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BOLTZMANN EQUATION WITH EXTERNAL FORCE AND VLASOV-POISSON-BOLTZMANN SYSTEM IN INFINITE VACUUM

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Abstract. In this paper, we study the Cauchy problem for the Boltzmann equation with an external force and the Vlasov-Poisson-Boltzmann system in infinite vacuum. The global existence of solutions is first proved for the Boltzmann equation with an external force which is integrable with respect to time in some sense under the smallness assumption on initial data in weighted norms. For the Vlasov-Poisson-Boltzmann system, the smallness assumption on initial data leads to the decay of the potential field which in turn gives the global existence of solutions by the result on the case with external forces and an iteration argument. The results obtained here generalize those previous works on these topics and they hold for a class of general cross sections including the hard-sphere model.

1. **Introduction.** For a rarefied gas in the whole space $\mathbf{R}^3_x$, let $f(t,x,v)$ be the distribution function for particles at time $t \geq 0$ with location $x = (x_1, x_2, x_3) \in \mathbf{R}^3_x$ and velocity $v = (v_1, v_2, v_3) \in \mathbf{R}^3_v$. In the presence of an external force, the time evolution of f is governed by the Boltzmann equation as a fundamental equation in statistical physics,

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = J(f,f), \tag{1.1}$$

with initial data

$$f(0,x,v) = f_0(x,v). \tag{1.2}$$

Here $E = E(t,x,v)$ is the external force. The collision operator $J(f,f)$ describing the binary elastic collision takes the form:

$$J(f,f) = Q(f,f) - fR(f), \tag{1.3}$$

with

$$Q(f,f)(t,x,v) = \int_D B(\theta, |v-v_1|) f(t,x,v') f(t,x,v_1')\, d\varepsilon d\theta dv_1, \tag{1.4}$$

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and

$$fR(f)(t,x,v) = f(t,x,v)\int_D B(\theta,|v-v_1|)f(t,x,v_1)\,d\varepsilon d\theta dv_1. \tag{1.5}$$

Here (v,v_1) and (v',v_1') are the pre-collision and post-collision velocities respectively, satisfying

$$v' = v - \langle v-v_1, w\rangle w,\ \ v_1' = v_1 + \langle v-v_1, w\rangle w, \tag{1.6}$$

by conservation of momentum and energy. ε and θ are, respectively, the polar and azimuthal angles when the relative velocity $v-v_1$ is taken as the z-direction in the spherical coordinates. $B(\theta,|v-v_1|)$ is the cross section characterizing the collision of the gas particles from some physical setting with various interaction potentials. $D=[0,2\pi]\times[0,\pi/2]\times\mathbf{R}^3_v$ is the integration domain of the variables (ε,θ,v_1).

For the Vlasov-Poisson-Boltzmann system, $E=E(t,x)$ independent of v is the self-induced force coupled with the distribution $f(t,x,v)$ by

$$E(t,x) = \nabla_x\phi(t,x),\ \triangle_x\phi(t,x) = \rho(t,x) = \int_{\mathbf{R}^3} f(t,x,v)\,dv. \tag{1.7}$$

Here, we normalize the physical constant in the Poisson equation to be unity without loss of generality in our discussion.

Throughout this paper, we assume that the cross section B is nonnegative and continuous in its arguments and satisfies the following condition:

$$\left|\frac{B(\theta,|v-v_1|)}{\sin\theta\cos\theta}\right| \le C\frac{1+|v-v_1|}{|v-v_1|^\delta},\quad 0\le\delta<1. \tag{1.8}$$

In particular, the case of hard-sphere model where

$$B(\theta,|v-v_1|) = C|v-v_1|\sin\theta\cos\theta, \tag{1.9}$$

satisfies (1.8) when $\delta=0$.

Now we review some previous works on the related topics and then give the main ideas in this paper. Some general knowledge on these topics can be found in the literature on the Boltzmann equation and the Vlasov-Poisson-Boltzmann system, such as [7, 8, 30]. The Cauchy problem and the initial boundary value problem for the Boltzmann equation in the absence of the external force have been extensively studied, see [11, 19, 20, 22, 24, 25, 27, 28] and references therein. To concentrate on the problems considered in this paper, in the following, we mainly mention some works on the Cauchy problem for the Boltzmann equation in infinite vacuum in the presence of a force field. In this direction, the first local existence theorem was given by Glikson [13, 14]. When the initial data can be arbitrarily large, the local existence of solutions to the Cauchy problem (also to the initial boundary value problem) was obtained by Asano [1]. Then Bellomo-Lachowicz-Palzewski-Toscani [5] gave a general framework on the global existence of mild solutions and also illustrated their theorem by using an example where the strength of the force is integrable in time in some sense up to subtraction of a constant. For classical solutions, the first existence result was obtained by Guo [17] for some rather soft potential when the external force is small and decays in time with some rates. In terms of the method used, this can be viewed as an extension of the well-known existence result by Illner-Shinbrot [19] on the Boltzmann equation in the absence of external forces. The global existence of mild solutions with arbitrary strong external forces was recently proved by Duan-Yang-Zhu in [12] under a constructive condition on the bi-characteristics. For solutions near a global Maxwellian, Ukai-Yang-Zhao [29] proved the stability of stationary Maxwellian solutions to the Boltzmann equation with an external force through the energy method. For this, please refer to some related results in [15, 16, 22] where the nonlinear energy method was used in the study on many aspects of the Boltzmann equation near a global Maxwellian or a solution profile.

For the Vlasov-Poisson-Boltzmann system, the large time asymptotic behavior of weak solutions with some extra regularity was studied by Desvillettes-Dolbeault [10]. The global existence of DiPerna-Lions renormalized solutions with arbitrary amplitude to the initial boundary value problem was given by Mischler [23]. Guo [17] obtained the global classical solutions in infinite vacuum for some soft potentials. The global existence of solutions near a global Maxwellian was also studied by Guo [16] and Yang-Zhao-Yu [31] respectively for the space periodic data and the Cauchy problem. Moreover, Bardos-Degond [2] considered the Vlasov-Poisson system near vacuum and used the dispersive property of the density to prove the global existence of smooth solutions for small initial data. See also [3, 6] for the other interesting topics.

In this paper, for the Cauchy problem of the Boltzmann equation with an external force and the Vlasov-Poisson-Boltzmann system in infinite vacuum, we prove the global existence of solutions. First, for the Boltzmann equation with an external force which is integrable with respect to time in some sense under the smallness assumption on the initial data in weighted norms, the global existence of the mild and classical solutions will be given by a contraction mapping argument. Then, based on the dispersive property of the local density, we obtain the global existence of classical solutions to the Vlasov-Poisson-Boltzmann system in infinite vacuum. In fact, the smallness assumption on initial data leads to the decay of the potential field which in turn gives the global existence of solutions by the result on the case with external forces and an iteration argument. Notice that here the results generalize those previous works on these topics and they hold for a class of general cross sections including the hard-sphere model.

In the proof, we use some known results given by previous works, like Lemmas 2.1-2.4 from [4, 26]. The key estimate in the analysis is

$$\left|\int_0^t J(f,g)^{\#}(s,x,v)ds\right| \leq Ch_\alpha(|x|)m_\beta(|v|)|||f|||\cdot|||g|||,$$

for the cross section B satisfying (1.8) and the external force $E = E(t,x)$ integrable in time in some sense, see Subsection 2.1 for details. Here the integration is along the bi-characteristics. For the case with external forces, this estimate yields the global existence by the contraction mapping theorem. For the Vlasov-Poisson-Boltzmann system, Lemmas 2.14 and 2.15 are the generalization of those corresponding results in [2] and they lead to the proof of the dispersive property of the density function which in turn gives the integrability of external forces in the approximate solution sequence.

The rest of the paper is organized as follows. In Section 2, we consider the Boltzmann equation with an external force in infinite vacuum. Based on some preliminary estimates given in Subsection 2.1, the global existence of the mild and classical solutions will be proved in Subsections 2.2 and 2.3 respectively. For the use in the study of the Vlasov-Poisson-Boltzmann system, in Subsection 2.4, we study characteristics generated by the external force $E = E(t,x)$ and generalize some results in [2] to deal with the pointwise estimates of a function $f(t,x,v)$ defined in some Banach spaces with weight function defined through the characteristics.

Based on the global existence of solutions to the Boltzmann equation with external forces, we study the Vlasov-Poisson-Boltzmann system in infinite vacuum in Section 3. As in [17], we will construct the approximate solution sequence in Subsection 3.1 and then obtain its compactness and convergence in Subsection 3.2. Finally, the existence and uniqueness of solutions will be given in Subsection 3.3.

Notation. Throughout this paper, C_i, $C_i(\cdot)$ and $C_i(\cdot,\cdot)$, $i \in \mathbf{N}$, denote the generic positive constants and may vary in different places. For any function $f = f(t,x,v)$, we shall denote by $\|f(t,\cdot,\cdot)\|_p$ and $\|f(t,x,\cdot)\|_p$ $(1 \le p \le \infty)$ the usual L^p norms. Suppose that U is an open subset in $\mathbf{R}^n$, $n \ge 1$ and $0 < \lambda \le 1$. The following function spaces are used. $C^0(U)$ denotes the space of all real, continuous functions on U. $C^0_b(U)$ denotes the space of all real, bounded and continuous functions on U. $C(\bar{U})$ denotes the space of all real, bounded and uniformly continuous functions on U. The definition of the function spaces $C^k(U)$, $C^k_b(U)$ and $C^k(\bar{U})$, $k \ge 1$ follows similarly. In the proof, we also need the usual Hölder continuous function spaces $C^{0,\lambda}(U)$ and $C^{0,\lambda}(\bar{U})$ which consist of locally and uniformly λ-order Hölder continuous functions respectively.

2. Boltzmann equation with external force.

2.1. Preliminaries.

As in the previous work, to prove the global existence of solutions to the Boltzmann equation in infinite vacuum, it is better to rewrite the equation along the bi-characteristics. For any $(x,v) \in \mathbf{R}^3_x \times \mathbf{R}^3_v$, the forward bi-characteristics is defined by

$$\begin{cases} \dfrac{dX^t(x,v)}{dt} = V^t(x,v), \quad \dfrac{dV^t(x,v)}{dt} = E(t, X^t(x,v), V^t(x,v)), \\ (X^t, V^t)_{t=0} = (x,v). \end{cases} \tag{2.1}$$

Then the mild form of the Boltzmann equation becomes

$$\begin{aligned} f^{\#}(t,x,v) = {} & f_0(x,v) \exp\left\{-\int_0^t R(f)^{\#}(\theta,x,v)\,d\theta\right\} \\ & + \int_0^t Q(f,f)^{\#}(s,x,v) \exp\left\{-\int_s^t R(f)^{\#}(\theta,x,v)\,d\theta\right\} ds, \end{aligned} \tag{2.2}$$

where as usual $h^{\#}(t,x,v) = h(t, X^t(x,v), V^t(x,v))$ for any function $h(t,x,v)$. On the other hand, for any fixed $(t,x,v) \in \mathbf{R}^+_t \times \mathbf{R}^3_x \times \mathbf{R}^3_v$, we can also define the backward bi-characteristics by

$$\begin{cases} \dfrac{dX(s;t,x,v)}{ds} = V(s;t,x,v), \ \dfrac{dV(s;t,x,v)}{ds} = E(s, X(s;t,x,v), V(s;t,x,v)), \\ (X(s;t,x,v), V(s;t,x,v))_{s=t} = (x,v), \end{cases} \tag{2.3}$$

so that the Boltzmann equation can also be rewritten as

$$\begin{aligned} f(t,x,v) = {} & f_0(X(0;t,x,v), V(0;t,x,v)) \\ & \times \exp\left\{-\int_0^t R(f)(\theta, X(\theta;t,x,v), V(\theta;t,x,v))d\theta\right\} \\ & + \int_0^t Q(f,f)(s, X(s;t,x,v), V(s;t,x,v)) \\ & \times \exp\left\{-\int_s^t R(f)(\theta, X(\theta;t,x,v), V(\theta;t,x,v))d\theta\right\} ds. \end{aligned} \tag{2.4}$$

Notice that if the mapping $(X^s, V^s) : \mathbf{R}^3_x \times \mathbf{R}^3_v \to \mathbf{R}^3_x \times \mathbf{R}^3_v$ is one-to-one and onto for any $s > 0$, then

$$(X^s, V^s)(X(0; t, x, v), V(0; t, x, v)) = (X(s; t, x, v), V(s; t, x, v))$$

for any $(t, x, v) \in \mathbf{R}^+_t \times \mathbf{R}^3_x \times \mathbf{R}^3_v$ and $s \in [0, t]$. In this case, the equations (2.2) and (2.4) are equivalent.

