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Nonlinear Analysis

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Optimal L^p $(1 \le p \le \infty)$ rates of decay to linear diffusion waves for nonlinear evolution equations with ellipticity and dissipation

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Abstract

In this paper, we are concerned with the large time decay estimates of solutions to the Cauchy problem of nonlinear evolution equations with ellipticity and damping. Different end states of initial data are considered. The optimal rate of decay to the linear diffusion waves is obtained. The optimal rate of decay of solutions to the linearized system plays a crucial role in the analysis.

MSC: 35K45; 35B40

Keywords: Evolution equations; Diffusion waves; Optimal decay rate

1. Introduction and main results

In this paper, we are concerned with the optimal decay estimates of solutions to the following nonlinear evolution equations with ellipticity and damping:

$$\begin{cases} \psi_t = -(1-\alpha)\psi - \theta_x + \alpha\psi_{xx}, \\ \theta_t = -(1-\alpha)\theta + \nu\psi_x + \alpha\theta_{xx} + 2\psi\theta_x, \end{cases} \tag{1.1}$$

with initial data

$$(\psi, \theta)(x, 0) = (\psi_0(x), \theta_0(x)) \to (\psi_\pm, \theta_\pm) \quad \text{as } x \to \pm\infty. \tag{1.2}$$

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Here $\psi = \psi(x,t)$ and $\theta = \theta(x,t)$, $x \in \mathbf{R}$, $t > 0$, are unknown functions, α and ν are positive constants, (ψ_+, θ_+) and (ψ_-, θ_-) are different end states of the initial data, i.e.,

$$(\psi_+, \theta_+) \neq (\psi_-, \theta_-). \tag{1.3}$$

System (1.1) was first proposed by Tang and Zhao in [7]. It is a set of simplified equations, which arise from physical and mechanical fields. As regards the complexity in system (1.1), readers can refer to [2–6]. The global existence, nonlinear stability and optimal rate of decay to zero equilibrium were established in [7], where the end states satisfy $(\psi_\pm, \theta_\pm) = (0, 0)$, which is a rigorous restriction for initial data $\psi_0(x)$ and $\theta_0(x)$.

When the initial data have different end states, i.e., the case (1.3) holds, the global existence and asymptotic behaviors of solutions were recently obtained by Duan and Zhu in [1]. They proved that under some small assumptions on initial data, the solutions to (1.1), (1.2) time-asymptotically behave as the following linear diffusion waves:

$$\begin{cases} \bar\psi(x,t) = \mathrm{e}^{-(1-\alpha)t}\left((\psi_+ - \psi_-)\displaystyle\int_{-\infty}^{x} G(y,t+1)\mathrm{d}y + \psi_-\right), \\ \bar\theta(x,t) = \mathrm{e}^{-(1-\alpha)t}\left((\theta_+ - \theta_-)\displaystyle\int_{-\infty}^{x} G(y,t+1)\mathrm{d}y + \theta_-\right), \end{cases} \tag{1.4}$$

where $G(x,t) = \frac{1}{\sqrt{4\pi\alpha t}}\exp(-\frac{x^2}{4\alpha t})$ is the heat kernel function. Diffusion waves (1.4) can be obtained by solving the following linear system:

$$\begin{cases} \bar\psi_t = -(1-\alpha)\bar\psi + \alpha\bar\psi_{xx}, \\ \bar\theta_t = -(1-\alpha)\bar\theta + \alpha\bar\theta_{xx}, \\ (\bar\psi(x,0), \bar\theta(x,0)) \to (\psi_\pm, \theta_\pm) \quad \text{as } x \to \pm\infty. \end{cases} \tag{1.5}$$

It is noted that the linear system (1.5) and the nonlinear system (1.1), (1.2) have the same end states, which assure that the energy method can be applied. In fact, let

$$\begin{cases} u(x,t) = \psi(x,t) - \bar\psi(x,t), \\ v(x,t) = \theta(x,t) - \bar\theta(x,t), \end{cases} \tag{1.6}$$

and then the problem (1.1), (1.2) can be reformulated as

$$\begin{cases} u_t = -(1-\alpha)u - v_x + \alpha u_{xx} - \bar\theta_x, \\ v_t = -(1-\alpha)v + \nu u_x + \alpha v_{xx} + 2uv_x + 2\bar\psi v_x + 2\bar\theta_x u + F(x,t), \end{cases} \tag{1.7}$$

