Optimal L^p-L^q Convergence Rates for the Compressible Navier-Stokes Equations with Potential Force

Renjun Duan[1], Hongxia Liu[2], Seiji Ukai[3], Tong Yang[1]

[1] Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong, P.R. China

[2] Department of Mathematics, Jinan University Guangzhou 510632, P.R. China

[3] Liu Bie Ju Centre for Mathematical Sciences City University of Hong Kong, Kowloon, Hong Kong, P.R. China

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Abstract

In this paper, we are concerned with the optimal L^p-L^q convergence rates for the compressible Navier-Stokes equations with a potential external force in the whole space. Under the smallness assumption on both the initial perturbation and the external force in some Sobolev spaces, the optimal convergence rates of the solution in L^q-norm with $2 \le q \le 6$ and its first order derivative in L^2-norm are obtained when the initial perturbation is bounded in L^p with $1 \le p < 6/5$. The proof is based on the energy estimates on the solution to the nonlinear problem and some L^p-L^q estimates on the semigroup generated by the corresponding linearized operator.

1 Introduction

Consider the initial value problem of the compressible Navier-Stokes equations with a potential force in the whole space:

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ u_t + (u \cdot \nabla) u + \dfrac{\nabla P(\rho)}{\rho} = \dfrac{\mu}{\rho} \Delta u + \dfrac{\mu + \mu'}{\rho} \nabla(\nabla \cdot u) - \nabla \phi(x), \\ (\rho, u)(0, x) = (\rho_0, u_0)(x) \to (\rho_\infty, 0), \quad \text{as} \quad |x| \to \infty. \end{cases} \tag{1.1}$$

Here, $t > 0$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. The unknown functions $\rho = \rho(t,x) > 0$ and $u = u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x))$ denote the density and velocity respectively. $P = P(\rho)$ is the pressure function, $-\nabla\phi(x)$ is the time independent potential force, μ, μ' are the viscosity coefficients, and $(\rho_\infty, 0)$ is the state of initial data at infinity. In the following discussion, it is assumed that μ and μ' satisfy the physical conditions $\mu > 0$ and $\mu' + \frac{2}{3}\mu \geq 0$, while ρ_∞ is a positive constant and $P(\rho)$ is smooth in a neighborhood of ρ_∞ with $P'(\rho_\infty) > 0$.

For the Navier-Stokes equations $(1.1)_1$-$(1.1)_2$ with potential force, the stationary solution (ρ_*, u_*) is given by $(\rho_*(x), 0)$, where $\rho_*(x)$ satisfies, cf. [14],

$$\int_{\rho_\infty}^{\rho_*(x)} \frac{P'(s)}{s} ds + \phi(x) = 0. \tag{1.2}$$

The global existence of solutions to the nonlinear problem (1.1) near the steady state $(\rho_*, 0)$ with initial perturbation in H^3 was proved by Matsumura and Nishida [14]. In this paper, we want to obtain the optimal convergence rate of the solution to the steady state when the initial perturbation is also bounded in L^p with $1 \leq p < 6/5$. Precisely, the result can be stated as follows.

Theorem 1.1. *Let (ρ, u) be a global classical solution in H^3 to the initial value problem (1.1), and $(\rho_*, 0)$ be the corresponding stationary solution. For given $1 \leq p < 6/5$, suppose that the potential function $\phi(x)$ and the initial perturbation satisfy*

$$\|\phi\|_{L^{\frac{2p}{2-p}} \cap L^\infty} + \sum_{k=1}^{4} \|(1+|x|)\nabla^k \phi\|_{L^{\frac{2p}{2-p}} \cap L^\infty} \leq \epsilon, \tag{1.3}$$

$$\|(\rho_0 - \rho_*, u_0)\|_{H^3} \leq \epsilon, \tag{1.4}$$

for some small constant $\epsilon > 0$, and

$$\|(\rho_0 - \rho_*, u_0)\|_{L^p} < +\infty. \tag{1.5}$$

Then, there exist constants $\epsilon_0 > 0$ and $C_0 > 0$ such that for any $0 < \epsilon \leq \epsilon_0$ such that

$$\|\nabla^k(\rho - \rho_*, u)(t)\|_{L^2} \leq C_0(1+t)^{-\sigma(p,2;1)}, \quad k = 1, 2, 3, \tag{1.6}$$

$$\|(\rho - \rho_*, u)(t)\|_{L^2} \leq C_0(1+t)^{-\sigma(p,2;0)}. \tag{1.7}$$

Furthermore,

$$\|(\rho - \rho_*, u)(t)\|_{L^q} \leq C_0(1+t)^{-\sigma(p,q;0)}, \quad 2 \leq q \leq 6. \tag{1.8}$$

Here

$$\sigma(p,q;k) = \frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{k}{2}. \tag{1.9}$$

