The Boltzmann Equation Near Equilibrium States in $\mathbb{R}^n$

Renjun Duan*

Department of Mathematics, City University of Hong Kong
Kowloon, Hong Kong, P.R. China

Abstract

In this paper, we review some recent results on the Boltzmann equation near the equilibrium states in the whole space $\mathbb{R}^n$. The emphasize is put on the well-posedness of the solution in some Sobolev space without time derivatives and its uniform stability and optimal decay rates, and also on the existence and asymptotical stability of the time-periodic solution. Most of results obtained here are proved by combining the energy estimates and the spectral analysis.

1 Introduction

1.1 The Boltzmann Equation. In the presence of an external force and a source, the Boltzmann equation in the whole space $\mathbb{R}^n$ takes the form

$$\partial_t f + \xi \cdot \nabla_x f + F \cdot \nabla_\xi f = Q(f,f) + S. \tag{1}$$

Here, the unknown $f = f(t,x,\xi)$ is a non-negative function standing for the number density of gas particles which have position $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and velocity $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$ at time $t \in \mathbb{R}$. The external force field $F = F(t,x,\xi) \in \mathbb{R}^n$ and the source term $S = S(t,x,\xi) \in \mathbb{R}$ are some given functions. Q is the usual bilinear collision operator defined by

$$Q(f,g) = \frac{1}{2}\int_{\mathbb{R}^n \times S^{n-1}} (f'g'_* + f'_*g' - fg_* - f_*g) B(|\xi - \xi_*|, \omega)\, d\omega d\xi_*,$$
$$f = f(t,x,\xi), \quad f' = f(t,x,\xi'), \quad f_* = f(t,x,\xi_*), \quad f'_* = f(t,x,\xi'_*),$$
$$\xi' = \xi - [(\xi-\xi_*)\cdot\omega]\omega, \quad \xi'_* = \xi_* + [(\xi-\xi_*)\cdot\omega]\omega, \quad \omega \in S^{n-1},$$

and likewise for g. B depending only on $|\xi - \xi_*|$ and $(\xi-\xi_*)\cdot\omega/|\xi-\xi_*|$ is called the collision kernel characterizing the collision of gas particles for various interaction potentials. One classical example is the hard sphere gas for which

$$B(|\xi - \xi_*|, \omega) = |(\xi - \xi_*)\cdot\omega|. \tag{2}$$

Later, we shall also mention the general collision kernel for the hard or Maxwell potentials satisfying the so-called Grad's cutoff assumption:

$$\begin{aligned} &\int_{S^{n-1}} B(z,\omega)\,d\omega \geq b_0 z^\gamma, \quad B(z,\omega) \leq b_1(1+z)^\gamma |\cos\theta|, \\ &b_0 > 0,\ b_1 > 0,\ z = |\xi - \xi_*|,\ \cos\theta = \frac{1}{z}(\xi-\xi_*)\cdot\omega,\ 0 \leq \gamma \leq 1. \end{aligned} \tag{3}$$

*Email: mathrjduan@hotmail.com. Date: December 12, 2007.

Notice that the hard sphere case (2) can be regarded as a special one of (3) with $\gamma = 1$. To the end, we suppose $n \geq 3$ if the spatial dimension n plays a role in the mathematical analysis. Otherwise, the only physical case $n = 3$ is considered for brevity.

1.2 Equilibrium states. Suppose $S \equiv 0$. Consider the equilibrium state of the Boltzmann equation when the external force is present or not, where by an equilibrium state it means the steady state satisfying the Boltzmann equation. Write

$$\mathbf{M} = \frac{1}{(2\pi)^{n/2}} \exp\left(-|\xi|^2/2\right),$$

which is a global Maxwellian normalized to have zero bulk velocity and unit density and temperature. Then it holds that

$$Q(\mathbf{M}, \mathbf{M}) = 0.$$

The equilibrium states of the Boltzmann equation can be described as follows:

F	Equilibrium states
$F = 0$	$\mathbf{M}$
$F = \nabla_x \phi(x)$	$e^{\phi(x)}\mathbf{M}$
$F = F(x)$	stationary solution to (1)

Here, we should point out that the existence of the stationary solution to the Boltzmann equation for the general force $F(x)$ is open in the physical three dimension, whereas this problem has been solved in [4] by considering the limiting case of the time-periodic solutions when the spatial dimension $n \geq 5$ holds.

1.3 Main goals. We devote ourselves to study the following issues about the Boltzmann equation near equilibrium states given as before.

• *Well-posedness.* It means the global existence and uniqueness of solutions to the Cauchy problem. The first result about the global existence was given by Ukai [23]. Recently, the nonlinear energy methods can be used to deal with this issue, which was initiated by Liu-Yu [15] and developed by Liu-Yang-Yu [14], and was independently found by Guo [9] in different ways. The method of the Green's function was founded by Liu-Yu [16] not only to establish the well-posedness, but also to give the wave structure of the convergence of solutions to the equilibrium state. We are interested to find the maximal spaces in which the Cauchy problem can be well-posed. By a maximal space, it means one with the lower regularity and integrability in the arguments. In fact, based on a refined energy method, Duan [1] obtained a solution space without any velocity weight and time derivative. Here, no time derivative implies that the solution space coincides with the initial space.

• *Uniform-in-time stability.* Suppose that the Banach space X with norm $\|\cdot\|_X$ is a solution space in which the Cauchy problem is well-posed. Let $f(t), g(t) \in X$, $t \geq 0$, be two solutions to the Cauchy problem corresponding to the given initial data $f(0), g(0) \in X$. To say that the solution to the Cauchy problem is stable uniformly in time, it means that there exist constants C and $\delta > 0$ independent of t such that

$$\|f(t) - g(t)\|_X \leq C\|f(0) - g(0)\|_X,$$

for any $t \geq 0$ and any $f(0), g(0) \in B_\delta$, where B_δ is a closed ball in X defined by

$$B_\delta = \{f \in X; \|f\|_X \leq \delta\}.$$

Here, generally δ is small enough since solutions considered are constructed in the framework of the small perturbations. The uniform-in-time stability in L^1 norm was firstly studied by Ha [12] for the Boltzmann equation without forces in infinite vacuum and later generalized by Duan-Yang-Zhu [6] to the case with general forces. Recently, Ha [13] obtained the uniform-in-time stability in L^2 norm for the Boltzmann equation near Maxwellians. Here, we are interested in the same issue but in the original solution space X. In fact, we will obtain a Lyapunov-type stability estimate as follows:

$$\|f(t) - g(t)\|_X + \int_0^t \|f(s) - g(s)\|_Y ds \leq C\|f(0) - g(0)\|_X,$$

where Y is a Banach space which shows the dissipative properties of solutions to the Boltzmann equation.

• *Optimal time decay estimates.* If there are no external forces and sources, the L^p-L^q time decay estimates on the solution semigroup of the linearized Boltzmann equation have been well established by the spectral analysis. When an external force is present, we will consider two kinds of situations: one is the optimal convergence rate of solutions to the equilibrium state on the Cauchy problem of the Boltzmann equation, and the other is the optimal L^p-L^q time decay estimates on the solution operator of the linearized Boltzmann equation. Actually, for the nonlinear problem in the latter situation, we need more restrictions on the force. In fact, we are interested to find weak conditions on external force F under which the solution has the same (hence optimal) decay-in-time rates as that when $F = S = 0$.

• *Existence and stability of time-periodic or stationary solutions for the case of general forces.* This issue is related to the generation and propagation of sound waves so that it has its physical importance besides its mathematical interest. In fact, for the time-periodic solution, the existence and stability have been studied for the compressible and incompressible Navier-Stokes equations by Serrin, Feireisal, Maremonti and Yamazaki; see [29] for an extensive review on this topic and reference therein. So far, the problem for the compressible Navier-Stokes equations driven by time-periodic forces is open in the whole three dimensional space. The same situation happens for the Boltzmann equation with time-periodic forces. However, there exists a time-periodic solution which is asymptotically stable if the force is small and time-periodic when the spatial dimension is not less than five.

1.4 Perturbations. For later use, here we shall write down the evolution equation of the perturbation corresponding to solutions around the equilibrium state.

