Time-periodic solutions of the Vlasov-Poisson-Fokker-Planck system

Renjun Duan

Department of Mathematics, The Chinese University of Hong Kong

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I. Problem and main result

Consider the Vlasov-Poisson-Fokker-Planck (VPFP) system

$$\partial_t F + \xi \cdot \nabla_x F + \nabla_x \Phi \cdot \nabla_\xi F = \nabla_\xi \cdot (\nabla_\xi F + \xi F),$$
$$\Delta_x \Phi = \int_{\mathbb{R}^3} F \, d\xi - \rho(t,x),$$

where

- **the unknown is** $F(t,x,\xi) \geq 0$ **for** $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, **and** $t \in \mathbb{R}$;
- $\Phi = \Phi(t,x)$ **is the self-consistent potential satisfying**

$$\lim_{|x| \to \infty} \Phi(t,x) = 0;$$

- **the background profile** $\rho(t,x)$ **is** T**-periodic in time for** $T \geq 0$.

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Whether a T-periodic driving force $\rho(t,x)$ is able to produce a time-periodic solution with the same period T?

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Define

$$\phi(t,x) = (-\Delta_x)^{-1}\big(\rho(t,x) - 1\big).$$

The above VPFP system can be also written as

$$\partial_t F + \xi\cdot\nabla_x F + \nabla_x(\Phi+\phi)\cdot\nabla_\xi F = \nabla_\xi\cdot(\nabla_\xi F + \xi F),$$
$$\Delta_x \Phi = \int_{\mathbb{R}^3} F\,d\xi - 1.$$

Define $M = (2\pi)^{-3/2}\exp\{-|\xi|^2/2\}$**, and set** $f = f(t,x,\xi)$ **by** $F = M + M^{1/2} f$**. Then,**

$$\partial_t f + \xi\cdot\nabla_x f + \nabla_x(\Phi+\phi)\cdot\nabla_\xi f$$
$$-\frac{1}{2}\xi\cdot\nabla_x(\Phi+\phi) f - \xi M^{1/2}\cdot\nabla_x(\Phi+\phi) = Lf,$$
$$\Delta_x\Phi = \int_{\mathbb{R}^3} M^{1/2} f\,d\xi.$$

Here

$$Lf = \frac{1}{M^{1/2}}\nabla_\xi\cdot\left[M\nabla_\xi\left(\frac{f}{M^{1/2}}\right)\right].$$

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$$\begin{aligned} &\partial_t f + \xi \cdot \nabla_x f + \nabla_x(\Phi + \phi) \cdot \nabla_\xi f \\ &\qquad - \frac{1}{2}\xi \cdot \nabla_x(\Phi + \phi) f - \xi M^{1/2} \cdot \nabla_x(\Phi + \phi) = Lf, \\ &\Delta_x \Phi = \int_{\mathbb{R}^3} M^{1/2} f \, d\xi. \end{aligned}$$

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We introduce the function space

$$X = \{f = f(x,\xi) \in L^2_\xi(H^3_x) : \|f\|_X < \infty, M + M^{1/2} f \geq 0, \iint_{\mathbb{R}^3\times\mathbb{R}^3} M^{1/2} f(x,\xi)\, d\xi dx = 0\}$$

with the norm $\|\cdot\|_X$ **defined by**

$$\|f\|_X^2 = \|f\|^2_{L^2_\xi(H^3_x)} + \|\nabla_x \Phi^f\|^2_{H^3_x}.$$

Here and in the sequel, for given $f(t,x,\xi)$**,** $\Phi^f = \Phi^f(t,x)$ **denotes**

$$\Phi^f(t,x) = -\frac{1}{4\pi}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{M^{1/2} f(t,y,\xi)}{|x-y|}\, d\xi dy.$$

Theorem (D.-Liu, 2015)

Assume that $\phi(t,x)$ is time-periodic with period $T>0$. There are $\epsilon>0$, $C>0$ such that if

$$\sup_{0\leq t\leq T}\|\nabla_x\phi(t)\|_{H^3_x}\leq \epsilon$$

then the reformulated VPFP system admits a unique time-periodic solution $f(t,x,\xi)\in X$ with the same period T and

$$\sup_{0\leq t\leq T}\|f(t)\|_X\leq C\sup_{0\leq t\leq T}\|\nabla_x\phi(t)\|_{H^3_x}.$$

II. Motivation and previous related work

- **Ukai (2006): For the Boltzmann equation**

$$\partial_t F + \xi \cdot \nabla_x F = Q(F,F) + S(t,x,\xi),$$

a small, T-periodic-in-time, microscopic, inhomogeneous source can induce a unique T-periodic mild solution with the time-period T.

