OPTIMAL CONVERGENCE RATES FOR THE COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH POTENTIAL FORCES

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For the viscous and heat-conductive fluids governed by the compressible Navier-Stokes equations with an external potential force, there exist non-trivial stationary solutions with zero velocity. By combining the $L^p - L^q$ estimates for the linearized equations and an elaborate energy method, the convergence rates are obtained in various norms for the solution to the stationary profile in the whole space, when the initial perturbation of the stationary solution and the potential force are small in some Sobolev norms. More precisely, the optimal convergence rates of the solution and its first order derivatives in L^2-norm are obtained when the L^1-norm of the perturbation is bounded.

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1. Introduction

The motion of compressible viscous and heat-conductive fluids in the whole space $\mathbb{R}^3$ can be described by the initial value problem of the compressible Navier-Stokes equations for the density ρ, velocity $u = (u^1, u^2, u^3)$ and temperature θ:

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ \rho\,[u_t + (u \cdot \nabla)u] + \nabla P(\rho, \theta) = \mu \Delta u + (\mu + \mu')\nabla(\nabla \cdot u) + \rho F, \\ \rho c_\nu [\theta_t + (u \cdot \nabla)\theta] + \theta P_\theta(\rho, \theta)\nabla \cdot u = \kappa \Delta \theta + \Psi(u). \end{cases} \tag{1.1}$$

In the following discussion, initial data satisfy

$$(\rho, u, \theta)(0, x) = (\rho_0, u_0, \theta_0)(x) \to (\rho_\infty, 0, \theta_\infty) \;\; \text{as} \;\; |x| \to \infty. \tag{1.2}$$

Here, $t \geq 0$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. $P = P(\rho, \theta)$, μ, μ', κ and c_ν are the pressure, the first and second viscosity coefficients, the coefficient of heat conduction and the specific heat at constant volume respectively. In addition, F is an external force and $\Psi = \Psi(u)$ is the classical dissipation function:

$$\Psi(u) = \frac{\mu}{2}\left(\partial_i u^j + \partial_j u^i\right)^2 + \mu'\left(\partial_j u^j\right)^2. \tag{1.3}$$

Throughout this paper, the above physical parameters and known functions are assumed to satisfy the following usual conditions:

A.1. μ, κ and c_ν are positive constants, while μ' is a constant satisfying $\mu' + \frac{2}{3}\mu \geq 0$.
A.2. ρ_∞ and θ_∞ are positive constants, and $P = P(\rho, \theta)$ is smooth in a neighborhood of $(\rho_\infty, \theta_\infty)$ satisfying $P_\rho(\rho_\infty, \theta_\infty) > 0$ and $P_\theta(\rho_\infty, \theta_\infty) > 0$.
A.3. There exists a function $\Phi \in H^5(\mathbb{R}^3)$ such that $F(x) = -\nabla \Phi(x)$.

As a sequence of the above assumptions, the stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\theta})(x)$ of (1.1)-(1.2) in a neighborhood of $(\rho_\infty, 0, \theta_\infty)$ is given by

$$\int_{\rho_\infty}^{\tilde{\rho}(x)} \frac{P_\rho(\eta, \theta_\infty)}{\eta} d\eta + \Phi(x) = 0, \quad \tilde{u}(x) = 0, \quad \tilde{\theta}(x) = \theta_\infty, \tag{1.4}$$

and satisfies

$$\|\tilde{\rho} - \rho_\infty\|_l \leq C\|\Phi\|_l, \quad 0 \leq l \leq 5, \tag{1.5}$$

$$\|\nabla \tilde{\rho}\|_{L^{6/5}} \leq C\|\nabla \Phi\|_{L^{6/5}}. \tag{1.6}$$

Under these assumptions, the nonlinear stability of the stationary solution to the problem (1.1)-(1.2) was proved by Matsumura and Nishida[15,16] as stated in the following proposition.

Proposition 1.1. *Under the assumptions A.1-A.3, there exist constants* $C_0 > 0$ *and* $\epsilon_0 > 0$ *such that if*

$$\|(\rho_0 - \rho_\infty, u_0, \theta_0 - \theta_\infty)\|_3 + \|\Phi\|_5 \leq \epsilon_0,$$

then the initial value problem (1.1)-(1.2) has a unique solution (ρ, u, θ) *globally in time and a unique stationary state* $(\tilde{\rho}, 0, \theta_\infty)$*, which satisfy*

$$\rho - \tilde{\rho} \in C^0(0, \infty; H^3(\mathbb{R}^3)) \cap C^1(0, \infty; H^2(\mathbb{R}^3)),$$
$$u, \theta - \theta_\infty \in C^0(0, \infty; H^3(\mathbb{R}^3)) \cap C^1(0, \infty; H^1(\mathbb{R}^3)),$$

and

$$\begin{aligned}
&\|(\rho-\tilde{\rho},u,\theta-\theta_\infty)(t)\|_3^2\\
&+\int_0^t (\|\nabla(\rho-\tilde{\rho},u,\theta-\theta_\infty)(s)\|_2^2+\|\nabla(u,\theta-\theta_\infty)(s)\|_3^2)\,ds\\
&\le C_0\|(\rho_0-\tilde{\rho},u_0,\theta_0-\theta_\infty)\|_3^2. \qquad (1.7)
\end{aligned}$$

Based on this stability result, the main purpose in this paper is to investigate the optimal convergence rates in time to the stationary solution. We remark that the convergence rate is an important topic in the study of the fluid dynamics for the purpose of the computation[5,6]. Since the background profile is non-trivial due to the effect of the external force, the analysis on the convergence rates is more delicate and difficult than the case without external forces. The main idea in this paper is to combine the L^p-L^q estimates for the linearized equations and an improved energy method which includes the estimation on the higher power of L^2-norm of solutions. By doing this, the optimal convergence rates for solutions to the nonlinear problem (1.1)-(1.2) in various norms can be obtained and are stated in the following theorem.

Theorem 1.1. *Let ϵ_0 be the constant defined in Proposition 1.1. There exist constants $\epsilon_1\in(0,\epsilon_0)$ and $C>0$ such that the following holds. For any $\epsilon\le\epsilon_1$, if*

$$\|(\rho_0-\rho_\infty,u_0,\theta_0-\theta_\infty)\|_3+\|\Phi\|_5+\|\nabla\Phi\|_{L^{6/5}}\le\epsilon, \qquad (1.8)$$

and

$$\rho_0-\rho_\infty,u_0,\theta_0-\theta_\infty\in L^1(\mathbb{R}^3), \qquad (1.9)$$

then, the solution (ρ,u,θ) in Proposition 1.1 enjoys the estimates

$$\|(\rho-\tilde{\rho},u,\theta-\theta_\infty)(t)\|_{L^p}\le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})},\quad 2\le p\le 6,\quad t\ge 0, \qquad (1.10)$$

$$\|(\rho-\tilde{\rho},u,\theta-\theta_\infty)(t)\|_{L^\infty}\le C(1+t)^{-\frac{5}{4}},\quad t\ge 0, \qquad (1.11)$$

$$\|\nabla(\rho-\tilde{\rho},u,\theta-\theta_\infty)(t)\|_2\le C(1+t)^{-\frac{5}{4}},\quad t\ge 0, \qquad (1.12)$$

$$\|(\rho_t,u_t,\theta_t)(t)\|\le C(1+t)^{-\frac{5}{4}},\quad t\ge 0. \qquad (1.13)$$

Remark 1.1. In Theorem 1.1, (1.8) together with (1.5)-(1.6) implies that

$$\|\tilde{\rho}-\rho_\infty\|_5+\|\nabla\tilde{\rho}\|_{L^{6/5}}\le C\epsilon. \qquad (1.14)$$

In addition, (1.9) shows that the perturbation of initial data around the constant state $(\rho_\infty,0,\theta_\infty)$ is bounded in L^1-norm, which need not be small.

Remark 1.2. The linearized equations of (1.1) around the constant state $(\rho_\infty,0,\theta_\infty)$ take the following form[12,16]:

$$\begin{cases}
\rho_t+\gamma\nabla\cdot u=0,\\
u_t-\mu_1\Delta u-\mu_2\nabla\nabla\cdot u+\gamma\nabla\rho+\lambda\nabla\theta=0,\\
\theta_t-\bar{\kappa}\Delta\theta+\lambda\nabla\cdot u=0,
\end{cases}$$

where μ_1, μ_2, γ, λ and $\bar{\kappa}$ are positive constants which will be given precisely in Section 2. Compared to the decay estimates of the solution to the above linearized equations by using Fourier analysis[12] stated in Lemma 2.1 in the next section, Theorem 1.1 gives the optimal decay rates for the solution in L^p-norm, $2 \leq p \leq 6$, and its first order derivatives in L^2-norm. However, notice that the convergence rates of the derivatives of higher order in L^2-norm and the solution in L^∞-norm are not the same as those for linearized equations. Even though it may not be feasible for the nonlinear system with external forces to have the same decay for higher order derivatives as the linearized one because the differentiation can be taken only on the stationary background profile, what are the optimal convergence rates for higher order derivatives is not known and is left for future study.