To apply the contraction mapping theorem for the existence, we now introduce some function spaces as in [5]. For any $f = f(t, x, v)$ and $f_0 = f_0(x, v)$, define

$$|||f||| = \sup_{t,x,v} \frac{|f^{\#}(t, x, v)|}{h_\alpha(|x|)m_\beta(|v|)}, \quad |f_0|_{\alpha,\beta,0} = \sup_{x,v} \frac{|f_0(x, v)|}{h_\alpha(|x|)m_\beta(|v|)}, \tag{2.5}$$

where the weight functions h_α and m_β have algebraic decay rates in the form of

$$h_\alpha(|x|) = (1 + |x|^2)^{-\alpha},\ \alpha > 0 \ \text{ and } \ m_\beta(|v|) = (1 + |v|^2)^{-\beta},\ \beta > 0. \tag{2.6}$$

Notice that even though the norm $||| \cdot |||$ depends on E, α and β, in the sequel, we omit them in the notation for simplicity without any confusion.

Set

$$L_0(E, \alpha, \beta) = \{f : f \in L^\infty(\mathbf{R}^+_t \times \mathbf{R}^3_x \times \mathbf{R}^3_v) \ \text{ with } \ |||f||| < \infty\},$$

and

$$C_0(E, \alpha, \beta) = \{f : f \in C^0_b(\mathbf{R}^+_t \times \mathbf{R}^3_x \times \mathbf{R}^3_v) \ \text{ with } \ |||f||| < \infty\}.$$

Then both $(L_0(E, \alpha, \beta), ||| \cdot |||)$ and $(C_0(E, \alpha, \beta), ||| \cdot |||)$ are Banach spaces. We remark that both $L_0(E, \alpha, \beta)$ and $C_0(E, \alpha, \beta)$ depend on the force field E because the norm $||| \cdot |||$ is defined along the bi-characteristics; see (2.1), (2.5) and the definition of $f^{\#}$. For the proof of the existence of classical solutions, we also need a norm including derivatives with respect to x and v, as in [17], by

$$|||f|||_E = |||f||| + |||\nabla_x f||| + |||(1 + t)^{-1}\nabla_v f||| \tag{2.7}$$

and

$$|f_0|_{\alpha,\beta,1} = |f_0|_{\alpha,\beta,0} + |\nabla_x f_0|_{\alpha,\beta,0} + |\nabla_v f_0|_{\alpha,\beta,0}. \tag{2.8}$$

Similarly, set

$$L_1(E, \alpha, \beta) = \{f : f, \nabla_x f, (1 + t)\nabla_v f \in L^\infty(\mathbf{R}^+_t \times \mathbf{R}^3_x \times \mathbf{R}^3_v) \ \text{ with } \ |||f|||_E < \infty\}$$

and

$$C_1(E, \alpha, \beta) = \{f : f, \nabla_x f, \nabla_v f \in C^0(\mathbf{R}^+_t \times \mathbf{R}^3_x \times \mathbf{R}^3_v) \ \text{ with } \ |||f|||_E < \infty\},$$

so that $(L_1(E, \alpha, \beta), ||| \cdot |||_E)$ and $(C_1(E, \alpha, \beta), ||| \cdot |||_E)$ are Banach spaces.

For later use, we now list some useful inequalities from papers [4, 5, 26]. Interested readers please refer to these papers for proofs.

Lemma 2.1. *For any $\alpha > 0$ and $(x, y) \in \mathbf{R}^3 \times \mathbf{R}^3$, we have*

$$(1 + |y| + |y|^2)^{-\alpha} \le \frac{h_\alpha(|x|)}{h_\alpha(|x + y|)} \le (1 + |y| + |y|^2)^\alpha,$$

and

$$h_\alpha(|x|)h_\alpha(|y|) \le 2^\alpha h_\alpha(|x + y|).$$

Lemma 2.2. *For any $\alpha > 0$, $t \in \mathbf{R}^+$, $x \in \mathbf{R}^3$ and $(u, v) \in \mathbf{R}^3 \times \mathbf{R}^3$ with $\langle u, v\rangle = 0$, we have*

$$h_\alpha(|x+tu|)h_\alpha(|x+tv|) \le h_\alpha(|x|) \{h_\alpha(|x + tu|) + h_\alpha(|x + tv|) + h_\alpha(|x + t(u + v)|)\}.$$

Lemma 2.3. *For any $\alpha > 1/2$, $x \in \mathbf{R}^3$ and $(u, v) \in \mathbf{R}^3 \times \mathbf{R}^3$ with $\langle u, v\rangle = 0$, we have*

$$\int_0^\infty h_\alpha(|x+tu|)h_\alpha(|x+tv|)\,dt \leq Ch_\alpha(|x|)\int_0^\infty h_\alpha(t\min\{|u|,|v|\})\,dt.$$

Lemma 2.4. *For any $\alpha > 1/2$, we have*

$$\sup_{x,u}\int_0^\infty |u|h_\alpha(|x+su|)\,ds \leq C_1(\alpha).$$

Lemma 2.5. *If the cross section B satisfies (1.8) with $0 \leq \delta < 1$, then the following integrals are bounded:*

$$\sup_v \int_D \frac{B(\theta, |v-v_1|)}{|v-v_1|\sin\theta\cos\theta} m_\beta(|v_1|)\,d\varepsilon d\theta dv_1 \leq C_2(\beta,\delta)$$

for any $\beta > 3/2$, and

$$\sup_v \int_D \frac{B(\theta, |v-v_1|)}{|v-v_1|\sin\theta\cos\theta}\frac{m_\beta(|v'|)m_\beta(|v_1'|)}{m_\beta(|v|)}\,d\varepsilon d\theta dv_1 \leq C_3(\beta,\delta)$$

for any $\beta > (3-\delta)/2$.

2.2. **Mild solution.** In this subsection, we will prove the global existence and uniqueness of the mild solution to the Cauchy problem (1.1)-(1.2) in the Banach spaces $L_0(E,\alpha,\beta)$ and $C_0(E,\alpha,\beta)$ if initial data is sufficiently small under some condition on the external force E.

To apply the contraction mapping theorem, for any $f = f(t,x,v)$, let's denote the function on the right hand of (2.4) by $(\mathbf{T}f)(t,x,v)$. Furthermore, throughout this subsection, we assume that the external force $E = E(t,x,v)$ satisfies the following two conditions:

- **(A1):** Both the forward and backward bi-characteristic equations (2.1) and (2.3) have global-in-time smooth solutions for any $(x,v) \in \mathbf{R}^3_x \times \mathbf{R}^3_v$ and $(t,x,v) \in \mathbf{R}^+_t \times \mathbf{R}^3_x \times \mathbf{R}^3_v$, respectively.
- **(A2):** There exists some constant $\varepsilon_0 > 0$ such that

$$\int_0^\infty \|E(t,\cdot,\cdot)\|_\infty\,dt \leq \varepsilon_0. \tag{2.9}$$

Notice that ε_0 need not be small.

The existence result on the mild solution is stated as follows.

Theorem 2.1. *Let the parameters in the weight functions $h_\alpha(|x|)$ and $m_\beta(|v|)$ satisfy $\alpha > 1/2$ and $\beta > 3/2$. Suppose that the cross section B satisfies (1.8) with $0 \leq \delta < 1$ and the external force E satisfies (A1) and (A2) with $\varepsilon_0 > 0$. If $0 \leq f_0(x,v) \in L^\infty(\mathbf{R}^3_x \times \mathbf{R}^3_v)$ with $|f_0|_{\alpha,\beta,0} \leq \delta_0$ where $\delta_0 > 0$ is sufficiently small, then there exists a unique solution $0 \leq f(t,x,v) \in L_0(E,\alpha,\beta)$ satisfying (2.4) with $|||f||| \leq 2\delta_0$. Furthermore, if $\beta > 2-\delta/2$ and $0 \leq f_0(x,v) \in C^0(\mathbf{R}^3_x \times \mathbf{R}^3_v)$ with $|f_0|_{\alpha,\beta,0} \leq \delta_0$ where $\delta_0 > 0$ is sufficiently small, then there exists a unique solution $0 \leq f(t,x,v) \in C_0(E,\alpha,\beta)$ satisfying (2.4) with $|||f||| \leq 2\delta_0$.*

To prove Theorem 2.1, we list the following lemmas which follow directly from (2.1), (2.3) and the assumption (A2) on E, see also the following Remark 2.1.

Lemma 2.6. *Suppose that the external force E satisfies (A1) and (A2) with $\varepsilon_0 > 0$. Then we have that for any $t \geq 0$, $s \geq 0$ and $(x,v) \in \mathbf{R}^3_x \times \mathbf{R}^3_v$,*

$$|V^t(x,v) - V^s(x,v)| \leq \varepsilon_0,$$

$$\left|\frac{1}{t}\int_0^t V^s(x,v)ds - v\right| \leq \varepsilon_0, \quad \left|\frac{1}{t}\int_0^t V^s(x,v)ds - V^t(x,v)\right| \leq 2\varepsilon_0.$$

In terms of the backward bi-characteristics, we have that for any $(t,x,v) \in \mathbf{R}^+_t \times \mathbf{R}^3_x \times \mathbf{R}^3_v$ and $0 \leq s_1, s_2 \leq t$,

$$|V(s_1;t,x,v) - V(s_2;t,x,v)| \leq \varepsilon_0,$$

$$\left|\frac{1}{t}\int_0^t V(s;t,x,v)ds - V(0;t,x,v)\right| \leq \varepsilon_0, \quad \left|\frac{1}{t}\int_0^t V(s;t,x,v)ds - v\right| \leq 2\varepsilon_0.$$

By Lemmas 2.1-2.5, the pointwise estimates on the gain term $Q(f,f)$ and the loss term $fR(f)$ in the Boltzmann equation are given by the following two lemmas.