Two key points in his proof:

- **obtain the extra time-decay of the semigroup e^{tB} for $B = L - \xi \cdot \nabla_x$;**
- **find the solution by establishing the contraction property of the mapping**

$$\Psi[f](t) = \int_{-\infty}^{t} e^{(t-s)B} N[f,S](s)\, ds,$$

in an appropriate function space.

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- **D.-Ukai-Yang-Zhao (2008):**

$$\partial_t F + \xi \cdot \nabla_x F + E(t,x) \cdot \nabla_\xi F = Q(F,F)$$

Q: Can the T-periodic external force $E(t,x)$ induce a time-periodic solution $F(t,x,\xi)$ with the same time-period?

A:
- **Yes if $n \geq 5$,**

Proof:
(i) Obtain the optimal time-decay estimates on the linearised equation;
(ii) Find the fixed point for certain nonlinear mapping Ψ:

$$\Psi[f](t) = \int_{-\infty}^{t} U_E(t,s) N[f,E](s) ds, \quad \forall t \in \mathbb{R}.$$

(Well-defined in case $n \geq 5$, as $U_E(t,s) \lesssim (1+t-s)^{-\frac{n}{4}}$)

- **Open for $1 \leq n \leq 4$, in particular, $n = 3$ (Physical).**

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- **DUYZ's result can also be directly applied to the Vlasov-Poisson-Boltzmann system with a T-periodic background profile $\rho(t,x)$ but still in case $n \geq 5$.**

For $n = 3$,

- Guo (2002): global solution around global Maxwellians in case when $\rho(t,x) \equiv 1$.
- D.-Yang (2009): global solution around a stationary local Maxwellian $e^{-\phi(x)-|\xi|^2/2}$ in case when $\rho(t,x) = \rho(x)$, independent of time, is sufficiently smooth and close to a positive constant.
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III. Proof

The proof is based on the Serrin's approach (ARMA, 1959). The key point is to solve the Cauchy problem in the following way:

Consider

$$\begin{cases} u_t = Au + f(t), \quad t > 0, \\ u|_{t=0} = u_0 \in X \supset Y. \end{cases}$$

The following theorem should be investigated.

Theorem. Denote by $f(t) \in Z$ the driving term and $u(t) \in Y$ a solution to the Cauchy problem with initial data $u \in X \supset Y$ where linear or nonlinear cases are included; X, Y and Z Banach spaces, with norms $\|\cdot\|_X$, $\|\cdot\|_Y$ and $\|\cdot\|_Z$ respectively. Furthermore,

$$\text{if } \sup_{t\geq 0} \|f(t)\|_Z < \infty, \text{ then } \sup_{t\geq 0} \|u(t)\|_Y < \infty.$$

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III.1 Cauchy problem

First consider the Cauchy problem on the reformulated VPFP system over $t > 0$, supplemented with initial data

$$f(0,x,\xi) = f_0(x,\xi).$$

Theorem

Assume that $f_0 \in X$, $\nabla_x \phi \in C(0,\infty; H_x^3)$ with

$$\|f_0\|_X + \sup_{t \geq 0} \|\nabla_x \phi(t)\|_{H_x^3}$$

sufficiently small. Then the Cauchy problem on the VPFP system admits a unique solution $f(t,x,\xi) \in X$ with

$$\sup_{t\geq 0} \|f(t)\|_X \leq C \left(\|f_0\|_X + \sup_{t\geq 0} \|\nabla_x \phi(t)\|_{H_x^3} \right).$$

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Theorem

Assume that $f_0 \in X$, $\nabla_x \phi \in C(0, \infty; H_x^3)$ *with*

$$\|f_0\|_X + \sup_{t \geq 0} \|\nabla_x \phi(t)\|_{H_x^3}$$

sufficiently small. Then the Cauchy problem on the VPFP system admits a unique solution $f(t, x, \xi) \in X$ *with*

$$\sup_{t \geq 0} \|f(t)\|_X \leq C \left(\|f_0\|_X + \sup_{t \geq 0} \|\nabla_x \phi(t)\|_{H_x^3} \right).$$