A lot of works have been done on the existence, stability, large time behavior and convergence rates of solutions to the compressible Navier-Stokes equations for either isentropic or nonisentropic (heat-conductive) case, cf.[15,16,8,11,17,23] and references therein; see also references[18,20] about some existence results of the stationary solutions to the compressible viscous Navier-Stokes equations with general external forces. In the following, we recall some studies on the convergence rates for the compressible Navier-Stokes equations in the whole space with or without external forces which are related to the results in this paper.

When there is no external force, Matsumura and Nishida[14] obtained the convergence rate for the compressible viscous and heat-conductive fluid in $\mathbb{R}^3$:

$$\|(\rho-\rho_\infty, u, \theta-\theta_\infty)(t)\|_2 \leq C(1+t)^{-\frac{3}{4}}, \quad t \geq 0,$$

if the small initial disturbance belongs to $H^4(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. For the same system, Ponce[19] gave the optimal L^p convergence rate

$$\|\nabla^l(\rho-\rho_\infty, u, \theta-\theta_\infty)(t)\|_{L^p} \leq C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{l}{2}}, \quad t \geq 0,$$

for $2 \leq p \leq \infty$ and $l = 0, 1, 2$, if the small initial disturbance belongs to $H^s(\mathbb{R}^n) \cap W^{s,1}(\mathbb{R}^n)$ with the integer $s \geq [n/2]+3$ and the space dimension $n = 2$ or 3. From the pointwise estimates through the study of the Green function, the optimal decay rates for the isentropic viscous fluid in $\mathbb{R}^n$, $n \geq 2$, were obtained by Hoff-Zumbrun[7] and Liu-Wang[13]:

$$\|(\rho-\rho_\infty, \rho u)(t)\|_{L^p} \leq \begin{cases} Ct^{-\frac{n}{2}(1-\frac{1}{p})}, & 2 \leq p \leq \infty, \\ Ct^{-\frac{n}{2}(1-\frac{1}{p})+\frac{n-1}{4}(\frac{2}{p}-1)} L_n(t), & 1 \leq p < 2, \end{cases}$$

for all large $t > 0$, if the small initial disturbance belongs to $H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ with the integer $s \geq [n/2]+3$, where $L_n(t)$ equals $\log(1+t)$ if $n = 2$ and 1 otherwise. This result was later generalized by Kobayashi-Shibata[12] and Kagei-Kobayashi[9,10] to the viscous and heat-conductive fluid and also to the half space problem but without the smallness of L^1-norm of the initial disturbance.

When there is an external potential force $F = -\nabla\Phi(x)$, there are also some results on the convergence rate for solutions to the compressible viscous Navier-Stokes equations. The difficulty in these analysis comes from the appearance of non-trivial stationary solutions. For this case, when the initial perturbation is not assumed in L^1, the analysis only on the Sobolev space $H^s(\mathbb{R}^3)$ yields a slower decay[2,3]. Recently, an almost optimal convergence rate in $L^2(\mathbb{R}^n)$, $n \geq 3$, was obtained by Ukai-Yang-Zhao[22]:

$$\|(\rho - \tilde{\rho}, u)(t)\|_3 \leq C(n,\delta)(1+t)^{-\frac{n}{4}+\delta}, \quad t \geq 0,$$

if the initial disturbance belongs to $H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, $s \geq [n/2]+2$ with only H^s-norm small, where $(\tilde{\rho}, 0)$ is the stationary solution and δ is any small positive constant. But $C(n,\delta)$ is a constant depending on n and δ which satisfies $C(n,\delta) \to \infty$ when δ tends to zero. The result in this paper generalizes the above one and improves its method. Finally, for the general (non-potential) external force $F = F(x)$, the velocity of the stationary solution may not be zero[20]. For this, the following convergence rate was obtained by Shibata-Tanaka[21] for the isentropic viscous fluid:

$$\|\nabla(\rho - \rho^*, u - u^*)(t)\|_2 \leq C(\delta)t^{-\frac{1}{2}+\delta}, \tag{1.15}$$

for any large $t > 0$, if the initial disturbance belongs to $H^3(\mathbb{R}^3) \cap L^{6/5}(\mathbb{R}^3)$ with only H^3-norm small, where (ρ^*, u^*) denotes the stationary solution and δ is any small positive constant.

Remark 1.3. Considers a potential force $F = -\nabla\Phi$ with Φ of the special form

$$\Phi(x) = \frac{\eta}{(1+|x|)^{1+r}},$$

where $\eta > 0$ is sufficiently small and r is a constant. Clearly, the condition (1.8) is satisfied if and only if $r > 1/2$ and then Theorem 1.1 holds on the optimal convergence rates for L^1 initial disturbance. On the other hand, it is seen that the argument of Ref. [21] is valid for this potential with $r \geq 0$ and the stationary solution defined by (1.4), to deduce the convergence rate (1.15) for $L^{6/5}$ initial disturbance. Thus, the optimal convergence rate depends on the spatial decay order of the potential function as well as the summability of the initial disturbance. A similar result holds for some negative r. This will be reported in a future.

To obtain the optimal convergence rates of the solution and its first order derivatives in L^2-norm for the case of the potential force in Theorem 1.1, besides the $L^p - L^q$ estimates on the solutions for the linearized system and the energy estimates for the nonlinear system, a new idea is to consider the estimates not only on the energy itself but also on its powers, which yields a uniform decay estimate when the power tends to infinity. That is, we are going to prove an estimate in the form of

$$(1+t)^{\frac{5p}{4}-\delta}\|\nabla(\sigma, w, z)(t)\|_2^p \leq C(\delta, p, \epsilon),$$

for all $\delta > 0, p \geq 2$ where (σ, w, z) is the solution to the reformulated problem (2.8) defined in the next section whereas ϵ is as in (1.8). Actually, we can only obtain the constant $C(\delta, p, \epsilon) > 0$ such that $C(\delta, p, \epsilon) \to \infty$ as $\delta \to 0$ for each fixed p, ϵ. In other words, this estimate cannot give the desired optimal convergence rate if p is fixed. However, for fixed $\delta > 0$, if we can show that

$$C(\delta, p, \epsilon)^{\frac{1}{p}} \to C(\delta, \epsilon) \text{ as } p \to \infty,$$

then, we get

$$\|\nabla(\sigma, w, z)(t)\|_2 \leq C(\delta, \epsilon)(1+t)^{-\frac{5}{4}},$$

which is the desired optimal estimate for the first order derivatives with respect to the space variable. This effective method to obtain the optimal convergence rate can be also applied to the other cases[4]. The optimal convergence rate of the solution in L^2-norm can then be obtained by using the integral formula of the solution to the nonlinear problem which will be given in Section 4.

The rest of the paper is organized as follows. In Section 2, the nonlinear problem is reformulated, and some basic inequalities and the decay properties of the linearized equations are given. Based on the $L^p - L^q$ and the energy estimates, some lemmas for obtaining optimal convergence rates are proved in Section 3. And then the main result Theorem 1.1 will be proved in the last section.

Notations. Throughout this paper, the norms in the Sobolev Spaces $H^m(\mathbb{R}^3)$ and $W^{m,p}(\mathbb{R}^3)$ are denoted respectively by $\|\cdot\|_m$ and $\|\cdot\|_{m,p}$ for $m \geq 0$, any $p \geq 1$. In particular, for $m = 0$, we will simply use $\|\cdot\|$ and $\|\cdot\|_{L^p}$. $\langle \cdot, \cdot \rangle$ denotes the inner-product in $L^2(\mathbb{R}^3)$. Moreover, C denotes a general constant which may vary in different estimates. If the dependence needs to be explicitly pointed out, then the notation $C(a, b, \ldots)$ is used. Finally,

$$\nabla = (\partial_1, \partial_2, \partial_3), \quad \partial_i = \partial_{x_i},\ i = 1, 2, 3,$$

and for any integer $l \geq 0$, $\nabla^l f$ denotes all derivatives up to l-order of the function f. And for multi-indices α and β

$$\alpha = (\alpha_1, \alpha_2, \alpha_3), \quad \beta = (\beta_1, \beta_2, \beta_3),$$

we use

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}, \quad |\alpha| = \sum_{i=1}^{3} \alpha_i,$$

and $C_\alpha^\beta = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$ when $\beta \leq \alpha$.

