indeed D. ('08) obtained such strong solution in the space**

$$C\Big(0,\infty; L^2(\mathbb{R}^3_v; H^N(\mathbb{R}^3_x))\Big), \quad N \geq 2.$$

- **AMUXY ('13) presents a result for local existence in a "larger" function space. In cutoff case the solution space may take**

$$L^\infty\Big(0, T_0; L^2(\mathbb{R}^3_v; H^s(\mathbb{R}^3_x))\Big), \quad s > \frac{3}{2},$$

where $T_0 > 0$ is a finite time.

- **The key motivation to consider $s = \frac{3}{2}$ is to apply the** $\mathrm{Chemin\text{-}Lerner}$ **space which has been extensively used to deal with the incompressible Navier-Stokes equations:**

$$u \in \widetilde{L}^\infty_T B^s_x, \ i.e. \sum_{q \geq -1} 2^{qs} \sup_{0 \leq t \leq T} \|\Delta_q u(t)\|_{L^2_x} < \infty.$$

Further recent progress:

- Yoshinori Morimoto, Shota Sakamoto, Global solutions in the critical Besov space for the non cutoff Boltzmann equation, arXiv:1512.00585.
- Zhengrong Liu, Hao Tang, On the Cauchy problem for the Boltzmann equation in Chemin-Lerner type spaces, DCDS 2015.

IV. The proof of the main result

Key points in the a priori estimates:

- **(K1) How does Besov meet Boltzmann?**

$(\partial_t + v\cdot\nabla_x + L)f = \Gamma(f,f)$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}2^{3q}\|\Delta_q f\|^2_{L^2_vL^2_x} + \lambda_0 2^{3q}\|\Delta_q\{\mathbf{I}-\mathbf{P}\}f\|^2_{L^2_{v,\nu}L^2_x} \le 2^{3q}|(\Delta_q\Gamma(f,f),\Delta_q\{\mathbf{I}-\mathbf{P}\}f)|.$$

$$\Rightarrow 2^{\frac{3q}{2}}\|\Delta_q f(t)\|_{L^2_vL^2_x} + \sqrt{\lambda_0}2^{\frac{3q}{2}}\left(\int_0^t \|\Delta_q\{\mathbf{I}-\mathbf{P}\}f\|^2_{L^2_{v,\nu}L^2_x}d\tau\right)^{1/2}$$

$$\le 2^{\frac{3q}{2}}\|\Delta_q f_0\|_{L^2_vL^2_x} + 2^{\frac{3q}{2}}\left(\int_0^t |(\Delta_q\Gamma(f,f),\Delta_q\{\mathbf{I}-\mathbf{P}\}f)|d\tau\right)^{1/2},$$

$$\Rightarrow \boxed{\sum_{q\ge -1} 2^{\frac{3q}{2}}\sup_{0\le t\le T}\|\Delta_q f(t)\|_{L^2_vL^2_x}}$$

$$+\sqrt{\lambda_0}\boxed{\sum_{q\ge -1} 2^{\frac{3q}{2}}\left(\int_0^T \|\Delta_q\{\mathbf{I}-\mathbf{P}\}f\|^2_{L^2_{v,\nu}L^2_x}dt\right)^{1/2}} \le \sum_{q\ge -1} 2^{\frac{3q}{2}}\|\Delta_q f_0\|_{L^2_vL^2_x}$$

$$+\boxed{\sum_{q\ge -1} 2^{\frac{3q}{2}}\left(\int_0^T |(\Delta_q\Gamma(f,f),\Delta_q\{\mathbf{I}-\mathbf{P}\}f)|dt\right)^{1/2}}.$$

- **(K2) Most key estimate: Trilinear estimate**

Lemma

Assume $s > 0$, $0 \le T \le +\infty$, it holds that

$$\sum_{q\geq -1} 2^{qs}\left[\int_0^T |(\Delta_q\Gamma(f,g),\Delta_q h)|\,dt\right]^{1/2} \lesssim \|h\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_{v,\nu}(B^s_x)}$$
$$\times\left[\|g\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_{v,\nu}(B^s_x)}\|f\|^{1/2}_{L^\infty_T L^2_v L^\infty_x} + \|f\|^{1/2}_{L^2_T L^2_{v,\nu} L^\infty_x}\|g\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^s_x)}\right.$$
$$\left. + \|f\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_{v,\nu}(B^s_x)}\|g\|^{1/2}_{L^\infty_T L^2_v L^\infty_x} + \|g\|^{1/2}_{L^2_T L^2_{v,\nu} L^\infty_x}\|f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^s_x)}\right].$$

