DECAY STRUCTURE OF TWO HYPERBOLIC RELAXATION MODELS WITH REGULARITY-LOSS

YOSHIHIRO UEDA, RENJUN DUAN, AND SHUICHI KAWASHIMA

ABSTRACT. The paper aims at investigating two types of decay structure for linear symmetric hyperbolic systems with non-symmetric relaxation. Precisely, the system is of the type (p, q) if the real part of all eigenvalues admits an upper bound $-c|\xi|^{2p}/(1 + |\xi|^2)^q$, where c is a generic positive constant and ξ is the frequency variable, and the system enjoys the regularity-loss property if $p < q$. It is well known that the standard type $(1, 1)$ can be assured by the classical Kawashima-Shizuta condition. A new structural condition was introduced in [33] to analyze the regularity-loss type $(1, 2)$ system with non-symmetric relaxation. In the paper, we construct two more complex models of the regularity-loss type corresponding to $p = m - 3$, $q = m - 2$ and $p = (3m - 10)/2$, $q = 2(m - 3)$, respectively, where m denotes phase dimensions. The proof is based on the delicate Fourier energy method as well as the suitable linear combination of series of energy inequalities. Due to arbitrary higher dimensions, it is not obvious to capture the energy dissipation rate with respect to the degenerate components. Thus, for each model, the analysis always starts from the case of low phase dimensions in order to understand the basic dissipative structure in the general case, and in the mean time, we also give the explicit construction of the compensating symmetric matrix K and skew-symmetric matrix S.

CONTENTS

1. Introduction

In the paper, we consider the Cauchy problem on the following linear symmetric hyperbolic system with relaxation (cf. [5]):

$$(1.1) \qquad u_t + A_m u_x + L_m u = 0$$

with

$$(1.2) \qquad u|_{t=0} = u_0.$$

Here $u = u(t, x) = (u_1, \cdots, u_m)^T(t, x) \in \mathbb{R}^m$ over $t > 0$, $x \in \mathbb{R}$ is an unknown function, $u_0 = u_0(x) \in \mathbb{R}^m$ over $x \in \mathbb{R}$ is a given function, and A_m and L_m are $m \times m$ real constant matrices. In general we assume A_m is symmetric and L_m is degenerately dissipative in the sense of $1 \le \dim(\ker L_m) \le m - 1$. As pointed out in [33], for a general linear degenerately dissipative system it is interesting to study its decay structure under additional conditions on the coefficient matrices and further investigate the corresponding time-decay property of solutions to the Cauchy problem at the linear level. The purpose of the paper is to present two concrete models of A_m and L_m, which do not satisfy the dissipative condition in [33], to derive the decay structure of the corresponding linear systems. We remark that the similar issue has been extensively investigated in Villani [37] for an infinite-dimensional dynamical system, for instance, in the content of kinetic theory.

In what follows let us explain the motivation of dealing with the problem considered here. More generally one may consider the system in multidimensional space $\mathbb{R}^n$:

$$(1.3) \qquad A_m^0 u_t + \sum_{j=1}^{n} A_m^j u_{x_j} + L_m u = 0,$$

where $u = u(t, x) \in \mathbb{R}^m$ over $t \ge 0$, $x \in \mathbb{R}^n$. When the degenerate relaxation matrix L_m is symmetric, Umeda-Kawashima-Shizuta [36] proved the large-time asymptotic stability of solutions for a class of equations of hyperbolic-parabolic type with applications to both electro-magneto-fluid dynamics and magnetohydrodynamics. The key idea in [36] and the later generalized work [31] that first introduced the so-called Kawashima-Shizuta (KS) condition is to construct the compensating matrix to capture the dissipation of systems over the degenerate kernel space of L_m. The typical feature of the time-decay property of solutions established in those work is that the high frequency part decays exponentially while the low frequency part decays polynomially with the same rate as the heat kernel. To precisely state these results, we apply Fourier transform to (1.3) (or (1.1)). Then we can obtain