with initial data

$$\begin{cases} u(x,0) = u_0(x) = \psi_0(x) - \bar\psi(x,0) \to 0, \quad x \to \pm\infty, \\ v(x,0) = v_0(x) = \theta_0(x) - \bar\theta(x,0) \to 0, \quad x \to \pm\infty, \end{cases} \tag{1.8}$$

where

$$F(x,t) = \nu\bar\psi_x + 2\bar\psi\bar\theta_x.$$

To state out main result, we give the following

Notation. Throughout this paper, we use $C(\cdot)$ to denote positive constants depending on the dummy variable, usually time, while C is used for generic constants. $L^p = L^p(\mathbf{R})$ $(1 \le p \le \infty)$ denotes the usual Lebesgue space on $\mathbf{R} = (-\infty, \infty)$ with its norm $\|f\|_{L^p} = \left(\int_{\mathbf{R}} |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$, $1 \le p < \infty$, $\|f\|_{L^\infty} = \sup_{x\in\mathbf{R}} |f(x)|$, and when $p = 2$, we write

$\|\cdot\|_{L^2(\mathbf{R})} = \|\cdot\|$. $H^l(\mathbf{R})$ denotes the usual l-th order Sobolev space with its norm $\|f\|_{H^l(\mathbf{R})} = \|f\|_l = (\sum_{i=0}^{l} \|\partial_x^i f\|^2)^{\frac{1}{2}}$.

Now the global existence of solutions to (1.1), (1.2) and the rate of decay to linear diffusion waves (1.4) can be stated as follows.

Theorem 1.1 (*See [1]*). *Let $0 < \alpha < 1$, $0 < \nu < 4\alpha(1-\alpha)$. Suppose that both $\delta = |\psi_+ - \psi_-| + |\theta_+ - \theta_-|$ and $\delta_0 = \|u_0\|_2^2 + \|v_0\|_2^2$ are sufficiently small. Then the Cauchy problem (1.7), (1.8) admits a unique global solution $(u(x,t), v(x,t))$ satisfying*

$$u(x,t), v(x,t) \in L^\infty(0,T; H^2(\mathbf{R})) \cap L^2(0,T; H^3(\mathbf{R})), \tag{1.9}$$

and

$$\|u(x,t)\|_2^2 + \|v(t)\|_2^2 + \int_0^t (\|u(x,\tau)\|_3^2 + \|v(x,\tau)\|_3^2) \mathrm{d}\tau \le C(\delta + \delta_0). \tag{1.10}$$

Furthermore, the following decay estimate holds:

$$\sum_{k=0}^{2} \|\partial_x^k u(x,t)\| + \sum_{k=0}^{2} \|\partial_x^k v(x,t)\| \le C \mathrm{e}^{-\frac{l}{2}t}, \tag{1.11}$$

for any $t > 0$, where l is some positive constant satisfying

$$0 < \frac{l}{2} < 1 - \alpha - \frac{\nu}{4\alpha}. \tag{1.12}$$

The proof of Theorem 1.1 can be found in [1]. The aim of this paper is to sharpen decay estimates (1.11) to get optimal decay rates. We can state our main result as follows.

Theorem 1.2 (*Main Result*). *Suppose that the assumptions in Theorem 1.1 hold. Furthermore suppose*

$$(u_0(x), v_0(x)) \in L^1(\mathbf{R}, \mathbf{R}^2) \cap L^2(\mathbf{R}, \mathbf{R}^2). \tag{1.13}$$

Let $\tau \in (0,1)$. Then

$$\sum_{k=0}^{2} \|\partial_x^k u(x,t)\|_{L^p} + \sum_{k=0}^{2} \|\partial_x^k v(x,t)\|_{L^p} \le C(\tau)(1+t)^{-\frac{1}{2}+\frac{1}{2p}} \mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})t} \tag{1.14}$$

for any $t \ge 2\tau$ and $1 \le p \le \infty$.