Remark 1.1. *For the linearized problem of (1.1)*

$$\begin{cases} \sigma_t + \gamma\nabla \cdot w = 0, \\ w_t - \mu_1 \Delta w - \mu_2 \nabla\nabla \cdot w + \gamma\nabla\sigma = 0, \\ (\sigma, w)(0,x) = (\sigma_0, w_0)(x), \end{cases}$$

where μ_1, μ_2 and γ are some positive constants, the precise convergence rates can be obtained through the Fourier transform:

$$\|(\sigma, w)(t)\|_{L^q} \leq C(1+t)^{-\sigma(p,q;0)}\|(\sigma_0, w_0)\|_{L^p\cap L^2},$$

$$\|\nabla(\sigma, w)(t)\|_{L^2} \leq C(1+t)^{-\sigma(p,2;1)}\|(\sigma_0, w_0)\|_{L^p\cap H^1}.$$

By comparison, the convergence rates for the nonlinear problem (1.1) given in (1.6) with $k = 1$ for the first order derivative in L^2-norm, and the one in (1.8) for the perturbation itself in L^q-norm with $2 \leq q \leq 6$, are the same as those for the linearized problem so that they are optimal.

Remark 1.2. *In Theorem 1.1, the constant C_0 depends only on ϵ_0 and p. It should be pointed out that C_0 may tend to infinity when the index p approaches $6/5$. In fact, when $p = 6/5$, a slower decay rate was obtained in [17] for a class of general (including non-potential) forces; see (1.10) in the below.*

Remark 1.3. *Consider the external force generated by the potential function ϕ in the form of*

$$\phi(x) = \frac{\epsilon}{(1+|x|)^{1+\delta}},$$

where ϵ and δ are small positive constants. Notice that the case when $\delta = 0$ corresponds to the Newton gravitational potential. It is straightforward to check that the condition (1.3) is satisfied if and only if

$$\delta > \frac{3}{p} - \frac{5}{2}.$$

Thus, $1 \leq p < 6/5$ implies $\delta > 0$. In fact, one can improve the argument used in [17] by employing the interpolation method on the Lorentz space to show that for $\delta = 3/p - 5/2$ with $1 \leq p < 6/5$, the decay estimates given in Theorem 1.1 still hold.

For the compressible Navier-Stokes equations, a lot of works have been done on the existence, stability and convergence rates of solutions for either isentropic or non-isentropic (heat-conductive) cases, cf. [2, 6, 9, 12, 13, 14] and references therein. In the following, we only review some studies on the convergence rates for the compressible Navier-Stokes equations in the whole space with or without external forces which are related to the results of this paper.

When there is no external force, the problem was studied by Matsumura and Nishida [12], Ponce [15], Deckelnick [2, 3], Hoff-Zumbrun [5], Liu-Wang [11], Kobayashi-Shibata [10], Kagei-Kobayashi [7, 8], and Kobayashi [9] in various settings. Recently, an almost optimal convergence rate in $L^2(\mathbb{R}^n)$, $n \geq 3$, was obtained by Ukai-Yang-Zhao [18] showing that

$$\|(\rho - \tilde{\rho}, u)(t)\|_3 \leq C(n, \kappa)(1+t)^{-\frac{n}{4}+\kappa},$$

when the initial perturbation belongs to $H^3(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Here $\kappa > 0$ is any positive constant which can be arbitrarily small, but $C(n, \kappa)$ may approach to ∞ when κ tends to

zero. The result was later generalized in [4] to obtain the optimal convergence rate under the same assumptions.

For the general external force, the following convergence rate was obtained by Shibata-Tanaka [16, 17]:

$$\|\nabla(\rho-\rho_*, u-u_*)(t)\|_2 \le C(\kappa)(1+t)^{-\frac{1}{2}+\kappa}, \tag{1.10}$$

for any small positive constant κ when the initial perturbation belongs to $H^3(\mathbb{R}^3)\cap L^{6/5}(\mathbb{R}^3)$. Similarly, $C(\kappa)$ may become ∞ when κ tends to zero. Notice that even when $\kappa=0$, the rate given in (1.10) is not optimal for the perturbation in the space $L^{6/5}$. For this reason, we study the optimal convergence rate problem first for the case with a potential force when initial perturbation in L^p with $1\le p<6/5$. However, the following analysis does not include the critical index $p=6/5$.

The idea of the proof can be outlined as follows. Firstly, we will obtain a Lyapunov-type energy inequality in the form of

$$\frac{dH(t)}{dt}+CH(t)\le C\|\nabla U(t)\|^2,$$

where $H(t)$ is an energy functional including all the derivatives of at least one order, and $U(t)$ denotes the perturbation. Then, based on the L^p-L^q estimates from spectral analysis on the linearized system, the first order derivative $\nabla U(t)$ can be bounded by $H(t)$ in some integral form with a small coefficient. Finally, combining these two types of estimates, the optimal convergence rates can be obtained by the Gronwall inequality.