$$(1.4) \qquad A_m^0 \hat{u}_t + i|\xi| A_m(\omega) \hat{u} + L_m \hat{u} = 0,$$

where $\xi \in \mathbb{R}^n$ denote the Fourier variable of $x \in \mathbb{R}^n$, $\omega = \xi/|\xi| \in S^{n-1}$, and $A_m(\omega) := \sum_{j=1}^{n} A_m^j \omega_j$. Moreover we prepare some notations. Given a real matrix X, we use X^{sy} and X^{asy} to denote the symmetric and skew-symmetric parts of X, respectively, namely, $X^{\mathrm{sy}} = (X + X^T)/2$ and $X^{\mathrm{asy}} = (X - X^T)/2$. Then the decay result in [36, 31] is stated as follows.

Proposition 1.1 (Decay property of the standard type ([36, 31])). *Consider* (1.3) *with the following condition:*

Condition (A)$_0$: A_m^0 *is real symmetric and positive definite,* A_m^j *for each* $1 \leq j \leq n$ *is real symmetric, and* L_m *is real symmetric and nonnegative definite with the nontrivial kernel.*

For this problem, assume that the following condition hold:

Condition (K): *There is a real compensating matrix* $K(\omega) \in C^\infty(S^{n-1})$ *with the properties:* $K(-\omega) = -K(\omega)$, $(K(\omega)A_m^0)^T = -K(\omega)A_m^0$ *and*

$$[K(\omega)A_m(\omega)]^{\mathrm{sy}} > 0 \quad on \quad \ker L_m$$

for each $\omega \in S^{n-1}$.

Then the Fourier image $\hat{u}$ *of the solution* u *to the equation* (1.3) *with initial data* $u(0,x) = u_0(x)$ *satisfies the pointwise estimate:*

$$(1.5) \qquad\qquad |\hat{u}(t,\xi)| \leq Ce^{-c\lambda(\xi)t}|\hat{u}_0(\xi)|,$$

where $\lambda(\xi) := |\xi|^2/(1+|\xi|^2)$. *Furthermore, let* $s \geq 0$ *be an integer and suppose that the initial data* u_0 *belong to* $H^s \cap L^1$. *Then the solution* u *satisfies the decay estimate:*

$$(1.6) \qquad \|\partial_x^k u(t)\|_{L^2} \leq C(1+t)^{-n/4-k/2}\|u_0\|_{L^1} + Ce^{-ct}\|\partial_x^k u_0\|_{L^2}$$

for $k \leq s$. *Here* C *and* c *are positive constants.*

Under the conditions (A)$_0$ and (K), we can construct the following energy inequality:

$$\frac{d}{dt}E + cD \leq 0,$$

where

$$(1.7)\ \ E = \langle A_m^0 \hat{u}, \hat{u}\rangle - \frac{\alpha|\xi|}{1+|\xi|^2}\delta\langle iK(\omega)A_m^0\hat{u}, \hat{u}\rangle, \quad D = \frac{|\xi|^2}{1+|\xi|^2}|\hat{u}|^2 + |(I-P)\hat{u}|^2,$$

α and δ are suitably small constants, and P denotes the orthogonal projection onto $\ker L_m$.

For the nonlinear system, the global existence of small-amplitude classical solutions was proved by Hanouzet-Natalini [11] in one space dimension and by Yong [38] in several space dimensions, provided that the system is strictly entropy dissipative and satisfies the KS condition. And later on, the large time behavior of solutions was obtained by Bianchini-Hanouzet-Natalini [3] and Kawashima-Yong [17] basing on the analysis of the Green function of the linearized problem. Those results show that solutions to such nonlinear system will not develop singularities (e.g., shock waves) in finite time for small smooth initial perturbations, cf. [5, 19]. Notice that the L^2-stability of a constant equilibrium state in a one-dimensional system of dissipative hyperbolic balance laws endowed with a convex entropy was also studied by Ruggeri-Serre [29]. Moreover, it would be an interesting and important topic to study the relaxation limit of general hyperbolic conservation laws with relaxations, see [4, 16] and reference therein.