Set

$$\begin{aligned} \mathbf{L}u &= \mathbf{M}^{-1/2}\left[Q(\mathbf{M}, \mathbf{M}^{1/2}u) + Q(\mathbf{M}^{1/2}u, \mathbf{M})\right], \\ \Gamma(u,u) &= \mathbf{M}^{-1/2}Q(\mathbf{M}^{1/2}u, \mathbf{M}^{1/2}u), \end{aligned}$$

where $\mathbf{L}$ is the linearized collision operator and Γ is the corresponding nonlinear collision operator. Suppose (3) hold. Recall some standard facts on $\mathbf{L}$ as follows:

(a) $(\mathbf{L}u)(\xi) = -\nu(\xi)u(\xi) + (Ku)(\xi)$;
(b) $\nu_0(1 + |\xi|)^\gamma \leq \nu(\xi) \leq \nu_0^{-1}(1 + |\xi|)^\gamma, \quad \nu_0 > 0$;

(c) K is a self-adjoint compact operator on $L^2(\mathbb{R}^n_\xi)$ with a real symmetric integral kernel $K(\xi,\xi_*)$ satisfying

$$\int_{\mathbb{R}^n}|K(\xi,\xi_*)|^2d\xi_*\leq C,$$

$$\int_{\mathbb{R}^n}|K(\xi,\xi_*)|(1+|\xi_*|)^{-\beta}d\xi_*\leq C(1+|\xi|)^{-\beta-1},\quad \beta\geq 0;$$

(d) $\mathcal{N}=Ker\mathbf{L}=\operatorname{span}\left\{\mathbf{M}^{1/2};\,\xi_i\mathbf{M}^{1/2},i=1,2,\cdots,n;\,|\xi|^2\mathbf{M}^{1/2}\right\}$;
(e) $\mathbf{L}$ is self-adjoint on $L^2(\mathbb{R}^n_\xi)$ with the domain

$$D(\mathbf{L})=\{u\in L^2(\mathbb{R}^n_\xi)|\nu(\xi)u\in L^2(\mathbb{R}^n_\xi)\}.$$

(f) $-\mathbf{L}$ is locally coercive: there is $\lambda>0$ such that

$$-\int_{\mathbb{R}^n}u\mathbf{L}u\,d\xi\geq\lambda\int_{\mathbb{R}^n}\nu(\xi)\left(\{\mathbf{I}-\mathbf{P}\}u\right)^2d\xi,\ \forall\,u\in D(\mathbf{L}),$$

where $\mathbf{P}$ is the projector from $L^2(\mathbb{R}^n_\xi)$ to $\mathcal{N}$.

Throughout this paper, C always denotes a large constant and λ denotes a positive small constant. They may take the different values at different places.

The perturbation $u=u(t,x,\xi)$ is defined in different ways depending on the explicit form of the external force.

P1: Consider the general force $F=F(t,x,\xi)$. Set

$$f=\mathbf{M}+\mathbf{M}^{1/2}u,$$

and then the Botlzmann equation for the perturbation u reads

$$\begin{aligned}\partial_t u+\xi\cdot\nabla_x u+F\cdot\nabla_\xi u-\frac{1}{2}\xi\cdot Fu&=\mathbf{L}u+\Gamma(u,u)+\widetilde{S},\\ \widetilde{S}&=\mathbf{M}^{-1/2}S+\mathbf{M}^{1/2}\xi\cdot F.\end{aligned}$$

P2: Consider the potential-type force

$$F(t,x,\xi)=\nabla_x\phi(x)+E(t,x,\xi).$$

Set

$$f=e^{\phi(x)}\mathbf{M}+\mathbf{M}^{1/2}u,$$

and then, u satisfies

$$\begin{aligned}\partial_t u+\xi\cdot\nabla_x u+F\cdot\nabla_\xi u-\frac{1}{2}\xi\cdot Fu&=e^{\phi(x)}\mathbf{L}u+\Gamma(u,u)+\widetilde{S},\\ \widetilde{S}&=\mathbf{M}^{-1/2}S+\mathbf{M}^{1/2}\xi\cdot Ee^{\phi(x)}.\end{aligned}$$

P3: Consider the same kind of the force as in P2. But let's set

$$f=e^{\phi(x)}\mathbf{M}+\left[e^{\phi(x)}\mathbf{M}\right]^{1/2}u.$$

Then u satisfies

$$\partial_t u + \xi \cdot \nabla_x u + F \cdot \nabla_\xi u - \frac{1}{2}\xi \cdot Eu = e^{\phi(x)}\mathbf{L}u + e^{\phi(x)/2}\Gamma(u,u) + \widetilde{S},$$
$$\widetilde{S} = \mathbf{M}^{-1/2}e^{-\phi(x)/2}S + \mathbf{M}^{1/2}\xi \cdot Ee^{\phi(x)/2}.$$

Remark I. *Let's point out the difference between the perturbation forms P2 and P3. For the time being, suppose that the external force is just a stationary potential, i.e.* $F = \nabla_x\phi(x)$ *with* $\phi(x) \to 0$ *as* $|x|$ *tends to infinity. In this case, when one tries to find the solutions near equilibrium states to the Boltzmann equation for the collision kernel satisfying* (3) *but with* $0 \le \gamma < 1$, *it seems necessary to use the perturbation in the form of P3, where* $E \equiv 0$ *holds and hence the term* $-\frac{1}{2}\xi \cdot Eu$ *disappears. Otherwise, if one uses the perturbation form P2, then from the property (f) above the dissipation rate from* $\mathbf{L}$ *when the energy estimates are applied is not enough to control the term* $-\frac{1}{2}\xi \cdot Fu$ *which increases in velocity with the rate* $|\xi|$.

2 Solutions near equilibrium states

Corresponding to those issues mentioned in the last section, we expose some main results obtained so far, along two different lines based on the analysis on the nonlinear or linearized equation respectively.

2.1 Well-posedness, stability and convergence rate. Suppose

$$F \equiv 0, \quad S \equiv 0, \quad n = 3, \tag{4}$$

i.e., there is no force and source and the physical three dimension is considered. Notice that all results about the existence and stability in this topic still hold for $n \ge 1$ since the proof is only based on the energy method. Under the above assumptions, we consider the Cauchy problem of the perturbation u:

$$\begin{cases} \partial_t u + \xi \cdot \nabla_x u = \mathbf{L}u + \Gamma(u,u), \\ u(0,x,\xi) = u_0(x,\xi). \end{cases} \tag{$(CP)_0$}$$

For fixed t, define the energy functional as

$$[[u(t)]]^2 \equiv \sum_{|\alpha| \le N} \|\partial_x^\alpha u(t)\|^2,$$

and the dissipation rate as

$$[[u(t)]]_\nu^2 \equiv \|\{\mathbf{I} - \mathbf{P}\}u(t)\|_\nu^2 + \sum_{0<|\alpha| \le N} \|\partial_x^\alpha u(t)\|_\nu^2,$$

where $N \ge 4$ is an integer. The following two results are proved in [1].

Theorem I (Well-posedness). *Suppose* (2) *and* (4) *hold. Let* $f_0(x,\xi) = \mathbf{M} + \sqrt{\mathbf{M}}u_0(x,\xi) \ge 0$. *There exist* $\delta_0 > 0$, $\lambda_0 > 0$ *and* $C_0 > 0$ *such that, if*

$$[[u(0)]] \le \delta_0,$$

then there exists a unique solution $u(t,x,\xi)$ *to* $(CP)_0$ *such that* $f(t,x,\xi)=\mathbf{M}+\sqrt{\mathbf{M}}u(t,x,\xi)\geq 0$, *and*

$$[[u(t)]]^2+\lambda_0\int_0^t [[u(s)]]_\nu^2 ds \leq C_0[[u(0)]]^2,\ \forall\ t\geq 0.$$

Theorem II (Uniform stability). *Let all conditions in Theorem I hold. Let*

$$f_0(x,\xi)=\mathbf{M}+\sqrt{\mathbf{M}}u_0(x,\xi)\geq 0,\quad g_0(x,\xi)=\mathbf{M}+\sqrt{\mathbf{M}}v_0(x,\xi)\geq 0.$$

There exist $\delta_1\in(0,\delta_0)$, $\lambda_1>0$ *and* $C_1>0$ *such that, if*

$$\max\{[[u(0)]],[[v(0)]]\}\leq \delta_1,$$

then the solutions $u(t,x,\xi)$, $v(t,x,\xi)$ *obtained in Theorem I satisfy*

$$[[u(t)-v(t)]]^2+\lambda_1\int_0^t [[u(s)-v(s)]]_\nu^2 ds \leq C_1[[u(0)-v(0)]]^2,\ \forall\ t\geq 0.$$

We remark that the above results also hold for the general collision kernel satisfying (3).

In fact, the global existence and uniqueness of solutions near Maxwellians as in Theorem I have been already shown in some other function spaces. Here, it is necessary to give some illuminations to show that the solution space $L^2(\mathbb{R}^3_\xi;H^N(\mathbb{R}^3_x))$ is new.

Up to now, there are three kinds of methods to deal with the well-posedness of the nonlinear Boltzmann equation as follows:

- Spectral analysis and bootstrap arguments;
- Macro-micro decomposition and L^2 energy method;
- Green's functions and pointwise estimates.