The rest of the paper is organized as follows. After stating some basic lemmas in Section 2, we give the proof of Theorem 1.1 in Section 3.

Notations. Throughout this paper, the norms in the Sobolev Spaces $H^m(\mathbb{R}^3)$ and $W^{m,p}(\mathbb{R}^3)$ are denoted respectively by $\|\cdot\|_m$ and $\|\cdot\|_{m,p}$ for integer $m\ge 0$, any $p\ge 1$. In particular, for $m=0$, we will simply use $\|\cdot\|$ and $\|\cdot\|_{L^p}$. And $\langle\cdot,\cdot\rangle$ denotes the inner-product in $L^2(\mathbb{R}^3)$. Moreover, C denotes a general constant which may vary in different estimates. If the dependence needs to be explicitly stressed, some notations like C_1, C_2, will be used. As usual,

$$\nabla=(\partial_1,\partial_2,\partial_3),\quad \partial_i=\partial_{x_i},\ i=1,2,3,$$

and for any integer $l\ge 0$, $\nabla^l f$ denotes all derivatives up to l-order of the function f. And for multi-indice $\alpha=(\alpha_1,\alpha_2,\alpha_3)$, $\beta=(\beta_1,\beta_2,\beta_3)$,

$$\partial_x^\alpha=\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3},\quad |\alpha|=\sum_{i=1}^3\alpha_i,$$

and $C_\beta^\alpha=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ when $\beta\le\alpha$.

2 Preliminary

In this section, we will first reformulate the problem and then give some decay-in-time estiamtes and Sobolev inequalities which will be used in the proof of Theorem 1.1.

Set

$$\tilde{\rho}(t,x) = \rho(t,x) - \rho_*(x), \quad \tilde{u}(t,x) = u(t,x),$$

and

$$\bar{\rho}(x) = \rho_*(x) - \rho_\infty. \tag{2.1}$$

Then (1.1) becomes

$$\begin{cases} \tilde{\rho}_t + \rho_\infty \nabla \cdot \tilde{u} = \tilde{S}_1, \\ \tilde{u}_t - \dfrac{\mu}{\rho_\infty}\Delta\tilde{u} - \dfrac{\mu+\mu'}{\rho_\infty}\nabla\nabla\cdot\tilde{u} + \dfrac{P'(\rho_\infty)}{\rho_\infty}\nabla\tilde{\rho} = \tilde{S}_2, \\ (\tilde{\rho},\tilde{u})(0,x) = (\rho_0 - \rho_*, u_0)(x) \to (0,0) \ \text{ as } \ |x| \to \infty, \end{cases}$$

where $\tilde{S}_1$ and $\tilde{S}_2$ are the source terms. Denote

$$\mu_1 = \frac{\mu}{\rho_\infty}, \quad \mu_2 = \frac{\mu+\mu'}{\rho_\infty}, \quad \gamma = \sqrt{P'(\rho_\infty)}.$$

By using the new unknown functions

$$\sigma(t,x) = \tilde{\rho}(t,x), \quad w(t,x) = \frac{\rho_\infty}{\sqrt{P'(\rho_\infty)}}\tilde{u}(t,x),$$

the initial value problem (1.1) is reformulated as

$$\begin{cases} \sigma_t + \gamma\nabla\cdot w = S_1, \\ w_t - \mu_1\Delta w - \mu_2\nabla\nabla\cdot w + \gamma\nabla\sigma = S_2, \\ (\sigma,w)(0,x) = (\sigma_0,w_0)(x), \end{cases} \tag{2.2}$$

where

$$S_1 = -\frac{\mu_1\gamma}{\mu}\nabla\cdot[(\sigma+\bar{\rho})w], \tag{2.3}$$

$$\begin{aligned} S_2 = &-\frac{\mu_1^2\gamma^2}{\mu^2}(w\cdot\nabla)w - \mu_1\frac{\sigma+\bar{\rho}}{\sigma+\rho_*}\Delta w - \mu_2\frac{\sigma+\bar{\rho}}{\sigma+\rho_*}\nabla\nabla\cdot w \\ &- \left[\frac{P'(\sigma+\rho_*)}{\sigma+\rho_*} - \frac{P'(\rho_*)}{\rho_*}\right]\nabla\bar{\rho} - \left[\frac{P'(\sigma+\rho_*)}{\sigma+\rho_*} - \frac{P'(\rho_\infty)}{\rho_\infty}\right]\nabla\sigma, \end{aligned} \tag{2.4}$$

and

$$(\sigma_0,w_0)(x) = \left(\rho_0 - \rho_*, \frac{\rho_\infty}{\sqrt{P'(\rho_\infty)}}u_0\right)(x) \to (0,0) \ \text{ as } \ |x| \to \infty.$$

We shall consider the convergence rates of the solution (ρ, u) to the steady state $(\rho_*, 0)$, that is, the decay rates of the perturbed solution (σ, w). For later use, the result on the global existence of solutions to (2.2) by [14] is stated as follows.