Lemma 2.7. *Let $\alpha > 0$ and $\beta > 0$. Suppose that the cross section B satisfies (1.8) with $0 \leq \delta < 1$ and the external force E satisfies (A1) and (A2) with $\varepsilon_0 > 0$. Then for any $f = f(t,x,v)$ and $g = g(t,x,v)$, we have*

$$\left|\int_0^t Q(f,g)^{\#}(s,x,v)\,ds\right| \leq A(t,x,v)|||f||| \cdot |||g|||.$$

Here $A(t,x,v)$ is defined by

$$A(t,x,v) = C_4(\alpha,\beta,\varepsilon_0) \sup_{|a|\leq 2\varepsilon_0} \int_0^t \int_D B(\theta, |v - v_1 - a|) h_\alpha(|y|) h_\alpha(|y_1|) \times m_\beta(|v'|) m_\beta(|v_1'|)\, d\varepsilon d\theta dv_1 ds,$$

where

$$\begin{cases} y = x + su_\parallel,\ y_1 = x + su_\perp,\ v' = v - u_\parallel,\ v_1' = v - u_\perp, \\ u_\parallel = \langle v - v_1, \omega\rangle\omega,\ u_\perp = v - v_1 - u_\parallel. \end{cases}$$

Furthermore, if $\alpha > 1/2$ and $\beta > (3-\delta)/2$, then for any $(t,x,v) \in \mathbf{R}^+_t \times \mathbf{R}^3_x \times \mathbf{R}^3_v$,

$$A(t,x,v) \leq C_5(\alpha,\beta,\delta,\varepsilon_0) h_\alpha(|x|) m_\beta(|v|).$$

Lemma 2.8. *Under the same conditions on the cross section and the external force as in Lemma 2.7, for any $g = g(t,x,v)$, we have*

$$\left|\int_0^t R(g)^{\#}(s,x,v)\,ds\right| \leq B(t,x,v)|||g|||.$$

Here $B(t,x,v)$ is defined by

$$B(t,x,v) = C_6(\alpha,\beta,\varepsilon_0) \sup_{|a|\leq 2\varepsilon_0} \int_0^t \int_D B(\theta, |v - v_1 - a|) h_\alpha(|x + s(v - v_1)|) m_\beta(|v_1|)\, d\varepsilon d\theta dv_1 ds.$$

Furthermore, if $\alpha > 1/2$ and $\beta > 3/2$, then for any $(t,x,v) \in \mathbf{R}^+_t \times \mathbf{R}^3_x \times \mathbf{R}^3_v$,

$$B(t,x,v) \leq C_7(\alpha,\beta,\delta,\varepsilon_0).$$

Thus, for any $f = f(t,x,v)$ and $g = g(t,x,v)$, we have

$$\left|\int_0^t f^{\#} R(g)^{\#}(s,x,v)\,ds\right| \leq C_7(\alpha,\beta,\delta,\varepsilon_0) h_\alpha(|x|) m_\beta(|v|)|||f||| \cdot |||g|||.$$

Remark 2.1. In the absence of external forces, Lemmas 2.7 and 2.8 were proved by Toscani and Bellomo in [26]. On the other hand, when the external force satisfies (A1) and (A2), Bellomo etc. [5] obtained the same estimates as in Lemmas 2.7 and 2.8 by using the following exponential and algebraic weight functions

$$h_\alpha(|x|)=\exp\{-\alpha|x|^2\} \quad \text{and} \quad m_\beta(|v|)=(1+|v|^2)^{-\beta}.$$

Following the argument in [5], it is straightforward to obtain Lemmas 2.7 and 2.8 for the algebraic weight functions so that we omit their proofs.

Based on Lemmas 2.6, 2.7 and 2.8, we have the following lemma.

Lemma 2.9. *Let $\alpha>1/2$ and $\beta>3/2$. Suppose that the cross section B satisfies (1.8) with $0\le\delta<1$ and the external force E satisfies (A1) and (A2) with $\varepsilon_0>0$. Then we have that for any $f\ge 0$ and $g\ge 0$,*

$$\begin{cases} |||\mathbf{T}f|||\le |f_0|_{\alpha,\beta,0}+C_5(\alpha,\beta,\delta,\varepsilon_0)|||f|||^2, \\ |||\mathbf{T}f-\mathbf{T}g|||\le C_8(\alpha,\beta,\delta,\varepsilon_0)(|||f|||+|||g|||+|||\mathbf{T}g|||)|||f-g|||, \end{cases} \tag{2.10}$$

where $C_8(\alpha,\beta,\delta,\varepsilon_0)=2C_5(\alpha,\beta,\delta,\varepsilon_0)+C_7(\alpha,\beta,\delta,\varepsilon_0)$.

Proof. Since $f\ge 0$, it follows from the definition (1.5) of the operator R that

$$\int_s^t R(f)^{\#}(\theta,x,v)d\theta\ge 0, \tag{2.11}$$

for any $0\le s\le t$. Thus, by Lemma 2.7, we have from the representation of $\mathbf{T}$ that

$$|(\mathbf{T}f)^{\#}(t,x,v)|\le |f_0(x,v)|+C_5(\alpha,\beta,\delta,\varepsilon_0)h_\alpha(|x|)m_\beta(|v|)|||f|||^2,$$

which gives $(2.10)_1$.

Next, for $(2.10)_2$, we first have the time evolution of $\mathbf{T}f-\mathbf{T}g$ as follows

$$\begin{aligned}\partial_t(\mathbf{T}f-\mathbf{T}g)+v\cdot\nabla_x(\mathbf{T}f-\mathbf{T}g)+E\cdot\nabla_v(\mathbf{T}f-\mathbf{T}g)+(\mathbf{T}f-\mathbf{T}g)R(f)\\ =Q(f-g,f)+Q(g,f-g)-(\mathbf{T}g)R(f-g),\end{aligned}$$

with initial data $(\mathbf{T}f-\mathbf{T}g)(0,x,v)=0$. Thus by expressing the above equation in the integral form as in (2.2) and then using (2.11) together with Lemmas 2.7 and 2.8, we have $(2.10)_2$. This completes the proof of the lemma. □

In the next lemma, we will show that $\mathbf{T}f$ is continuous if both f_0 and f are continuous under some conditions on the weight functions.

Lemma 2.10. *Let $\alpha>1/2$ and $\beta>2-\delta/2$. Suppose that the cross section B satisfies (1.8) with $0\le\delta<1$ and the external force E satisfies (A1) and (A2) with $\varepsilon_0>0$. If $0\le f_0\in C^0(\mathbf{R}^3_x\times\mathbf{R}^3_v)$ and $0\le f\in C^0(\mathbf{R}^+_t\times\mathbf{R}^3_x\times\mathbf{R}^3_v)$ with $|||f|||<\infty$. Then it holds that*

$$0\le\mathbf{T}f\in C^0(\mathbf{R}^+_t\times\mathbf{R}^3_x\times\mathbf{R}^3_v).$$

Proof. As in [17], first notice that $Q(f,g)$ and $R(g)$ can be written as

$$\begin{aligned}Q(f,g)(t,x,v)&=\int_D B(\theta,|u|)f(t,x,v-u_\parallel)g(t,x,v-u_\perp)\,d\varepsilon d\theta du,\\ R(f)(t,x,v)&=\int_D B(\theta,|u|)f(t,x,v-u)\,d\varepsilon d\theta du.\end{aligned}$$

Set

$$Q(f(t_1,x_1,v_1),g(t_2,x_2,v_2)) = \int_D B(\theta,|u|)f(t_1,x_1,v_1-u_\parallel) \times g(t_2,x_2,v_2-u_\perp)\,d\varepsilon d\theta du,$$

$$R(f)(t_1,x_1,v_1) = \int_D B(\theta,|u|)f(t_1,x_1,v_1-u)\,d\varepsilon d\theta du.$$

Hence, Lemmas 2.1 and 2.6 give

$$\begin{aligned}
&|B(\theta,|u|)f(t_1,x_1,v_1-u_\parallel)g(t_2,x_2,v_2-u_\perp)| \\
&\le B(\theta,|u|)|||f|||\cdot|||g|||m_\beta(|V(0;t_1,x_1,v_1-u_\parallel)|)m_\beta(|V(0;t_2,x_2,v_2-u_\perp)|) \\
&\le (1+\varepsilon_0+\varepsilon_0^2)^{2\beta}B(\theta,|u|)|||f|||\cdot|||g|||m_\beta(|v_1-u_\parallel)|)m_\beta(|v_2-u_\perp|) \\
&\le \frac{8^\beta(1+\varepsilon_0+\varepsilon_0^2)^{2\beta}|||f|||\cdot|||g|||}{m_\beta(|v_1|)m_\beta(|v_2|)}\frac{1+|u|}{|u|^\delta}m_\beta(|u|).
\end{aligned}$$

Similarly, we have

$$|B(\theta,|u|)f(t_1,x_1,v_1-u)| \le \frac{2^\beta(1+\varepsilon_0+\varepsilon_0^2)^\beta|||f|||}{m_\beta(|v_1|)}\frac{1+|u|}{|u|^\delta}m_\beta(|u|).$$

Since $0\le\delta<1$ and $\beta>2-\delta/2$, i.e., $2\beta+(\delta-1)>3$, we have

$$\int_{\mathbf{R}^3}\frac{1+|u|}{|u|^\delta}m_\beta(|u|)\,du \le C_9(\beta,\delta).$$

Therefore, if $f,g\in C^0(\mathbf{R}_t^+\times\mathbf{R}_x^3\times\mathbf{R}_v^3)$ where $|||f|||$, $|||g|||<\infty$, it follows from the dominated convergence theorem that both functions $Q(f(t_1,x_1,v_1),g(t_2,x_2,v_2))$ and $R(f)(t_1,x_1,v_1)$ are continuous with respect to $(t_i,x_i,v_i)\in\mathbf{R}_t^+\times\mathbf{R}_x^3\times\mathbf{R}_v^3$, $i=1,2$.

Now from the representation of the mapping $\mathbf{T}$, it is easy to see that $\mathbf{T}f\ge0$ if $f_0\ge0$ and $f\ge0$. On the other hand, if $f_0\ge0$ and $f\ge0$ are continuous, the above argument combined with the continuity of the backward bi-characteristics $[X(s;t,x,v),V(s;t,x,v)]$ yields the continuity of $\mathbf{T}f$ in (t,x,v) over $\mathbf{R}_t^+\times\mathbf{R}_x^3\times\mathbf{R}_v^3$. Hence, it completes the proof of the lemma. □

Finally, at the end of this subsection, we give the proof of Theorem 2.1.

Proof of Theorem 2.1. First, for the existence of solutions in the space $L_0(E,\alpha,\beta)$, it is sufficient to show that $\mathbf{T}$ is a contraction mapping from SL_0 to SL_0 if $\delta_0>0$ is sufficiently small, where

$$SL_0=\{f: f\ge0, f\in L_0(E,\alpha,\beta), |||f|||\le2\delta_0\}$$

is a closed subset of the Banach space $L_0(E,\alpha,\beta)$. It is a standard argument from the representation of the mapping $\mathbf{T}$ and Lemma 2.9. Furthermore, if $\beta>2-\delta/2$ and $0\le f_0(x,v)\in C^0(\mathbf{R}_x^3\times\mathbf{R}_v^3)$ with $|f_0|_{\alpha,\beta,0}\le\delta_0$, then it follows from Lemma 2.10 that $\mathbf{T}f\in C^0(\mathbf{R}_t^+\times\mathbf{R}_x^3\times\mathbf{R}_v^3)$. Hence, $\mathbf{T}$ is also a contraction mapping from SC_0 to SC_0 if δ_0 is sufficiently small, where

$$SC_0=\{f: f\ge0, f\in C_0(E,\alpha,\beta), |||f|||\le2\delta_0\}$$

is a closed subset of the Banach space $C_0(E,\alpha,\beta)$. The proof of Theorem 2.1 is then completed.