Next we present the solutions spaces in which these methods work. The first global existence theorem was established in the space

$$L^\infty_{\beta_1}(\mathbb{R}^3_\xi;H^k(\mathbb{R}^3_x)),\quad \beta_1>\frac{5}{2},\ k\geq 2, \tag{5}$$

by using the spectral analysis [19, 24, 25], where

$$L^\infty_{\beta_1}(\mathbb{R}^3_\xi)\equiv\{u;(1+|\xi|)^{\beta_1}u\in L^\infty(\mathbb{R}^3_\xi)\}.$$

The same result was obtained in [21] for the torus case with the space

$$L^\infty_{\beta_1}(\mathbb{R}^3_\xi;C^k(\mathbb{T}^3_x)),\quad \beta_1>\frac{5}{2},\ k=0,1,\cdots. \tag{6}$$

Recently, [26] presented a function space

$$L^2(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)\cap L^\infty_{\beta_2}(\mathbb{R}^3_\xi;L^\infty(\mathbb{R}^3_x)),\quad \beta_2>\frac{3}{2}, \tag{7}$$

in which the Cauchy problem is globally well-posed in a mild sense without any regularity conditions. Notice that if the spatial regularity is neglected for the moment, the solution space $L^2_\xi(H^N_x)$ in Theorem I is larger than (5), (6) and (7) in the sense that the velocity integrability in $L^2_\xi(H^N_x)$ is the lowest among them since the following strict inclusion relations hold

$$L^\infty_{\beta_1}(\mathbb{R}^3_\xi) \subsetneqq L^\infty_{\beta_2}(\mathbb{R}^3_\xi) \subsetneqq L^2(\mathbb{R}^3_\xi),$$

where it has been supposed that β_1 and β_2 are sufficiently close to $5/2$ and $3/2$, respectively. On the other hand, by means of the classical energy method [9, 15, 14], the well-posedness was also established in the Sobolev space

$$H^{N(n_1,n_2,n_3)}_{t,x,\xi}(\mathbb{R}^3_x \times \mathbb{R}^3_\xi), \tag{8}$$

which denotes a set of all functions whose derivatives of all variables t, x and ξ up to N order are integrable in $L^2(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)$, where

$$N = N(n_1, n_2, n_3) \equiv n_1 + n_2 + n_3 \geq 4.$$

In particular, for the case without any external force, n_3 can be taken as zero, which means that the velocity derivatives need not be considered [9, 14, 15, 30], while they have to be included for the case with forcing [3, 4, 10, 11, 22, 28]. Compared with (8), the solution space $L^2_\xi(H^N_x)$ in Theorem I is again in the weak form

$$L^2_\xi(H^N_x) = H^{N(0,N,0)}_{t,x,\xi}(\mathbb{R}^3_x \times \mathbb{R}^3_\xi),$$

with $n_1 = n_3 = 0$. Finally, in terms of the Green's function to the linearized Boltzmann equation, the well-posedness is firstly established by Liu-Yu [16] in the space

$$L^\infty(\mathbb{R}^3_x, w(t,x)dx; L^\infty_{\beta_3}(\mathbb{R}^3_\xi)), \quad \beta_3 \geq 3,$$

where the pointwise weight function,

$$w(t,x) = e^{-C(|x|+\beta_3 t)} + \frac{e^{-\frac{(|x|-\sqrt{5/3}t)^2}{Ct}}}{(1+t)^2} + \{\textit{acoustic cone}\},$$

exposes the wave structure of convergence of solutions to the Maxwellian. Notice that in this case, the initial perturbation u_0 should decay exponentially in x.

The key point for the proof of Theorem I as in [8] to deal with the diffusive limit of the Boltzmann equation is to introduce some temporal interactive energy functionals between the microscopic part $\{\mathbf{I}-\mathbf{P}\}u$ and the macroscopic part $\mathbf{P}u$:

$$\mathcal{I}^a_{\alpha,i}(u(t)),\ \mathcal{I}^b_{\alpha,i}(u(t)),\ \mathcal{I}^c_{\alpha,i}(u(t)),\ \ \mathcal{I}^{ab}_{\alpha,i}(u(t)),$$

which turn out to be the inner products between the microscopic and macroscopic parts, where a, b, c are the coefficients of $\mathbf{P}u$, i.e.

$$\mathbf{P}u = \left\{ a(t,x) + \sum_{i=1}^3 b_i(t,x)\xi_i + c(t,x)|\xi|^2 \right\} \sqrt{\mathbf{M}}.$$

By the energy estimates, one can obtain a Lyapunov-type inequality

$$\frac{d}{dt}\mathcal{E}_M(u(t)) + C\mathcal{D}(u(t)) \leq C\mathcal{E}_M(u(t))\mathcal{D}(u(t)),$$

where the constant $M > 0$ is chosen to be large enough and

$$\begin{aligned}
\mathcal{E}_M(u(t)) &\equiv \frac{M}{2}[[u(t)]]^2 + 2\mathcal{I}(u(t)),\\
\mathcal{I}(u(t)) &\equiv \sum_{|\alpha|\leq N-1}\sum_{i=1}^{3}\left[\mathcal{I}^a_{\alpha,i}(u(t)) + \mathcal{I}^b_{\alpha,i}(u(t)) + \mathcal{I}^c_{\alpha,i}(u(t)) + \mathcal{I}^{ab}_{\alpha,i}(u(t))\right],\\
\mathcal{D}(u(t)) &\equiv \sum_{|\alpha|\leq N}\|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\|_\nu^2 + \sum_{|\alpha|\leq N-1}\|\nabla_x\partial_x^\alpha(a,b,c)\|^2.
\end{aligned}$$

Notice that the interactive functional $\mathcal{I}(u(t))$ can be bounded by $C[[u(t)]]^2$ for some constant C. Then $\mathcal{E}(u(t))$ and $\mathcal{D}(u(t))$ are indeed the equivalent energy functional and dissipation rate.

Remark II. *In [1], it was shown that by combining the macroscopic equations and the local macroscopic conservation laws, the Boltzmann equation can be written as the linearized viscous compressible Navier-Stokes equations with the remaining terms only related to the microscopic part* $\{\mathbf{I}-\mathbf{P}\}u$ *and the nonlinear term* $\Gamma(u,u)$*. Precisely,* $a = a(t,x), b = b(t,x)$ *and* $c = c(t,x)$ *satisfy*

$$\begin{aligned}
&\partial_t(a+3c) + \nabla_x\cdot b = 0,\\
&\partial_t b + \nabla_x(a+3c) + 2\nabla_x c - \Delta_x b - \frac{1}{3}\nabla_x\nabla_x\cdot b = R^b,\\
&\partial_t c + \frac{1}{3}\nabla_x\cdot b - \Delta_x c = R^c,
\end{aligned}$$

where R^b, R^c *are the remaining terms generated by* $\{\mathbf{I}-\mathbf{P}\}u$ *and* $\Gamma(u,u)$*. Therefore by the classical theory on the Navier-Stokes equations [18], it is reasonable to have the dissipation rate of the macroscopic part of higher order.*

The proof of Theorem II about the stability of solutions is almost the same as one of Theorem I. We mention the recent work [13] about the uniform stability in $L^2_{x,\xi} \equiv L^2_\xi(L^2_x)$ for solutions satisfying some general framework conditions.

Theorem I and Theorem II can be generalized to the case when an external force is present. For simplicity, let's suppose

$$F = \nabla_x\phi(x), \quad S = 0, \quad n = 3. \tag{9}$$

Similarly $n = 3$ can be replaced by $n \geq 1$. For the case of the general force, see the discussions in Subsection 4.1. Set the perturbation u by

$$f = e^{\phi(x)}\mathbf{M} + \sqrt{\mathbf{M}}u.$$

Consider the Cauchy problem corresponding to u:

$$\begin{cases} \partial_t u + \xi\cdot\nabla_x u + \nabla_x\phi(x)\cdot\nabla_\xi u - \frac{1}{2}\xi\cdot\nabla_x\phi(x)u = e^{\phi(x)}\mathbf{L}u + \Gamma(u,u),\\ \qquad\qquad u(0,x,\xi) = u_0(x,\xi). \end{cases} \tag{$(CP)_1$}$$

Define the energy functional

$$[[u(t)]]^2 \equiv \sum_{|\alpha|+|\beta|\leq N}\|\partial_x^\alpha\partial_\xi^\beta u(t)\|^2,$$

and the dissipation rate

$$[[u(t)]]_\nu^2 \equiv \|\{\mathbf{I}-\mathbf{P}\}u(t)\|_\nu^2 + \sum_{0<|\alpha|\le N} \|\partial_x^\alpha u(t)\|_\nu^2 + \sum_{|\alpha|+|\beta|\le N,\ |\beta|>0} \|\partial_x^\alpha \partial_\xi^\beta \{\mathbf{I}-\mathbf{P}\}u(t)\|_\nu^2.$$

Then we have the following

Theorem III (Global existence and uniform stability). *Suppose* (2) *and* (9) *hold. For the Cauchy problem* $(CP)_1$*, the statements in Theorem I and II still hold in terms of the above new energy functional and dissipation rate, provided that* $\phi \in L_x^\infty$ *and*

$$\|(1+|x|)^2\nabla_x\phi\|_{L_x^\infty} + \sum_{2\le|\alpha|\le N} \|(1+|x|)\partial_x^\alpha\phi\|_{L_x^\infty}$$

is sufficiently small.