Proposition 2.1. *There exist constants $\epsilon_1 > 0$ and $C_1 > 0$ such that if (1.3) and (1.4) hold for any $\epsilon \le \epsilon_1$, then the initial value problem (2.2) has a unique solution (σ, w) globally in time, satisfying*

$$\sigma \in C^0(0,\infty; H^3(\mathbb{R}^3)) \cap C^1(0,\infty; H^2(\mathbb{R}^3)),$$
$$w \in C^0(0,\infty; H^3(\mathbb{R}^3)) \cap C^1(0,\infty; H^1(\mathbb{R}^3)),$$

and

$$\|(\sigma,w)(t)\|_3^2 + \int_0^t (\|\nabla(\sigma,w)(s)\|_2^2 + \|\nabla w(s)\|_3^2)\, ds \le C_1 \|(\sigma_0, w_0)\|_3^2.$$

To use the L^p-L^q estimates of the linear problem for the nonlinear problem (2.4), we write the solution of (2.2) as

$$U(t) = E(t)U_0 + \int_0^t E(t-s)F(U(s))ds. \tag{2.5}$$

From now on, we use the notations

$$U = [\sigma, w]^T, \quad U_0 = [\sigma_0, w_0]^T, \quad F = [S_1, S_2]^T, \tag{2.6}$$

and $E(t)$ is the solution semi-group defined by $E(t) = e^{-tA}$, $t \ge 0$, with A being a matrix-valued differential operator given by

$$A = \begin{pmatrix} 0 & \gamma \mathrm{div} \\ \gamma\nabla & -\mu_1\Delta - \mu_2 \nabla \mathrm{div} \end{pmatrix}.$$

The semigroup $E(t)$ has the following properties on the decay in time, cf. [9, 10].

Lemma 2.1. *Let $k \ge 0$ be an integer and $1 \le p \le 2 \le q < \infty$. Then for any $t \ge 0$, it holds that*

$$\|\nabla^k E(t)U_0\|_{L^q} \le C(1+t)^{-\sigma(p,q;k)} \|U_0\|_{L^p \cap H^k},$$

where $\sigma(p,q;k)$ is defined by (1.9).

For later use, let us also list some classical Sobolev inequalities and integral inequalities as follows, cf. [1, 2] for the proofs.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^3$ be the whole space $\mathbb{R}^3$, or half space $\mathbb{R}^3_+$ or the exterior domain of a bounded region with smooth boundary. Then*

(i) $\|f\|_{L^6(\Omega)} \le C\|\nabla f\|_{L^2(\Omega)}$, *for* $f \in H^1(\Omega)$.

(ii) $\|f\|_{C^0(\bar\Omega)} \le C\|f\|_{W^{1,6}(\Omega)} \le C\|\nabla f\|_{H^1(\Omega)}$, *for* $f \in H^2(\Omega)$.

Lemma 2.3. *For Ω defined in Lemma 2.2, we have*

(i) $|\int_\Omega f \cdot g \cdot h\, dx| \le \varepsilon\|\nabla f\|^2 + \frac{C}{\varepsilon}\|g\|_1^2\|h\|^2$ *for* $\varepsilon > 0$, $f, g \in H^1(\Omega)$, $h \in L^2(\Omega)$.

(ii) $|\int_\Omega f \cdot g \cdot h\, dx| \le \varepsilon\|g\|^2 + \frac{C}{\varepsilon}\|\nabla f\|_1^2\|h\|^2$ *for* $\varepsilon > 0$, $f \in H^2(\Omega)$, $g, h \in L^2(\Omega)$.

3 Convergence rate

We are now ready to prove Theorem 1.1. As in [18], the strategy is to first obtain a Lyapunov-type energy inequality of derivatives of the order from the first to the third. And then based on the L^p-L^q estimates, for the initial perturbation being bounded in L^p with $1 \le p < 6/5$, we will show that the decay rate of the first order derivative is of order $(1+t)^{-\sigma(p,2;1)}$ up to an error related to the derivatives of higher order. Combining these two estimations yields that all higher order derivatives decay at the same rate as the first order derivative and then gives the proof of the theorem.

3.1 Energy estimate for derivatives

As in [4, 18], based on Proposition 2.1 on the existence of the global-in-time solution, we will derive a Lyapunov-type inequality by the energy estimates.