- Let $\sigma(\xi) = 1 + |\xi|^2$. Denote $|\cdot|_\sigma$ by

$$|f|_\sigma^2 = \int_{\mathbb{R}^3} \left[|\nabla_\xi f|^2 + \sigma(\xi)|f|^2 \right] d\xi, \quad f = f(\xi).$$

  For $f = f(x, \xi)$, $\|f\|_\sigma^2$ stands for the spatial integration of $|f(x, \cdot)|_\sigma^2$ over $\mathbb{R}^3$.
- Recall there is $\lambda_0 > 0$ such that

$$-\int_{\mathbb{R}^3} f L f \, d\xi \geq \lambda_0 |\{\mathbf{I} - \mathbf{P}_0\} f|_\sigma^2,$$

  where $\mathbf{P}_0 f = a^f M^{1/2}$, and $a^f(t,x) = \int_{\mathbb{R}^3} M^{1/2} f(t,x,\xi)\, d\xi$.
- We also introduce the velocity orthogonal projection $\mathbf{P} : L_\xi^2 \to \mathrm{span}\{M^{1/2}, \xi M^{1/2}\}$ by $\mathbf{P} = \mathbf{P}_0 \oplus \mathbf{P}_1$ with $\mathbf{P}_1 f = b^f \cdot \xi M^{1/2}$ and $b^f(t,x) = \int_{\mathbb{R}^3} \xi M^{1/2} f(t,x,\xi)\, d\xi$.

- Let $\sigma(\xi) = 1 + |\xi|^2$. Denote $|\cdot|_\sigma$ by

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- **Let** $\sigma(\xi) = 1 + |\xi|^2$**. Denote** $|\cdot|_\sigma$ **by**

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- *Zero-order estimate:*

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|f\|^2+\|\nabla_x\Phi^f\|^2)+\lambda_0\|\{\mathbf{I}-\mathbf{P}_0\}f\|_\sigma^2\\
&\leq C\{\eta+\sup_x\{|\nabla_x\Phi|,|\nabla_x\phi|\}\|f\|_\sigma^2+C_\eta\|\nabla_x\phi\|^2.
\end{aligned}
$$

- *Higher-order estimate:* **We introduce an equivalent energy functional**

$$
\mathcal{E}(f)\sim\|f\|^2_{L^2_\xi(H^3_x)}+\|\nabla_x\Phi^f\|^2_{H^3_x}.
$$

Then,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{1\leq|\alpha|\leq 3}(\|\partial^\alpha f\|^2+\|\partial^\alpha\nabla_x\Phi^f\|^2)+\lambda_0\sum_{1\leq|\alpha|\leq 3}\|\{\mathbf{I}-\mathbf{P}_0\}\partial^\alpha f\|_\sigma^2\\
&\leq C(\eta+\sqrt{\mathcal{E}(f)}+\|\nabla_x\phi\|_{H^3})\sum_{|\alpha|\leq 3}(\|\partial^\alpha f\|_\sigma^2+\|\partial^\alpha\nabla_x\Phi^f\|^2)\\
&\quad+C_\eta\|\nabla_x\phi\|^2_{H^3}.
\end{aligned}
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- *Dissipation of a^f and $\nabla_x \Phi^f$:*

$$\partial_t a^f + \nabla_x \cdot b^f = 0,$$
$$\partial_t b^f + \nabla_x a^f + \nabla_x \cdot \Gamma(\{\mathbf{I} - \mathbf{P}\}f) = -b^f + (1 + a^f)\nabla_x(\Phi^f + \phi),$$
$$\Delta_x \Phi^f = a^f,$$

where $\Gamma = (\Gamma_{ij})_{1 \le i,j \le 3}$ is the moment functional defined by

$$\Gamma_{ij}(f) = \int_{\mathbb{R}^3} (\xi_i \xi_j - 1) M^{1/2} f \, d\xi.$$

Then, for $|\alpha| \le 3$,

$$\|\partial^\alpha \nabla_x \Phi^f\|^2 + \|\partial^\alpha a^f\|^2$$
$$= \int_{\mathbb{R}^3} \partial_t \partial^\alpha b^f \cdot \partial^\alpha \nabla_x \Phi^f \, dx + \cdots$$
$$= \frac{d}{dt} \int_{\mathbb{R}^3} \partial^\alpha b^f \cdot \partial^\alpha \nabla_x \Phi^f \, dx + \int_{\mathbb{R}^3} |\nabla_x \Delta_x^{-1} \partial^\alpha \nabla_x \cdot b^f|^2 \, dx + \cdots$$