□

2.3. **Classical solution.** In this subsection, we will prove the global existence of classical solutions to the Cauchy problem (1.1)-(1.2) in the Banach spaces $L_1(E,\alpha,\beta)$ and $C_1(E,\alpha,\beta)$ by using the similar approach as that used in Subsection 2.2.

Here we need the following stronger assumption on the external force field $E = E(t,x,v)$:

(B1): $E(\cdot,\cdot,\cdot)\in C_b^0(\mathbf{R}_t^+\times\mathbf{R}_x^3\times\mathbf{R}_v^3)$ and $\nabla_x E(t,\cdot,\cdot)$, $\nabla_v E(t,\cdot,\cdot)\in C_b^0(\mathbf{R}_x^3\times\mathbf{R}_v^3)$ for any fixed $t>0$.

(B2): There exist constants $\varepsilon_0>0$ and $0<\varepsilon_1<1$ such that

$$\begin{cases} \displaystyle\int_0^\infty \|E(t,\cdot,\cdot)\|_\infty dt \le \varepsilon_0, \\ \displaystyle\int_0^\infty \|\nabla_v E(t,\cdot,\cdot)\|_\infty dt + \int_0^\infty (1+t)\|\nabla_x E(t,\cdot,\cdot)\|_\infty dt \le \varepsilon_1. \end{cases} \tag{2.12}$$

The global existence result on the classical solutions can be stated as follows.

Theorem 2.2. *Let $\alpha>1/2$ and $\beta>3/2$. Suppose that the cross section B satisfies (1.8) with $0\le\delta<1$ and the external force E satisfies (B1) and (B2) with $\varepsilon_0>0$ and $0<\varepsilon_1<1$. If $0\le f_0(x,v)\in W^{1,\infty}(\mathbf{R}_x^3\times\mathbf{R}_v^3)$ with $|f_0|_{\alpha,\beta,1}\le\delta_1$ where $\delta_1>0$ is sufficiently small, then there exists a unique solution $0\le f(t,x,v)\in L_1(E,\alpha,\beta)$ satisfying (2.4) with $|||f|||_E\le 2\delta_1/(1-\varepsilon_1)$. Furthermore, if $\beta>2-\delta/2$ and $0\le f_0(x,v)\in C^1(\mathbf{R}_x^3\times\mathbf{R}_v^3)$ with $|f_0|_{\alpha,\beta,1}\le\delta_1$ where $\delta_1>0$ is sufficiently small, then there exists a unique solution $0\le f(t,x,v)\in C_1(E,\alpha,\beta)$ satisfying (2.4) with $|||f|||_E\le 2\delta_1/(1-\varepsilon_1)$.*

Remark 2.2. The assumption (B1) on the external force E guarantees that both the forward and backward bi-characteristic equations (2.1) and (2.3) have unique C^1 solutions globally in time, which are C^1 differentiable with respect to the initial data (x,v) and (t,x,v) respectively. As in [17], the condition $(2.12)_2$ in the assumption (B2) requires the integrability of the external force to balance $\nabla_v f(t,x,v)$ which may grow linearly in time when estimating the derivatives.

Similar to the proof of Theorem 2.1, let's define

$$\begin{aligned} SL_1 &= \{f: f\ge 0, f\in L_1(E,\alpha,\beta), |||f|||_E\le 2\delta_1/(1-\varepsilon_1)\}, \\ SC_1 &= \{f: f\ge 0, f\in C_1(E,\alpha,\beta), |||f|||_E\le 2\delta_1/(1-\varepsilon_1)\}, \end{aligned}$$

which are the closed subsets of $L_1(E,\alpha,\beta)$ and $C_1(E,\alpha,\beta)$ respectively. We will prove that if $\delta_1>0$ is sufficiently small, then $\mathbf{T}$ is a contraction mapping both on the closed subsets SL_1 and SC_1 respectively. For this purpose, we need a series of lemmas.

To the end, fix any $f,g\in SL_1$ and denote $F=\mathbf{T}f$ and $G=\mathbf{T}g$. First $|||F|||_E$ is estimated as follows.

Lemma 2.11. *Let $\alpha>1/2$ and $\beta>3/2$. Suppose that the cross section B satisfies (1.8) with $0\le\delta<1$ and the external force E satisfies (B1) and $(2.12)_1$ with $\varepsilon_0>0$. Then we have*

$$|||F|||\le |f_0|_{\alpha,\beta,0}+C_5(\alpha,\beta,\delta,\varepsilon_0)|||f|||^2, \tag{2.13}$$

$$\begin{aligned} |||\nabla_x F||| \le{}& |\nabla_x f_0|_{\alpha,\beta,0} + |||(1+t)^{-1}\nabla_v F|||\int_0^\infty (1+s)\|\nabla_x E(s,\cdot,\cdot)\|_\infty\, ds \\ &+\{2C_5(\alpha,\beta,\delta,\varepsilon_0)|||f|||+C_7(\alpha,\beta,\delta,\varepsilon_0)|||F|||\}\,|||\nabla_x f|||, \end{aligned} \tag{2.14}$$

and

$$
\begin{aligned}
|||(1+t)^{-1}\nabla_v F||| \le |\nabla_v f_0|_{\alpha,\beta,0} &+ |||\nabla_x F||| + \{2C_5(\alpha,\beta,\delta,\varepsilon_0)|||f||| \\
&+ C_7(\alpha,\beta,\delta,\varepsilon_0)|||F|||\}|||(1+t)^{-1}\nabla_x f||| \\
&+ |||(1+t)^{-1}\nabla_v F||| \int_0^\infty \|\nabla_v E(s,\cdot,\cdot)\|_\infty \, ds. \qquad (2.15)
\end{aligned}
$$

Proof. The estimate (2.13) follows immediately from Lemma 2.9. To prove (2.14) and (2.15), since $F = \mathbf{T}f$, we have

$$\partial_t F + v \cdot \nabla_x F + E(t,x,v) \cdot \nabla_v F + FR(f) = Q(f,f).$$

By taking x_i and v_i derivatives of the above equation, we have

$$
\begin{aligned}
&\partial_t(\partial_{x_i} F) + v \cdot \nabla_x(\partial_{x_i} F) + E(t,x,v) \cdot \nabla_v(\partial_{x_i} F) + (\partial_{x_i} F)R(f) \\
&= Q(\partial_{x_i} f, f) + Q(f, \partial_{x_i} f) - \{(1+t)\partial_{x_i} E\} \cdot \left\{(1+t)^{-1}\nabla_v F\right\} - FR(\partial_{x_i} f)
\end{aligned}
$$

and

$$
\begin{aligned}
&\partial_t(\partial_{v_i} F) + v \cdot \nabla_x(\partial_{v_i} F) + E(t,x,v) \cdot \nabla_v(\partial_{v_i} F) + (\partial_{v_i} F)R(f) \\
&= (1+t)Q((1+t)^{-1}\partial_{v_i} f, f) + (1+t)Q(f, (1+t)^{-1}\partial_{v_i} f) - \partial_{x_i} F \\
&\quad - (1+t)\partial_{v_i} E \cdot \{(1+t)^{-1}\nabla_v F\} - (1+t)FR((1+t)^{-1}\partial_{v_i} f).
\end{aligned}
$$

Thus the proof of this lemma follows by integrating the above two equations along the forward bi-characteristics as in (2.2), summing them over $i = 1,2,3$ respectively and then using Lemmas 2.7 and 2.8 together with (2.11). □

A direct consequence of Lemma 2.11 is the following corollary.

Corollary 2.1. *Suppose that the assumptions of Lemma 2.11 hold. If the external force E satisfies $(2.12)_2$ with $0 < \varepsilon_1 < 1$, then we have*

$$|||F|||_E \le \frac{1}{1-\varepsilon_1}|f_0|_{\alpha,\beta,1} + \left(\frac{2C_5(\alpha,\beta,\delta,\varepsilon_0)}{1-\varepsilon_1}|||f||| + \frac{C_7(\alpha,\beta,\delta,\varepsilon_0)}{1-\varepsilon_1}|||F|||\right)|||f|||_E.$$

Next, we estimate $|||F-G|||_E$. Similar to Lemma 2.9 and Corollary 2.1, we have the following lemma with its proof omitted for brevity.

Lemma 2.12. *Let $\alpha > 1/2$ and $\beta > 3/2$. Suppose that the cross section B satisfies (1.8) with $0 \le \delta < 1$ and the external force E satisfies (B1) and (B2) with $\varepsilon_0 > 0$ and $0 < \varepsilon_1 < 1$. Then we have*

$$|||F-G|||_E \le \frac{1}{1-\varepsilon_1}\lambda(|||f|||_E, |||g|||_E, |||G|||)|||f-g|||_E,$$

where $\lambda = \lambda(s_1,s_2,s_3)$ is a polynomial function with positive coefficients depending only on α, β, δ and ε_0, and satisfying

$$\lambda(s_1,s_2,s_3) \to 0 \quad as \quad |s_1| + |s_2| + |s_3| \to 0.$$

Finally, we will show that $\mathbf{T}f$ is nonnegative and continuously differentiable if the initial data f_0 and f is nonnegative and continuously differentiable.

Lemma 2.13. *Let $\alpha > 1/2$ and $\beta > 2 - \delta/2$. Suppose that the cross section B satisfies (1.8) with $0 \le \delta < 1$ and the external force E satisfies (B1) and $(2.12)_1$ with $\varepsilon_0 > 0$. If $0 \le f_0 \in C^1(\mathbf{R}^3_x \times \mathbf{R}^3_v)$ and $0 \le f \in C_1(E,\alpha,\beta)$ with $|||f|||_E < \infty$, then*

$$0 \le \mathbf{T}f \in C_1(E,\alpha,\beta).$$

Proof. By Lemma 2.10, $Q(f,f)(t,x,v)$ is continuous in (t,x,v) over $\mathbf{R}_t^+\times\mathbf{R}_x^3\times\mathbf{R}_v^3$. In fact, we claim that $Q(f,f)(t,x,v)$ is C^1 in (x,v) over $\mathbf{R}_x^3\times\mathbf{R}_v^3$ for any $t>0$. Fix any $(t,x,v)\in\mathbf{R}_t^+\times\mathbf{R}_x^3\times\mathbf{R}_v^3$, as in [17], for any $h\in\mathbf{R}$ with $h\neq 0$, we have

$$
\begin{aligned}
&\frac{1}{h}\left[Q(f,f)(v+he_i)-Q(f,f)(v)\right]\\
&=Q\left(\frac{f(v+he_i)-f(v)}{h},f(v)\right)+Q\left(f(v),\frac{f(v+he_i)-f(v)}{h}\right)\\
&=(1+t)Q\left((1+t)^{-1}\partial_{v_i}f(\bar v),f(v)\right)+(1+t)Q\left(f(v),(1+t)^{-1}\partial_{v_i}f(\bar v)\right),
\end{aligned}
$$

where e_i is a unit vector in $\mathbf{R}^3$, $\bar v$ is the vector between v and $v+he_i$ through the mean value theorem and $f(v)$ represents $f(t,x,v)$ for simplicity. The similar equation holds for the variable x. Thus our claim follows from the dominated convergence theorem. Moreover, by the same method, we have that $R(f)(t,x,v)$ is continuous in (t,x,v) over $\mathbf{R}_t^+\times\mathbf{R}_x^3\times\mathbf{R}_v^3$ and C^1 in (x,v) over $\mathbf{R}_x^3\times\mathbf{R}_v^3$ for any $t>0$.