Recently it has been found that there exist physical systems which violate the KS condition but still have some kind of time-decay properties. For instance, for the dissipative Timoshenko system [13, 14] and the Euler-Maxwell system [7, 35, 34], the linearized relaxation matrix L_m has a nonzero skew-symmetric part while it was still proved that solutions decay in time in some different way. Besides those, there are two related works dealing with general partially dissipative hyperbolic

systems with zero-order source when the KS condition is not satisfied. Beauchard-Zuazua [2] first observed the equivalence of the KS condition with the Kalman rank condition in the context of the control theory. They extended the previous analysis to some other situations beyond the KS condition, and established the explicit estimate on the solution semigroup in terms of the frequency variable and also the global existence of near-equilibrium classical solutions for some nonlinear balance laws without the KS condition. In the mean time, Mascia-Natalini [25] also made a general study of the same topic for a class of systems without the KS condition. The typical situation considered in [25] is that the non-dissipative components are linearly degenerate which indeed does not hold under the KS condition (see also [15]). Notice that both in [2] and [25], the rate of convergence of solutions to the equilibrium states for the nonlinear Cauchy problem is still left unknown.

In [33], the same authors of this paper introduced a new structural condition which is a generalization of the KS condition, and also analyzed the corresponding weak dissipative structure called the regularity-loss type for general systems with non-symmetric relaxation which includes the Timoshenko system and the Euler-Maxwell system as two concrete examples. Precisely, one has the following result.

Proposition 1.2 (Decay property of the regularity-loss type ([33])). *Consider* (1.3) *with the condition:*

> **Condition (A):** A_m^0 *is real symmetric and positive definite,* A_m^j *for each* $1 \leq j \leq n$ *is real symmetric, while* L_m *is not necessarily real symmetric but is nonnegative definite with the nontrivial kernel.*

For this problem, assume the previous condition (K) *and the following condition hold:*

> **Condition (S):** *There is a real matrix* S *such that* $(SA_m^0)^T = SA_m^0$, *and*
>
> $$[SL_m]^{\mathrm{sy}} + [L_m]^{\mathrm{sy}} \geq 0 \quad \text{on} \quad \mathbb{C}^m, \quad \ker\left([SL_m]^{\mathrm{sy}} + [L_m]^{\mathrm{sy}}\right) = \ker L_m,$$
>
> *and moreover, for each* $\omega \in S^{n-1}$,

$$(1.8) \qquad\qquad i[SA_m(\omega)]^{\mathrm{asy}} \geq 0 \quad on \quad \ker[L_m]^{\mathrm{sy}}.$$

Then the Fourier image $\hat{u}$ *of the solution* u *to the equation* (1.3) *with initial data* $u(0,x) = u_0(x)$ *satisfies the pointwise estimate:*

$$(1.9) \qquad\qquad |\hat{u}(t,\xi)| \leq C e^{-c\lambda(\xi)t}|\hat{u}_0(\xi)|,$$

where $\lambda(\xi) := |\xi|^2/(1+|\xi|^2)^2$. *Moreover, let* $s \geq 0$ *be an integer and suppose that the initial data* u_0 *belong to* $H^s \cap L^1$. *Then the solution* u *satisfies the decay estimate:*

$$(1.10) \qquad \|\partial_x^k u(t)\|_{L^2} \leq C(1+t)^{-n/4-k/2}\|u_0\|_{L^1} + C(1+t)^{-\ell/2}\|\partial_x^{k+\ell}u_0\|_{L^2}$$

for $k + \ell \leq s$. *Here* C *and* c *are positive constants.*

Observe that $\lambda(\xi)$ in (1.9) behaves as $|\xi|^2$ as $|\xi| \to 0$ but behaves as $1/|\xi|^2$ as $|\xi| \to \infty$. Thus those estimates (1.9) and (1.10) are weaker than (1.5) and (1.6), respectively. In particular, the decay estimate (1.9) is said to be of the regularity-loss type. Similar decay properties of the regularity-loss type have been recently observed for several interesting systems. We refer the reader to [13, 14, 23] (cf. [1, 28]) for the dissipative Timoshenko system, [7, 35, 34] for the Euler-Maxwell system, [12, 18] for a hyperbolic-elliptic system in radiation gas dynamics, [20, 21, 22, 24, 32] for a dissipative plate equation, and [6, 9] for various kinetic-fluid models.