Some remarks are given as follows.

Remark III. *(a) Theorem III shows that the Cauchy problem* $(CP)_1$ *for the Boltzmann equation with forces is well-posed in the space*

$$H_{x,\xi}^N = H^N(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)$$

and the solution is stable uniformly in time under the conditions of smallness and spatial-decay on the force. In order to prove Theorem III, we use the same method as for Theorem I and Theorem II but with an additional care for the estimates on the velocity derivatives as in [4, 22]. Furthermore, to avoid the time derivatives, we use the local macroscopic balance laws and the Hardy inequality to replace them by the spatial derivatives. The whole proof of Theorem III can be found in [2].

(b) For the case of the collision kernel satisfying (3) *with* $0 \le \gamma < 1$*, similar results hold but under the perturbation in another form*

$$f = e^{\phi(x)}\mathbf{M} + \sqrt{e^{\phi(x)}\mathbf{M}}u.$$

As before, the corresponding equation for u *reads*

$$\partial_t u + \xi\cdot\nabla_x u + \nabla_x\phi(x)\cdot\nabla_\xi u = e^{\phi(x)}\mathbf{L}u + e^{\phi(x)/2}\Gamma(u,u).$$

Thus the energy method still works.

For the solution to the Cauchy problem $(CP)_1$ obtained in Theorem III, we are also concerned with the decay-in-time estimates. Actually, by combing the energy estimates and the spectral analysis, we can obtain the optimal convergence rates, which is stated as follows.

Theorem IV (Optimal convergence rates). *Let all conditions in Theorem III hold. Further assume that* $\|u_0\|_{Z_1}$ *is bounded and*

$$\|(1+|x|)\phi\|_{L_x^\infty} + \||x|\nabla_x\phi\|_{L_x^2}$$

is small enough. Then the solution u *obtained in Theorem III satisfies*

$$[[u(t)]] \le C(1+t)^{-\frac{3}{4}}([[u_0]] + \|u_0\|_{Z_1}), \quad \forall\ t \ge 0.$$

Let's give a remark to explain what it means by the optimal convergence rate. When $F \equiv 0$, i.e., there is not any external force, the solution semigroup $\{e^{\mathbf{B}t}\}_{t\geq 0}$, where

$$\mathbf{B} = -\xi \cdot \nabla_x + \mathbf{L}$$

corresponds to the evolution operator for the linearized Boltzmann equation, decays with an algebraic rate. Precisely, it holds that

$$\|\nabla_x^m e^{\mathbf{B}t} g\|_{L^2_{x,\xi}} \leq C(q,m)(1+t)^{-\sigma_{q,m}}(\|g\|_{Z_q} + \|\nabla_x^m g\|_{L^2_{x,\xi}}), \tag{10}$$

for the integer $m \geq 0$, $q \in [1,2]$ and any function $g = g(x,\xi)$, where the decay rate is measured by

$$\sigma_{q,m} = \frac{n}{2}\left(\frac{1}{q} - \frac{1}{2}\right) + \frac{m}{2},$$

with the spatial dimension $n \geq 3$, and $Z_q = L^2_\xi(L^q_x)$. The decay-in-time estimate (10) was proved by the spectral analysis due to Ukai [23, 24] and Nishida-Imai [19]. By the optimal convergence rate, it means that the decay rate is the same as that of $e^{\mathbf{B}t}$ when $F = 0$, at the level of zero order since $\sigma_{1,0} = 3/4$ in the three dimensional case.

We also review some known results on the convergence rates. For the Boltzmann equation without forces, the exponential convergence rate in bounded domain and torus was given by Ukai , Giraud, Shizuta-Asano; Algebraic convergence rate in unbounded domain by Ukai, Nishida-Imai, Ukai-Asano; Almost exponential convergence rate by Strain-Guo, Desvillettes-Villani; Optimal convergence rate with extra time decay by Ukai-Yang. For the case with external forces, the convergence rate in L^∞ framework was obtained by Asano; Convergence rate in L^2 framework by Ukai-Yang-Zhao; Almost exponential rate in Torus case by Mouhout-Neumann. Here, the list of references is skipped for simplicity and interested readers can turn to [2, 3, 4] and references therein.

Theorem IV has been obtained in [3] in a slightly different form, where the proof is based on the macro-micro decomposition near the local Maxwellian. In the next section, we shall give the sketch of another proof in terms of the energy estimates method as in [4]; the detailed proof can be found in [2]. It should be pointed out that, it is not trivial to directly get the optimal convergence rates from the spectral analysis because of the increasing for the large velocity in the collision kernel. Actually, the proof of Theorem IV strongly depends on the mathematical analysis of the transport part in the semigroup $e^{\mathbf{B}t}$. For this, define $\Psi[h](t,x,\xi)$ as the solution to the the following Cauchy problem on the transport equation with zero initial data and the nonhomogeneous source term νh:

$$\begin{cases} \partial_t u + \xi \cdot \nabla_x u + \nu(\xi) u = \nu(\xi) h(t,x,\xi), \\ u|_{t=0} = 0, \end{cases} \tag{11}$$

where we recall that in the case of the hard sphere gas, there is a constant $\nu_0 > 0$ such that

$$\nu_0(1+|\xi|) \leq \nu(\xi) \leq \frac{1}{\nu_0}(1+|\xi|). \tag{12}$$

Proposition I. *For any $\lambda \in (0,\nu_0)$, there exists C such that*

$$\int_0^t e^{-\lambda(t-s)} \|\Psi[h](s)\|^2_{L^2_{x,\xi}} ds \leq C \int_0^t e^{-\lambda(t-s)} \|h(s)\|^2_{L^2_{x,\xi}} ds.$$

This proposition plays an important role in making estimates on some remaining terms which increase in velocity variable with the algebraic rate $|\xi|$ in the case of the hard sphere. For completeness, the proof of Proposition I is also given in the next section.

2.2 Time decay estimates on the linearized Boltzmann equation with time-dependent forces, and time-periodic solutions. The results in this subsection come from [4]. Here we remark that the proof of [4] can be fully simplified by using the method developed in [1] combined with the idea of the proof of Theorem IV given in the next section. A related problem about more general linearized Boltzmann equation with variable coefficients and external force depending on all time, spatial and velocity variables will be discussed in the final section.

In order to find the time-periodic solution, we first start from the linearized Boltzmann equation in the whole space $\mathbb{R}^n$:

$$\partial_t u + \xi \cdot \nabla_x u + F \cdot \nabla_\xi u - \frac{1}{2}\xi \cdot F u = \mathbf{L}u, \qquad (LBE)$$

where

$$F = F(t,x)$$

is the external force depending only on time and spatial variables, and the dimension n will go into the later mathematical analysis. Fix any $s \in \mathbb{R}$ and set the initial data

$$u(t,x,\xi)|_{t=s} = u_s(x,\xi).$$

The solution to (LBE) is denoted by

$$U(t,s)u_s, \quad -\infty < s \le t < \infty,$$

where $U(t,s)$ is called the solution operator for the linearized equation (LBE). For any number k, define a weighted norm $[[\cdot]]_k$ over the Sobolev space $H^\ell(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ by

$$[[u]]_k = \sum_{0\le|\alpha|+|\beta|\le\ell} \|\nu^k \partial_x^\alpha \partial_\xi^\beta u\|.$$

Theorem V (Time decay estimates). *Suppose that*

(i) the integers $n \ge 3$, $\ell \ge 2$ *and the number* $1 \le q < \frac{2n}{n+2}$*;*

(ii) there a small constant $\delta > 0$ *such that*

$$\sum_{0\le|\beta|\le\ell} \left\|(1+|x|)\partial_x^\beta F(t,x)\right\|_{L^\infty_{t,x}} + \sum_{0\le|\beta|\le\ell-1} \left\|(1+|x|)\partial_t\partial_x^\beta F(t,x)\right\|_{L^\infty_{t,x}} \le \delta,$$

and

$$\||x|F(t,x)\|_{L^\infty_t\left(L_x^{2q/(2-q)}\right)} \le \delta.$$

Then for any $k \ge 1$*, there exist a constant* C *such that,* $U(t,s)$*,* $-\infty < s \le t < \infty$ *satisfies*

$$[[U(t,s)u_s]]_k \le C(1+t-s)^{-\sigma_{q,0}}([[u_s]]_k + \|u_s\|_{Z_q}),$$

for any $u_s = u_s(x,\xi)$*, where the constant* C *depends only on* n*,* ℓ*,* q*,* k *and* δ*.*

To our knowledge, Theorem V is the first result about the time decay estimates for the linearized Boltzmann equation with general forces. The method of the direct spectral analysis by Fourier transformation seems very difficult to apply to this case.