Now recall the representation of $\mathbf{T}f$ and the inequality (2.11). Notice that $\mathbf{T}f\geq 0$ if $f_0\geq 0$ and $f\geq 0$, and that the bi-characteristics $[X(s;t,x,v),V(s;t,x,v)]$ is C^1 in (t,x,v) over $\mathbf{R}_t^+\times\mathbf{R}_x^3\times\mathbf{R}_v^3$. Hence, the differentiability of f_0 and the dominated convergence theorem give that $\mathbf{T}f$, $\nabla_x\mathbf{T}f$ and $\nabla_v\mathbf{T}f$ are continuous in (t,x,v) over $\mathbf{R}_t^+\times\mathbf{R}_x^3\times\mathbf{R}_v^3$. Finally, from Corollary 2.1, we have $|||\mathbf{T}f|||_E<\infty$ and thus $\mathbf{T}f\in C_1(E,\alpha,\beta)$. The proof of Lemma 2.13 is then completed. □

Therefore, by the standard contraction argument as in the proof of Theorem 2.1 together with Lemmas 2.11-2.13 and Corollary 2.1, we end the proof of Theorem 2.2.

2.4. **Characteristics and pointwise estimates.** In this subsection, we will give some properties of the backward bi-characteristics and some pointwise estimates on the functions in the Banach space $L_1(E,\alpha,\beta)$ for later use in the study of the Vlasov-Poisson-Boltzmann system. For this purpose, throughout this subsection, the external force $E=E(t,x)$ depending only on t and x is supposed to satisfy the following assumption:

(E): $E(\cdot,\cdot)\in C_b^0(\mathbf{R}_t^+\times\mathbf{R}_x^3)$ and $\nabla_xE(t,\cdot)\in C_b^0(\mathbf{R}_x^3)$ for any fixed $t>0$ with the bound

$$\int_0^\infty(1+t)\|\nabla_xE(t,\cdot)\|_\infty dt\leq\varepsilon_1,\quad \varepsilon_1>0. \tag{2.16}$$

The following two lemmas on the bi-characteristics $[X(s;t,x,v),V(s;t,x,v)]$ are analogous to those given in [2], where the external force $E(t,x)$ satisfies a stronger assumption than (E), that is,

$$\|\nabla_xE(t,\cdot)\|_\infty\leq C(1+t)^{-5/2}, \tag{2.17}$$

for some sufficiently small constant $C>0$ and any $t\geq 0$. It is obvious that (2.17) implies (2.16). Since the proofs are easy to be modified, we omit them for brevity.

Lemma 2.14. *Suppose that the external force $E=E(t,x)$ satisfies the assumption (E) with $0<\varepsilon_1<1/4$. Then for any $(t,x,v)\in\mathbf{R}_t^+\times\mathbf{R}_x^3\times\mathbf{R}_v^3$ and any $s\in[0,t]$, we have*

$$\left|\frac{\partial X}{\partial v}(s;t,x,v)-(s-t)Id\right|+\left|\frac{\partial V}{\partial v}(s;t,x,v)-Id\right|\leq 4\varepsilon_1(t-s),$$

and

$$\left|\frac{\partial X}{\partial x}(s;t,x,v) - Id\right| + \left|\frac{\partial V}{\partial x}(s;t,x,v)\right| \le 2\varepsilon_1,$$

where Id denotes the identity matrix in $\mathbf{R}^3$.

Lemma 2.15. *Suppose that the external force $E = E(t,x)$ satisfies the assumption (E) where $\varepsilon_1 > 0$ is sufficiently small. Then*

(i) for any $(t,x,v) \in \mathbf{R}_t^+ \times \mathbf{R}_x^3 \times \mathbf{R}_v^3$ and $s \in [0,t]$, we have

$$\left|\det\left(\frac{\partial X}{\partial v}(s;t,x,v)\right)\right| \ge \frac{(t-s)^3}{2}.$$

(ii) for any fixed $(s,t,x) \in \mathbf{R}_t^+ \times \mathbf{R}_t^+ \times \mathbf{R}_x^3$ with $0 \le s < t$, the mapping

$$v \longmapsto X(s;t,x,v)$$

is one-to-one from $\mathbf{R}^3$ to $\mathbf{R}^3$.

Now we use Lemma 2.15 to obtain some pointwise estimates for a function $f = f(t,x,v)$ in the Banach space $L_1(E,\alpha,\beta)$, which play an important role in the proof of the construction of the approximate solution sequence for the Vlasov-Poisson-Boltzmann system in the next section. These estimates essentially come from the dispersive property of functions in $L_1(E,\alpha,\beta)$ with respect to x and v.

Theorem 2.3. *Suppose that $\alpha > 3/2$, $\beta > 3/2$ and the external force $E = E(t,x)$ satisfies the assumption (E) with $\varepsilon_1 > 0$ being sufficiently small. Moreover, suppose that*

$$\int_0^\infty \|E(t,\cdot)\|_\infty dt \le \varepsilon_0, \quad \varepsilon_0 > 0.$$

Let $1 \le p \le \infty$ and $0 < T < \infty$. Then for any function $f = f(t,x,v) \in L_1(E,\alpha,\beta)$, we have that

$$f \in L^\infty((0,T); W^{1,p}(\mathbf{R}_x^3 \times \mathbf{R}_v^3)) \cap L^\infty((0,T)\times\mathbf{R}_x^3; W^{1,p}(\mathbf{R}_v^3))$$

with the following estimates

$$\|f(t,\cdot,\cdot)\|_p + \|\nabla_x f(t,\cdot,\cdot)\|_p + (1+t)^{-1}\|\nabla_v f(t,\cdot,\cdot)\|_p \le C_{10}(\alpha,\beta,\varepsilon_0)|||f|||_E, \quad (2.18)$$

$$\begin{aligned}\|f(t,x,\cdot)\|_p + \|\nabla_x f(t,x,\cdot)\|_p &+ (1+t)^{-1}\|\nabla_v f(t,x,\cdot)\|_p \\ &\le C_{11}(\alpha,\beta,\varepsilon_0)|||f|||_E(1+t)^{-3/p},\end{aligned} \quad (2.19)$$

for $0 \le t < \infty$ and $x \in \mathbf{R}_x^3$.

Proof. In fact, it suffices to obtain the L^∞ and L^1 estimates in the spaces $\mathbf{R}_x^3 \times \mathbf{R}_v^3$ and $\mathbf{R}_v^3$ respectively, which yield the L^p estimates by the interpolation.

We only prove (2.19) since (2.18) can be proved similarly. Fix any $0 \le t < \infty$ and $x \in \mathbf{R}_x^3$, the estimate (2.19) when $p = \infty$ comes directly from the definition of $|||f|||_E$. Next, consider the case when $p = 1$ as follows. Notice that

$$\|f(t,x,\cdot)\|_1 \le |||f||| \int_{\mathbf{R}^3} h_\alpha(|X(0;t,x,v)|)m_\beta(|V(0;t,x,v)|)dv.$$

It follows from Lemmas 2.1 and 2.6 that

$$\|f(t,x,\cdot)\|_1 \le C(\beta,\varepsilon_0)|||f||| \int_{\mathbf{R}^3} m_\beta(|v|)dv \le C(\beta,\varepsilon_0)|||f|||. \quad (2.20)$$

When $t \geq 1$, by the change of variable $X(0; t, x, v) = \bar{v}$, we have from Lemma 2.15 that

$$\begin{aligned} \|f(t,x,\cdot)\|_1 &\leq |||f||| \int_{\mathbf{R}^3} h_\alpha(|X(0;t,x,v)|)dv \\ &= |||f||| \int_{\mathbf{R}^3} h_\alpha(|\bar{v}|) \left|\det\left(\frac{\partial X}{\partial v}(0;t,x,v)\right)\right|^{-1} d\bar{v} \\ &\leq C(\alpha)|||f|||t^{-3}. \end{aligned} \tag{2.21}$$

Combining (2.20) and (2.21) yields

$$\|f(t,x,\cdot)\|_1 \leq C(\alpha,\beta,\varepsilon_0)|||f|||(1+t)^{-3}, \quad t \geq 0,\ x \in \mathbf{R}^3_x.$$

Similarly, we can obtain the estimates on $\nabla_x f$ and $\nabla_v f$. Thus (2.19) is proved. The proof of Theorem 2.3 is completed. □

3. Vlasov-Poisson-Boltzmann system.

3.1. **Approximate solution sequence.** From now on, we consider the global existence of solutions to the Cauchy problem (1.1), (1.2) and (1.7) for the Vlasov-Poisson-Boltzmann system. Let's first construct the approximate solution sequence $\{[f^{n+1}, E^{n+1}]\}_{n=0}^\infty$ by the following iterative scheme as in [17]:

$$\begin{cases} \partial_t f^{n+1} + v\cdot\nabla_x f^{n+1} + E^n(t,x)\cdot\nabla_v f^{n+1} = J(f^{n+1}, f^{n+1}), \\ E^{n+1}(t,x) = \nabla_x\phi^{n+1}(t,x),\ \Delta_x\phi^{n+1}(t,x) = \rho^{n+1} = \displaystyle\int_{\mathbf{R}^3} f^{n+1}(t,x,v)dv, \\ f^{n+1}(0,x,v) = f_0(x,v), \quad n = 0,1,2,\ldots \end{cases} \tag{3.1}$$

Set beginning condition $E^0(t,x) \equiv 0$.

We claim that for each n, the solution $[f^{n+1}, E^{n+1}]$ to (3.1) is well-defined as stated in the following theorem.

Theorem 3.1. *Suppose that the cross section B satisfies (1.8) with $0 \leq \delta < 1$. Let $\alpha > 2$, $\beta > 4-\delta$ and $\delta_1 > 0$ be defined in Theorem 2.2. Fix any $\varepsilon_0 > 0$ and $\varepsilon_1 \in (0,1)$ with ε_1 sufficiently small such that Lemma 2.15 holds, and let $C_1 = 2/(1-\varepsilon_1)$. If $0 \leq f_0(x,v) \in W^{1,\infty}(\mathbf{R}^3_x \times \mathbf{R}^3_v)$ with $|f_0|_{\alpha,\beta,1} \leq \delta_2 < \delta_1$ where $\delta_2 > 0$ is sufficiently small, then for each $n = 0,1,2,\ldots,$ the solution $[f^{n+1}, E^{n+1}]$ to (3.1) is well-defined which satisfies (H1)$_n$ and (H2)$_n$ as follows:*

(H1)$_n$ $0 \leq f^{n+1}(t,x,v) \in L_1(E^n,\alpha,\beta)$ with uniform bound

$$|||f^{n+1}|||_{E^n} \leq C_1\delta_2. \tag{3.2}$$

(H2)$_n$ $E^n(\cdot,\cdot) \in C^0_b(\mathbf{R}^+_t \times \mathbf{R}^3_x)$ and $\nabla_x E^n(t,\cdot) \in C^0_b(\mathbf{R}^3_x)$ for any $t > 0$, and they have the following uniform bound

$$\int_0^\infty \|E^n(t,\cdot)\|_\infty dt \leq \varepsilon_0 \quad \text{and} \quad \int_0^\infty (1+t)\|\nabla_x E^n(t,\cdot)\|_\infty dt \leq \varepsilon_1. \tag{3.3}$$

To prove the above theorem, we first borrow a lemma from [2].