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$$
\begin{aligned}
&-\frac{d}{dt}\sum_{|\alpha|\leq 3}\int_{\mathbb{R}^3}\partial^\alpha b^f\cdot\partial^\alpha\nabla_x\Phi^f\,dx \\
&\qquad+\lambda(\|\nabla_x\Phi^f\|_{H^3}^2+\|a^f\|_{H^3}^2)\leq C\|\{\mathbf{I}-\mathbf{P}_0\}f\|^2_{L^2_\xi(H^3_x)} \\
&\qquad\qquad+C(\|a^f\|_{H^3}+\|\nabla_x\phi\|_{H^3})(\|a^f\|_{H^3}^2+\|\nabla_x\Phi^f\|_{H^3}^2).
\end{aligned}
$$

- **We now define**

$$\mathcal{E}(f) = \|f\|^2_{L^2_\xi(H^3_x)} + \|\nabla_x \Phi^f\|^2_{H^3_x} - \kappa \sum_{|\alpha|\leq 3} \int_{\mathbb{R}^3} \partial^\alpha b^f \cdot \partial^\alpha \nabla_x \Phi^f \, dx,$$

with the constant $\kappa > 0$ small enough. **Notice that $\mathcal{E}(f) \sim \|f\|^2_X$ and**

$$\mathcal{E}(f) \leq C \sum_{|\alpha|\leq 3} \|\{\mathbf{I} - \mathbf{P}_0\}\partial^\alpha f\|^2_\sigma + C(\|\nabla_x \Phi^f\|^2_{H^3} + \|a^f\|^2_{H^3}).$$

One has

$$\frac{d}{dt}\mathcal{E}(f) + \lambda \mathcal{E}(f) \leq C\|\nabla_x \phi\|^2_{H^3},$$

Gronwall's inequality implies

$$\|f(t)\|_X \leq C(\|f_0\|_X + \sup_{t\geq 0} \|\nabla \phi(t)\|_{H^3}),$$

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III.2 Time-periodic solutions

- **Assume that $\phi(t,x)$ is T-periodic in time, and $\delta_\phi := \sup_{0\le t\le T} \|\nabla_x \phi(t)\|_{H^3}$ is sufficiently small.**

- *Step 1: Find special initial data.* **Let $f(t,\cdot,\cdot) \in X$ $(t \ge 0)$ be the solution by solving the Cauchy problem with arbitrary initial data $f_0(x,\xi)$ with $\|f_0\|_X \le \delta_0$ for $\delta_0 > 0$ small enough. Take integers $m \ge k \ge 1$, and define**

$$g(t,x,\xi) = f(t + (m-k)T, x, \xi).$$

As $\phi(t,x)$ is T-periodic, it is direct to see that $g(t,x,\xi)$ solves the same VPFP system

$$\begin{aligned}
&\partial_t g + \xi\cdot\nabla_x g + \nabla_x(\Phi^g + \phi)\cdot\nabla_\xi g \\
&\qquad -\frac{1}{2}\xi\cdot\nabla_x(\Phi^g+\phi) g - \xi M^{1/2}\cdot\nabla_x(\Phi^g+\phi) = Lg, \\
&\Delta_x \Phi^g = \int_{\mathbb{R}^3} M^{1/2} g \, d\xi,
\end{aligned}$$

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As $\phi(t,x)$ is T-periodic, it is direct to see that $g(t,x,\xi)$ solves the same VPFP system

$$\begin{aligned} &\partial_t g + \xi \cdot \nabla_x g + \nabla_x(\Phi^g + \phi) \cdot \nabla_\xi g \\ &\qquad -\frac{1}{2}\xi \cdot \nabla_x(\Phi^g + \phi) g - \xi M^{1/2} \cdot \nabla_x(\Phi^g + \phi) = Lg, \\ &\Delta_x \Phi^g = \int_{\mathbb{R}^3} M^{1/2} g \, d\xi, \end{aligned}$$