Remark 1.3. We not only obtain the L^p $(1 \le p \le \infty)$ rates of decay to linear diffusion waves but also sharpen decay estimates (1.11). Indeed, let $p = 2$ in (1.14) and then we have

$$\sum_{k=0}^{2} \|\partial_x^k u(x,t)\|_{L^2} + \sum_{k=0}^{2} \|\partial_x^k v(x,t)\|_{L^2} \le C(\tau) \mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})t}.$$

Since (1.12) holds, the decay rates in (1.11) are improved. On the other hand, the decay rates in (1.14) are optimal, which comes from the decay properties (2.15) of the linearized system (2.11) in Section 2. The proof of Theorem 1.2 is based on Theorem 1.1. The integral forms of solutions to system (1.7) and (1.8) can be used to sharpen decay estimates (1.11).

This paper is arranged as follows. After this introduction and the statement of our main results, which constitutes Section 1, we give some preliminary lemmas in Section 2. The proof of Theorem 1.2 is given in Section 3.

2. Preliminary lemmas

In this section, we cite some fundamental results and give some basic estimates for our later use.

First we cite the following four well-known results.

Lemma 2.1 (*Young's Inequality*). *If* $f \in L^p(\mathbf{R})$, $g \in L^r(\mathbf{R})$, $1 \le p, r \le \infty$, *and* $\frac{1}{p} + \frac{1}{r} \ge 1$, *then* $h = f * g \in L^q(\mathbf{R})$ *with* $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$. *Furthermore, it holds that*

$$\|h\|_{L^q(\mathbf{R})} \le \|f\|_{L^p(\mathbf{R})}\|g\|_{L^r(\mathbf{R})}. \tag{2.1}$$

Lemma 2.2 (*Interpolation Inequality for* $L^p(\mathbf{R})$). *Assume* $1 \le p \le q \le r \le \infty$ *and* $\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}$. *Suppose also* $f \in L^p(\mathbf{R}) \cap L^r(\mathbf{R})$. *Then* $f \in L^q(\mathbf{R})$, *and*

$$\|f\|_{L^q(\mathbf{R})} \le \|f\|^{\lambda}_{L^p(\mathbf{R})}\|f\|^{1-\lambda}_{L^r(\mathbf{R})}. \tag{2.2}$$

Lemma 2.3 (*See [8]*). *Let* a, b *be positive numbers,* $0 < \tau < 1$, $t \ge 2\tau$. *Then*

$$\int_\tau^t (1+t-s)^{-a}(1+s)^{-b}\,\mathrm{d}s \le C(1+t)^{-\min\{a,b\}} \tag{2.3}$$

if $\max\{a, b\} > 1$. *In particular,*

$$\int_\tau^t (1+t-s)^{-a}\mathrm{e}^{-bs}\mathrm{d}s \le C(1+t)^{-a}. \tag{2.4}$$

Lemma 2.4 (*Generalized Gronwall Inequality*). *Suppose that the nonnegative continuous functions* $g(t)$ *and* $h(t)$ *satisfy*

$$g(t) \le N_1\mathrm{e}^{-r(t-\tau)}(1+t-\tau)^{-a} + N_2\int_\tau^t (1+t-s)^{-a}\mathrm{e}^{-r(t-s)}g(s)h(s)\mathrm{d}s \tag{2.5}$$

with N_1, N_2, a *and* r *nonnegative constants and*

$$\int_\tau^\infty h(s)\mathrm{d}s \le N(\tau) < \infty. \tag{2.6}$$

Then it holds that

$$\begin{aligned} g(t) &\le N_1(1+t-\tau)^{-a}\mathrm{e}^{-r(t-\tau)}\exp\left\{N_2\int_\tau^\infty h(s)\mathrm{d}s\right\} \\ &\le C(\tau)(1+t)^{-a}\mathrm{e}^{-rt}. \end{aligned} \tag{2.7}$$

On the other hand, we give some basic estimates which will play an essential role in our proving Theorem 1.2. First for the heat kernel function $G(x, t)$, we have the following property.

Lemma 2.5. *When* $1 \le p \le \infty$, $0 \le l,\ k < \infty$, *we have*

$$\|\partial_t^l \partial_x^k G(x,t)\|_{L^p} \le C t^{-\frac{1}{2}(1-\frac{1}{p})-l-\frac{k}{2}}. \tag{2.8}$$

For the linear diffusion waves $\bar{\psi}(x,t)$ and $\bar{\theta}(x,t)$, we have the following asymptotic behavior.