2. Preliminaries

In this section, the initial problem (1.1)-(1.2) will be reformulated as follows. Set

$$\rho'(t,x)=\rho(t,x)-\tilde\rho(x),\quad u'(t,x)=u(t,x),\quad \theta'(t,x)=\theta(t,x)-\theta_\infty,$$

and

$$\bar\rho(x)=\tilde\rho(x)-\rho_\infty. \tag{2.1}$$

By (1.4), it holds that

$$F(x)=-\nabla\Phi(x)=\frac{\nabla P(\tilde\rho,\theta_\infty)}{\tilde\rho}.$$

Thus (1.1) becomes

$$\begin{cases} \rho'_t+\rho_\infty\nabla\cdot u'=F'_1,\\ u'_t-\dfrac{\mu}{\rho_\infty}\Delta u'-\dfrac{\mu+\mu'}{\rho_\infty}\nabla\nabla\cdot u'+\dfrac{P_\rho(\rho_\infty,\theta_\infty)}{\rho_\infty}\nabla\rho'+\dfrac{P_\theta(\rho_\infty,\theta_\infty)}{\rho_\infty}\nabla\theta'=F'_2,\\ \theta'_t-\dfrac{\kappa}{c_\nu\rho_\infty}\Delta\theta'+\dfrac{\theta_\infty P_\theta(\rho_\infty,\theta_\infty)}{c_\nu\rho_\infty}\nabla\cdot u'=F'_3, \end{cases} \tag{2.2}$$

where

$$F'_1=-\nabla\cdot(\rho'u')-\nabla\cdot(\bar\rho u'), \tag{2.3}$$

$$\begin{aligned} F'_2=&-(u'\cdot\nabla)u-\left[\frac{P_\rho(\rho'+\tilde\rho,\theta'+\theta_\infty)}{\rho'+\tilde\rho}-\frac{P_\rho(\rho_\infty,\theta_\infty)}{\rho_\infty}\right]\nabla\rho'\\ &-\left[\frac{P_\theta(\rho'+\tilde\rho,\theta'+\theta_\infty)}{\rho'+\tilde\rho}-\frac{P_\theta(\rho_\infty,\theta_\infty)}{\rho_\infty}\right]\nabla\theta'\\ &-\left[\frac{P_\rho(\rho'+\tilde\rho,\theta'+\theta_\infty)}{\rho'+\tilde\rho}-\frac{P_\rho(\rho_\infty,\theta_\infty)}{\rho_\infty}\right]\nabla\bar\rho\\ &+\left[\frac{\mu}{\rho'+\tilde\rho}-\frac{\mu}{\rho_\infty}\right]\Delta u'+\left[\frac{\mu+\mu'}{\rho'+\tilde\rho}-\frac{\mu+\mu'}{\rho_\infty}\right]\nabla(\nabla\cdot u'), \end{aligned} \tag{2.4}$$

$$\begin{aligned} F'_3=&-(u'\cdot\nabla)\theta'-\left[\frac{(\theta'+\theta_\infty)P_\theta(\rho'+\tilde\rho,\theta'+\theta_\infty)}{c_\nu(\rho'+\tilde\rho)}-\frac{\theta_\infty P_\theta(\rho_\infty,\theta_\infty)}{c_\nu\rho_\infty}\right]\nabla\cdot u'\\ &+\left[\frac{\kappa}{c_\nu(\rho'+\tilde\rho)}-\frac{\kappa}{c_\nu\rho_\infty}\right]\Delta\theta'+\left[\frac{1}{c_\nu(\rho'+\tilde\rho)}-\frac{1}{c_\nu\rho_\infty}\right]\Psi(u')\\ &+\frac{1}{c_\nu\rho_\infty}\Psi(u'). \end{aligned} \tag{2.5}$$

Denote the scaled parameters and constants by

$$\mu_1=\frac{\mu}{\rho_\infty},\quad \mu_2=\frac{\mu+\mu'}{\rho_\infty},\quad \gamma=\sqrt{P_1\rho_\infty},\quad \lambda=\sqrt{P_2P_3},\quad \bar\kappa=\frac{\kappa}{c_\nu}\sqrt{\frac{P_2}{P_1P_3\rho_\infty}}.$$

Then by defining

$$\sigma(t,x)=\rho'(t,x),\quad w(t,x)=\sqrt{\frac{\rho_\infty}{P_1}}u'(t,x),\quad z(t,x)=\sqrt{\frac{P_2\rho_\infty}{P_1P_3}}\theta'(t,x),$$

where

$$P_1 = \frac{P_\rho(\rho_\infty, \theta_\infty)}{\rho_\infty}, \quad P_2 = \frac{P_\theta(\rho_\infty, \theta_\infty)}{\rho_\infty}, \quad P_3 = \frac{\theta_\infty P_\theta(\rho_\infty, \theta_\infty)}{c_\nu \rho_\infty},$$

the initial value problem (1.1)-(1.2) is reformulated into

$$\begin{cases} \sigma_t + \gamma \nabla \cdot w = F_1, \\ w_t - \mu_1 \Delta w - \mu_2 \nabla \nabla \cdot w + \gamma \nabla \sigma + \lambda \nabla z = F_2, \\ z_t - \bar{\kappa} \Delta z + \lambda \nabla \cdot w = F_3, \end{cases} \tag{2.6}$$

with initial data

$$(\sigma, w, z)(0, x) = (\sigma_0, w_0, z_0)(x) \to (0, 0, 0) \quad \text{as} \quad |x| \to \infty. \tag{2.7}$$

Here, F_1, F_2, F_3 are F_1', F_2', F_3' respectively in terms of (σ, w, z), and

$$(\sigma_0, w_0, z_0)(x) = \left(\rho_0(x) - \tilde{\rho}(x), \sqrt{\frac{\rho_\infty}{P_1}} u_0(x), \sqrt{\frac{P_2 \rho_\infty}{P_1 P_3}} (\theta_0(x) - \theta_\infty) \right).$$

Use $\mathbb{A}$ to denote the following matrix-valued differential operator

$$\mathbb{A} = \begin{pmatrix} 0 & \gamma \mathrm{div} & 0 \\ \gamma \nabla & -\mu_1 \Delta - \mu_2 \nabla \mathrm{div} & \lambda \nabla \\ 0 & \lambda \mathrm{div} & -\bar{\kappa} \Delta \end{pmatrix}.$$

Then the corresponding semigroup generated by the linear operator $-\mathbb{A}$ is

$$E(t) = e^{-t\mathbb{A}}, \quad t \geq 0.$$

For simplicity of notations, set

$$U = (\sigma, w, z) \quad \text{and} \quad F(U) = (F_1, F_2, F_3)(U).$$

Then the reformulated problem (2.6)-(2.7) can be written both as

$$U_t + \mathbb{A} U = F(U) \quad \text{in } (0, \infty) \times \mathbb{R}^3, \quad U(0) = U_0 \quad \text{in } \mathbb{R}^3,$$

and the integral form

$$U(t) = E(t) U_0 + \int_0^t E(t-s) F(U)(s) ds, \quad t \geq 0. \tag{2.8}$$

To fully use the decay estimates of the semigroup $E(t)$, the following $L^p - L^q$ estimates[12] will be applied to the integral formula (2.13).

Lemma 2.1. *Let $l \geq 0$, $m \geq 0$ be integers and $1 \leq q \leq 2 \leq p < \infty$. Then for any $t > 0$, we have*

$$\begin{aligned} \|\partial_t^m \nabla^l E(t) U_0\|_{L^p} \leq & C(m,l,p,q)(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})-\frac{m+l}{2}} \|U_0\|_{L^q} \\ & + C(m,l,p,q) e^{-ct} \left[t^{-\frac{n_1}{2}} \|\sigma_0\|_{(2m+l-n_1-1)^+,p} + \|\sigma_0\|_{l,p} \right] \\ & + C(m,l,p,q) e^{-ct} \left[t^{-\frac{n_2}{2}} \|(w_0,z_0)\|_{(2m+l-n_2)^+,p} + \|(w_0,z_0)\|_{(l-1)^+,p} \right], \end{aligned}$$

where $n_1 \geq 0$ and $n_2 \geq 0$ are integers, $c > 0$ is a positive constant, and $(k)^+ = k$ if $k \geq 0$ and 0 otherwise. In particular, it holds that

$$\|\nabla^l E(t)U_0\| \leq C(l)(1+t)^{-\frac{3}{4}-\frac{l}{2}}(\|U_0\|_{L^1} + \|U_0\|_l),$$

and

$$\|\partial_t E(t)U_0\| \leq C(1+t)^{-\frac{5}{4}}\left[\|U_0\|_{L^1} + \left(1+t^{-\frac{1}{2}}\right)\|U_0\|_1\right].$$

For later use and clear reference, some Sobolev inequalities[1,2] are listed as follows.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^3$ be any domain with smooth boundary. Then*
(i) $\|f\|_{C^0(\bar{\Omega})} \leq C\|f\|_{W^{m,p}(\Omega)}$, for $f \in W^{m,p}(\Omega)$, $(m-1)p < 3 < mp$.
(ii) $\|f\|_{L^p(\Omega)} \leq C\|f\|_{H^1(\Omega)}$, for $f \in H^1(\Omega)$, $2 \leq p \leq 6$.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^3$ be the whole space $\mathbb{R}^3$, or half space $\mathbb{R}^3_+$ or the exterior domain of a bounded region with smooth boundary. Then*
(i) $\|f\|_{L^6(\Omega)} \leq C\|\nabla f\|_{L^2(\Omega)}$, for $f \in H^1(\Omega)$.
(ii) $\|f\|_{C^0(\bar{\Omega})} \leq C\|f\|_{W^{1,p}(\Omega)} \leq C\|\nabla f\|_{H^1(\Omega)}$, for $f \in H^2(\Omega)$.