Some remarks follow. Firstly, the proof of Theorem V is similar to one for Theorem IV and the time decay rate $\sigma_{q,0}$ is optimal at the level of zero order. Secondly, in Theorem V, there is not any decay-in-time assumption on $F(t,x)$ and the space-weighted norm of F can be replaced by

$$\|\partial_x^\beta F(t,x)\|_{L_t^\infty(L_x^3)},\ etc.$$

Thirdly, if $F \neq 0$, the optimal convergence rate is not known for $\frac{2n}{n+2} \leq q \leq 2$ or for the general general collision kernel satisfying (3) but with $0 \leq \gamma < 1$. Finally, the condition $k \geq 1$ can be relaxed to $k \geq 0$ by using the same proof as in the next section.

Now we turn to the existence of the time-periodic solution to the nonlinear Boltzmann equation

$$\partial_t u + \xi \cdot \nabla_x u + F \cdot \nabla_\xi u - \frac{1}{2}\xi \cdot F u = \mathbf{L}u + \Gamma(u) + \widetilde{S}, \qquad (BE)_T$$

where let's recall that $\widetilde{S}$ is given by

$$\widetilde{S} = \mathbf{M}^{-1/2} S + \mathbf{M}^{1/2} \xi \cdot F.$$

For this purpose, for given $k \geq 1$ and fixed period T, define the norm $|||\cdot|||_{k,T}$ by

$$|||u|||_{k,T}^2 = \sup_{0\leq t\leq T} [[u(t)]]_k^2 + \int_0^T [[u(s)]]_{k+1}^2 ds.$$

for any T-periodic function $u(t,x,\xi)$. Then we have the following

Theorem VI (Existence of time-periodic solutions). *Suppose that*

(i) the integers $n \geq 5$, $\ell \geq [n/2]+2$;

(ii) the functions $F = F(t,x)$ and $S = S(t,x,\xi)$ are time periodic with period T, satisfying

$$F \in C_b^i\left(\mathbb{R}_t; H^{\ell-i}(\mathbb{R}_x^n)\right),\quad i=0,1,\quad S \in C_b^0\left(\mathbb{R}_t; H^\ell(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)\right);$$

(iii) there are constants $\delta > 0$ and $k \geq 1$ such that F and S are bounded in the sense that

$$\begin{aligned}&\sum_{0\leq|\beta|\leq\ell} \left\|(1+|x|)\partial_x^\beta F(t,x)\right\|_{L_{t,x}^\infty}\\ &+\sum_{0\leq|\beta|\leq\ell-1} \left\|(1+|x|)\partial_t\partial_x^\beta F(t,x)\right\|_{L_{t,x}^\infty} + \left\||x|F(t,x)\right\|_{L_t^\infty(L_x^2)} \leq \delta,\\ &\sup_{t\in\mathbb{R}}\left\{\|F(t)\|_{H_x^\ell\cap L_x^1} + [[\mathbf{M}^{-1/2}S(t)]]_{0,k-1/2} + \left\|\mathbf{M}^{-1/2}S(t)\right\|_{Z_1}\right\} \leq \delta.\end{aligned}$$

Then the equation $(BE)_T$ has a unique time periodic solution

$$u^* \in C_b^i\left(\mathbb{R}_t; H^{\ell-i}(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)\right),\quad i=0,1,$$

with the same period T, such that $|||u|||_{k,T} < \infty$.

If $F \equiv 0$, the existence of time-periodic solution when $n \geq 3$ was obtained by Ukai [23, 25] in the space $L^2_\beta(\mathbb{R}^3_\xi, H^\ell_x)$ and Ukai-Yang [26] in the space $L^2_{x,\xi} \cap L^\infty_\beta$. Notice that if F and S do not depend on t, then the periodic solution reduces to the stationary solution. So far, the physical three dimensional case is open. Finally it could be also possible to relax $k \geq 1$ to $k \geq 0$ as remarked for Theorem V, in particular including the case $k = 0$, in terms of the aforementioned new solution space.

Arguments for the proof of Theorem VI follow from [26]. In fact, on one hand, if a T-periodic solution $u(t)$ exists, then by the Duhamel's principle it holds that

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)N(\tau)d\tau, \quad t \geq s,$$

where the nonlinear term is in the form

$$N(\tau) = \Gamma(u(\tau), u(\tau)) + \widetilde{S}(\tau).$$

Particularly one can take $s = -mT$ for any integer m. Because of the periodicity assumption of $u(t)$, one has

$$u(s) = u(0).$$

Applying the time decay estimates on the solution operator $U(t,s)$ of the linearized Boltzmann equation, one has

$$[[U(t,s)u(s)]]_k \leq C(1+t+mT)^{-\sigma_{q,0}}([[u(0)]]_k + \|u(0)\|_{Z_q}).$$

Let $m \to \infty$. Then formally $u(t)$ satisfies

$$u(t) = \Phi(u)(t), \ \Phi(u)(t) \equiv \int_{-\infty}^t U(t,s)N(s)ds.$$

On the other hand, if F and S (thus $\widetilde{S}$) and u are T-periodic, so is $\Phi(u)(t)$ since the periodicity of F and S implies

$$U(t+T, s+T) = U(t,s), \ t \geq s,$$

and

$$\begin{aligned}
\Phi(u)(t+T) &= \int_{-\infty}^{t+T} U(t+T,s)N(s)ds \\
&= \int_{-\infty}^{t} U(t+T,s+T)N(s+T)ds \\
&= \int_{-\infty}^{t} U(t,s)N(s)ds = \Phi(u)(t).
\end{aligned}$$

Then if the uniqueness is supposed, the fixed point of Φ must be the desired T-periodic solution. Therefore, in a word, to obtain the time periodic solution, it suffices to show that the nonlinear functional Φ has a unique fixed point in the space

$$\mathbb{S} = \left\{ u = u(t,x,\xi); \begin{array}{l} u \text{ is time-periodic with period } T, \\ u \in C^0_b(\mathbb{R}_t; H^\ell(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)), \ |||u|||_* \leq C \end{array} \right\}.$$

Actually it is true when $\sigma_{1,0} = n/4 > 1$, i.e. $n \geq 5$.

In order to study the stability of the time periodic solution u^* obtained in Theorem VI, we shall consider the Cauchy problem

$$\begin{cases} \partial_t u + \xi \cdot \nabla_x u + F \cdot \nabla_\xi u - \frac{1}{2}\xi \cdot F u = \mathbf{L}u + \Gamma(u) + \widetilde{S}, \\ u(t,x,\xi)|_{t=t_0} = u_0(x,\xi), \end{cases}$$

for some $t_0 \in \mathbb{R}$, where $u = u(t,x,\xi)$, $(t,x,\xi) \in (t_0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n$. It it noticed that the initial time t_0 can be chosen arbitrarily. By putting

$$v = u - u^*,$$

the above initial value problem can be rewritten as

$$\begin{cases} \partial_t v + \xi \cdot \nabla_x v + F \cdot \nabla_\xi v - \frac{1}{2}\xi \cdot F v = \mathbf{L}v + \Gamma(v,v) + 2\Gamma(u^*,v), \\ v(t,x,\xi)|_{t=t_0} = v_0(x,\xi), \end{cases} \tag{$(BE)_{TS}$}$$

where

$$v_0(x,\xi) \equiv u_0(x,\xi) - u^*(t_0,x,\xi).$$

Theorem VII (Asymptotical stability of time periodic solutions). *Let all assumptions in Theorem VI hold and u^* be the corresponding time periodic solution obtained. Moreover, suppose that $u_0 \in H^\ell(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ and there are constants $\delta > 0$ and $k \geq 2$ such that*

$$[[v_0]]_k + \|v_0\|_{Z_1} \leq \delta.$$

Then the Cauchy problem $(BE)_{TS}$ has a unique global solution

$$v \in C^i_b\left([t_0,\infty); H^{\ell-i}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)\right), \quad i = 0,1,$$

with

$$[[v(t)]]_k \leq C\delta(1+t-t_0)^{-\sigma_{1,0}},$$

for some constant C.