Lemma 3.1. *Let $\rho(x) \in L^1(\mathbf{R}^3) \cap W^{1,\infty}(\mathbf{R}^3)$ and $\phi(x) = 1/|x| * \rho$. Then one has the following estimates*

$$\|\phi\|_\infty \leq C\|\rho\|_\infty^{1/3}\|\rho\|_1^{2/3}, \quad \|\nabla_x\phi\|_\infty \leq C\|\rho\|_\infty^{2/3}\|\rho\|_1^{1/3},$$
$$\|D_x^2\phi\|_\infty \leq C(\lambda)\|\rho\|_\infty^{1-4\lambda}\|\nabla_x\rho\|_\infty^{3\lambda}\|\rho\|_1^{\lambda},$$

where $0 < \lambda < 1/4$ and $C(\lambda)$ is some positive constant depending only on λ.

Based on Theorem 2.3 and Lemma 3.1, Theorem 3.1 can be proved as follows.

Proof of Theorem 3.1. We do it by induction. Since $E^0(t,x)\equiv 0$, it is obvious that $(\mathrm{H1})_0$ and $(\mathrm{H2})_0$ hold by Theorem 2.2. Suppose that $(\mathrm{H1})_n$ and $(\mathrm{H2})_n$ hold. Then it suffices to prove $(\mathrm{H2})_{n+1}$ because $(\mathrm{H1})_{n+1}$ follows from Theorem 2.2 on the Boltzmann equation with the external force $E^{n+1}(t,x)$.

Under the assumptions $(\mathrm{H1})_n$ and $(\mathrm{H2})_n$, it follows from Theorem 2.3 that for any $t\geq 0$ and $1\leq p\leq\infty$,

$$\|\rho^{n+1}(t,\cdot)\|_p+\|\nabla_x\rho^{n+1}(t,\cdot)\|_p\leq C_{12}(\alpha,\beta,\varepsilon_0)C_1\delta_2(1+t)^{-3(1-1/p)},\tag{3.4}$$

where

$$\rho^{n+1}(t,x)=\int_{\mathbf{R}^3}f^{n+1}(t,x,v)dv.$$

By Lemma 3.1, we have from (3.4) that

$$\|E^{n+1}(t,\cdot)\|_\infty=\|\nabla_x\phi^{n+1}(t,\cdot)\|_\infty\leq C_{13}(\alpha,\beta,\varepsilon_0)C_1\delta_2(1+t)^{-2},\tag{3.5}$$

$$\|\nabla_xE^{n+1}(t,\cdot)\|_\infty=\|D_x^2\phi^{n+1}(t,\cdot)\|_\infty\leq C_{14}(\alpha,\beta,\varepsilon_0)C(\lambda)C_1\delta_2(1+t)^{-3(1-\lambda)}.\tag{3.6}$$

If we choose $\delta_2>0$ sufficiently small such that

$$C_{13}(\alpha,\beta,\varepsilon_0)C_1\delta_2\leq\varepsilon_0\quad\text{and}\quad\frac{C(\lambda)}{1-3\lambda}C_{14}(\alpha,\beta,\varepsilon_0)C_1\delta_2\leq\varepsilon_1,$$

for some fixed $\lambda\in(0,1/4)$, then we have (3.3) from (3.5) and (3.6).

For the regularity of E^{n+1}, we claim that up to the possible re-definition on a set of zero measure, E^{n+1} and ∇_xE^{n+1} are continuous, and what's more, $E^{n+1}(\cdot,\cdot)\in C_b^0(\mathbf{R}_t^+\times\mathbf{R}_x^3)$ and for any fixed $t>0$, $\nabla_xE^{n+1}(t,\cdot)\in C_b^0(\mathbf{R}_x^3)$. Indeed, notice that

$$W^{1,\infty}((0,T)\times\mathbf{R}_x^3\times\mathbf{R}_v^3)\hookrightarrow C^{0,\mu}(\overline{(0,T)\times\mathbf{R}_x^3\times\mathbf{R}_v^3})$$

for some $0<\mu<1$. Since $f^{n+1}\in W^{1,\infty}((0,T)\times\mathbf{R}_x^3\times\mathbf{R}_v^3)$, f^{n+1} is Hölder continuous with the exponent μ, i.e., $f^{n+1}\in C^{0,\mu}(\overline{(0,T)\times\mathbf{R}_x^3\times\mathbf{R}_v^3})$ after possibly being re-defined on a set of zero measure. Recall (3.1) and

$$|f^{n+1}(t,x,v)|\leq C(\beta,\varepsilon_0)C_1\delta_2m_\beta(|v|).$$

By the dominated convergence theorem, we have that $\rho^{n+1}(\cdot,\cdot)\in C^0(\mathbf{R}_t^+\times\mathbf{R}_x^3)$. Moreover, (3.4) shows that $\rho^{n+1}(\cdot,\cdot)\in C_b^0(\mathbf{R}_t^+\times\mathbf{R}_x^3)$. On the other hand, for any $t>0$, x_1 and x_2, we have

$$\begin{aligned}\left|\rho^{n+1}(t,x_1)-\rho^{n+1}(t,x_2)\right|&\leq|x_1-x_2|\int_{\mathbf{R}^3}\left|\nabla_xf^{n+1}(t,\xi,v)\right|dv\\&\leq C(\beta,\varepsilon_0)C_1\delta_2|x_1-x_2|,\end{aligned}$$

where $\xi\in\mathbf{R}^3$ is between x_1 and x_2. Hence, $\rho^{n+1}(t,\cdot)\in C^{0,1}(\mathbf{R}_x^3)$ uniformly for all $t>0$. Since $\Delta_x\phi^{n+1}(t,x)=\rho^{n+1}(t,x)$, $\phi^{n+1}(t,\cdot)\in C^2(\mathbf{R}_x^3)$ for any $t>0$ so that $E^{n+1}(t,\cdot)=\nabla_x\phi^{n+1}(t,\cdot)\in C^1(\mathbf{R}_x^3)$ for any $t>0$. Furthermore, (3.5) and (3.6) show that $E^{n+1}(t,\cdot)\in C_b^1(\mathbf{R}_x^3)$ for any $t>0$.

Finally, $E^{n+1}(\cdot,\cdot)=\nabla_x\phi^{n+1}(\cdot,\cdot)\in C_b^0(\mathbf{R}_t^+\times\mathbf{R}_x^3)$ just comes from the continuity and integrability of ρ^{n+1}. In fact, (3.4) gives

$$\rho^{n+1}\in L^\infty(\mathbf{R}_t^+;W^{1,p}(\mathbf{R}_x^3))$$

for any $1\leq p\leq\infty$ with the uniform bound $C(\alpha,\beta,\varepsilon_0)C_1\delta_2$. Noticing that

$$E^{n+1}(t,x)=\nabla_x\phi^{n+1}(t,x)=\int_{\mathbf{R}^3}\frac{y}{|y|^3}\rho^{n+1}(t,x-y)dy,$$

then we have

$$\begin{aligned}
&|E^{n+1}(t,x)-E^{n+1}(t_0,x_0)|\\
&=\left|\int_{\mathbf{R}^3}\frac{y}{|y|^3}\left(\rho^{n+1}(t,x-y)-\rho^{n+1}(t_0,x_0-y)\right)dy\right|\\
&\le\left(\int_{|y|\le r}+\int_{|y|\ge r}\right)\frac{1}{|y|^2}\left|\rho^{n+1}(t,x-y)-\rho^{n+1}(t_0,x_0-y)\right|dy\\
&\le\int_{|y|\le r}\frac{1}{|y|^2}\left|\rho^{n+1}(t,x-y)-\rho^{n+1}(t_0,x_0-y)\right|dy+\frac{2C(\alpha,\beta,\varepsilon_0)C_1\delta_2}{r^2}.
\end{aligned}$$

Hence $E^{n+1}(\cdot,\cdot)\in C^0(\mathbf{R}_t^+\times\mathbf{R}_x^3)$. It then follows from (3.5) that $E^{n+1}(\cdot,\cdot)\in C_b^0(\mathbf{R}_t^+\times\mathbf{R}_x^3)$. Thus the proof of Theorem 3.1 is completed. □

3.2. **Compactness and convergence.** In this subsection, we will give the uniform estimates and convergence of the approximate solution sequence $\{[f^n,E^n]\}$ in some Sobolev spaces. First, the following lemma follows directly from Theorems 2.3 and 3.1 so that we omit its proof for brevity.

Lemma 3.2. *Let $1\le p\le\infty$ and $0<T<\infty$. Under the conditions of Theorem 3.1, we have*

(i) $\{f^n(t,x,v)\}$ is bounded in the space

$$L^\infty((0,T);W^{1,p}(\mathbf{R}_x^3\times\mathbf{R}_v^3))\cap L^\infty((0,T)\times\mathbf{R}_x^3;W^{1,p}(\mathbf{R}_v^3)),$$

with uniform estimates

$$\|f^n(t,\cdot,\cdot)\|_p+\|\nabla_x f^n(t,\cdot,\cdot)\|_p+(1+t)^{-1}\|\nabla_v f^n(t,\cdot,\cdot)\|_p\le C_{10}(\alpha,\beta,\varepsilon_0)C_1\delta_2,$$

and

$$\begin{aligned}
&\|f^n(t,x,\cdot)\|_p+\|\nabla_x f^n(t,x,\cdot)\|_p+(1+t)^{-1}\|\nabla_v f^n(t,x,\cdot)\|_p\\
&\qquad\qquad\le C_{11}(\alpha,\beta,\varepsilon_0)C_1\delta_2(1+t)^{-3/p},
\end{aligned}$$

for any $t\ge0$ and $x\in\mathbf{R}_x^3$.

(ii) $\{\rho^n(t,x)\}$ is bounded in $L^\infty(\mathbf{R}_t^+;W^{1,p}(\mathbf{R}_x^3))$ with uniform estimates

$$\|\rho^n(t,\cdot)\|_p+\|\nabla_x\rho^n(t,\cdot)\|_p\le C_{12}(\alpha,\beta,\varepsilon_0)C_1\delta_2(1+t)^{-3(1-1/p)}.$$

(iii) $\{E^n(t,x)\}$ is bounded in $L^\infty(\mathbf{R}_t^+;W^{1,\infty}(\mathbf{R}_x^3))$ with uniform estimates

$$\begin{cases}\|E^n(t,\cdot)\|_\infty\le C_{13}(\alpha,\beta,\varepsilon_0)C_1\delta_2\varepsilon_0(1+t)^{-2},\\ \|\nabla_x E^n(t,\cdot)\|_\infty\le C_{14}(\alpha,\beta,\varepsilon_0)C(\lambda)C_1\delta_2\varepsilon_1(1+t)^{-3(1-\lambda)},\end{cases}$$

for $0<\lambda<1/4$ and $t\ge0$.

Next we give the uniform estimates on $\partial_t f^n$ in the spaces $L^p(\mathbf{R}_v^3)$ and $L^p(\mathbf{R}_x^3\times\mathbf{R}_v^3)$ respectively, which essentially come from ones on $\nabla_x f^n$ and $\nabla_v f^n$. For this, we need the pointwise estimate like

$$\left|F^\#(t,x,v)\right|\le|||F|||h_\alpha(|x|)m_\beta(|v|)$$

for the collision term $F(t,x,v)=J(f^n,f^n)(t,x,v)$. In fact, we will give a better estimate so that the decay rate of $J(f^n,f^n)^\#(t,x,v)$ with respect to time t is also obtained.