Lemma 3.1. *Under the assumptions of Theorem 1.1, let $U = (\sigma, w)^T$ be the solution to the initial value problem (2.2). Then there are constants $C_2 > 0$ and $D_2 > 0$ such that if $\epsilon > 0$ in (1.3) and (1.4) is small enough, then for any $t \ge 0$, it holds that*

$$\frac{dH(t)}{dt} + D_2 H(t) \le C_2 \|\nabla U(t)\|^2, \tag{3.1}$$

where the energy functional $H(t)$ is equivalent to $\|\nabla U(t)\|_2^2$, that is, there exists a positive constant $C_3 > 0$ such that

$$\frac{1}{C_3}\|\nabla U(t)\|_2^2 \le H(t) \le C_3 \|\nabla U(t)\|_2^2, \quad t \ge 0. \tag{3.2}$$

Proof. For each multi-index α with $1 \le |\alpha| \le 3$, by applying ∂_x^α to (2.2), multiplying by $\partial_x^\alpha \sigma$, $\partial_x^\alpha w$ respectively, and then integrating over $\mathbb{R}^3$, we have from the sum of $(2.2)_1$-$(2.2)_2$ that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha U(t)\|^2 + \mu_1 \|\nabla \partial_x^\alpha w(t)\|^2 + \mu_2 \|\nabla \cdot \partial_x^\alpha w(t)\|^2 \\ &= \langle \partial_x^\alpha \sigma(t), \partial_x^\alpha S_1(t)\rangle + \langle \partial_x^\alpha w(t), \partial_x^\alpha S_2(t)\rangle. \end{aligned} \tag{3.3}$$

Before estimating the two terms on the right hand side of (3.3), we notice from (2.3) and (2.4) that the source term (S_1, S_2) has the following equivalent properties

$$S_1 \sim \partial_i \sigma w^i + \sigma \partial_i w^i + \partial_i \bar{\rho} w^i + \bar{\rho} \partial_i w^i, \tag{3.4}$$

$$\begin{aligned} S_2 \sim\ & w^i \partial_i w^j + \sigma \partial_i \partial_i w^j + \sigma \partial_j \partial_i w^i + \sigma \partial_j \sigma \\ & + \bar{\rho} \partial_i \partial_i w^j + \bar{\rho} \partial_j \partial_i w^i + \partial_j \bar{\rho} \sigma + \bar{\rho} \partial_j \sigma. \end{aligned} \tag{3.5}$$

Then for the first term on the right hand side of (3.3), (3.4) implies that

$$\begin{aligned} \langle \partial_x^\alpha \sigma(t), \partial_x^\alpha S_1(t)\rangle \le\ & C|\langle \partial_x^\alpha \sigma(t), \partial_x^\alpha \partial_i \sigma(t) w^i(t)\rangle| \\ & + C \textstyle\sum_{|\beta| \le |\alpha|-1} C_\beta^\alpha |\langle \partial_x^\alpha \sigma(t), \partial_x^\beta \partial_i \sigma(t) \partial_x^{\alpha-\beta} w^i(t)\rangle| \\ & + C \textstyle\sum_{|\beta| \le |\alpha|} C_\beta^\alpha |\langle \partial_x^\alpha \sigma(t), \partial_x^\beta \sigma(t) \partial_x^{\alpha-\beta} \partial_i w^i(t)\rangle| \\ & + C \textstyle\sum_{|\beta| \le |\alpha|} C_\beta^\alpha |\langle \partial_x^\alpha \sigma(t), \partial_x^\beta \partial_i \bar{\rho} \partial_x^{\alpha-\beta} w^i(t)\rangle| \\ & + C \textstyle\sum_{|\beta| \le |\alpha|} C_\beta^\alpha |\langle \partial_x^\alpha \sigma(t), \partial_x^\beta \bar{\rho} \partial_x^{\alpha-\beta} \partial_i w^i(t)\rangle|. \end{aligned} \tag{3.6}$$

Let I_j denote the j-th ($j = 1, 2, ..., 5$) term on the right hand side of (3.6). As in [4, 18], by Proposition 2.1, Lemmas 2.2 and 2.3, we have

$$\sum_{j=1}^{3} I_j \leq C\epsilon \sum_{1\leq|\alpha|\leq 3} \|\partial_x^\alpha \sigma(t)\|^2 + C\epsilon \sum_{1\leq|\alpha|\leq 4} \|\partial_x^\alpha w(t)\|^2. \tag{3.7}$$

For I_4 and I_5, by (1.2), (1.3) and (2.1) imply

$$\|\rho_* - \rho_\infty\|_{L^{\frac{2p}{2-p}}\cap L^\infty} + \sum_{k=1}^{4} \|(1+|x|)\nabla^k(\rho_* - \rho_\infty)\|_{L^{\frac{2p}{2-p}}\cap L^\infty} \leq \epsilon. \tag{3.8}$$