Recall

$$
\begin{aligned}
\Gamma(f,g) =& \mu^{-1/2}(v) Q\left[\mu^{1/2} f, \mu^{1/2} g\right] \\
=& \Gamma_{gain}(f,g) - \Gamma_{loss}(f,g) \\
=& \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\omega \, |v-v_*|^\gamma B_0(\theta) \mu^{1/2}(v_*) f(v_*') g(v') \\
& - g(v) \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\omega \, |v-v_*|^\gamma B_0(\theta) \mu^{1/2}(v_*) f(v_*).
\end{aligned}
$$

- **Proof of Trilinear Estimate-1:**

$$I_0 = \boxed{\sum_{q\geq -1} 2^{qs}\left[\int_0^T |(\Delta_q\Gamma(f,g),\Delta_q h)|\,dt\right]^{1/2}}$$

- **Elementary observation-1: ($(A+B)^{1/2}\leq A^{1/2}+B^{1/2}$)**

$$\left[\int_0^T |(\Delta_q\Gamma(f,g),\Delta_q h)|\,dt\right]^{\frac{1}{2}}$$

$$\leq \left[\int_0^T |(\Delta_q\Gamma_{gain}(f,g),\Delta_q h)|\,dt\right]^{\frac{1}{2}} + \left[\int_0^T |(\Delta_q\Gamma_{loss}(f,g),\Delta_q h)|\,dt\right]^{\frac{1}{2}}$$

with

$$\Delta_q\Gamma_{gain}(f,g) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\omega\,|v-v_*|^{\gamma} B_0(\theta)\mu^{1/2}(v_*)\Delta_q[f(v'_*)g(v')],$$

$$\Delta_q\Gamma_{loss}(f,g) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\omega\,|v-v_*|^{\gamma} B_0(\theta)\mu^{1/2}(v_*)\Delta_q[f(v_*)g(v)].$$

- **Proof of Trilinear Estimate-2:**

$$I_0 = \boxed{\sum_{q \geq -1} 2^{qs} \left[\int_0^T |(\Delta_q \Gamma(f,g), \Delta_q h)| \, dt \right]^{1/2}}$$

- **Elementary observation-2:**

$$I_0 \lesssim \sum_{q \geq -1} 2^{qs} \left[\left(\int_0^T dt \int_{\mathbb{R}^9 \times \mathbb{S}^2} dx dv dv_* d\omega \, |v' - v'_*|^\gamma \mu^{1/2}(v'_*) \, |\Delta_q [f_* g]|^2 \right)^{1/2} \right]^{1/2}$$

$$\times \left[\left(\int_0^T dt \int_{\mathbb{R}^9} dx dv dv_* d\omega \, |v - v_*|^\gamma \mu^{1/2}(v_*) \, |\Delta_q h|^2 \right)^{1/2} \right]^{1/2}$$

$$+ \sum_{q \geq -1} 2^{qs} \left[\left(\int_0^T dt \int_{\mathbb{R}^9 \times \mathbb{S}^2} dx dv dv_* d\omega \, |v - v_*|^\gamma \mu^{1/2}(v_*) \, |\Delta_q [f_* g]|^2 \right)^{1/2} \right]^{1/2}$$

$$\times \left[\left(\int_0^T dt \int_{\mathbb{R}^9} dx dv dv_* d\omega \, |v - v_*|^\gamma \mu^{1/2}(v_*) \, |\Delta_q h|^2 \right)^{1/2} \right]^{1/2}$$