Lemma 2.6. *The functions* $\bar{\psi}(x,t)$ *and* $\bar{\theta}(x,t)$ *defined by* (1.4) *satisfy*

$$\|\partial_t^l \bar{\psi}(t)\|_{L^\infty} \le C\mathrm{e}^{-(1-\alpha)t}, \quad \|\partial_t^l \bar{\theta}(t)\|_{L^\infty} \le C\mathrm{e}^{-(1-\alpha)t} \tag{2.9}$$

for $l = 0, 1, 2, \ldots$, *and*

$$\begin{cases} \|\partial_t^l \partial_x^k \bar{\psi}(t)\|_{L^p} \le C|\psi_+ - \psi_-|\mathrm{e}^{-(1-\alpha)t}(1+t)^{\frac{1}{2p}-\frac{k}{2}}, \\ \|\partial_t^l \partial_x^k \bar{\theta}(t)\|_{L^p} \le C|\theta_+ - \theta_-|\mathrm{e}^{-(1-\alpha)t}(1+t)^{\frac{1}{2p}-\frac{k}{2}} \end{cases} \tag{2.10}$$

for $k = 1, 2, \ldots, l = 0, 1, 2, \ldots$, *and any* p *with* $1 \le p \le \infty$.

In order to obtain the optimal rate of decay to the linear diffusion wave, we must consider the linearized system

$$\begin{cases} \hat{\psi}_t = -(1-\alpha)\hat{\psi} - \hat{\theta}_x + \alpha\hat{\psi}_{xx}, \\ \hat{\theta}_t = -(1-\alpha)\hat{\theta} + \nu\hat{\psi}_x + \alpha\hat{\theta}_{xx}, \end{cases} \tag{2.11}$$

with initial data

$$(\hat{\psi}(x,0), \hat{\theta}(x,0)) = (\hat{\psi}_0(x), \hat{\theta}_0(x)). \tag{2.12}$$

As in [7], by using the method of Fourier transformation, we can get the solution of the linearized system (2.11) and (2.12)

$$\begin{aligned} \hat{\psi}(x,t) &= \frac{1}{2\mathrm{i}\sqrt{\nu}}\left[K_1(x,t) * (\hat{\theta}_0 + \mathrm{i}\sqrt{\nu}\hat{\psi}_0) - K_2(x,t) * (\hat{\theta}_0 - \mathrm{i}\sqrt{\nu}\hat{\psi}_0)\right], \\ \hat{\theta}(x,t) &= \frac{1}{2}\left[K_2(x,t) * (\hat{\theta}_0 - \mathrm{i}\sqrt{\nu}\hat{\psi}_0) + K_1(x,t) * (\hat{\theta}_0 + \mathrm{i}\sqrt{\nu}\hat{\psi}_0)\right], \end{aligned} \tag{2.13}$$

where the kernel functions are

$$\begin{cases} K_1(x,t) = \dfrac{1}{\sqrt{4\pi\alpha t}} \exp\left\{-\left(1-\alpha-\dfrac{\nu}{4\alpha}\right)t\right\} \exp\left\{-\dfrac{\mathrm{i}\sqrt{\nu}}{2\alpha} - \dfrac{x^2}{4\alpha t}\right\}, \\ K_2(x,t) = \dfrac{1}{\sqrt{4\pi\alpha t}} \exp\left\{-\left(1-\alpha-\dfrac{\nu}{4\alpha}\right)t\right\} \exp\left\{\dfrac{\mathrm{i}\sqrt{\nu}}{2\alpha} - \dfrac{x^2}{4\alpha t}\right\}. \end{cases} \tag{2.14}$$

The following estimates for the kernel functions $K_1(x,t)$ and $K_2(x,t)$ play an important role in the proof of Theorem 1.2, cf. [7].

Lemma 2.7. *For* $i = 1,\ 2$, $1 \le p \le \infty$, *we have*

$$\|\partial_x^k K_i(x,t)\|_{L^p} \le C t^{-\frac{1}{2}+\frac{1}{2p}} \mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})t}. \tag{2.15}$$