Hence, it follows again from Lemmas 2.2 and 2.3 that

$$\begin{aligned}
I_4 &= C\left\{\sum_{|\beta|=|\alpha|} + \sum_{0\leq|\beta|\leq|\alpha|-1}\right\} |\langle \partial_x^\alpha \sigma(t), \partial_x^\beta \partial_i \bar{\rho} \partial_x^{\alpha-\beta} w^i(t)\rangle| \\
&\leq \epsilon \|\partial_x^\alpha \sigma(t)\|^2 + \frac{C}{\epsilon}\|\partial_x^\alpha \partial_i \bar{\rho} w^i(t)\|^2 + \frac{C}{\epsilon} \sum_{0\leq|\beta|\leq|\alpha|-1} \|\partial_x^\beta \partial_i \bar{\rho} \partial_x^{\alpha-\beta} w^i(t)\|^2 \\
&\leq \epsilon \|\partial_x^\alpha \sigma(t)\|^2 + \frac{C}{\epsilon}\|(1+|x|)\partial_x^\alpha \partial_i \bar{\rho}\|_{L^\infty}^2 \left\|\frac{w^i(t)}{1+|x|}\right\|^2 \\
&\quad + \frac{C}{\epsilon} \sum_{0\leq|\beta|\leq|\alpha|-1} \|\partial_x^\beta \partial_i \bar{\rho}\|_{L^\infty}^2 \|\partial_x^{\alpha-\beta} \partial_i w^i(t)\|^2 \\
&\leq \epsilon \sum_{1\leq|\alpha|\leq 3} \|\partial_x^\alpha \sigma(t)\|^2 + C\epsilon \sum_{1\leq|\alpha|\leq 4} \|\partial_x^\alpha w(t)\|^2,
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
I_5 &\leq C \sum_{0\leq|\beta|\leq|\alpha|} |\langle \partial_x^\alpha \sigma(t), \partial_x^\beta \bar{\rho} \partial_x^{\alpha-\beta} \partial_i w^i(t)\rangle| \\
&\leq \epsilon \|\partial_x^\alpha \sigma(t)\|^2 + \frac{C}{\epsilon} \sum_{0\leq|\beta|\leq|\alpha|} \|\partial_x^\beta \bar{\rho} \partial_x^{\alpha-\beta} \partial_i w^i(t)\|^2 \\
&\leq \epsilon \|\partial_x^\alpha \sigma(t)\|^2 + \frac{C}{\epsilon} \sum_{0\leq|\beta|\leq|\alpha|} \|\partial_x^\beta \bar{\rho}\|_{L^\infty}^2 \|\partial_x^{\alpha-\beta} \partial_i w^i(t)\|^2 \\
&\leq \epsilon \sum_{1\leq|\alpha|\leq 3} \|\partial_x^\alpha \sigma(t)\|^2 + C\epsilon \sum_{1\leq|\alpha|\leq 4} \|\partial_x^\alpha w(t)\|^2.
\end{aligned} \tag{3.10}$$

Here, we have used the following Hardy inequality

$$\left\|\frac{w^i(t)}{1+|x|}\right\| \leq \|\nabla w^i(t)\|.$$

Thus, (3.7), (3.9) and (3.10) give that

$$\langle \partial_x^\alpha \sigma(t), \partial_x^\alpha S_1(t)\rangle \leq \epsilon \sum_{1\leq|\alpha|\leq 3} \|\partial_x^\alpha \sigma(t)\|^2 + C\epsilon \sum_{1\leq|\alpha|\leq 4} \|\partial_x^\alpha w(t)\|^2. \tag{3.11}$$

Similar argument gives

$$\langle \partial_x^\alpha w(t), \partial_x^\alpha S_2(t)\rangle \le \epsilon \sum_{1\le|\alpha|\le 3} \|\partial_x^\alpha \sigma(t)\|^2 + C\epsilon \sum_{1\le|\alpha|\le 4} \|\partial_x^\alpha w(t)\|^2. \tag{3.12}$$

Hence, (3.3) together with (3.11)-(3.12) yield

$$\begin{aligned}\frac{d}{dt}\sum_{1\le|\alpha|\le 3} \|\partial_x^\alpha U(t)\|^2 + C\sum_{1\le|\alpha|\le 3} \|\nabla\partial_x^\alpha w(t)\|^2 \\ \le C\epsilon \sum_{1\le|\alpha|\le 3} \|\partial_x^\alpha \sigma(t)\|^2 + C\epsilon \sum_{1\le|\alpha|\le 4} \|\partial_x^\alpha w(t)\|^2.\end{aligned} \tag{3.13}$$

Furthermore, to include the estimation on $\|\nabla\partial_x^\alpha\sigma(t)\|^2$ when $1\le|\alpha|\le 2$, from $(2.4)_2$, we have

$$\gamma\nabla\sigma = -w_t + \mu_1\Delta w + \mu_2\nabla(\nabla\cdot w) + S_2.$$