Lemma 2.4. *For Ω defined in Lemma 2.3, we have*
(i) $|\int_\Omega f \cdot g \cdot h\, dx| \leq \varepsilon\|\nabla f\|^2 + \frac{C}{\varepsilon}\|g\|_1^2\|h\|^2$ for $\varepsilon > 0$, $f, g \in H^1(\Omega)$, $h \in L^2(\Omega)$.
(ii) $|\int_\Omega f \cdot g \cdot h\, dx| \leq \varepsilon\|g\|^2 + \frac{C}{\varepsilon}\|\nabla f\|_1^2\|h\|^2$ for $\varepsilon > 0$, $f \in H^2(\Omega)$, $g, h \in L^2(\Omega)$.

Finally, the following elementary inequality will also be used.

Lemma 2.5. *If $r_1 > 1$ and $r_2 \in [0, r_1]$, then it holds that*

$$\int_0^t (1+t-s)^{-r_1}(1+s)^{-r_2} ds \leq C_1(r_1, r_2)(1+t)^{-r_2}, \tag{2.9}$$

where $C_1(r_1, r_2)$ is defined by

$$C_1(r_1, r_2) = \frac{2^{r_2+1}}{r_1 - 1}. \tag{2.10}$$

Proof. The integral in (2.9) is estimated by

$$\left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t\right)(1+t-s)^{-r_1}(1+s)^{-r_2} ds = \mathrm{I} + \mathrm{II}. \tag{2.11}$$

For the second part, it is easy to see

$$\mathrm{II} \leq \left(1+\frac{t}{2}\right)^{-r_2}\int_{\frac{t}{2}}^t (1+t-s)^{-r_1} ds \leq \frac{1}{r_1 - 1}\left(1+\frac{t}{2}\right)^{-r_2}. \tag{2.12}$$

For the first part, when $r_2 \neq 1$, it holds that

$$\begin{aligned} \mathrm{I} &= \frac{(1+\frac{t}{2})^{-r_1}}{1-r_2}\left[\left(1+\frac{t}{2}\right)^{1-r_2}-1\right] \\ &= \left(1+\frac{t}{2}\right)^{-r_2}\frac{(1+\frac{t}{2})^{1-r_1}-(1+\frac{t}{2})^{r_2-r_1}}{1-r_2}. \end{aligned}$$

Define an auxiliary function

$$G(x) = \frac{(1+x)^{1-r_1}-(1+x)^{r_2-r_1}}{1-r_2}.$$

Direct calculations gives $0 \leq G(x) \leq G(x_*)$, where x_* satisfies

$$1+x_* = \left(\frac{r_1-1}{r_1-r_2}\right)^{\frac{1}{r_2-1}}.$$

Thus,

$$\begin{aligned} G(x_*) &= \frac{1}{1-r_2}\left[\left(\frac{r_1-1}{r_1-r_2}\right)^{\frac{1-r_1}{r_2-1}}-\left(\frac{r_1-1}{r_1-r_2}\right)^{\frac{r_2-r_1}{r_2-1}}\right] \\ &= \left(\frac{r_1-1}{r_1-r_2}\right)^{\frac{r_2-r_1}{r_2-1}}\frac{1}{r_1-1} \leq \frac{1}{r_1-1}, \end{aligned}$$

where we have used the inequality

$$\left(\frac{r_1-1}{r_1-r_2}\right)^{\frac{r_2-r_1}{r_2-1}} \leq 1$$

because $r_1 > 1$ and $0 \leq r_2 \leq r_1$. Thus for the first part, it also holds that

$$\mathrm{I} \leq \frac{1}{r_1-1}\left(1+\frac{t}{2}\right)^{-r_2}. \tag{2.13}$$

Hence putting (2.12) and (2.13) into (2.11) yields (2.9). This completes the proof of the lemma. □

3. Basic estimates

In this section, we shall prove two basic inequalities for the proof of the optimal convergence rates in Section 4. One inequality is based on the $L^p - L^q$ estimates of solutions to the linearized equations, while the other comes from the use of the energy method.

Lemma 3.1. *Let $U = (\sigma, w, z)$ be the solution to the problem (2.6)-(2.7). Under the assumptions of Theorem 1.1, we have*

$$\|\nabla U(t)\| \leq CE_0(1+t)^{-\frac{5}{4}} + C\epsilon\int_0^t (1+t-s)^{-\frac{5}{4}}\|\nabla U(s)\|_2 ds, \tag{3.1}$$

where $E_0 = \|U_0\|_{H^3 \cap L^1}$ *is finite by (1.8) and (1.9).*

Proof. From (2.8) and Lemma 2.1, we have

$$\begin{aligned}\|\nabla U(t)\| &\le CE_0(1+t)^{-\frac{5}{4}} \\ &\quad + C\int_0^t (1+t-s)^{-\frac{5}{4}} \left(\|F(U)(s)\|_{L^1} + \|F(U)(s)\|_1\right) ds, \end{aligned} \tag{3.2}$$

where $F(U)$ given in (2.3)-(2.5) has the following equivalence properties:

$$\begin{aligned} F_1 &\sim \partial_i \sigma w^i + \sigma \partial_i w^i + \partial_i \bar\rho w^i + \bar\rho \partial_i w^i, \\ F_2^j &\sim w^i \partial_i w^j + \sigma \partial_i \partial_i w^j + \sigma \partial_j \partial_i w^i + \sigma \partial_j \sigma + z \partial_j \sigma + \sigma \partial_j z + z \partial_j z \\ &\quad + \bar\rho \partial_i \partial_i w^j + \bar\rho \partial_j \partial_i w^i + \bar\rho \partial_j \sigma + \sigma \partial_j \bar\rho + z \partial_j \bar\rho + \bar\rho \partial_j z, \\ F_3 &\sim w^i \partial_i z + \sigma \partial_i \partial_i z + \sigma \partial_i w^i + z \partial_i w^i + \sigma \Psi(w) + \Psi(w) \\ &\quad + \bar\rho \partial_i \partial_i z + \bar\rho \partial_i w^i + \bar\rho \Psi(w). \end{aligned}$$

And $\Psi(w)$ is given by (1.3). Thus, it follows from the Hölder inequality, Proposition 1.1, Lemma 2.2, (2.1) and (1.14) that

$$\begin{aligned} \|F(U)(t)\|_{L^1} &\le C(\|U(t)\| + \|\bar\rho\|)\|\nabla U(t)\|_1 + C\|\nabla\bar\rho\|_{L^{6/5}}\|U(t)\|_{L^6} \\ &\quad + C(\|\nabla\sigma(t)\|_1 + \|\nabla\bar\rho\|_1)\|\nabla w(t)\|^2 \\ &\le C\epsilon\|\nabla U(t)\|_1, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \|F(U)(t)\|_1 &\le C(\|U(t)\|_{W^{1,\infty}} + \|\bar\rho\|_{W^{1,\infty}})\|\nabla U(t)\|_2 + C\|\nabla\bar\rho\|_1\|U(t)\|_{L^\infty} \\ &\quad + C\|\sigma(t)\|_{W^{1,\infty}}\|\nabla w(t)\|_{L^\infty}\|\nabla w(t)\|_1 \\ &\le C(\|\nabla U(t)\|_2 + \|\nabla\bar\rho\|_2)\|\nabla U(t)\|_2 + C\|\nabla\bar\rho\|_1\|\nabla U(t)\|_1 \\ &\quad + C\|\nabla\sigma(t)\|_2\|\nabla^2 w(t)\|_1\|\nabla w(t)\|_1 \\ &\le C\epsilon\|\nabla U(t)\|_2. \end{aligned} \tag{3.4}$$

Putting (3.3) and (3.4) into (3.2) yields (3.1) and hence this completes the proof of the lemma. □