Further by using the discrete version of Cauchy-Schwarz inequality to two summations $\sum_{q\geq -1}$ above, one obtains that

$$
\begin{aligned}
I_0 \lesssim & \left[\sum_{q\geq -1} 2^{qs}\left(\int_0^T dt\int_{\mathbb{R}^9} dxdvdv_*\,|v-v_*|^{\gamma}\,|\Delta_q[f_*g]|^2\right)^{1/2}\right]^{1/2}\\
& \times\left[\sum_{q\geq -1} 2^{qs}\left(\int_0^T dt\int_{\mathbb{R}^9} dxdvdv_*\,|v-v_*|^{\gamma}\mu^{1/2}(v_*)\,|\Delta_q h|^2\right)^{1/2}\right]^{1/2}\\
& := I^{1/2}\times II^{1/2},
\end{aligned}
$$

where $|v'-v'_*|=|v-v_*|$, $\mu^{1/2}(v'_*)\leq 1$ and $\int_{\mathbb{S}^2} d\omega = 4\pi$ have been used.

It is straightforward to see

$$II \leq \|h\|_{\widetilde{L}^2_T \widetilde{L}^2_{v,\nu}(B^s_x)}$$

due to

$$\int_{\mathbb{R}^3} dv_* \, |v - v_*|^\gamma \mu^{1/2}(v_*) \sim (1 + |v|)^\gamma \sim \nu(v).$$

It remains to estimate

$$I = \sum_{q \geq -1} 2^{qs} \left(\int_0^T dt \int_{\mathbb{R}^9} dx dv dv_* \, |v - v_*|^\gamma \, |\Delta_q [f_* g]|^2 \right)^{1/2}$$

- **Proof of Trilinear Estimate-3:**

Idea for estimating I:

- **Bony's decomposition:**

$$f_* g = \mathcal{T}_{f_*} g + \mathcal{T}_g f_* + \mathcal{R}(f_*, g),$$

with

$$\mathcal{T}_u v = \sum_j S_{j-1} u \Delta_j v, \quad \mathcal{R}(u, v) = \sum_{|j'-j| \le 1} \Delta_{j'} u \Delta_j v.$$

- **Use the basic properties: for $1 \le p \le \infty$,**

$$\|\Delta_q \cdot \|_{L^p_x} \le C \| \cdot \|_{L^p_x}, \quad \|S_q \cdot \|_{L^p_x} \le C \| \cdot \|_{L^p_x}.$$

- **Obtain paraproduct property by defining a new ℓ^1 sequence. For instance, we define**

$$c_1(j) = 2^{js} \left(\int_0^T dt \int_{\mathbb{R}^3} |v|^\gamma \, \|\Delta_j g\|^2_{L^2_x} \, dv \right)^{1/2} / \|g\|_{\widetilde{L}^2_T \widetilde{L}^2_{v,\nu} B^s_x}.$$

One can see that $\|c_1(j)\|_{\ell^1} = 1$.

Indeed,

$$
\begin{aligned}
I \leq & \sum_{q\geq -1} 2^{qs} \left(\int_0^T dt \int_{\mathbb{R}^9} dxdvdv_* |v-v_*|^\gamma \left| \sum_j \Delta_q [S_{j-1} f_* \Delta_j g] \right|^2 \right)^{1/2} \\
& + \sum_{q\geq -1} 2^{qs} \left(\int_0^T dt \int_{\mathbb{R}^9} dxdvdv_* |v-v_*|^\gamma \left| \sum_j \Delta_q [S_{j-1} g \Delta_j f_*] \right|^2 \right)^{1/2} \\
& + \sum_{q\geq -1} 2^{qs} \left(\int_0^T dt \int_{\mathbb{R}^9} dxdvdv_* |v-v_*|^\gamma \left| \sum_{|j-j'|\leq 1} \Delta_q [\Delta_j f_* \Delta_{j'} g] \right|^2 \right)^{1/2} \\
& := I_1 + I_2 + I_3.
\end{aligned}
$$

For instance, to estimate I_1, notice that

$$
\sum_j \Delta_q [S_{j-1} f_* \Delta_j g] = \sum_{|j-q|\leq 4} \Delta_q [S_{j-1} f_* \Delta_j g].
$$