3. Optimal L^p ($1 \le p \le \infty$) decay rate

In this section, we give the proof of Theorem 1.2, which follows from a series of lemmas. To this end, we rewrite Eqs. (1.7) in an integral form as

$$\begin{cases} u(x,t) = G(x,t)*u_0(x) - (1-\alpha)\int_0^t G(x,t-s)*u(x,s)\mathrm{d}s \\ \quad + \int_0^t G_x(x,t-s)*v(x,s)\mathrm{d}s - \int_0^t G(x,t-s)*\bar{\theta}_x(x,s)\mathrm{d}s, \\ v(x,t) = G(x,t)*v_0(x) - (1-\alpha)\int_0^t G(x,t-s)*v(x,s)\mathrm{d}s \\ \quad -\nu\int_0^t G_x(x,t-s)*u(x,s)\mathrm{d}s \\ \quad +2\int_0^t G(x,t-s)*(uv_x)(x,s)\mathrm{d}s \\ \quad -2\int_0^t G_x(x,t-s)*\left(\bar{\psi}v\right)(x,s)\mathrm{d}s - 2\int_0^t G(x,t-s) \\ \quad * \left(\bar{\psi}_x v\right)(x,s)\mathrm{d}s + 2\int_0^t G(x,t-s)*\left(\bar{\theta}_x u\right)(x,s)\mathrm{d}s \\ \quad + \int_0^t G(x,t-s)* F(x,s)\mathrm{d}s, \end{cases} \tag{3.1}$$

where the convolutions are taken with respect to the space variable x. On the basis of the integral form above, we have

Lemma 3.1. *Under the assumptions of Theorem 1.2, it holds that*

$$\|(u(x,\tau),v(x,\tau))\|_{L^1(\mathbf{R},\mathbf{R}^2)} \le C(\tau) \tag{3.2}$$

for any $\tau \in (0,1)$.

Proof. From (1.11), Lemmas 2.1, 2.5 and 2.6, we deduce that

$$\begin{aligned} \|u(x,\tau)\|_{L^1(\mathbf{R})} &\le \|G(x,\tau)\|_{L^1(\mathbf{R})}\|u_0(x)\|_{L^1(\mathbf{R})} \\ &\quad + (1-\alpha)\int_0^\tau \|G(x,\tau-s)\|_{L^1(\mathbf{R})}\|u(x,s)\|_{L^1(\mathbf{R})}\mathrm{d}s \\ &\quad + \int_0^\tau \|G_x(x,\tau-s)\|_{L^1(\mathbf{R})}\|v(x,s)\|_{L^1(\mathbf{R})}\mathrm{d}s \\ &\quad + \int_0^\tau \|G(x,\tau-s)\|_{L^1(\mathbf{R})}\|\bar{\theta}_x(x,s)\|_{L^1(\mathbf{R})}\mathrm{d}s \\ &\le \|u_0(x)\|_{L^1(\mathbf{R})} + (1-\alpha)\int_0^\tau \|u(x,s)\|_{L^1(\mathbf{R})}\mathrm{d}s \\ &\quad + C\int_0^\tau (\tau-s)^{-\frac{1}{2}}\|v(x,s)\|_{L^1(\mathbf{R})}\mathrm{d}s + C\int_0^\tau \mathrm{e}^{-(1-\alpha)s}\mathrm{d}s. \end{aligned} \tag{3.3}$$

Similarly, we have

$$\|v(x,\tau)\|_{L^1(\mathbf{R})} \le \|G(x,\tau)\|_{L^1(\mathbf{R})}\|v_0(x)\|_{L^1(\mathbf{R})}$$