Lemma 3.2. *Let* $U = (\sigma, w, z)$ *be the solution to the problem (2.6)-(2.7). Under the assumptions of Theorem 1.1, if* $\epsilon > 0$ *is sufficiently small, then it holds that*

$$\frac{dH(t)}{dt} + \left(\|\nabla^2\sigma(t)\|_1^2 + \|\nabla^2(w,z)(t)\|_2^2\right) \le C\epsilon\|\nabla U(t)\|^2, \tag{3.5}$$

where $H(t)$ *is equivalent to* $\|\nabla U(t)\|_2^2$*, that is, there exists a positive constant* C_2 *such that*

$$\frac{1}{C_2}\|\nabla U(t)\|_2^2 \le H(t) \le C_2\|\nabla U(t)\|_2^2, \quad t \ge 0. \tag{3.6}$$

Proof. For each multi-index α with $1 \le |\alpha| \le 3$, by applying ∂_x^α to (2.6), multiplying it by $\partial_x^\alpha \sigma$, $\partial_x^\alpha w$, $\partial_x^\alpha z$ respectively and then integrating it over $\mathbb{R}^3$, we have from $(2.6)_1$-$(2.6)_3$ that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha U(t)\|^2 + \mu_1\|\nabla\partial_x^\alpha w(t)\|^2 + \mu_2\|\nabla\cdot\partial_x^\alpha w(t)\|^2 + \bar{\kappa}\|\nabla\partial_x^\alpha z(t)\|^2\\
&= \langle\partial_x^\alpha\sigma(t), \partial_x^\alpha F_1(t)\rangle + \langle\partial_x^\alpha w(t), \partial_x^\alpha F_2(t)\rangle + \langle\partial_x^\alpha z(t), \partial_x^\alpha F_3(t)\rangle\\
&= I_1(t) + I_2(t) + I_3(t),
\end{aligned} \tag{3.7}$$

where $I_i(t)$, $i = 1, 2, 3$, are the corresponding terms in the above equation which will be estimated as follows.

Firstly, for $I_1(t)$, it holds that

$$\begin{aligned}
I_1(t) \le\ & C\left\{|\langle\partial_x^\alpha\sigma(t), \partial_x^\alpha(\partial_i\sigma w^i)(t)\rangle| + |\langle\partial_x^\alpha\sigma(t), \partial_x^\alpha(\sigma\partial_i w^i)(t)\rangle|\right.\\
&\left.+|\langle\partial_x^\alpha\sigma(t), \partial_x^\alpha(\partial_i\bar{\rho}w^i)(t)\rangle| + |\langle\partial_x^\alpha\sigma(t), \partial_x^\alpha(\bar{\rho}\partial_i w^i)(t)\rangle|\right\}\\
\le\ & C|\langle\partial_x^\alpha\sigma(t), \partial_x^\alpha\partial_i\sigma(t)w^i(t)\rangle|\\
&+C\textstyle\sum_{|\beta|\le|\alpha|-1} C_\alpha^\beta|\langle\partial_x^\alpha\sigma(t), \partial_x^\beta\partial_i\sigma(t)\partial_x^{\alpha-\beta}w^i(t)\rangle|\\
&+C\textstyle\sum_{|\beta|\le|\alpha|} C_\alpha^\beta|\langle\partial_x^\alpha\sigma(t), \partial_x^\beta\sigma(t)\partial_x^{\alpha-\beta}\partial_i w^i(t)\rangle|\\
&+C\textstyle\sum_{|\beta|\le|\alpha|} C_\alpha^\beta|\langle\partial_x^\alpha\sigma(t), \partial_x^\beta\partial_i\bar{\rho}\partial_x^{\alpha-\beta}w^i(t)\rangle|\\
&+C\textstyle\sum_{|\beta|\le|\alpha|} C_\alpha^\beta|\langle\partial_x^\alpha\sigma(t), \partial_x^\beta\bar{\rho}\partial_x^{\alpha-\beta}\partial_i w^i(t)\rangle|.
\end{aligned} \tag{3.8}$$

For the first term on the right hand side of (3.8), Lemma 2.3 and Proposition 1.1 give

$$\begin{aligned}
&|\langle\partial_x^\alpha\sigma(t), \partial_x^\alpha\partial_i\sigma(t)w^i(t)\rangle| = \tfrac{1}{2}|\langle(\partial_x^\alpha\sigma(t))^2, \partial_i w^i(t)\rangle|\\
&\le C\|\partial_i w^i(t)\|_{L^\infty}\|\partial_x^\alpha\sigma(t)\|^2 \le C\|\nabla^2 w(t)\|_1\|\partial_x^\alpha\sigma(t)\|^2\\
&\le C\epsilon\|\partial_x^\alpha\sigma(t)\|^2.
\end{aligned}$$

Furthermore, for the second term, it follows from Lemma 2.4 and Proposition 1.1 that

$$\begin{aligned}
&\sum_{|\beta|\le|\alpha|-1}|\langle\partial_x^\alpha\sigma(t), \partial_x^\beta\partial_i\sigma(t)\partial_x^{\alpha-\beta}w^i(t)\rangle|\\
&= \left\{\textstyle\sum_{|\beta|=0} + \sum_{1\le|\beta|\le|\alpha|-1}\right\}|\langle\partial_x^\alpha\sigma(t), \partial_x^\beta\partial_i\sigma(t)\partial_x^{\alpha-\beta}w^i(t)\rangle|\\
&\le 2\epsilon\|\partial_x^\alpha\sigma(t)\|^2 + \tfrac{C}{\epsilon}\|\nabla\partial_i\sigma(t)\|_1^2\|\partial_x^\alpha w^i(t)\|^2\\
&\quad+\tfrac{C}{\epsilon}\textstyle\sum_{1\le|\beta|\le|\alpha|-1}\|\nabla\partial_x^{\alpha-\beta}w^i(t)\|_1^2\|\partial_x^\beta\partial_i\sigma(t)\|^2\\
&\le C\epsilon\textstyle\sum_{1\le|\alpha|\le3}\|\partial_x^\alpha\sigma(t)\|^2 + C\epsilon\sum_{1\le|\alpha|\le4}\|\partial_x^\alpha w(t)\|^2.
\end{aligned}$$

The other terms on the right hand side of (3.8) can be estimated similarly. Thus,

$$I_1(t) \le C\epsilon\sum_{1\le|\alpha|\le3}\|\partial_x^\alpha\sigma(t)\|^2 + C\epsilon\sum_{1\le|\alpha|\le4}\|\partial_x^\alpha w(t)\|^2. \tag{3.9}$$

Moreover, $I_2(t)$ and $I_3(t)$ can be estimated similarly by using Lemma 2.4:

$$I_2(t) + I_3(t) \le C\epsilon \sum_{1\le|\alpha|\le 3} \|\partial_x^\alpha \sigma(t)\|^2 + C\epsilon \sum_{1\le|\alpha|\le 4} \|\partial_x^\alpha (w,z)(t)\|^2. \tag{3.10}$$

Hence (3.7) together with (3.9) and (3.10) yields

$$\begin{aligned}\frac{d}{dt}\sum_{1\le|\alpha|\le 3} \|\partial_x^\alpha U(t)\|^2 + &\sum_{1\le|\alpha|\le 3} \|\nabla\partial_x^\alpha (w,z)(t)\|^2 \\ &\le C\epsilon \sum_{1\le|\alpha|\le 3} \|\partial_x^\alpha \sigma(t)\|^2 + C\epsilon \sum_{1\le|\alpha|\le 4} \|\partial_x^\alpha (w,z)(t)\|^2. \end{aligned}\tag{3.11}$$

Next we shall estimate $\|\nabla\partial_x^\alpha\sigma(t)\|^2$ when $1\le|\alpha|\le 2$. Since $(2.6)_2$ gives

$$\gamma\nabla\sigma = -w_t + \mu_1\Delta w + \mu_2\nabla(\nabla\cdot w) - \lambda\nabla z + F_2,$$

we have that for $1\le|\alpha|\le 2$,

$$\begin{aligned}\gamma\|\nabla\partial_x^\alpha\sigma(t)\|^2 = &-\langle\partial_x^\alpha w_t(t), \nabla\partial_x^\alpha\sigma(t)\rangle + \mu_1\langle\partial_x^\alpha\Delta w(t), \nabla\partial_x^\alpha\sigma(t)\rangle \\ &+\mu_2\langle\partial_x^\alpha\nabla(\nabla\cdot w)(t), \nabla\partial_x^\alpha\sigma(t)\rangle - \lambda\langle\partial_x^\alpha\nabla z(t), \nabla\partial_x^\alpha\sigma(t)\rangle \\ &+\langle\partial_x^\alpha F_2(t), \nabla\partial_x^\alpha\sigma(t)\rangle.\end{aligned}\tag{3.12}$$