By $|v-v_*|^\gamma \le |v|^\gamma + |v_*|^\gamma$ **and Minkowski's inequality,**

$$\begin{aligned} I_1 \le & \sum_{q\ge -1}\sum_{|j-q|\le 4} 2^{qs}\left(\int_0^T dt \int_{\mathbb{R}^9} dx dv dv_* \, |v|^\gamma \left|\Delta_q[S_{j-1}f_*\Delta_j g]\right|^2\right)^{1/2} \\ & + \sum_{q\ge -1}\sum_{|j-q|\le 4} 2^{qs}\left(\int_0^T dt \int_{\mathbb{R}^9} dx dv dv_* \, |v_*|^\gamma \left|\Delta_q[S_{j-1}f_*\Delta_j g]\right|^2\right)^{1/2} \\ & := I_{1,1} + I_{1,2}. \end{aligned}$$

Here $I_{1,1}$ is bounded as

$$\begin{aligned} I_{1,1} &\leq \sum_{q\geq -1}\sum_{|j-q|\leq 4} 2^{qs}\left(\int_0^T dt\int_{\mathbb{R}^3}\|f_*\|^2_{L^\infty_x}dv_*\int_{\mathbb{R}^3}|v|^\gamma\|\Delta_j g\|^2_{L^2_x}\,dv\right)^{1/2} \\ &\leq \sum_{q\geq -1}\sum_{|j-q|\leq 4} 2^{qs}\left(\sup_{0\leq t\leq T}\int_{\mathbb{R}^3}\|f_*\|^2_{L^\infty_x}dv_*\int_0^T dt\int_{\mathbb{R}^3}|v|^\gamma\|\Delta_j g\|^2_{L^2_x}\,dv\right)^{1/2} \\ &\leq \sum_{q\geq -1}\sum_{|j-q|\leq 4} 2^{qs}\left(\int_0^T dt\int_{\mathbb{R}^3}|v|^\gamma\|\Delta_j g\|^2_{L^2_x}\,dv\right)^{1/2}\|f\|_{L^\infty_T L^2_v L^\infty_x} \\ &\leq \sum_{q\geq -1}\sum_{|j-q|\leq 4} 2^{(q-j)s}c_1(j)\|g\|_{\widetilde{L}^2_T\widetilde{L}^2_{v,\nu}(B^s_x)}\|f\|_{L^\infty_T L^2_v L^\infty_x}, \end{aligned}$$

where $c_1(j)$ is defined as

$$c_1(j) = \frac{2^{js}\left(\int_0^T dt\int_{\mathbb{R}^3}|v|^\gamma\|\Delta_j g\|^2_{L^2_x}\,dv\right)^{1/2}}{\|g\|_{\widetilde{L}^2_T\widetilde{L}^2_{v,\nu}(B^s_x)}}.$$

Using the convolution inequality for series

$$\sum_{q\geq -1}\sum_{|j-q|\leq 4} 2^{(q-j)s} c_1(j) = \sum_{q\geq -1} \left[\left(\mathbf{1}_{|j|\leq 4} 2^{js} \right) * c_1(j) \right] (q)$$
$$\leq \|\mathbf{1}_{|j|\leq 4} 2^{js}\|_{\ell^1} \|c_1(j)\|_{\ell^1} < +\infty,$$

we further get that

$$I_{1,1} \lesssim \|g\|_{\widetilde{L}^2_T \widetilde{L}^2_{v,\nu}(B^s_x)} \|f\|_{L^\infty_T L^2_v L^\infty_x}.$$

Other terms can be estimated in a similar way. □

Remark: The other two occasions where Besov meets Boltzmann

- **Arsénio-Masmoudi (JMPA '13): Velocity averaging lemmas.**
- **Sohinger-Strain (AM, '14): Time-decay rate for $f_0 \in B^s_{2,\infty} L^2_v$ with some $s < 0$.**
- **Bedrossian-Masmoudi-Mouhot (arXiv, '13): a simpler proof of nonlinear Landau damping for Gevrey data through the Bony decomposition.**