After applying ∂_x^α with $1\le|\alpha|\le 2$ to the above equation, combining $(2.2)_1$ and performing the computations similar to (3.9) and (3.10) lead to

$$\begin{aligned}&\frac{\gamma}{2}\sum_{1\le|\alpha|\le 2} \|\nabla\partial_x^\alpha\sigma(t)\|^2 + \frac{d}{dt}\sum_{1\le|\alpha|\le 2} \langle \partial_x^\alpha w(t), \nabla\partial_x^\alpha\sigma(t)\rangle \\ &\le C\sum_{1\le|\alpha|\le 2} \|\partial_x^\alpha\nabla w(t)\|_1^2 + C\epsilon\sum_{1\le|\alpha|\le 3} \|\partial_x^\alpha\sigma(t)\|^2 + C\epsilon\sum_{1\le|\alpha|\le 4} \|\partial_x^\alpha w(t)\|^2.\end{aligned} \tag{3.14}$$

Define

$$H(t) = D_1\sum_{1\le|\alpha|\le 3} \|\partial^\alpha U(t)\|^2 + \sum_{1\le|\alpha|\le 2} \langle \partial_x^\alpha w(t), \nabla\partial_x^\alpha\sigma(t)\rangle.$$

By choosing the constant $D_1 > 0$ sufficiently large and $\epsilon > 0$ small enough, the linear combination of (3.13) and (3.14) leads to

$$\frac{dH(t)}{dt} + C\left(\|\nabla^2\sigma(t)\|_1^2 + \|\nabla^2 w(t)\|_2^2\right) \le C\epsilon\|\nabla U(t)\|^2,$$

which implies

$$\frac{dH(t)}{dt} + D_2\|\nabla^2 U(t)\|_1^2 \le C\epsilon\|\nabla U(t)\|^2, \tag{3.15}$$

where D_2 is a positive constant independent of ϵ. Adding $D_2\|\nabla U(t)\|^2$ to both sides of (3.15) yields (3.1) and this completes the proof of the lemma.

□

3.2 Estimates on the first order derivative

In this subsection, based on the L^p-L^q estimates for the linear problem, we shall use the formula (2.5) to estimate the decay rate of the first order derivative.

Lemma 3.2. *Under the assumptions of Theorem 1.1, let $U = (\sigma, w)^T$ be the solution to the initial value problem (2.4). Then we have*

$$\|\nabla U(t)\| \le CK_0(1+t)^{-\sigma(p,2;1)} + C\epsilon \int_0^t (1+t-s)^{-\sigma(p,2;1)} \|\nabla U(s)\|_2 ds, \tag{3.16}$$

where $K_0 = \|U_0\|_{L^p \cap H^1}$ is finite by (1.4) and (1.5).

Proof. From the integral formula (2.5) and Lemma 2.1, we have

$$\|\nabla U(t)\| \le CK_0(1+t)^{-\sigma(p,2;1)} + C \int_0^t (1+t-s)^{-\sigma(p,2;1)} \|F(U)(s)\|_{L^p \cap H^1} ds, \tag{3.17}$$

where $F(U)$ is given in (2.6). To derive (3.16), we need to control $\|F(U(t))\|_{L^p \cap H^1}$ by the L^2-norm of the derivatives of at least first order.

Firstly, we estimate those terms including $\bar{\rho}$. By (3.8), it follows that

$$\|\nabla \bar{\rho} \cdot w\|_{L^p} \le \|(1+|x|)\nabla \bar{\rho}\|_{L^{\frac{2p}{2-p}}} \left\| \frac{w}{1+|x|} \right\| \le C\epsilon \|\nabla w\|,$$
$$\|\bar{\rho} \nabla \cdot w\|_{L^p} \le \|\bar{\rho}\|_{L^{\frac{2p}{2-p}}} \|\nabla \cdot w\| \le C\epsilon \|\nabla w\|,$$

and

$$\|\nabla \bar{\rho} \cdot w\| \le \|(1+|x|)\nabla \bar{\rho}\|_{L^\infty} \left\| \frac{w}{1+|x|} \right\| \le C\epsilon \|\nabla w\|,$$
$$\|\bar{\rho} \nabla \cdot w\| \le \|\bar{\rho}\|_{L^\infty} \|\nabla \cdot w\| \le C\epsilon \|\nabla w\|.$$

Thus, the above inequalities give that

$$\|\nabla \cdot (\bar{\rho} w)\|_{L^p},\ \|\nabla \cdot (\bar{\rho} w)\| \le C\epsilon \|\nabla w\|.$$

Similarly, it holds that

$$\|\nabla \cdot (\bar{\rho} \sigma)\|_{L^p},\ \|\nabla \cdot (\bar{\rho} \sigma)\| \le O(1)\epsilon \|\nabla \sigma\|,$$
$$\|\nabla \cdot (\bar{\rho} w)\|_1 \le O(1)\epsilon \|\nabla w\|_1,\ \|\nabla \cdot (\bar{\rho} \sigma)\|_1 \le O(1)\epsilon \|\nabla \sigma\|_1,$$
$$\|\nabla^2 (\bar{\rho} w)\|_{L^p} \le O(1)\epsilon \|\nabla w\|_1,\ \|\nabla^2 (\bar{\rho} w)\|_1 \le O(1)\epsilon \|\nabla w\|_2.$$