In fact, one can show that Proposition 1.1 can be regarded as a corollary of Proposition 1.2 after replacing (1.8) in condition (S) by a stronger condition:

$$i[SA_m(\omega)]^{\mathrm{asy}} \geq 0 \quad \text{on} \quad \mathbb{C}^m.$$

for each $\omega \in S^{n-1}$. The key point for the proof of (1.9) is to derive the matrices S and $K(\omega)$ such that the coercive estimate:

$$(1.11) \qquad \delta[K(\omega)A_m(\omega)]^{\mathrm{sy}} + [SL_m]^{\mathrm{sy}} + [L_m]^{\mathrm{sy}} > 0 \quad \text{on} \quad \mathbb{C}^m$$

holds true for suitably small $\delta > 0$. Indeed, under the conditions (A), (S) and (K), the estimate (1.11) is satisfied. Then, using (1.11), we get the following energy equality

$$(1.12) \qquad \frac{d}{dt}E + cD \leq 0,$$

where

$$
(1.13) \qquad
\begin{aligned}
E &= \langle A_m^0 \hat{u}, \hat{u}\rangle + \frac{\alpha_1}{1+|\xi|^2}\left(\langle SA_m^0 \hat{u}, \hat{u}\rangle - \frac{\alpha_2|\xi|}{1+|\xi|^2}\delta\langle iK(\omega)A_m^0 \hat{u}, \hat{u}\rangle\right), \\
D &= \frac{|\xi|^2}{(1+|\xi|^2)^2}|\hat{u}|^2 + \frac{1}{1+|\xi|^2}|(I-P)\hat{u}|^2 + |(I-P_1)\hat{u}|^2,
\end{aligned}
$$

α_1 and α_2 are suitably small constants, and P and P_1 denote the orthogonal projections onto $\ker L_m$ and $\ker [L_m]^{\mathrm{sy}}$. Interested readers may refer to [33] for more details of this issue and also for the construction of S and $K(\omega)$ for the Timoshenko system and the Euler-Maxwell system. Therefore, those conditions in Proposition 1.2 are generalizations of the classical KS conditions. We finally remark that it should be interesting to further investigate the nonlinear stability of constant equilibrium states of the system of the regularity-loss type under the structural condition postulated in Proposition 1.2.

Inspired by the previous work [33], the goal of the paper is to construct much more complex models (1.1) with given A_m and L_m such that they enjoy some new dissipative structure of the regularity-loss type. Here we recall a notion of the *uniform dissipativity* of the system (1.1) introduced in [33]. Consider the eigenvalue problem for the system (1.1):

$$(\eta A_m^0 + i\xi A_m + L_m)\phi = 0,$$

where $\eta \in \mathbb{C}$ and $\phi \in \mathbb{C}^m$. The corresponding characteristic equation is given by

$$(1.14) \qquad \det(\eta A_m^0 + i\xi A_m + L_m) = 0.$$

The solution $\eta = \eta(i\xi)$ of (1.14) is called the eigenvalue of the system (1.1).

Definition 1.3. *The system* (1.1) *is called uniformly dissipative of the type* (p,q) *if the eigenvalue* $\eta = \eta(i\xi)$ *satisfies*

$$\Re\,\eta(i\xi) \leq -c|\xi|^{2p}/(1+|\xi|^2)^q$$

for all $\xi \in \mathbb{R}^n$, *where* c *is a positive constant and* (p,q) *is a pair of positive integers.*

Note that as proved in [33, Theorem 4.2], one has $\Re\,\eta(i\xi) \leq -c\lambda(\xi)$ whenever the pointwise estimates in the form of (1.5) or (1.9) hold true. Therefore, we can determine the type (p,q) for a uniformly dissipative system (1.1) in terms of the function $\lambda(\xi)$ obtained from the pointwise estimate on $\hat{u}(t,\xi)$:

$$(1.15) \qquad |\hat{u}(t,\xi)| \leq Ce^{-c\lambda(\xi)t}|\hat{u}_0(\xi)|.$$

For example, under the assumption in Propositions 1.1 or 1.2, the system (1.1) is uniformly dissipative of the type $(1,1)$ or $(1,2)$, respectively. Notice that the regularity-loss type corresponds to the situation when p is strictly less than q, i.e., $p < q$.