$$
\begin{aligned}
&+ (1-\alpha)\int_0^\tau \|G(x,\tau-s)\|_{L^1(\mathbf{R})}\|v(x,s)\|_{L^1(\mathbf{R})}\mathrm{d}s\\
&+ \nu\int_0^\tau \|G_x(x,\tau-s)\|_{L^1(\mathbf{R})}\|u(x,s)\|_{L^1(\mathbf{R})}\mathrm{d}s\\
&+ 2\int_0^\tau \|G(x,\tau-s)\|_{L^1(\mathbf{R})}\|u(x,s)\|_{L^2(\mathbf{R})}\|v_x(x,s)\|_{L^2(\mathbf{R})}\mathrm{d}s\\
&+ 2\int_0^\tau \|G_x(x,\tau-s)\|_{L^1(\mathbf{R})}\|\bar\psi(x,s)\|_{L^\infty(\mathbf{R})}\|v(x,s)\|_{L^1(\mathbf{R})}\mathrm{d}s\\
&+ 2\int_0^\tau \|G(x,\tau-s)\|_{L^1(\mathbf{R})}\|\bar\psi_x(x,s)\|_{L^2(\mathbf{R})}\|v(x,s)\|_{L^2(\mathbf{R})}\mathrm{d}s\\
&+ 2\int_0^\tau \|G(x,\tau-s)\|_{L^1(\mathbf{R})}\|\bar\theta_x(x,s)\|_{L^2(\mathbf{R})}\|u(x,s)\|_{L^2(\mathbf{R})}\mathrm{d}s\\
&+ \int_0^\tau \|G(x,\tau-s)\|_{L^1(\mathbf{R})}\|\nu\bar\psi_x(x,s)+2(\bar\psi\bar\theta_x)(x,s)\|_{L^1(\mathbf{R})}\mathrm{d}s\\
\le{}& \|v_0(x)\|_{L^1(\mathbf{R})} + (1-\alpha)\int_0^\tau \|v(x,s)\|_{L^1(\mathbf{R})}\mathrm{d}s\\
&+ C\nu\int_0^\tau (\tau-s)^{-\frac12}\|u(x,s)\|_{L^1(\mathbf{R})}\mathrm{d}s + C\int_0^\tau \mathrm{e}^{-ls}\mathrm{d}s\\
&+ C\int_0^\tau (\tau-s)^{-\frac12}\mathrm{e}^{-(1-\alpha)s}\|v(x,s)\|_{L^1(\mathbf{R})}\mathrm{d}s\\
&+ C\int_0^\tau \mathrm{e}^{-(1-\alpha)s-\frac{l}{2}s}\mathrm{d}s + C\int_0^\tau \mathrm{e}^{-(1-\alpha)s}\mathrm{d}s.
\end{aligned}
\tag{3.4}
$$

Let $\tau \in (0,1)$. It follows from (3.3) and (3.4) that

$$
\begin{aligned}
\|(u(x,\tau),v(x,\tau))\|_{L^1(\mathbf{R},\mathbf{R}^2)} \le{}& \|(u_0(x),v_0(x))\|_{L^1(\mathbf{R},\mathbf{R}^2)} + C\\
&+ C\int_0^\tau \left[1+(\tau-s)^{-\frac12}\right]\|(u(x,s),v(x,s))\|_{L^1(\mathbf{R},\mathbf{R}^2)}\mathrm{d}s.
\end{aligned}
$$

Then (3.2) is verified with Gronwall's inequality. □

Lemma 3.2. *Let $\tau \in (0,1)$. Under the assumptions of Theorem 1.2, it holds that*

$$
\|\partial_x^k(u(x,t),v(x,t))\|_{L^1(\mathbf{R},\mathbf{R}^2)} \le C(\tau)\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})t}
\tag{3.5}
$$

for any $t \ge \tau$ and $k = 0,1,2$.

Proof. As in [7], taking the Fourier transform of (1.7) and then taking the inverse Fourier transform, we have

$$
\begin{cases}
u(x,t) = \dfrac{1}{2\mathrm{i}\sqrt{\nu}}\left[K_1(x,t)*(v_0+\mathrm{i}\sqrt{\nu}u_0) - K_2(x,t)*(v_0-\mathrm{i}\sqrt{\nu}u_0)\right]\\
\qquad - \dfrac{1}{2\mathrm{i}\sqrt{\nu}}\displaystyle\int_0^t [K_2(x,t-s)-K_1(x,t-s)]*\bar\theta_x(x,s)\mathrm{d}s,\\
v(x,t) = \dfrac12\left[K_1(x,t)*(v_0-\mathrm{i}\sqrt{\nu}u_0) + K_2(x,t)*(v_0+\mathrm{i}\sqrt{\nu}u_0)\right]\\
\qquad + \dfrac12\displaystyle\int_0^t [K_1(x,t-s)+K_2(x,t-s)]\\
\qquad * (2uv_x + 2\bar\psi v_x + 2\bar\theta_x u + F)(x,s)\mathrm{d}s.
\end{cases}
\tag{3.6}
$$

By the Sobolev inequality $\|f\|_{L^\infty} \le \|f\|^{\frac{1}{2}}\|f_x\|^{\frac{1}{2}}$, we have from (1.11) that

$$\|(u(x,t), v(x,t), u_x(x,t), v_x(x,t))\|_{L^\infty(\mathbf{R},\mathbf{R}^4)} \le C\mathrm{e}^{-\frac{l}{2}t}. \tag{3.7}$$