Lemma 3.3. *Suppose that the conditions in Theorem 3.1 hold. Fix any γ with $2-\delta/2<\gamma<\beta-(2-\delta/2)$. Then for any $(t,x,v)\in\mathbf{R}_t^+\times\mathbf{R}_x^3\times\mathbf{R}_v^3$, we have*

$$|J(f^n,f^n)^{\#}(t,x,v)|\leq\frac{C_{15}(\alpha,\beta,\gamma,\delta,\varepsilon_0)|||f^n|||^2}{(1+t)^2}h_{\alpha-1/2}(|x|)m_{\beta-\gamma-(1-\delta)/2}(|v|).$$

Proof. Fix $(x,v)\in\mathbf{R}_x^3\times\mathbf{R}_v^3$. By Lemmas 2.7 and 2.8, we have

$$|J(f^n,f^n)^{\#}(t,x,v)|\leq C|||f^n|||^2\sup_{|a|\leq2\varepsilon_0}\{I_1(a)+I_2(a)\},\tag{3.7}$$

where

$$\begin{aligned}I_1(a)=\int_D&B(\theta,|u-a|)h_\alpha(|x+tu_\parallel|)h_\alpha(|x+tu_\perp|)\\&\times m_\beta(|v-u_\parallel|)m_\beta(|v-u_\perp|)d\varepsilon d\theta du,\end{aligned}$$

$$I_2(a)=\int_D B(\theta,|u-a|)h_\alpha(|x|)h_\alpha(|x+tu|)m_\beta(|v|)m_\beta(|v-u|)d\varepsilon d\theta du.$$

For fixed $a\in\mathbf{R}^3$ with $|a|\leq2\varepsilon_0$, we first estimate $I_2(a)$ as follows:

$$I_2(a)\leq C(\beta,\varepsilon_0)h_\alpha(|x|)m_\beta(|v|)\int_D B(\theta,|u|)m_\beta(|v-u|)d\varepsilon d\theta du.\tag{3.8}$$

Notice that

$$\int_{\mathbf{R}^3}\frac{1+|u|}{|u|^\delta}m_\beta(|v-u|)du\leq C(\beta,\delta)(1+|v|^2)^{\frac{1-\delta}{2}},\tag{3.9}$$

where we have used $0\leq\delta<1$, $\beta>3/2$ and $2\beta-(1-\delta)>3$ for $\beta>4-\delta$. Hence, it follows from (3.8) and (3.9) that for any $t\geq0$,

$$I_2(a)\leq C(\beta,\delta,\varepsilon_0)h_\alpha(|x|)m_{\beta-(1-\delta)/2}(|v|).\tag{3.10}$$

Furthermore, when $t\geq1$, we can use the integration of $h_\alpha(|x+tu|)$ to get the decay in time:

$$I_2(a)\leq\frac{h_\alpha(|x|)m_\beta(|v|)}{t^{3-\delta}}\int_{\mathbf{R}^3}\frac{1+|u|}{|u|^\delta}h_\alpha(|x+at+u|)du.$$

Similar to (3.9), we also have

$$\int_{\mathbf{R}^3}\frac{1+|u|}{|u|^\delta}h_\alpha(|x+at+u|)du\leq C(\alpha,\delta,\varepsilon_0)t^{1-\delta}(1+|x|^2)^{\frac{1-\delta}{2}},$$

where we have used $0\leq\delta<1$, $\alpha>3/2$ and $2\alpha-(1-\delta)>3$ for $\alpha>2$. Hence, we have that for any $t\geq1$,

$$I_2(a)\leq\frac{C(\alpha,\delta,\varepsilon_0)}{t^2}h_{\alpha-(1-\delta)/2}(|x|)m_\beta(|v|).\tag{3.11}$$

Thus, combining (3.10) and (3.11) yields

$$I_2(a)\leq\frac{C(\alpha,\beta,\delta,\varepsilon_0)}{(1+t)^2}h_{\alpha-(1-\delta)/2}(|x|)m_{\beta-(1-\delta)/2}(|v|),\tag{3.12}$$

for any $t\geq0$ and $a\in\mathbf{R}^3$ with $|a|\leq2\varepsilon_0$.

$I_1(a)$ can be estimated similarly. In fact, by Lemmas 2.1 and 2.2, we have

$$m_\beta(|v-u_\parallel|)m_\beta(|v-u_\perp|)\leq3m_{\beta-\gamma}(|v|)m_\gamma(|2v-u|),$$

and

$$h_\alpha(|x+tu_\parallel|)h_\alpha(|x+tu_\perp|)\leq3h_\alpha(|x|).$$

Hence, as for (3.10), we have that for any $t \geq 0$,

$$
\begin{aligned}
I_1(a) &\leq C h_\alpha(|x|) m_{\beta-\gamma}(|v|) \int_D B(\theta, |u-a|) m_\gamma(|2v-u|) d\varepsilon d\theta du \\
&\leq C(\gamma, \delta, \varepsilon_0) h_\alpha(|x|) m_{\beta-\gamma-(1-\delta)/2}(|v|),
\end{aligned} \tag{3.13}
$$

where we have used $0 \leq \delta < 1$, $\gamma > 3/2$ and $2\gamma - (1-\delta) > 3$ for $\gamma > 2 - \delta/2$.

When $t \geq 1$, we claim that $I_1(a)$ decays like t^{-2}. Indeed, by Lemma 2.2, we have

$$
\begin{aligned}
I_1(a) \leq h_\alpha(|x|) \int_D & B(\theta, |u-a|) \{h_\alpha(|x+tu_\parallel|) + h_\alpha(|x+tu_\perp|) + h_\alpha(|x+tu|)\} \\
& \qquad\qquad \times m_\beta(|v-u_\parallel|) m_\beta(|v-u_\perp|) d\varepsilon d\theta du \\
=: & \, I_{11}(a) + I_{12}(a) + I_{13}(a),
\end{aligned} \tag{3.14}
$$

where I_{1i}, $i = 1, 2, 3$ denotes each term in the integral respectively.

Now we estimate $I_{1i}(a)\,(i = 1, 2, 3)$ as follows. First by (3.11), for $I_{13}(a)$, we have

$$
I_{13}(a) \leq \frac{C(\alpha, \beta, \delta, \varepsilon_0)}{t^2} h_{\alpha-(1-\delta)/2}(|x|) m_\beta(|v|). \tag{3.15}
$$

Second, for $I_{11}(a)$, we have

$$
\begin{aligned}
I_{11}(a) &\leq 3 h_\alpha(|x|) m_{\beta-\gamma}(|v|) \int_D B(\theta, |u-a|) h_\alpha(|x+tu_\parallel|) m_\gamma(|2v-u|) d\varepsilon d\theta du \\
&\leq C h_\alpha(|x|) m_{\beta-\gamma}(|v|) \\
&\quad \times \int_{\mathbf{R}^3} \int_0^{\frac{\pi}{2}} \frac{1+|u-a|}{|u-a|^\delta} h_\alpha(|t|u|\cos\theta - |x||) m_\gamma(|2v-u|) \sin\theta \cos\theta d\theta du.
\end{aligned}
$$

By using the change of variable $z = t|u|\cos\theta - |x|$, we have that $I_{11}(a)$ is bounded by

$$
\begin{aligned}
& C h_\alpha(|x|) m_{\beta-\gamma}(|v|) \int_{\mathbf{R}^3} \int_{-|x|}^{t|u|-|x|} \frac{(1+|u-a|)(z+|x|)}{t^2 |u-a|^\delta |u|^2} h_\alpha(|z|) m_\gamma(|2v-u|) dz du \\
\leq & \frac{C h_\alpha(|x|) m_{\beta-\gamma}(|v|)}{t^2} \int_{\mathbf{R}^3} \frac{1+|u-a|}{|u-a|^\delta |u|^2} m_\gamma(|2v-u|) du \\
& \qquad\qquad \times \left(\int_{\mathbf{R}} |z| h_\alpha(|z|) dz + |x| \int_{\mathbf{R}} h_\alpha(|z|) dz \right) \\
\leq & \frac{C(\alpha) h_{\alpha-1/2}(|x|) m_{\beta-\gamma}(|v|)}{t^2} \int_{\mathbf{R}^3} \frac{1+|u-a|}{|u-a|^\delta |u|^2} m_\gamma(|2v-u|) du,
\end{aligned} \tag{3.16}
$$

where we have used $2\alpha - 1 > 1$ and $2\alpha > 1$ for $\alpha > 2$. By Lemma 2.1, since $0 \leq \delta < 1$ and $2\gamma > 3$, we then have

$$
\begin{aligned}
& \int_{\mathbf{R}^3} \frac{1+|u-a|}{|u-a|^\delta |u|^2} m_\gamma(|2v-u|) du \\
= & \left(\int_{|u-a| \geq |u|} + \int_{|u-a| < |u|} \right) \frac{1+|u-a|}{|u-a|^\delta |u|^2} m_\gamma(|2v-u|) du \\
\leq & \, C(\gamma, \varepsilon_0) \int_{\mathbf{R}^3} \frac{1+|u|}{|u|^{\delta+2}} m_\gamma(|2v-u|) du \\
\leq & \, C(\gamma, \delta, \varepsilon_0).
\end{aligned} \tag{3.17}
$$

Hence, from (3.16) and (3.17), we have that for any $t \geq 1$,

$$I_{11}(a) \leq \frac{C(\alpha,\gamma,\delta,\varepsilon_0)}{t^2} h_{\alpha-1/2}(|x|) m_{\beta-\gamma}(|v|). \tag{3.18}$$

Finally, for $I_{12}(a)$, by the change of variable $z = t|u|\sin\theta - |x|$, the similar argument leads to

$$I_{12}(a) \leq \frac{C(\alpha,\gamma,\delta,\varepsilon_0)}{t^2} h_{\alpha-1/2}(|x|) m_{\beta-\gamma}(|v|). \tag{3.19}$$

Thus, combining (3.13), (3.15), (3.18) and (3.19) yields

$$I_1(a) \leq \frac{C(\alpha,\beta,\gamma,\delta,\varepsilon_0)}{(1+t)^2} h_{\alpha-1/2}(|x|) m_{\beta-\gamma-(1-\delta)/2}(|v|), \tag{3.20}$$

for any $t \geq 0$ and $a \in \mathbf{R}^3$ with $|a| \leq 2\varepsilon_0$.

Therefore, both (3.12) and (3.20) together with (3.7) end the proof of Lemma 3.3. □

Remark 3.1. The similar estimate like (3.7) was obtained in [18] for the case without external forces. Precisely, when the external force $E(t,x) \equiv 0$ and the cross section B satisfies the inverse power law with the angular cut-off assumption, i.e.,

$$\left| \frac{B(\theta,|v-v_1|)}{\sin\theta\cos\theta} \right| \leq C|v-v_1|^{\delta}, \quad -2 < \delta \leq 1,$$

the following decay estimate for the gain term $Q(f,f)$ along the characteristics was given in [18]:

$$|Q(f,f)(t,x+tv,v)| \leq \frac{C|||f|||^2}{(1+t)^{\min\{\delta+3,2\}}} h_{(\alpha-\delta)/2}(|x|) m_{\beta-2}(|v|).$$

Hence, (3.7) is an extension of the above estimate to the case with the integrable external force and more general cross section B.

Based on Lemma 3.3, we have the following two corollaries.