Historically, Shizuta-Kawashima [32] showed that, under the condition $(A)_0$, the strict dissipativity $\Re\eta(i\xi) < 0$ for $\xi \neq 0$ is equivalent to the uniform dissipativity of the type $(1,1)$. Moreover, they showed the pointwise estimate (1.5) by using only one compensating skew-symmetric matrix $K(\omega)$ (see (1.7)). On the other hand, the authors formulated in [33] a class of systems whose dissipativity is of the type $(1,2)$ and got Proposition 1.2. Notice that, in this cace, we need to use one compensating symmetric matrix S and one compensating skew-symmetric matrix $K(\omega)$ to get the desired pointwise estimate (1.9) (see (1.13)). We note that the dissipative Timoshenko system and the Euler-Maxwell system studied in [13] and [34], respectively, are included in the class of systems with the type $(1,2)$ which was formulated in [33]. However, to get the optimal dissipative estimate for these two examples, we need to use one S and two different $K(\omega)$ (see [26, 34]).

On the other hand, more complicated concrete models are found in these years. Indeed, Mori-Kawashima [27] considered the Timoshenko-Cattaneo system with heat conduction and showed that its dissipativity is of the type $(2,3)$. Moreover, they proved the optimal dissipative estimate by using four different S and four different $K(\omega)$. This means that Proposition 1.2 and the class formulated in [33] is not enough to analyze the dissipativity of general systems (1.3), and we have to study other concrete models.

In this paper, we will present a study of two concrete models of the system (1.1) related to the above general issue. For the Model I, one has

$$p = m - 3, \quad q = m - 2,$$

see (2.2) in Theorem 2.1. While for the Model II, we let m be even and one has

$$p = \frac{1}{2}(3m - 10), \quad q = 2(m - 3),$$

see (3.2) in Theorem 3.1. In both cases we see $p < q$ and hence two models that we consider are of the regularity-loss type. Compared with the energy inequality (1.12), the energy inequalities of the Model I and II are much more complicated. More precisely, to control the dissipation term, we must employ a lot of compensating symmetric matrices and skew-symmetric matrices whose numbers depend on the dimension m of the coefficient matrices. Therefore we can not apply Proposition 1.2 to the Model I and II, and need direct calculations (see in Section 2 and 3).

The proof of the estimate in the form of (1.15) is based on the Fourier energy method, and in the mean time we also give the explicit construction of matrices S and K as used in Proposition 1.2. As seen later on, a series of energy estimates are derived and their appropriate linear combination leads to a Lyapunov-type inequality of the time-frequency functional equivalent with $|\hat{u}(t,\xi)|^2$, which hence implies (1.15). The most difficult point is that it is priorly unclear to justify whether one choice of (p,q) is optimal; see more discussions in Section 4.1. For that purpose, we also present an alternative approach to find out the value of (p,q) for both Model I and Model II, and the detailed strategy of the approach is to be given later on.

The rest of the paper is organized as follows. In Section 2 and Section 3, we study Model I and Model II, respectively. In each section, for the given model, we

first state the main results on the dissipative structure and the decay property of the system (1.1), give the proof by the energy method in the case $m = 6$ which indeed corresponds to some existing physical models, show the proof in the general case $m \geq 6$ still using the energy method, and finally give the explicit construction of matrices S and K. The matrices S and K constructed in Subsection 2.3 and 3.3 have a very important role in obtaining the coercive estimate similar to (1.11). Consequently, by employing these matrices, we can derive the desired pointwise estimates through (2.47) and (3.63) to be verified later on. In the last Section 4, we provide another approach to justify the dissipative structure of the system (1.1).