For the proof of Theorem VII, because of appearance of the linear term $\Gamma(u^*,v)$ in $(BE)_{TS}$, one has to consider the time-decay estimates on the solution operator of the corresponding linearized equation of $(BE)_{TS}$. It turns out that the similar decay estimates as in Theorem V still hold. Thus the solution v to $(BE)_{TS}$ can be found as a fixed point of the nonlinear mapping in some space with time weight.

3 Proof of Theorem IV

In this section, we devote ourselves to the proof of Theorem IV. For this purpose, let's first prove Proposition I in Subsection 2.1, which plays a key role in the estimates on the optimal convergence rates.

Proof of Proposition I: Recall the definition (11) for the linear damped transport operator $\Psi[\cdot]$. Then one has the explicit representation

$$\Psi[h](t,x,\xi) = \int_0^t e^{-\nu(\xi)(t-\theta)}\nu(\xi)h(\theta, x-(t-\theta)\xi, \xi)d\theta.$$

Compute

$$\begin{aligned}\|\Psi[h](s)\|^2_{L^2_{x,\xi}} &= \int_{\mathbb{R}^3_x\times\mathbb{R}^3_\xi} |\Psi[h](s,x,\xi)|^2 dxd\xi \\ &= \sum_{R=0}^{\infty}\int_{\mathbb{R}^3_x\times\{R\le|\xi|<R+1\}} |\Psi[h](s,x,\xi)|^2 dxd\xi \\ &= \sum_{R=0}^{\infty} \|\Psi[h](s)\|^2_{L^2(\mathbb{R}^3_x\times\Omega_\xi(R))},\end{aligned}$$

where R is an integer and for each $R=0,1,2,\cdots$, $\Omega_\xi(R)$ is denoted by

$$\Omega_\xi(R) = \{\xi\in\mathbb{R}^3; R\le|\xi|<R+1\}.$$

For any $(x,\xi)\in\mathbb{R}^3_x\times\Omega_\xi(R)$, it holds that

$$\begin{aligned}&|\Psi[h](s,x,\xi)| \\ &= \left|\int_0^s e^{-\nu(\xi)(s-\theta)}\nu(\xi)h(\theta,x-(s-\theta)\xi,\xi)d\theta\right| \\ &\le \int_0^s e^{-\nu_0(1+R)(s-\theta)}\frac{1}{\nu_0}(2+R)|h(\theta,x-(s-\theta)\xi,\xi)|d\theta,\end{aligned}$$

where we used the pointwise property (12) for the collision frequency $\nu(\xi)$ in the case of the hard sphere gas. The Minkowski inequality and the Hölder inequality give

$$\begin{aligned}&\|\Psi[h](s)\|_{L^2(\mathbb{R}^3_x\times\Omega_\xi(R))} \\ &\le \int_0^s \left\|e^{-\nu_0(1+R)(s-\theta)}\frac{2+R}{\nu_0}h(\theta,x-(s-\theta)\xi,\xi)\right\|_{L^2(\mathbb{R}^3_x\times\Omega_\xi(R))} d\theta, \\ &= \frac{2+R}{\nu_0}\int_0^s e^{-\nu_0(1+R)(s-\theta)}\|h(\theta,x-(s-\theta)\xi,\xi)\|_{L^2(\mathbb{R}^3_x\times\Omega_\xi(R))}d\theta \\ &= \frac{2+R}{\nu_0}\int_0^s e^{-\nu_0(1+R)(s-\theta)}\|h(\theta,x,\xi)\|_{L^2(\mathbb{R}^3_x\times\Omega_\xi(R))}d\theta \\ &\le \frac{2+R}{\nu_0}\left[\int_0^s e^{-\nu_0(1+R)(s-\theta)}d\theta\right]^{1/2} \\ &\quad\times\left[\int_0^s e^{-\nu_0(1+R)(s-\theta)}\|h(\theta,x,\xi)\|^2_{L^2(\mathbb{R}^3_x\times\Omega_\xi(R))}d\theta\right]^{1/2}.\end{aligned}$$

Noticing that

$$\int_0^s e^{-\nu_0(1+R)(s-\theta)}d\theta = \frac{1}{\nu_0(1+R)}\left[1-e^{-\nu_0(1+R)}\right] \le \frac{1}{\nu_0(1+R)}.$$

one has

$$\|\Psi[h](s)\|^2_{L^2(\mathbb{R}^3_x\times\Omega_\xi(R))} \le \frac{(2+R)^2}{\nu_0^3(1+R)}\int_0^s e^{-\nu_0(1+R)(s-\theta)}\|h(\theta)\|^2_{L^2(\mathbb{R}^3_x\times\Omega_\xi(R))}d\theta.$$

which further implies

$$\begin{aligned}
&\int_0^t e^{-\lambda(t-s)}\|\Psi[h](s)\|_{L^2_{x,\xi}}^2 ds\\
&=\sum_{R=0}^{\infty}\int_0^t e^{-\lambda(t-s)}\|\Psi[h](s)\|_{L^2(\mathbb{R}^3_x\times\Omega_\xi(R))}^2 ds\\
&\leq \sum_{R=0}^{\infty}\frac{(2+R)^2}{\nu_0^3(1+R)}\int_0^t e^{-\lambda(t-s)}\left[\int_0^s e^{-\nu_0(1+R)(s-\theta)}\,\|h(\theta)\|_{L^2(\mathbb{R}^3_x\times\Omega_\xi(R))}^2\,d\theta\right]ds.
\end{aligned}$$

It follows from the Fubini Theorem that

$$\begin{aligned}
&\int_0^t e^{-\lambda(t-s)}\left[\int_0^s e^{-\nu_0(1+R)(s-\theta)}\,\|h(\theta)\|_{L^2(\mathbb{R}^3_x\times\Omega_\xi(R))}^2\,d\theta\right]ds\\
&=\int_0^t e^{-\lambda t+\nu_0(1+R)\theta}\,\|h(\theta)\|_{L^2(\mathbb{R}^3_x\times\Omega_\xi(R))}^2\left[\int_\theta^t e^{-[\nu_0(1+R)-\lambda]s}ds\right]d\theta.
\end{aligned}$$

Notice that since $0<\lambda<\nu_0$, then $\nu_0(1+R)-\lambda>0$ holds for any nonnegative integer R. Thus for any $\theta\in[0,t]$,

$$\begin{aligned}
\int_\theta^t e^{-[\nu_0(1+R)-\lambda]s}ds&=\frac{1}{\nu_0(1+R)}\left[e^{-[\nu_0(1+R)-\lambda]\theta}-e^{-[\nu_0(1+R)-\lambda]t}\right]\\
&\leq\frac{1}{\nu_0(1+R)}e^{-[\nu_0(1+R)-\lambda]\theta}.
\end{aligned}$$

Therefore, using the above inequalities, one can obtain

$$\begin{aligned}
&\int_0^t e^{-\lambda(t-s)}\|\Psi[h](s)\|_{L^2_{x,\xi}}^2 ds\\
&\leq\sum_{R=0}^{\infty}\frac{(2+R)^2}{\nu_0^4(1+R)^2}\int_0^t e^{-\lambda(t-\theta)}\,\|h(\theta)\|_{L^2(\mathbb{R}^3_x\times\Omega_\xi(R))}^2\,d\theta\\
&\leq\frac{4}{\nu_0^4}\sum_{R=0}^{\infty}\int_0^t e^{-\lambda(t-\theta)}\,\|h(\theta)\|_{L^2(\mathbb{R}^3_x\times\Omega_\xi(R))}^2\,d\theta\\
&=\frac{4}{\nu_0^4}\lim_{M\to\infty}A_M,
\end{aligned}$$

where A_M denotes the partial summation of the nonnegative series and it can be bounded in such a way that for each M,

$$\begin{aligned}
A_N&\equiv\sum_{R=0}^{N}\int_0^t e^{-\lambda(t-\theta)}\,\|h(\theta)\|_{L^2(\mathbb{R}^3_x\times\Omega_\xi(R))}^2\,d\theta\\
&=\int_0^t e^{-\lambda(t-\theta)}\sum_{R=0}^{N}\|h(\theta)\|_{L^2(\mathbb{R}^3_x\times\Omega_\xi(R))}^2\,d\theta\\
&=\int_0^t e^{-\lambda(t-\theta)}\,\|h(\theta)\|_{L^2(\mathbb{R}^3_x\times\{|\xi|<N+1\})}^2\,d\theta\\
&\leq\int_0^t e^{-\lambda(t-\theta)}\,\|h(\theta)\|_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)}^2\,d\theta.
\end{aligned}$$

Thus one has

$$\int_0^t e^{-\lambda(t-s)}\|\Psi[h](s)\|^2_{L^2_{x,\xi}}\,ds \le \frac{4}{\nu_0^4}\int_0^t e^{-\lambda(t-\theta)}\,\|h(\theta)\|^2_{L^2_{x,\xi}}\,d\theta.$$

This completes the proof of Proposition I.