The estimation on the other terms in $F(U(t))$ are easier so that we omit it for brevity. Hence, by Proposition 2.1, we have

$$\begin{aligned} \|F(U)(t)\|_{L^p} &\le C\|U(t)\| \|\nabla U(t)\|_1 \\ &\quad + C(\|\nabla \cdot (\bar{\rho} w)\|_{L^p} + \|\nabla \cdot (\bar{\rho} \sigma)\|_{L^p} + \|\nabla^2 (\bar{\rho} w)\|_{L^p}) \\ &\le C\epsilon \|\nabla U(t)\|_1, \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} \|F(U)(t)\|_{H_1} &\le C\|U(t)\|_{W^{1,\infty}} \|\nabla U(t)\|_2 \\ &\quad + C(\|\nabla \cdot (\bar{\rho} w)\|_1 + \|\nabla \cdot (\bar{\rho} \sigma)\|_1 + \|\nabla^2 (\bar{\rho} w)\|_1) \\ &\le C\epsilon \|\nabla U(t)\|_2. \end{aligned} \tag{3.19}$$

Putting (3.18) and (3.19) into (3.17) gives the desired inequality (3.16) and this completes the proof of the lemma.

□

3.3 Proof of Theorem 1.1

To prove Theorem 1.1, define

$$M(t) = \sup_{0\le s\le t} (1+s)^{2\sigma(p,2;1)} H(s). \tag{3.20}$$

Notice that $M(t)$ is non-decreasing, and

$$\|\nabla U(s)\|_2 \le C\sqrt{H(s)} \le C(1+s)^{-\sigma(p,2;1)}\sqrt{M(t)}, \quad 0\le s\le t.$$

Then it follows from (3.16) that

$$\begin{aligned}\|\nabla U(t)\| &\le CK_0(1+t)^{-\sigma(p,2;1)}\\ &\quad +C\epsilon\int_0^t (1+t-s)^{-\sigma(p,2;1)}(1+s)^{-\sigma(p,2;1)}ds\sqrt{M(t)}\\ &\le C(1+t)^{-\sigma(p,2;1)}\left(K_0+\epsilon\sqrt{M(t)}\right),\end{aligned} \tag{3.21}$$

because $\sigma(p,2;1)>1$ when $1\le p<6/5$. Hence, by the Gronwall's inequality, (3.1) and (3.21) give

$$\begin{aligned}H(t) &\le H(0)e^{-D_2t} + C_2\int_0^t e^{-D_2(t-s)}\|\nabla U(s)\|^2 ds\\ &\le H(0)e^{-D_2t} + C\int_0^t e^{-D_2(t-s)}(1+s)^{-2\sigma(p,2;1)}ds\left(K_0^2+\epsilon^2M(t)\right)\\ &\le C(1+t)^{-2\sigma(p,2;1)}\left(H(0)+K_0^2+\epsilon^2M(t)\right).\end{aligned} \tag{3.22}$$

In terms of $M(t)$, we have from (3.22) that

$$M(t) \le C\left(H(0)+K_0^2\right) + C\epsilon^2 M(t),$$

which implies that if $\epsilon>0$ is small enough, then

$$M(t) \le C\left(H(0)+K_0^2\right).$$

In return this gives (1.6) by noticing (3.2) and (3.20).

Next, by Lemma 2.1, (3.18) and (3.19), it follows from the integral formula (2.5) that

$$\begin{aligned}\|U(t)\| &\le CK_0(1+t)^{-\sigma(p,2;0)} + C\int_0^t (1+t-s)^{-\sigma(p,2;0)}\|F(U(s))\|_{L^p\bigcap L^2}ds\\ &\le CK_0(1+t)^{-\sigma(p,2;0)} + C\epsilon\int_0^t (1+t-s)^{-\sigma(p,2;0)}\|\nabla U(s))\|_2 ds\\ &\le CK_0(1+t)^{-\sigma(p,2;0)}\\ &\quad +C\epsilon\left(H(0)+K_0^2\right)^{1/2}\int_0^t (1+t-s)^{-\sigma(p,2;0)}(1+s)^{-\sigma(p,2;1)}ds\\ &\le C(1+t)^{-\sigma(p,2;0)},\end{aligned}$$

where we used again $\sigma(p,2;1)>1$. Hence (1.7) is proved.

Finally, by the interpolation, for any $2\le q\le 6$, combining (1.6) and (1.7) yields

$$\|U(t)\|_{L^q} \le \|U(t)\|_{L^6}^{\theta}\|U(t)\|^{1-\theta} \le C\|\nabla U(t)\|^{\theta}\|U(t)\|^{1-\theta} \le C(1+t)^{-\sigma(p,q;0)},$$

where $\theta=\frac{3(q-2)}{2q}$. And this gives (1.8) and then completes the proof of Theorem 1.1.

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