Notations. For a nonnegative integer k, we denote by ∂_x^k the totality of all the k-th order derivatives with respect to $x = (x_1, \cdots, x_n)$. Let $1 \leq p \leq \infty$. Then $L^p = L^p(\mathbb{R}^n)$ denotes the usual Lebesgue space over $\mathbb{R}^n$ with the norm $\| \cdot \|_{L^p}$. For a nonnegative integer s, $H^s = H^s(\mathbb{R}^n)$ denotes the s-th order Sobolev space over $\mathbb{R}^n$ in the L^2 sense, equipped with the norm $\| \cdot \|_{H^s}$. We note that $L^2 = H^0$. Finally, in this paper, we use C or c to denote various positive constants without confusion.

2. Model I

2.1. **Main result I.** In this section, we consider the Cauchy problem (1.1), (1.2) with coefficient matrices given by

$$(2.1) \quad A_m = \begin{pmatrix} 0 & 1 & 0 & 0 & & & & \\ 1 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & a_4 & 0 & & & \\ 0 & 0 & a_4 & 0 & a_5 & 0 & & \\ & & & 0 & a_5 & 0 & a_6 & \\ & & & & 0 & a_6 & \ddots & \\ & O & & & & & & a_m \\ & & & & & & a_m & 0 \end{pmatrix}, \quad L_m = \begin{pmatrix} 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ -1 & 0 & 0 & 0 & & & & \\ & & & & 0 & & & \\ & & & & & \ddots & & \\ & O & & & & & 0 & \\ & & & & & & & \gamma \end{pmatrix},$$

where integer $m \geq 6$ is even, $\gamma > 0$, and all elements a_j $(4 \leq j \leq m)$ are nonzero. We note that the system (1.1), (2.1) with $m = 6$ is the Timoshenko system with the heat conduction via Cattaneo law (cf. [10, 30]). For this problem, we can derive the following decay structure.

Theorem 2.1. *The Fourier image $\hat{u}$ of the solution u to the Cauchy problem (1.1)-(1.2) with (2.1) satisfies the pointwise estimate:*

$$(2.2) \qquad\qquad |\hat{u}(t,\xi)| \leq Ce^{-c\lambda(\xi)t}|\hat{u}_0(\xi)|,$$

where $\lambda(\xi) := \xi^{2(m-3)}/(1 + \xi^2)^{m-2}$. Furthermore, let $s \geq 0$ be an integer and suppose that the initial data u_0 belong to $H^s \cap L^1$. Then the solution u satisfies the decay estimate:

$$(2.3) \qquad \|\partial_x^k u(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2(m-3)}(\frac{1}{2}+k)}\|u_0\|_{L^1} + C(1+t)^{-\frac{\ell}{2}}\|\partial_x^{k+\ell} u_0\|_{L^2}$$

for $k + \ell \leq s$. Here C and c are positive constants.

We remark that the estimates (2.2) and (2.3) with $m = 6$ is not optimal. Indeed, Mori-Kawashima [27] showed more sharp estimates.

The decay estimate (2.3) is derived by the pointwise estimate (2.2) in Fourier space immediately. Thus readers may refer to the same authors' paper [33] (see

also [8]) and we omit the proof of (2.3) for brevity. In order to make the proof more precise, we first consider the special case $m = 6$ in Section 2.2, and then generalize it to the case $m \geq 6$ in Section 2.3. The proof of (2.2) is given in the following two subsections.

2.2. **Energy method in the case $m = 6$.** In this subsection we first consider the case $m = 6$. In such case, the system (1.1) with (2.1) is described as

$$
\begin{aligned}
&\partial_t \hat{u}_1 + i\xi \hat{u}_2 + \hat{u}_4 = 0, \\
&\partial_t \hat{u}_2 + i\xi \hat{u}_1 = 0, \\
&\partial_t \hat{u}_3 + i\xi a_4 \hat{u}_4 = 0, \\
&\partial_t \hat{u}_4 + i\xi (a_4 \hat{u}_3 + a_5 \hat{u}_5) - \hat{u}_1 = 0, \\
&\partial_t \hat{u}_5 + i\xi (a_5 \hat{u}_4 + a_6 \hat{u}_6) = 0, \\
&\partial_t \hat{u}_6 + i\xi a_6 \hat{u}_5 + \gamma \hat{u}_6 = 0.
\end{aligned}
$$
(2.4)

For this system we are going to apply the energy method to derive Theorem 2.1 in the case $m = 6$. The proof is organized by the following three steps.