Corollary 3.1. *Under the conditions in Lemma 3.3, we have for any* $(t,x,v) \in \mathbf{R}_t^+ \times \mathbf{R}_x^3 \times \mathbf{R}_v^3$,

$$\begin{aligned}\left|(\partial_t f^n)^{\#}(t,x,v)\right| \leq C_{16}(\alpha,\beta,\gamma,\delta,\varepsilon_0)\left(|||f^n|||^2 + |||\nabla_x f^n||| + |||(1+t)^{-1}\nabla_v f^n|||\right)\\ \times h_{\alpha-1/2}(|x|) m_{\beta-\gamma-(1-\delta)/2}(|v|).\end{aligned}$$

Corollary 3.2. *Under the conditions in Lemma 3.3, we have*

(i) $\{\partial_t f^n(t,x,v)\}$ *is bounded in the space* $L^\infty(\mathbf{R}_t^+; L^p(\mathbf{R}_x^3 \times \mathbf{R}_v^3)) \cap L^\infty(\mathbf{R}_t^+ \times \mathbf{R}_x^3; L^p(\mathbf{R}_x^3))$ *with uniform estimate*

$$\|\partial_t f^n(t,\cdot,\cdot)\|_p + (1+t)^{3/p}\|\partial_t f^n(t,x,\cdot)\|_p \leq C(\alpha,\beta,\gamma,\varepsilon_0)\delta_2,$$

for any $1 \leq p \leq \infty$.

(ii) $\{\partial_t \rho^n(t,x)\}$ *is bounded in* $L^\infty(\mathbf{R}_t^+; L^p(\mathbf{R}_x^3))$ *with uniform estimate*

$$\|\partial_t \rho^n(t,\cdot)\|_p \leq C(\alpha,\beta,\gamma,\delta_0)\delta_2(1+t)^{-3(1-1/p)},$$

for any $1 \leq p \leq \infty$. *Furthermore, if* $p \geq 4/3$, *then* $\{\rho^n(t,x)\}$ *is bounded in* $W^{1,p}(\mathbf{R}_t^+ \times \mathbf{R}_x^3)$ *with uniform estimate*

$$\|\rho^n(\cdot,\cdot)\|_p + \|\partial_t \rho^n(\cdot,\cdot)\|_p + \|\nabla_x \rho^n(\cdot,\cdot)\|_p \leq C(\alpha,\beta,\gamma,\delta_0)\delta_2.$$

Remark 3.2. Notice that the uniform L^1 estimate on $f^n(t,x,v)$ over $\mathbf{R}_v^3$ and $\mathbf{R}_x^3 \times \mathbf{R}_v^3$ can be also obtained by using the nonnegativity of $f^n(t,x,v)$ and directly integrating the Boltzmann equation since

$$\int_{\mathbf{R}^3} J(f^n, f^n) dv = 0.$$

Based on Lemma 3.2 and Corollary 3.2, the convergence of the approximate solution sequence $[f^n, E^n]$ can be obtained as follows. First, for the sequences $\{\rho^n\}$ and $\{\phi^n\}$, by the standard arguments in the Sobolev space and the regularity of the solution to the Poisson equation, we have the following two theorems and their proofs are omitted.

Theorem 3.2. *Suppose that the conditions of Theorem 3.1 hold. Let* $0 < T < \infty$.
(i) If $4 < p \leq \infty$ *and* $1 < q \leq \infty$, *then there exists*

$$\rho(t,x) \in C^{0,\mu}(\overline{\mathbf{R}_t^+ \times \mathbf{R}_x^3}) \cap W^{1,p}(\mathbf{R}_t^+ \times \mathbf{R}_x^3) \cap L^\infty((0,T);\ W^{1,q}(\mathbf{R}_x^3)),$$

for some $0 < \mu < 1$ *such that*

$$\rho^n \to \rho \quad in \quad C^0(\overline{\mathbf{R}_t^+ \times \mathbf{R}_x^3}) \quad as \quad n \to \infty,$$

up to a subsequence. Furthermore, $\rho(t,x)$ *satisfies*

$$\|\rho(t,\cdot)\|_q + \|\partial_t \rho(t,\cdot)\|_q + \|\nabla_x \rho(t,\cdot)\|_q \leq C(1+t)^{-3(1-1/q)}, \tag{3.21}$$

for $t \geq 0$. *Here, for the first term* $\|\rho(t,\cdot)\|_q$, q *may also take* 1.
(ii) There exists $\phi \in L^\infty(\mathbf{R}_t^+;\ W^{2,\infty}(\mathbf{R}_x^3))$ *such that*

$$\phi^n \to \phi \quad in \quad L^\infty(\mathbf{R}_t^+;\ W^{1,\infty}(\mathbf{R}_x^3)) \quad as \quad n \to \infty$$

up to a subsequence. Furthermore, for all $(t,x) \in \mathbf{R}_t^+ \times \mathbf{R}_x^3$, *it holds that*

$$\Delta_x \phi(t,x) = \rho(t,x),$$

where $\phi(t,x)$ *has the regularities*

$$\phi(\cdot,\cdot),\ \Delta_x \phi(\cdot,\cdot) \in C_b^0(\mathbf{R}_t^+ \times \mathbf{R}_x^3) \quad and \quad \phi(t,\cdot) \in C^{2,\mu}(\mathbf{R}_x^3) \cap C_b^2(\mathbf{R}_x^3)$$

for any $t \geq 0$, *and uniform estimates*

$$(1+t)\|\phi(t,\cdot)\|_\infty + (1+t)^2 \|\nabla_x \phi(t,\cdot)\|_\infty \leq C,$$
$$(1+t)^{3(1-\lambda)} \|D_x^2 \phi(t,\cdot)\|_\infty \leq C(\lambda),$$

for $0 < \lambda < 1/4$ *and* $t \geq 0$.

Remark 3.3. Let's define $E(t,x) = \nabla_x \phi(t,x)$. Then it follows from (ii) of Theorem 3.2 that

$$E^n \to E \quad \text{in} \quad L^\infty(\mathbf{R}_t^+; W^{1,\infty}(\mathbf{R}_x^3)),$$

up to a subsequence. Furthermore, $E(\cdot,\cdot) \in C_b^0(\mathbf{R}_t^+ \times \mathbf{R}_x^3)$ and $E(t,\cdot) \in C^{1,\mu}(\mathbf{R}_x^3) \cap C_b^1(\mathbf{R}_x^3)$ for any $t > 0$ with the following estimates

$$\|E(t,\cdot)\|_\infty \leq C\varepsilon_0 (1+t)^{-2} \quad \text{and} \quad \|\nabla_x E(t,\cdot)\|_\infty \leq C\varepsilon_1 (1+t)^{-3(1-\lambda)}, \tag{3.22}$$

for some fixed $0 < \lambda < 1/4$.

Theorem 3.3. *Under the conditions of Theorem 3.1, for* $0 < T < \infty$, *there exists*

$$f(t,x,v) \in W^{1,\infty}((0,T) \times \mathbf{R}_x^3 \times \mathbf{R}_v^3) \cap C^{0,\nu}(\overline{(0,T) \times \mathbf{R}_x^3 \times \mathbf{R}_v^3}),$$

for some $0 < \nu < 1$, *such that*

$$f^n \to f \quad in \quad C^0(\overline{(0,T) \times \mathbf{R}_x^3 \times \mathbf{R}_v^3}),$$

$$\partial f^n \rightharpoonup \partial f \quad \textit{weakly-star in} \quad L^\infty((0,T)\times \mathbf{R}_x^3 \times \mathbf{R}_v^3),$$

up to a subsequence, where ∂ *denote* ∂_t*,* ∂_{x_i} *or* ∂_{v_i}*,* $i=1,2,3$*. Furthermore,* $f(t,x,v)$ *satisfies*

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + E(t,x)\cdot\nabla_v f = J(f,f), \quad a.e.\ (t,x,v)\in(0,T)\times\mathbf{R}_x^3\times\mathbf{R}_v^3,\\ E(t,x)=\nabla_x\phi(t,x),\ \Delta_x\phi(t,x)=\rho=\displaystyle\int_{\mathbf{R}^3} f(t,x,v)dv,\ (t,x)\in(0,T)\times\mathbf{R}_x^3,\\ f(0,x,v)=f_0(x,v), \quad (x,v)\in\mathbf{R}_x^3\times\mathbf{R}_v^3. \end{cases}$$

3.3. **Existence.** In this final subsection, we will give the global existence and uniqueness of solutions to the Cauchy problem of the Vlasov-Poisson-Boltzmann system in infinity vacuum for the cross section B satisfying (1.8) including the hard-sphere model.

Theorem 3.4. *Suppose that the cross section* B *satisfies (1.8) with* $0\le\delta<1$*. Let* $\alpha>2$ *and* $\beta>4-\delta$*. Fix any* $\varepsilon_0>0$ *and* $\varepsilon_1\in(0,1)$ *with* ε_1 *sufficiently small such that Lemma 2.15 holds, and let* $C_1=2/(1-\varepsilon_1)$*. If* $0\le f_0(x,v)\in W^{1,\infty}(\mathbf{R}_x^3\times\mathbf{R}_v^3)$ *with* $|f_0|_{\alpha,\beta,1}\le\delta_2$ *where* $\delta_2>0$ *is sufficiently small, then there exists a unique solution* $[f(t,x,v),E(t,x)]$ *to the Cauchy problem (1.1), (1.2) and (1.7) such that*

(i) $f\ge 0$*,* $f\in C^{0,\nu}(\mathbf{R}_t^+\times\mathbf{R}_x^3\times\mathbf{R}_v^3)\cap W^{1,\infty}(\mathbf{R}_{\mathrm{loc}}^+\times\mathbf{R}_x^3\times\mathbf{R}_v^3)$ *for some* $0<\nu<1$*, and* $E(\cdot,\cdot)\in C_b^0(\mathbf{R}_t^+\times\mathbf{R}_x^3)$*,* $E(t,\cdot)\in C_b^1(\mathbf{R}_x^3)$ *for any fixed* $t>0$ *with the following estimates*

$$\begin{cases} |||f|||_E \le C_1\delta_2,\\ \displaystyle\int_0^\infty \|E(t,\cdot)\|_\infty dt \le \varepsilon_0 \quad \textit{and} \quad \int_0^\infty (1+t)\|\nabla_x E(t,\cdot)\|_\infty dt\le\varepsilon_1. \end{cases}$$

(ii) ρ *and* E *satisfy the decay estimates (3.21) and (3.22) respectively.*

Based on Theorems 3.1, 3.2 and 3.3, the global existence of the solution is obtained by the uniform estimates on the approximate solution sequence and the continuity argument. Finally, it is noticed that since the solution obtained in Theorem 3.3 as a limit of an approximate sequence exists in the classical Sobolev space, the uniqueness follows from the standard arguments and its proof is omitted for brevity. Thus we are done.

Remark 3.4. Notice that Theorem 3.4 also holds when the Poisson equation has a minus sign. That is when the equation (1.7) is replaced by

$$E(t,x)=\nabla_x\phi(t,x),\ -\triangle_x\phi(t,x)=\rho(t,x)=\int_{\mathbf{R}^3} f(t,x,v)\,dv.$$

In this case, the force is attractive in stead of repulsive. It is known that there exists non-trivial stationary profile to this case in infinite vacuum. However, one can show that the total mass of this kind of non-trivial profile in infinite vacuum can not be arbitrarily small. Therefore, a sufficiently small perturbation of vacuum does not generate such a non-trivial profile time asymptotically. Instead, the solution will tend to zero as time approaches to infinity implied by the above analysis. On the other hand, it will be very interesting as in [9, 21] to study the stability of the non-trivial stationary solution profiles which is not in the scope of this paper.

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