Proof of Theorem IV: The proof is divided by five steps.

Step 1. Let $u(t,x,\xi)$ be the solution to the Cauchy problem $(CP)_1$ which is obtained in Theorem III. Denote δ_0, K_0, ϵ, respectively, by

$$\delta_0 = [[u_0]],\ K_0 = \|u_0\|_{Z_1},\ \epsilon = \sum_{|\alpha|\le N}\|(1+|x|)\partial_x^\alpha\phi\|_{L^\infty_x} + \||x|\nabla_x\phi\|_{L^2_x}.$$

Notice that δ_0, ϵ can be small enough, and K_0 is finite.

Step 2. From the energy estimates of higher order as in [4], one can obtain

$$\frac{d}{dt}\mathcal{E}_M(u(t)) + \lambda[[u(t)]]^2_\nu \le C\|\nabla_x u\|^2.$$

Here M is a fixed large constant, and $\mathcal{E}_M(u(t))$ is equivalent with $[[u(t)]]^2_H$ in the sense that

$$\frac{1}{C}[[u(t)]]^2_H \le \mathcal{E}_M(u(t)) \le C[[u(t)]]^2_H,$$

where $[[u(t)]]^2_H$ is given by

$$[[u(t)]]^2_H \equiv \|\{\mathbf{I}-\mathbf{P}\}u(t)\|^2 + \sum_{0<|\alpha|\le N}\|\partial_x^\alpha u(t)\|^2 + \sum_{|\alpha|+|\beta|\le N,\ |\beta|>0}\|\partial_x^\alpha\partial_\xi^\beta\{\mathbf{I}-\mathbf{P}\}u(t)\|^2.$$

With the observation that

$$[[u(t)]]_H \le C[[u(t)]]_\nu,$$

one has

$$\frac{d}{dt}\mathcal{E}_M(u(t)) + \lambda\mathcal{E}_M(u(t)) \le C\|\nabla_x u\|^2. \tag{13}$$

Step 3. The solution u to the Cauchy problem $(CP)_1$ can be written as the following mild form

$$u(t) = e^{\mathbf{B}t}u_0 + \int_0^t e^{\mathbf{B}(t-s)}\mathbf{S}[u](s)ds,$$

where $\mathbf{S}[u]$ is the source term given by

$$\mathbf{S}[u] = -\nabla_x\phi\cdot\nabla_\xi u + \frac{1}{2}\nabla_x\phi\cdot\xi u + (e^\phi - 1)\mathbf{L}u + \Gamma(u,u).$$

For this time being, let's recall a very useful proposition about the decomposition of the semigroup $e^{\mathbf{B}t}$ and the corresponding decay estimates, which was proved in [26].

Proposition II. *The semigroup $e^{\mathbf{B}t}$ can be decomposed as*

$$e^{\mathbf{B}t} = \mathbf{E}_0(t) + \mathbf{E}_1(t) + \mathbf{E}_2(t).$$

Here, $\mathbf{E}_0(t)$ *is the linear transport operator with the collision frequency* $\nu(\xi)$*, and defined by*

$$\mathbf{E}_0(t)u \equiv e^{-\nu(\xi)t}u(x-\xi t,\xi).$$

$\mathbf{E}_1(t)\nu$ *has an algebraic decay while* $\mathbf{E}_2(t)\nu$ *has an exponential decay. Precisely, it holds that*

$$\begin{aligned}\|\nabla_x^m \mathbf{E}_1(t)\nu u\|_{Z_2} &\le C(1+t)^{-\sigma_{q,m}}\|u\|_{Z_q},\\ \|\nabla_x^m \mathbf{E}_1(t)\{\mathbf{I}-\mathbf{P}\}\nu u\|_{Z_2} &\le C(1+t)^{-\sigma_{q,m}}\|u\|_{Z_q},\\ \|\nabla_x^m \mathbf{E}_2(t)\nu u\|_{Z_2} &\le Ce^{-\lambda t}\|\nabla_x^m u\|_{Z_2},\end{aligned}$$

for any nonnegative integer m *and* $q\in[1,2]$.

Thus one can rewrite the solution u as the summation of two parts as follows

$$\begin{aligned} u(t) &= I_1[u](t)+I_2[u](t),\\ I_1[u](t) &= e^{\mathbf{B}t}u_0+\int_0^t\{\mathbf{E}_1(t-s)+\mathbf{E}_2(t-s)\}\mathbf{S}[u](s)ds,\\ I_2[u](t) &= \int_0^t \mathbf{E}_0(t-s)\mathbf{S}[u](s)ds.\end{aligned}$$

Define

$$\mathcal{E}(t)=\sup_{0\le s\le t}(1+s)^{\frac{5}{2}}\mathcal{E}_M(u(s)),$$

where it is noticed that $5/2$ corresponds to $2\sigma_{1,0}$ in the case of the three dimensional case and also that $\mathcal{E}(t)$ is an increasing function. From Proposition II, it is direct to show as in [4] that

$$\|\nabla_x I_1[u](t)\|^2 \le C(1+t)^{-\frac{5}{2}}[\delta_0^2+K_0^2+(\delta_0^2+\epsilon^2)\mathcal{E}(t)]. \tag{14}$$

Step 4. We need carefully consider the estimate on $\|\nabla_x I_2[u](t)\|$ since the same decay estimate as in (14) does not hold for $I_2[u]$ any more. Instead, the time integration can be controlled. In fact, it follows from (13) and (14) that

$$\begin{aligned}\mathcal{E}_M(u(t)) &\le e^{-\lambda t}\mathcal{E}_M(u(0))+\int_0^t e^{-\lambda(t-s)}\|\nabla_x u(s)\|^2_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)}ds\\ &\le C(1+t)^{-\frac{5}{2}}[\delta_0^2+K_0^2+(\delta_0^2+\epsilon^2)\mathcal{E}(t)]\\ &\quad+\int_0^t e^{-\lambda(t-s)}\|\nabla_x I_2[u](t)\|^2_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)}ds.\end{aligned}$$

One can write

$$\nabla_x I_2[u](t)=\int_0^t \mathbf{E}_0(t-s)\nabla_x\mathbf{S}[u](s)ds=\Psi[\nabla_x\mathbf{S}[u]/\nu].$$

The direct calculations yield

$$\|\nabla_x\mathbf{S}[u]/\nu\|^2_{L^2_{x,\xi}}\le(\delta_0^2+\epsilon^2)\mathcal{E}_M(u(t)).$$

Thus by using Proposition I, one has

$$\begin{aligned}\mathcal{E}_M(u(t)) &\leq C(1+t)^{-\frac{5}{2}}[\delta_0^2 + K_0^2 + (\delta_0^2+\epsilon^2)\mathcal{E}(t)]\\ &\quad + \int_0^t e^{-\lambda(t-s)}\|\Psi[\nabla_x \mathbf{S}[u]/\nu]\|^2_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)}ds\\ &\leq C(1+t)^{-\frac{5}{2}}[\delta_0^2 + K_0^2 + (\delta_0^2+\epsilon^2)\mathcal{E}(t)].\end{aligned}$$

If δ_0 and ϵ are small enough, then $\mathcal{E}(t)$ is bounded uniformly in t and hence the following decay estimates on the energy of higher order hold:

$$\mathcal{E}_M(u(t)) \leq C(\delta_0^2 + K_0^2)(1+t)^{-\frac{5}{2}}. \tag{15}$$

Step 5. By using (15), one can obtain the decay estimate on $\|u(t)\|_{L^2_{x,\xi}}$ at the zero order in the same way as that in the last two steps. In fact, it follows from the standard energy estimates that

$$\frac{d}{dt}\mathcal{E}_{M_1}(u(t)) + \lambda[[u(t)]]^2_\nu \leq 0,$$

where as before M_1 is a large constant, and $\mathcal{E}_{M_1}(u(t))$ is an equivalent energy with $[[u(t)]]^2$ in the sense that

$$\frac{1}{C}[[u(t)]]^2 \leq \mathcal{E}_{M_1}(u(t)) \leq C[[u(t)]]^2.$$

Adding $\|\mathbf{P}u(t)\|^2_{L^2_{x,\xi}}$ to both sides, one has

$$\frac{d}{dt}\mathcal{E}_{M_1}(u(t)) + \lambda\mathcal{E}_{M_1}(u(t)) \leq C\|u(t)\|^2_{L^2_{x,\xi}}.$$

The Gronwall's inequality gives

$$\mathcal{E}_{M_1}(u(t)) \leq e^{-\lambda t}\mathcal{E}_{M_1}(u(0)) + C\int_0^t e^{-\lambda(t-s)}\|u(s)\|^2_{L^2_{x,\xi}}ds.$$

With the help of (15), the completely same procedure as in Step 2 and Step 3 can be repeated to obtain

$$\mathcal{E}_{M_1}(u(t)) \leq C(\delta_0^2 + K_0^2)(1+t)^{-\frac{3}{2}}.$$

This completes the proof of Theorem IV.