On the other hand, it follows from $(2.6)_1$ that

$$\begin{aligned}&\frac{d}{dt}\langle\partial_x^\alpha w(t), \nabla\partial_x^\alpha\sigma(t)\rangle \\ &= \langle\partial_x^\alpha w_t(t), \nabla\partial_x^\alpha\sigma(t)\rangle + \langle\partial_x^\alpha w(t), \nabla\partial_x^\alpha\sigma_t(t)\rangle \\ &= \langle\partial_x^\alpha w_t(t), \nabla\partial_x^\alpha\sigma(t)\rangle - \gamma\langle\partial_x^\alpha w(t), \nabla\partial_x^\alpha\nabla\cdot w(t)\rangle + \langle\partial_x^\alpha w(t), \nabla\partial_x^\alpha F_1(t)\rangle.\end{aligned}\tag{3.13}$$

Adding (3.12) and (3.13) gives

$$\begin{aligned}&\gamma\|\nabla\partial_x^\alpha\sigma(t)\|^2 + \frac{d}{dt}\langle\partial_x^\alpha w(t), \nabla\partial_x^\alpha\sigma(t)\rangle \\ &= \mu_1\langle\partial_x^\alpha\Delta w(t), \nabla\partial_x^\alpha\sigma(t)\rangle + \mu_2\langle\partial_x^\alpha\nabla(\nabla\cdot w)(t), \nabla\partial_x^\alpha\sigma(t)\rangle \\ &\quad -\gamma\langle\partial_x^\alpha w(t), \nabla\partial_x^\alpha\nabla\cdot w(t)\rangle - \lambda\langle\partial_x^\alpha\nabla z, \nabla\partial_x^\alpha\sigma(t)\rangle \\ &\quad +\langle\partial_x^\alpha w(t), \nabla\partial_x^\alpha F_1(t)\rangle + \langle\nabla\partial_x^\alpha\sigma(t), \partial_x^\alpha F_2(t)\rangle,\end{aligned}$$

which implies

$$\begin{aligned}&\frac{\gamma}{2}\|\nabla\partial_x^\alpha\sigma(t)\|^2 + \frac{d}{dt}\langle\partial_x^\alpha w(t), \nabla\partial_x^\alpha\sigma(t)\rangle \\ &\le C(\|\partial_x^\alpha\nabla^2 w(t)\|^2 + \|\partial_x^\alpha\nabla\cdot w(t)\|^2 + \|\partial_x^\alpha\nabla z(t)\|^2) \\ &\quad +C|\langle\partial_x^\alpha w(t), \nabla\partial_x^\alpha F_1(t)\rangle| + C|\langle\nabla\partial_x^\alpha\sigma(t), \partial_x^\alpha F_2(t)\rangle|.\end{aligned}\tag{3.14}$$

Similar to the estimation on $I_1(t)$, we have

$$\begin{aligned}&|\langle\partial_x^\alpha w(t),\ \nabla\partial_x^\alpha F_1(t)\rangle| + |\langle\nabla\partial_x^\alpha\sigma(t), \partial_x^\alpha F_2(t)\rangle| \\ &\qquad \le C\epsilon \sum_{1\le|\alpha|\le 3} \|\partial_x^\alpha \sigma(t)\|^2 + C\epsilon \sum_{1\le|\alpha|\le 4} \|\partial_x^\alpha (w,z)(t)\|^2.\end{aligned}\tag{3.15}$$

Putting (3.15) into (3.14) gives that for $1 \leq |\alpha| \leq 2$,

$$\begin{aligned}\frac{\gamma}{2}\sum_{1\leq|\alpha|\leq 2}\|\nabla\partial_x^\alpha\sigma(t)\|^2 &+ \frac{d}{dt}\sum_{1\leq|\alpha|\leq 2}\langle\partial_x^\alpha w(t), \nabla\partial_x^\alpha\sigma(t)\rangle \\ &\leq C\sum_{1\leq|\alpha|\leq 2}(\|\partial_x^\alpha\nabla w(t)\|_1^2 + \|\partial_x^\alpha\nabla z(t)\|^2) \\ &\quad + C\epsilon\sum_{1\leq|\alpha|\leq 3}\|\partial_x^\alpha\sigma(t)\|^2 + C\epsilon\sum_{1\leq|\alpha|\leq 4}\|\partial_x^\alpha(w,z)(t)\|^2. \end{aligned} \tag{3.16}$$

Define

$$H(t) = D_1\sum_{1\leq|\alpha|\leq 3}\|\partial^\alpha U(t)\|^2 + \sum_{1\leq|\alpha|\leq 2}\langle\partial_x^\alpha w(t), \nabla\partial_x^\alpha\sigma(t)\rangle.$$

By choosing D_1 sufficiently large and $\epsilon > 0$ sufficiently small, (3.11) and (3.16) give

$$\frac{dH(t)}{dt} + D_2\left(\|\nabla^2\sigma(t)\|_1^2 + \|\nabla^2(w,z)(t)\|_2^2\right) \leq C\epsilon\|\nabla U(t)\|^2,$$

where D_2 is a positive constant independent of ϵ. And this completes the proof of the lemma. □

4. Optimal convergence rates

The optimal convergence rates will be proved by first improving the estimates given in Lemmas 3.1 and 3.2 to the estimates on the L^2-norms of solutions to higher power and then letting the power tend to infinity. By the inequality (3.1), we have the following lemma.

Lemma 4.1. *Let $U = (\sigma, w, z)$ be the solution to the problem (2.6)-(2.7). Under the assumptions of Theorem 1.1, if $\epsilon > 0$ is sufficiently small, then for any integer $n \geq 1$, we have*

$$\int_0^t (1+s)^k\|\nabla U(s)\|^{2n}ds \leq (CE_0)^{2n} + (C\epsilon)^{2n}\int_0^t(1+s)^k\|\nabla^2 U(s)\|_1^{2n}ds, \tag{4.1}$$

where $k = 0, 1, \cdots, N = [5n/2 - 2]$ and the constant E_0 is given in Lemma 3.1.

Proof. Fix any integer $n \geq 1$. By taking (3.1) to power $2n$ and multiplying it by $(1+t)^k$, $k = 0, 1, \cdots, N$, the integration over $[0, t]$ gives

$$\begin{aligned}\int_0^t (1+\tau)^k\|\nabla U(\tau)\|^{2n}ds &\leq (CE_0)^{2n}\int_0^t(1+\tau)^{-(\frac{5}{2}n-k)}d\tau \\ &+ (C\epsilon)^{2n}\int_0^t(1+\tau)^k\left[\int_0^\tau(1+\tau-s)^{-\frac{5}{4}}\|\nabla U(s)\|_2 ds\right]^{2n}d\tau. \end{aligned} \tag{4.2}$$

It follows from the Hölder inequality that

$$\begin{aligned}
&\left[\int_0^\tau (1+\ \tau-s)^{-\frac{5}{4}}\|\nabla U(s)\|_2 ds\right]^{2n} \\
&\quad\le \left[\int_0^\tau (1+\tau-s)^{-r_1}(1+s)^{-r_2}ds\right]^{2n-1} \\
&\qquad\times \int_0^\tau (1+\tau-s)^{-\frac{4}{3}}(1+s)^k\|\nabla U(s)\|_2^{2n}ds, \qquad (4.3)
\end{aligned}$$

where

$$r_1=\left(\frac{5}{4}-\frac{2}{3n}\right)\frac{2n}{2n-1} \quad\text{and}\quad r_2=\frac{k}{2n-1}.$$

Since $7/6 \le r_1 \le 5/4$ and $r_2 \in [0, r_1]$ for $n \ge 1$ and $0 \le k \le N$, Lemma 2.5 gives

$$\int_0^\tau (1+\tau-s)^{-r_1}(1+s)^{-r_2}ds \le C_1(r_1,r_2)(1+\tau)^{-r_2} \le C(1+\tau)^{-r_2}, \qquad (4.4)$$

where $C_1(r_1, r_2)$ given by (2.10) is bounded uniformly for $n \ge 1$. Hence, (4.2) together with (4.3) and (4.4) implies

$$\begin{aligned}
&\int_0^t (1+\tau)^k\|\nabla U(\tau)\|^{2n}d\tau \\
&\le (CE_0)^{2n}\frac{1}{5n/2-k-1} \\
&\quad +(C\epsilon)^{2n}\int_0^t (1+s)^k\|\nabla U(s)\|_2^{2n}\int_s^t (1+\tau-s)^{-\frac{4}{3}}d\tau ds \\
&\le (CE_0)^{2n}+(C\epsilon)^{2n}\int_0^t (1+s)^k\|\nabla U(s)\|_2^{2n}ds \\
&\le (CE_0)^{2n}+(C\epsilon)^{2n}\int_0^t (1+s)^k(\|\nabla U(s)\|^{2n}+\|\nabla^2 U(s)\|_1^{2n})\,ds. \qquad (4.5)
\end{aligned}$$

Here we have used

$$\frac{5n}{2}-k-1 \ge \frac{5n}{2}-\left(\frac{5n}{2}-2\right)-1=1.$$

Thus if $\epsilon > 0$ is sufficiently small such that $(C\epsilon)^{2n} \le 1/2$ in the final inequality of (4.5), then (4.5) gives (4.1). The proof of the lemma is complete. □

Based on the inequality (3.5), the following lemma is about the estimates on $H(t)^n$ weighted with the function $(1+t)^k$.