- **(K3) Applying the Trilinear Estimate to**

$$\Gamma(f,f) = \Gamma(\mathbf{P}f,\mathbf{P}f) + \Gamma(\mathbf{P}f,\{\mathbf{I}-\mathbf{P}\}f) + \Gamma(\{\mathbf{I}-\mathbf{P}\}f,\mathbf{P}f) + \Gamma(\{\mathbf{I}-\mathbf{P}\}f,\{\mathbf{I}-\mathbf{P}\}f),$$

together with

$$\|\mathbf{P}f\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v\dot{B}^{3/2}_x} \lesssim \|(a,b,c)\|^{1/2}_{\widetilde{L}^2_T\dot{B}^{3/2}_x} \sim \|\nabla_x(a,b,c)\|^{1/2}_{\widetilde{L}^2_T\dot{B}^{1/2}_x} \lesssim \|\nabla_x(a,b,c)\|^{1/2}_{\widetilde{L}^2_T B^{1/2}_x} \lesssim \sqrt{\mathcal{D}_T(f)},$$

$$B^{3/2}_x \subset L^\infty_x, \quad \dot{B}^{3/2}_x \subset L^\infty_x,$$

one has

$$\boxed{\|f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_x)} + \sqrt{\lambda_0}\|\{\mathbf{I}-\mathbf{P}\}f\|_{\widetilde{L}^2_T\widetilde{L}^2_{v,\nu}(B^{3/2}_x)}} \lesssim \|f_0\|_{\widetilde{L}^2_v(B^{3/2}_x)} + \sqrt{\mathcal{E}_T(f)}\mathcal{D}_T(f).$$

• **(K4) Macroscopic Dissipation: The macroscopic component** $\mathbf{P}f = \left\{a(t,x) + v\cdot b(t,x) + \left(|v|^2-3\right)c(t,x)\right\}\sqrt{\mu}$ **satisfies the fluid-type system**

$$\begin{cases} \partial_t a + \nabla_x \cdot b = 0, \\ \partial_t b + \nabla_x (a+2c) + \nabla_x \cdot \Theta(\{\mathbf{I}-\mathbf{P}\}f) = 0, \\ \partial_t c + \dfrac{1}{3}\nabla_x \cdot b + \dfrac{1}{6}\nabla_x \cdot \Lambda(\{\mathbf{I}-\mathbf{P}\}f) = 0, \\ \partial_t[\Theta_{im}(\{\mathbf{I}-\mathbf{P}\}f) + 2c\delta_{im}] + \partial_i b_m + \partial_m b_i = \Theta_{im}(\mathbb{r}+\mathbb{h}), \\ \partial_t \Lambda_i(\{\mathbf{I}-\mathbf{P}\}f) + \partial_i c = \Lambda_i(\mathbb{r}+\mathbb{h}), \end{cases}$$

where

$$\Theta_{im}(f) = \left((v_i v_m - 1)\mu^{1/2}, f\right), \quad \Lambda_i(f) = \frac{1}{10}\left((|v|^2-5)v_i\mu^{1/2}, f\right).$$

The energy estimate in Besov space gives

$$\|\nabla_x(a,b,c)\|_{\widetilde{L}^2_T B^{1/2}_x} \lesssim \|f_0\|_{\widetilde{L}^2_v B^{3/2}_x} + \mathcal{E}_T(f) + \|\{\mathbf{I}-\mathbf{P}\}f\|_{\widetilde{L}^2_T \widetilde{L}^2_{v,\nu} B^{3/2}_x} + \mathcal{E}_T(f)\mathcal{D}_T(f),$$

• The global a priori estimate follows from the linear combination:

$$\begin{aligned}\|f\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_x)} &- \kappa_3 \mathcal{E}_T(f) \\ &+ \lambda \left\{ \|\nabla_x(a,b,c)\|_{\widetilde{L}^2_T(B^{1/2}_x)} + \|\{\mathbf{I}-\mathbf{P}\}f\|_{\widetilde{L}^2_T \widetilde{L}^2_{v,\nu}(B^{3/2}_x)} \right\} \\ &\lesssim \|f_0\|_{\widetilde{L}^2_v(B^{3/2}_x)} + \left\{ \sqrt{\mathcal{E}_T(f)} + \mathcal{E}_T(f) \right\} \mathcal{D}_T(f),\end{aligned}$$

with

$$\|f\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_x)} - \kappa_3 \mathcal{E}_T(f) \sim \mathcal{E}_T(f),$$

since $\kappa_3 > 0$ can be small enough.