In terms of the decay rate (2.15) on kernel functions $K_1(x,t)$, $K_2(x,t)$, the exponential decay rate (1.11), (3.7) of solutions $u(x,t)$, $v(x,t)$ and the exponential decay rate (2.9), (2.10) for linear diffusion waves $\bar{\psi}(x,t)$, $\bar{\theta}(x,t)$, we have from (3.6) that

$$\begin{aligned}
&\|\partial_x^j(u(x,t), v(x,t))\|_{L^1(\mathbf{R},\mathbf{R}^2)} \\
&\quad \le C\|\partial_x^j(K_1(x,t-\tau), K_2(x,t-\tau))\|_{L^1(\mathbf{R},\mathbf{R}^2)}\|(u(x,\tau), v(x,\tau))\|_{L^1(\mathbf{R},\mathbf{R}^2)} \\
&\qquad + C\int_\tau^t \|\partial_x^j(K_1(x,t-s), K_2(x,t-s))\|_{L^1(\mathbf{R},\mathbf{R}^2)} \\
&\qquad \times \left(\|\bar{\theta}_x(x,s)\|_{L^1(\mathbf{R})} + \|F(x,s)\|_{L^1(\mathbf{R})}\right)\mathrm{d}s \\
&\qquad + C\int_\tau^t \|\partial_x(K_1(x,t-s), K_2(x,t-s))\|_{L^1(\mathbf{R},\mathbf{R}^2)}\|\partial_x^{j-1}\left(uv_x + \bar{\psi}v_x\right)\|_{L^1(\mathbf{R})}\mathrm{d}s \\
&\qquad + C\int_\tau^t \|(K_1(x,t-s), K_2(x,t-s))\|_{L^1(\mathbf{R},\mathbf{R}^2)}\|\partial_x^j\left(\bar{\theta}_x u\right)\|_{L^1(\mathbf{R})}\mathrm{d}s \\
&\quad \le C\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})(t-\tau)} + C\int_\tau^t \mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})(t-s)}\mathrm{e}^{-(1-\alpha)s}\mathrm{d}s \\
&\qquad + C\int_\tau^t \mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})(t-s)}\left[\mathrm{e}^{-\frac{l}{2}s} + \mathrm{e}^{-(1-\alpha)s}\right]\sum_{m=0}^{j}\|\partial_x^m v(x,s)\|_{L^1(\mathbf{R})}\mathrm{d}s \\
&\qquad + C\int_\tau^t \mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})(t-s)}\mathrm{e}^{-(1-\alpha)s}\sum_{m=0}^{j}\|\partial_x^m u(x,s)\|_{L^1(\mathbf{R})}\mathrm{d}s
\end{aligned} \tag{3.8}$$

for $j = 0, 1, 2$. Let

$$Z(t) = \sum_{m=0}^{2}\|\partial_x^m(u(x,t), v(x,t))\|_{L^1(\mathbf{R},\mathbf{R}^2)} \tag{3.9}$$

and then summing (3.8) over $j = 0, 1, 2$ yields

$$Z(t) \le C(\tau)\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})(t-\tau)} + C\int_\tau^t \mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})(t-s)}\mathrm{e}^{-\frac{l}{2}s}Z(s)\mathrm{d}s, \tag{3.10}$$

where (1.12) has been used. With help of Lemma 2.4, (3.5) follows easily from (3.10). This completes the proof of Lemma 3.2. □

Lemma 3.3. *Under the assumptions of Theorem 1.2, it holds that*

$$\|\partial_x^k(u(x,\tau), v(x,\tau))\|_{L^\infty(\mathbf{R},\mathbf{R}^2)} \le C(\tau)(1+t)^{-\frac{1}{2}}\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})t} \tag{3.11}$$

for any $t \ge 2\tau$ and $k = 0, 1, 2$.