Lemma 4.2. *Let $U = (\sigma, w, z)$ be the solution to the problem (2.6)-(2.7) and $H(t)$ be defined in Lemma 3.2. Under the assumptions of Theorem 1.1, if $\epsilon > 0$ is sufficiently small, then for any integer $n \ge 1$, it holds that*

$$\begin{aligned}
&(1+t)^kH(t)^n+n\int_0^t (1+s)^kH(s)^{n-1}\|\nabla^2U(s)\|_1^2ds \\
&\le 2H(0)^n+(C_3E_0)^{2n}+10C_2n\int_0^t (1+s)^{k-1}H(s)^{n-1}\|\nabla^2U(s)\|_1^2ds, \qquad (4.6)
\end{aligned}$$

where $k = 0, 1, \cdots, N = [5n/2-2]$, C_2 *is defined in Lemma 3.2 and* C_3 *is independent of* ϵ *and* n.

Proof. Multiplying (3.5) by $n(1+t)^k H(t)^{n-1}$ for $k = 0, 1, \cdots, N$ and integrating it over $[0, t]$ give

$$\begin{aligned}(1+t)^k H(t)^n &+ n\int_0^t (1+s)^k H(s)^{n-1}\|\nabla^2 U(s)\|_1^2 ds\\ &\le H(0)^n + C\epsilon n\int_0^t (1+s)^k H(s)^{n-1}\|\nabla U(s)\|^2 ds\\ &\quad +k\int_0^t (1+s)^{k-1}H(s)^n ds.\end{aligned}\tag{4.7}$$

For the second term on the right hand side of (4.7), by the Young inequality and (3.6), we have that for any $\delta > 0$,

$$\begin{aligned}&\epsilon n\int_0^t (1+s)^k H(s)^{n-1}\|\nabla U(s)\|^2 ds\\ &\le \epsilon n\int_0^t (1+s)^k\left[\frac{n-1}{n}\delta H(s)^n + \frac{1}{n}\frac{1}{\delta^{n-1}}\|\nabla U(s)\|^{2n}\right] ds\\ &\le \epsilon n C_2\delta\int_0^t (1+s)^k H(s)^{n-1}(\|\nabla U(s)\|^2 + \|\nabla^2 U(s)\|_1^2) ds\\ &\quad +\epsilon\delta^{1-n}\int_0^t (1+s)^k\|\nabla U(s)\|^{2n} ds,\end{aligned}$$

which together with Lemma 4.1 implies that

$$\begin{aligned}&\epsilon n\int_0^t (1+s)^k H(s)^{n-1}\|\nabla U(s)\|^2 ds\\ &\le \epsilon n C_2\delta\int_0^t (1+s)^k H(s)^{n-1}\|\nabla U(s)\|^2 ds\\ &\quad +\epsilon n C_2\delta\int_0^t (1+s)^k H(s)^{n-1}\|\nabla^2 U(s)\|_1^2 ds\\ &\quad +\epsilon\delta^{1-n}\left[(CE_0)^{2n} + (C\epsilon)^{2n}\int_0^t (1+s)^k\|\nabla^2 U(s)\|_1^{2n} ds\right]\\ &\le \epsilon\delta^{1-n}(CE_0)^{2n} + \epsilon n C_2\delta\int_0^t (1+s)^k H(s)^{n-1}\|\nabla U(s)\|^2 ds\\ &\quad +\epsilon n C_2\delta\int_0^t (1+s)^k H(s)^{n-1}\|\nabla^2 U(s)\|_1^2 ds\\ &\quad +\epsilon\delta^{1-n}(C\epsilon)^{2n}C_2^{n-1}\int_0^t (1+s)^k H(s)^{n-1}\|\nabla^2 U(s)\|_1^2 ds.\end{aligned}\tag{4.8}$$

Choose $\delta = \frac{1}{2C_2}$ in (4.8). We have

$$\begin{aligned}
&\epsilon n \int_0^t (1+s)^k H(s)^{n-1} \|\nabla U(s)\|^2 ds \\
&\le 2\epsilon (2C_2)^{n-1} (CE_0)^{2n} \\
&\quad + \epsilon n \left[1 + \frac{2}{n} (C\epsilon)^{2n} (2C_2^2)^{n-1} \right] \int_0^t (1+s)^k H(s)^{n-1} \|\nabla^2 U(s)\|_1^2 ds \\
&\le \epsilon (CE_0)^{2n} + \epsilon n \left[1 + (C\epsilon)^{2n} \right] \int_0^t (1+s)^k H(s)^{n-1} \|\nabla^2 U(s)\|_1^2 ds. \qquad (4.9)
\end{aligned}$$

Thus, if $\epsilon > 0$ is sufficiently small such that $C\epsilon \le 1$ in (4.9), then $(C\epsilon)^{2n} \le 1$ for any $n \ge 1$. And (4.9) gives

$$\begin{aligned}
\epsilon n \int_0^t & (1+s)^k H(s)^{n-1} \|\nabla U(s)\|^2 ds \\
&\le (CE_0)^{2n} + 2\epsilon n \int_0^t (1+s)^k H(s)^{n-1} \|\nabla^2 U(s)\|_1^2 ds. \qquad (4.10)
\end{aligned}$$

Similar to the proof of (4.8), the third term on the right hand side of (4.7) can be estimated by:

$$\begin{aligned}
&k \int_0^t (1+s)^{k-1} H(s)^n ds \\
&\le k C_2 \int_0^t (1+s)^{k-1} H(s)^{n-1} (\|\nabla U(s)\|^2 + \|\nabla^2 U(s)\|_1^2) ds \\
&\le k C_2 \delta \int_0^t (1+s)^{k-1} H(s)^n ds + k C_2 \delta^{1-n} \int_0^t (1+s)^{k-1} \|\nabla U(s)\|^{2n} ds \\
&\quad + k C_2 \int_0^t (1+s)^{k-1} H(s)^{n-1} \|\nabla^2 U(s)\|_1^2 ds, \qquad (4.11)
\end{aligned}$$

where $\delta > 0$ is to be determined later. By Lemma 4.1, (4.11) gives

$$\begin{aligned}
&k\int_0^t (1+s)^{k-1}H(s)^n ds\\
&\le kC_2\delta\int_0^t (1+s)^{k-1}H(s)^n ds + kC_2\delta^{1-n}(CE_0)^{2n}\\
&\quad +kC_2\delta^{1-n}(C\epsilon)^{2n}\int_0^t (1+s)^k\|\nabla^2 U(s)\|_1^{2n}ds\\
&\quad +kC_2\int_0^t (1+s)^{k-1}H(s)^{n-1}\|\nabla^2 U(s)\|_1^2 ds\\
&\le kC_2\delta^{1-n}(CE_0)^{2n} + kC_2\delta\int_0^t (1+s)^{k-1}H(s)^n ds\\
&\quad +kC_2(C\epsilon)^{2n}\left(\frac{C_2}{\delta}\right)^{n-1}\int_0^t (1+s)^k H(s)^{n-1}\|\nabla^2 U(s)\|_1^2 ds\\
&\quad +kC_2\int_0^t (1+s)^{k-1}H(s)^{n-1}\|\nabla^2 U(s)\|_1^2 ds. \qquad (4.12)
\end{aligned}$$

By taking $\delta = \frac{1}{2C_2}$ again, (4.12) gives

$$\begin{aligned}
&k\int_0^t (1+s)^{k-1}H(s)^n ds\\
&\le 2kC_2(2C_2)^{n-1}(CE_0)^{2n}\\
&\quad +2kC_2(C\epsilon)^{2n}(2C_2^2)^{n-1}\int_0^t (1+s)^k H(s)^{n-1}\|\nabla^2 U(s)\|_1^2 ds\\
&\quad +2kC_2\int_0^t (1+s)^{k-1}H(s)^{n-1}\|\nabla^2 U(s)\|_1^2 ds\\
&\le (CE_0)^{2n} + n\epsilon(C\epsilon)^{2n-1}\int_0^t (1+s)^k H(s)^{n-1}\|\nabla^2 U(s)\|_1^2 ds\\
&\quad +2kC_2\int_0^t (1+s)^{k-1}H(s)^{n-1}\|\nabla^2 U(s)\|_1^2 ds. \qquad (4.13)
\end{aligned}$$