Local existence: **The construction of the local solution is based on**

$$\begin{cases} \{\partial_t + v \cdot \nabla_x\} F^{n+1} + F^{n+1}(v) \displaystyle\int_{\mathbb{R}^3 \times \mathbb{S}^2} |v - v_*|^\gamma B_0(\theta) F^n(v_*)\, dv_* d\omega \\ \qquad = \displaystyle\int_{\mathbb{R}^3 \times \mathbb{S}^2} |v - v_*|^\gamma B_0(\theta) F^n(v'_*) F^n(v')\, dv_* d\omega, \\ F^{n+1}(0, x, v) = F_0(x, v), \end{cases}$$

starting with $F^0(t, x, v) = F_0(x, v)$**.**

Noticing that $F^{n+1} = \mu + \mu^{1/2} f^{n+1}$**, equivalently we need to solve** f^{n+1} **such that**

$$\{\partial_t + v \cdot \nabla_x + \nu\} f^{n+1} - K f^n = \Gamma_{gain}(f^n, f^n) - \Gamma_{loss}(f^n, f^{n+1}),$$
$$f^{n+1}(0, x, v) = f_0(x, v).$$

Lemma

The solution sequence $\{f^n\}_{n=1}^{\infty}$ *is well defined. For a sufficiently small constant* $M_0 > 0$, *there exists* $T^* = T^*(M_0) > 0$ *such that if*

$$\|f_0\|_{\widetilde{L}^2_v B^{3/2}_x} \leq M_0,$$

then for any n, *it holds that*

$$\widetilde{Y}_T(f^n) := \mathcal{E}_T(f^n) + \widetilde{\mathcal{D}}_T(f^n) \leq 2M_0, \quad \forall\, T \in [0, T^*),$$

where $\widetilde{\mathcal{D}}_T(f)$ *is defined by* $\widetilde{\mathcal{D}}_T(f) = \|f\|_{\widetilde{L}^2_T \widetilde{L}^2_{v,\nu} B^{3/2}_x}$.

Theorem

For a sufficiently small $M_0 > 0$*, there exists* $T^* = T^*(M_0) > 0$ *such that if*

$$\|f_0\|_{\widetilde{L}^2_v B^{3/2}_x} \leq M_0,$$

then there is a unique strong solution $f(t,x,v)$ *to the Boltzmann equation in* $(0,T^*) \times \mathbb{R}^3_x \times \mathbb{R}^3_v$*, such that*

$$\widetilde{Y}_T(f) \leq 2M_0,$$

for any $T \in [0,T^*)$*. Moreover* $\widetilde{Y}_T(f)$ *is continuous in* T *over* $[0,T^*)$*, and if* $F_0(x,v) = \mu + \mu^{1/2} f_0 \geq 0$*, then*

$$F(t,x,v) = \mu + \mu^{1/2} f(t,x,v) \geq 0$$

holds true.

Key points of the local existence:

- **Energy estimate for uniform bound of sequence of approximate solutions.**
- **Energy estimate for uniqueness of solutions in the space.**
- **A new inequality is observed to prove the continuity of the super-time norm $\widetilde{Y}_T(f)$:**
  - **Show $t \mapsto \mathcal{E}(f(t)) := \sum_{q \geq -1} 2^{\frac{3q}{2}} \|\Delta_q f(t)\|_{L^2_{x,v}}$ is continuous on $[0, T^*)$, in terms of**

$$|\mathcal{E}(f(t_2)) - \mathcal{E}(f(t_1))| \lesssim (\sqrt{M_0} + 1) \sum_{q \geq -1} 2^{\frac{3q}{2}} \left(\int_{t_1}^{t_2} \|\Delta_q f\|^2_{L^2_{v,\nu} L^2_x} dt \right)^{1/2}.$$

  - **Show $T \mapsto \widetilde{Y}_T(f)$ is continuous.** □

Thank you!