Proof. Using the integral (3.6) and Lemma 3.2, we have

$$\begin{aligned}
&\|\partial_x^j(u(x,t), v(x,t))\|_{L^\infty(\mathbf{R},\mathbf{R}^2)} \\
&\quad \le C\|\partial_x^j(K_1(x,t-\tau), K_2(x,t-\tau))\|_{L^\infty(\mathbf{R},\mathbf{R}^2)}\|(u(x,\tau), v(x,\tau))\|_{L^1(\mathbf{R},\mathbf{R}^2)}
\end{aligned}$$

$$
\begin{aligned}
&+C\int_\tau^t \|\partial_x^j(K_1(x,t-s),K_2(x,t-s))\|_{L^\infty(\mathbf{R},\mathbf{R}^2)}\\
&\times\left(\|\bar\theta_x(x,s)\|_{L^1(\mathbf{R})}+\|F(x,s)\|_{L^1(\mathbf{R})}+\|\bar\psi(x,s)\|_{L^\infty(\mathbf{R})}\|v_x(x,s)\|_{L^1(\mathbf{R})}\right)\mathrm{d}s\\
&+C\int_\tau^t \|\partial_x^j(K_1(x,t-s),K_2(x,t-s))\|_{L^\infty(\mathbf{R},\mathbf{R}^2)}\\
&\times\left(\|u(x,s)\|_{L^\infty(\mathbf{R})}\|v_x(x,s)\|_{L^1(\mathbf{R})}+\|\bar\theta_x(x,s)\|_{L^1(\mathbf{R})}\|u(x,s)\|_{L^\infty(\mathbf{R})}\right)\mathrm{d}s\\
\le{}& C(1+t-\tau)^{-\frac12}\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})(t-\tau)}\\
&+C\int_\tau^t(1+t-s)^{-\frac12}\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})(t-s)}\mathrm{e}^{-(1-\alpha)s}\mathrm{d}s\\
&+C\int_\tau^t(1+t-s)^{-\frac12}\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})(t-s)}\left[\mathrm{e}^{-(1-\alpha)s}+\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})s}\right]\\
&\times\|\partial_x^j(u(x,s),v(x,s))\|_{L^\infty(\mathbf{R},\mathbf{R}^2)}\mathrm{d}s
\end{aligned}\tag{3.12}
$$

for $j=0,1,2$. By virtue of Lemma 2.3, it holds that

$$
\begin{aligned}
&\int_\tau^t(1+t-s)^{-\frac12}\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})(t-s)}\mathrm{e}^{-(1-\alpha)s}\mathrm{d}s\\
&=\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})t}\int_\tau^t(1+t-s)^{-\frac12}\mathrm{e}^{-\frac{\nu}{4\alpha}s}\mathrm{d}s\\
&\le C\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})t}(1+t)^{-\frac12}\\
&\le C(\tau)(1+t-\tau)^{-\frac12}\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})(t-\tau)}
\end{aligned}\tag{3.13}
$$

for $t\ge 2\tau$. Let

$$
\widetilde{Z}(t)=\sum_{m=0}^{2}\|\partial_x^m(u(x,t),v(x,t))\|_{L^\infty(\mathbf{R},\mathbf{R}^2)}.\tag{3.14}
$$

Summing (3.12) over $j=0,1,2$, we have from (3.12) and (3.13) that

$$
\begin{aligned}
\widetilde{Z}(t)\le{}& C(1+t-\tau)^{-\frac12}\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})(t-\tau)}\\
&+C\int_\tau^t(1+t-s)^{-\frac12}\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})(t-s)}\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})s}\widetilde{Z}(s)\mathrm{d}s.
\end{aligned}\tag{3.15}
$$

By Lemma 2.4, (3.15) implies that (3.11) holds. The proof of Lemma 3.3 is complete. □

The proof of Theorem 1.2. By using Lemmas 2.2, 3.2 and 3.3, we have

$$
\begin{aligned}
&\|\partial_x^k(u(x,t),v(x,t))\|_{L^p(\mathbf{R},\mathbf{R}^2)}\\
&\le\|\partial_x^k(u(x,t),v(x,t))\|_{L^\infty(\mathbf{R},\mathbf{R}^2)}^{(p-1)/p}\|\partial_x^k(u(x,t),v(x,t))\|_{L^1(\mathbf{R},\mathbf{R}^2)}^{1/p}\\
&\le C(\tau)(1+t)^{-\frac12+\frac{1}{2p}}\mathrm{e}^{-(1-\alpha-\frac{\nu}{4\alpha})t}
\end{aligned}\tag{3.16}
$$

for $k=0,1,2$ and $1<p<\infty$. Together with Lemmas 3.2 and 3.3, (3.16) implies (1.14) holds. This proves Theorem 1.2.

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