4 Discussions

In this section, based on the works introduced as before, we shall discuss some problems related to the Boltzmann equation near the equilibrium states in the whole space. Among these problems to be discussed, some could be direct to solve by carefully carrying out the further computations along the well-established line of proof, but some others could be difficult to deal with, which should depend on finding some new ideas to overcome the main difficulties.

4.1 Optimal decay rates in $H^N_{x,\xi}$ for the linearized Boltzmann equation with variable coefficient linear sources. It is natural to consider the following more general linearized Boltzmann equation with linear and variable coefficient sources

$$\begin{aligned}\partial_t u + \xi\cdot\nabla_x u - \mathbf{L}u &= A_0 K u + \sum_{|\alpha|+|\beta|\leq 1} A_{\alpha\beta}\partial_x^\alpha\partial_\xi^\beta u,\\ &\equiv \{A_0 K + A_{00}\}u + A_{10}\cdot\nabla_x u + A_{01}\cdot\nabla_\xi u,\end{aligned}$$

where

$$A_0 = A_0(t, x, \xi), \quad A_{\alpha\beta} = A_{\alpha\beta}(t, x, \xi)$$

satisfy some conditions on smallness in some space and may increase in $|\xi|$. This problem can be solved if the growth speed of velocity in all variable coefficients of sources when $|\xi|$ tends to infinity is not larger than one of $\nu(\xi)$ which is the collision frequency corresponding to the linearized collision operator $\mathbf{L}$. Otherwise, so far it is not known how to deal with this problem without any restriction on the growth speed in velocity of coefficients. A physical force inducing the above equation is in the form:

$$F(t, x, \xi) = E(t, x) + \xi \times B(t, x),$$

where E and B are the time-space dependent electric field and magnetic field, respectively.

4.2 Stability of the stationary solution to the Vlasov-Poisson-Boltzmann system with a nontrivial background density. The Vlasov-Poisson-Boltzmann system (VPB for simplicity) for particles of one species is written as

$$\begin{cases} \partial_t f + \xi \cdot \nabla_x f + \nabla_x \Phi \cdot \nabla_\xi f = Q(f, f), \\ \Delta \Phi = \int_{\mathbb{R}^3} f d\xi - \bar{\rho}(x), \end{cases} \tag{VPB}$$

where the Boltzmann equation is coupled with the Poisson equation with a given background density $\bar{\rho}(x)$. It is supposed that $\bar{\rho}(x)$ is near a positive constant state $\rho_\infty > 0$. Then, from [7, 5] there exists a smooth stationary solution $e^{\phi(x)}\mathbf{M}$ to (VPB) where $\phi(x)$ is determined by a second order elliptic equation with the exponential nonlinearity:

$$\Delta \phi(x) = e^{\phi(x)} - \bar{\rho}(x).$$

Set the perturbation u by

$$f = e^{\phi(x)}\mathbf{M} + \sqrt{\mathbf{M}}u.$$

Then u satisfies

$$\begin{cases} \partial_t u + \xi \cdot \nabla_x u + \nabla_x[\Psi + \phi(x)] \cdot \nabla_\xi u - \xi\sqrt{\mathbf{M}} \cdot e^{\phi(x)}\nabla_x \Psi - e^{\phi(x)}\mathbf{L}u \\ \qquad\qquad = \frac{1}{2}\xi \cdot \nabla_x[\Psi + \phi(x)]u + \Gamma(u, u), \\ \Delta \Psi = \int_{\mathbb{R}^3} \xi\sqrt{\mathbf{M}}u d\xi. \end{cases}$$

We can obtain the global existence and uniqueness results by carrying out the same energy estimates mentioned as before together with some ideas coming from Strain [22] to care about estimates on potential force when considering the more general two species Vlasov-Maxwell-Boltzmann system around the global Maxwellian. In fact, it is easy to get the energy estimates of zero order as follows:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\iint |u|^2 + \int e^{\phi(x)}|\nabla_x \Psi|^2\right) - \langle e^{\phi(x)}\mathbf{L}u, u\rangle \\ &= -\iint \Psi\nabla_x e^{\phi(x)} \cdot \xi\sqrt{\mathbf{M}}u - \int \Psi\nabla_x e^{\phi(x)} \cdot \partial_t \nabla_x \Psi \\ &\quad + \frac{1}{2}\langle \xi \cdot \nabla_x[\Psi + \phi(x)]u, u\rangle + \langle \Gamma(u, u), u\rangle. \end{aligned}$$

Energy estimates of higher order similarly hold. For the dissipation rate of the macroscopic part $\mathbf{P}u$, we can apply the standard elliptic-type energy estimates to all components

a, $b = (b_1, b_2, b_3)$, c of $\mathbf{P}u$. The important point is to obtain a Lyapunov-type energy inequality with both the energy functional and the dissipation rate not including the time derivatives. On the other hand, it is still an open problem to obtain the optimal convergence rates of solutions to the VPB system, for which see [27] for some investigations on this issue.

4.3 Time-periodic solution to the Boltzmann equation with general forces. As mentioned before, it is interesting and challenging to consider the existence of the time-periodic solution to the Boltzmann equation driven by a time-periodic external force in the case of the physical three dimension. Even though it has been solved for the dimension at least five which produces a strong enough decay-in-time rate such that the time-periodic solution can be obtained by finding the fixed point of a nonlinear mapping through the well-defined infinite time integral, it seems that in the physical three dimension one should meet with the essential difficulty so that some new idea need be used to overcome it.

As the first step in this direction, it is natural to consider the same case of the full compressible Navier-Stokes equations, for which the existence of time-periodic solutions in three-dimension is also open. However, there are some results known for the incompressible Navier-Stokes equations:

$$\begin{gathered}\partial_t u + u \cdot \nabla u + \nabla \pi = \Delta u + F(t,x), \quad t \in \mathbb{R}, x \in \mathbb{R}^3, \\ \nabla \cdot u = 0, \quad u \to 0 \ \text{ as } |x| \to \infty.\end{gathered}$$

Maremonti [17] in 1991 firstly proved that, roughly speaking, if $F(t,x)$ is T-periodic with

$$\begin{gathered}F(t,x) = \nabla \times \psi(t,x), \\ \sup_{t\geq 0} \|F(t)\|_{L^3_x} + \sup_{t\geq 0} \|\psi(t)\|_{L_x^{r/(r-1)}} \ll 1, \quad 3 < r < 6,\end{gathered}$$

then there exists a unique T-periodic solution u and it is also asymptotically stable in L^2-norm. As pointed out in [17], the problem of finding a time-periodic solution is generally equivalent to solving the Cauchy problem in such a way that if the system is driven by a time-dependent external force uniformly bounded in time, then it can be proved that there exists a global solution uniformly bounded in time. In the latter case, it implies that there could exist a steady state which is obtained by establishing the large time behavior of the solution to the Cauchy problem, and hence a time-periodic solution could be found by solving a special Cauchy problem with that steady state as given initial data. This is the so-called Serrin's method [20] to find the time-periodic solution. So, the key point in the whole thing is to find an appropriate solution space to prove this kind of theorem for the Cauchy problem. A possible choice is the Sobolev space with spatial weight functions.

4.4 Well-posedness of the Cauchy problem near Maxwellians in a space with mild regularity and integrability in the arguments. Different from the last issue, this problem is directly concerned with the Boltzmann equation without any force and source. Recall that we have given some solution spaces in which the Cauchy problem is well-posed. Even though there is no actual inclusion relation among them, they show the different regularity and integrability in temporal, velocity or spatial variable, respectively. As mentioned in Section 1, we shall be interested to find solutions spaces as larger as possible in which the Cauchy problem can be well-posed. By a large space it means one with mild regularity and integrability in all arguments. In particular, the following question is proposed for the future study.

Question: *Could the Cauchy problem* $(CP)_0$ *be well-posed in the spaces*

$$L^2(\mathbb{R}^3_x \times \mathbb{R}^3_\xi) \cap L^\infty(\mathbb{R}^3_x; L^2(\mathbb{R}^3_\xi))$$

or

$$L^\infty(\mathbb{R}^3_x, w(x)dx; L^2(\mathbb{R}^3_\xi))$$

for some appropriate algebraic weight function $w(x)$*?*

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