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We define

$$h(t,x,\xi) = g(t,x,\xi) - f(t,x,\xi), \quad \Phi^h(t,x) = \Phi^g(t,x) - \Phi^f(t,x).$$

Then $h(t,x,\xi)$ satisfies

$$\begin{aligned} &\partial_t h + \xi \cdot \nabla_x h + \nabla_x(\Phi^h + \phi) \cdot \nabla_\xi h \\ &\qquad\qquad - \frac{1}{2}\xi \cdot \nabla_x(\Phi^h + \phi)h - \xi M^{1/2} \cdot \nabla_x \Phi^h = Lh + R, \\ &\Delta_x \Phi^h = \int_{\mathbb{R}^3} M^{1/2} h \, d\xi, \end{aligned}$$

where R is denoted by

$$R = \frac{1}{2}\xi \cdot \nabla_x \Phi^f h - \nabla_x \Phi^f \cdot \nabla_\xi h + \frac{1}{2}\xi \cdot \nabla_x \Phi^h f - \nabla_x \Phi^h \cdot \nabla_\xi f.$$

Repeat the similar proofs in solving the Cauchy problem, so

$$\frac{d}{dt}\mathcal{E}(h) + \lambda\mathcal{E}(h) \le 0,$$

which implies

$$\|h(t)\|_X \le C\mathcal{E}(h(t)) \le C\mathcal{E}(h(0))e^{-\lambda t} \le C\|h(0)\|_X e^{-\lambda t},$$

for all $t \ge 0$**. Then,**

$$\begin{aligned}\|f(t+(m-k)T) - f(t)\|_X &\le C\|f((m-k)T) - f(0)\|_X e^{-\lambda t}\\ &\le C(\|f((m-k)T)\|_X + \|f(0)\|_X)e^{-\lambda t}\\ &\le C(\|f(0)\|_X + \sup_{t\ge 0}\|\nabla_x\phi(t)\|_{H^3_x})e^{-\lambda t}.\end{aligned}$$

Taking $t = kT$**, one has**

$$\|f(mT) - f(kT)\|_X \le C(\delta_0 + \delta_\phi)e^{-\lambda kT},$$

for all integers $m \ge k \ge 1$**. As** $e^{-\lambda kT} \to 0$ **as** $k \to \infty$**, it shows that** $\{f(kT,\cdot,\cdot)\}_{k\ge 1} \subset X$ **is Cauchy w.r.t.** $\|\cdot\|_X$**, and the limit function denoted by** $f_0^* = f_0^*(x,\xi) \in X$ **satisfies**

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•*Step 2: Solve the Cauchy problem on the VPFP system with initial data* f_0^*.

As both δ_0 and δ_ϕ are small enough, so is $\|f_0^*\|_X$. Again applying the existence result for the Cauchy problem with initial data given by $f_0^*(x,\xi) \in X$, one can obtain a solution $f^*(t,x,\xi)$.

•*Step 2: Solve the Cauchy problem on the VPFP system with initial data* f_0^*.

As both δ_0 and δ_ϕ are small enough, so is $\|f_0^*\|_X$. Again applying the existence result for the Cauchy problem with initial data given by $f_0^*(x,\xi) \in X$, one can obtain a solution $f^*(t,x,\xi)$.

Claim#1. $f^*(t,x,\xi)$ **is T-periodic in time.**

Indeed, for an integer $n \geq 1$, we define

$$\tilde{h}(t,x,\xi) = f(t+nT,x,\xi) - f^*(t,x,\xi),$$

where $f(t,x,\xi)$ is the solution used for obtaining $f_0^*(x,\xi)$ in the previous step. By estimating $\tilde{h}(t,x,\xi)$,

$$\|f(t+nT) - f^*(t)\|_X \leq C\|f(nT) - f^*(0)\|_X e^{-\lambda t},$$

for all $t \geq 0$. Letting $t = T$,

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Claim#2.

$$\sup_{0\le t\le T}\|f(t)\|_X \le C\sup_{0\le t\le T}\|\nabla_x\phi(t)\|_{H^3_x}.$$

Indeed, using the estimates in solving the Cauchy problem,

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Problems:

- **Existence of large-amplitude T-periodic solution;**
- **Existence of small-amplitude T-periodic solution to the Vlasov-Poisson-Boltzmann system;**
- **...**

Thanks a lot for your attention!