Therefore, (4.7) together with (4.10) and (4.13) gives

$$\begin{aligned}
&(1+t)^k\ H(t)^n + n\int_0^t (1+s)^k H(s)^{n-1}\|\nabla^2 U(s)\|_1^2 ds\\
&\quad\le H(0)^n + (CE_0)^{2n} + 2kC_2\int_0^t (1+s)^{k-1}H(s)^{n-1}\|\nabla^2 U(s)\|_1^2 ds\\
&\qquad +n\epsilon\left[C+(C\epsilon)^{2n-1}\right]\int_0^t (1+s)^k H(s)^{n-1}\|\nabla^2 U(s)\|_1^2 ds. \qquad (4.14)
\end{aligned}$$

By choosing $\epsilon > 0$ sufficiently small such that for any $n \ge 1$,

$$\epsilon\left[C+(C\epsilon)^{2n-1}\right] \le \frac{1}{2},$$

we have from (4.14) that

$$
\begin{aligned}
&(1+t)^k\ H(t)^n + n\int_0^t (1+s)^k H(s)^{n-1}\|\nabla^2 U(s)\|_1^2 ds \\
&\qquad \le 2H(0)^n + (C_3E_0)^{2n} + 4kC_2\int_0^t (1+s)^{k-1}H(s)^{n-1}\|\nabla^2 U(s)\|_1^2 ds.
\end{aligned}
$$

This implies (4.6) because $k \le N \le 5n/2$ and then it completes the proof of the lemma. □

<u>Proof of Theorem 1.1.</u> First, let $\epsilon > 0$ be sufficiently small such that Lemma 4.2 holds for any $n \ge 2$. The proof of the theorem can be given in two steps.

Step 1. For any fixed integer $n \ge 2$, Lemma 4.2 implies that the inequality (4.6) holds for any $k = 0, 1, \cdots, N$. When $k = 1$, (4.6) becomes

$$
\begin{aligned}
&(1+t)H(t)^n + n\int_0^t (1+s)H(s)^{n-1}\|\nabla^2 U(s)\|_1^2 ds \\
&\qquad \le 2H(0)^n + (C_3E_0)^{2n} + 10C_2 n\int_0^t H(s)^{n-1}\|\nabla^2 U(s)\|_1^2 ds. \qquad (4.15)
\end{aligned}
$$

By (1.7) in Proposition 1.1, it holds that

$$
\begin{aligned}
&\int_0^t H(s)^{n-1}\|\nabla^2 U(s)\|_1^2 ds \le \left[\sup_{s\ge 0} H(s)\right]^{n-1}\int_0^t \|\nabla^2 U(s)\|_1^2 ds \\
&\le (C_2C_0\epsilon^2)^{n-1}C_0\epsilon^2 \le (C_2C_0\epsilon^2)^n. \qquad (4.16)
\end{aligned}
$$

Thus (4.15) and (4.16) imply

$$
\begin{aligned}
&(1+t)H(t)^n + n\int_0^t (1+s)H(s)^{n-1}\|\nabla^2 U(s)\|_1^2 ds \\
&\qquad \le 2H(0)^n + (C_3E_0)^{2n} + n(10C_2)(C_2C_0\epsilon^2)^n. \qquad (4.17)
\end{aligned}
$$

For $1 \le k \le N$, the following estimate can be proved by induction.

$$
\begin{aligned}
&(1+t)^k H(t)^n + n\int_0^t (1+s)^k H(s)^{n-1}\|\nabla^2 U(s)\|_1^2 ds \\
&\le \left[2H(0)^n + (C_3E_0)^{2n}\right]\sum_{l=1}^k (10C_2)^{l-1} + n(10C_2)^k(C_2C_0\epsilon^2)^n. \qquad (4.18)
\end{aligned}
$$

In fact, suppose that (4.18) holds for $1 \leq k \leq N-1$. Then it follows from (4.6) that

$$
\begin{aligned}
&(1+t)^{k+1}H(t)^n + n\int_0^t (1+s)^{k+1}H(s)^{n-1}\|\nabla^2 U(s)\|_1^2 ds \\
&\leq 2H(0)^n + (C_3E_0)^{2n} + 10C_2 n\int_0^t (1+s)^k H(s)^{n-1}\|\nabla^2 U(s)\|_1^2 ds \\
&\leq \left[2H(0)^n + (C_3E_0)^{2n}\right] \\
&\quad +10C_2\left\{\left[2H(0)^n + (C_3E_0)^{2n}\right]\sum_{l=1}^{k}(10C_2)^{l-1} + n(10C_2)^k(C_2C_0\epsilon^2)^n\right\} \\
&\leq \left[2H(0)^n + (C_3E_0)^{2n}\right]\sum_{l=1}^{k+1}(10C_2)^{l-1} + n(10C_2)^{k+1}(C_2C_0\epsilon^2)^n. \qquad (4.19)
\end{aligned}
$$

Thus, (4.19) together with (4.17) shows that (4.18) holds for any $1 \leq k \leq N$. In particular,

$$
(1+t)^N H(t)^n \leq \left[2H(0)^n + (C_3E_0)^{2n}\right]\frac{(10C_2)^N - 1}{10C_2 - 1} + n(10C_2)^N(C_2C_0\epsilon^2)^n.
$$

Since

$$
\frac{5n}{2} - 3 \leq N = \left[\frac{5n}{2} - 2\right] \leq \frac{5n}{2} - 1,
$$

we have

$$
(1+t)^{\frac{5}{2}n-3}H(t)^n \leq C^{\frac{5n}{2}}\left[H(0)^n + E_0^{2n} + \epsilon^{2n}\right],
$$

which implies

$$
H(t)^{\frac{1}{2}} \leq C\left[H(0)^n + E_0^{2n} + \epsilon^{2n}\right]^{\frac{1}{2n}}(1+t)^{-\frac{5}{4}+\frac{3}{2n}}. \qquad (4.20)
$$

Notice that $H(0)$, E_0 and ϵ are independent of n. Hence,

$$
\left[H(0)^n + E_0^{2n} + \epsilon^{2n}\right]^{\frac{1}{2n}} \to \max\left\{\sqrt{H(0)}, E_0, \epsilon\right\},
$$

as n tends to infinity. Thus, taking $n \to \infty$ in (4.20) gives

$$
H(t)^{\frac{1}{2}} \leq C\max\left\{\sqrt{H(0)}, E_0, \epsilon\right\}(1+t)^{-\frac{5}{4}},
$$

that is,

$$
\|\nabla U(t)\|_2 \leq C\max\left\{\sqrt{H(0)}, E_0, \epsilon\right\}(1+t)^{-\frac{5}{4}}.
$$

This together with Lemma 2.2 gives (1.11) and (1.12).

Step 2. To estimate (1.10), the integral formula (2.8) and Lemma 2.1 yield

$$\begin{aligned}\|U(t)\| &\leq CE_0(1+t)^{-\frac{3}{4}} + C\int_0^t (1+t-s)^{-\frac{3}{4}}(\|F(U)(s)\|_{L^1} + \|F(U)(s)\|)ds \\ &\leq CE_0(1+t)^{-\frac{3}{4}} + C\epsilon\int_0^t (1+t-s)^{-\frac{3}{4}}\|\nabla U(s)\|_1 ds \\ &\leq CE_0(1+t)^{-\frac{3}{4}} + C\epsilon\int_0^t (1+t-s)^{-\frac{3}{4}}(1+s)^{-\frac{5}{4}} ds \\ &\leq C(1+t)^{-\frac{3}{4}},\end{aligned}$$

where (1.12) was also used. Thus, for any $2 \leq p \leq 6$, (1.10) holds by the interpolation. For (1.13), first, the boundedness of $\|\partial_t U(t)\|$ is directly from (2.6). Then, again by Lemma 2.1, we have for $t \geq 1$,

$$\begin{aligned}\|\partial_t U(t)\| &\leq CE_0(1+t)^{-\frac{3}{4}} + C\int_0^t (1+t-s)^{-\frac{5}{4}}\|F(U)(s)\|_{L^1} ds \\ &\quad + C\int_0^t (1+t-s)^{-\frac{5}{4}}\left[1+(t-s)^{-\frac{1}{2}}\right]\|F(U)(s)\|_1 ds \\ &\leq CE_0(1+t)^{-\frac{3}{4}} + C\epsilon\int_0^t (1+t-s)^{-\frac{5}{4}}(1+s)^{-\frac{5}{4}} ds \\ &\quad + C\epsilon\int_0^t (1+t-s)^{-\frac{5}{4}}\left[1+(t-s)^{-\frac{1}{2}}\right](1+s)^{-\frac{5}{4}} ds \\ &\leq C(1+t)^{-\frac{5}{4}}.\end{aligned}$$

Thus, (1.13) is proved and this completes the proof of the theorem.

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