Step 1. We first derive the basic energy equality for the system (2.4) in the Fourier space. We multiply the all equations of (2.4) by $\bar{\hat{u}} = (\bar{\hat{u}}_1, \bar{\hat{u}}_2, \bar{\hat{u}}_3, \bar{\hat{u}}_4, \bar{\hat{u}}_5, \bar{\hat{u}}_6)^T$, respectively, and combine the resultant equations. Then we obtain

$$
\sum_{j=1}^{6} \bar{\hat{u}}_j \partial_t \hat{u}_j + 2i\xi \Re(\hat{u}_1 \bar{\hat{u}}_2) + 2i\xi \sum_{j=3}^{5} a_{j+1} \Re(\hat{u}_j \bar{\hat{u}}_{j+1}) + 2i\mathrm{Im}(\hat{u}_4 \bar{\hat{u}}_1) + \gamma |\hat{u}_6|^2 = 0.
$$

Thus, taking the real part for the above equality, we arrive at the basic energy equality

$$
\frac{1}{2} \partial_t |\hat{u}|^2 + \gamma |\hat{u}_6|^2 = 0.
$$
(2.5)

Here we use the simple relation $\partial_t(\hat{u}_j^2) = 2\Re(\bar{\hat{u}}_j \partial_t \hat{u}_j)$ for any j. Next we create the dissipation terms.

Step 2. We first construct the dissipation for $\hat{u}_1$. We multiply the first and fourth equations in (2.4) by $-\bar{\hat{u}}_4$ and $-\bar{\hat{u}}_1$, respectively. Then, combining the resultant equations and taking the real part, we have

(2.6) $-\partial_t \Re(\hat{u}_1 \bar{\hat{u}}_4) + |\hat{u}_1|^2 - |\hat{u}_4|^2 - \xi \Re(i\hat{u}_2 \bar{\hat{u}}_4) + a_4 \xi \Re(i\hat{u}_1 \bar{\hat{u}}_3) + a_5 \xi \Re(i\hat{u}_1 \bar{\hat{u}}_5) = 0.$

On the other hand, we multiply the second and third equations in (2.4) by $-a_4 \bar{\hat{u}}_3$ and $-a_4 \bar{\hat{u}}_2$, respectively. Then, combining the resultant equations and taking the real part, we have

$$
-a_4 \partial_t \Re(\hat{u}_2 \bar{\hat{u}}_3) - a_4 \xi \Re(i\hat{u}_1 \bar{\hat{u}}_3) + a_4^2 \xi \Re(i\hat{u}_2 \bar{\hat{u}}_4) = 0.
$$

Therefore, combining the above two equalities, we obtain

(2.7) $\quad - \partial_t \Re(\hat{u}_1 \bar{\hat{u}}_4 + a_4 \hat{u}_2 \bar{\hat{u}}_3) + |\hat{u}_1|^2 - |\hat{u}_4|^2$
$$
+ (a_4^2 - 1)\xi \Re(i\hat{u}_2 \bar{\hat{u}}_4) + a_5 \xi \Re(i\hat{u}_1 \bar{\hat{u}}_5) = 0.
$$

Furthermore, we multiply the second equation and fifth equation in (2.4) by $-\bar{\hat{u}}_5$ and $-\bar{\hat{u}}_2$, respectively. Then, combining the resultant equations and taking the real part, we have

(2.8) $\quad -\partial_t \Re(\hat{u}_2 \bar{\hat{u}}_5) - \xi \Re(i\hat{u}_1 \bar{\hat{u}}_5) + a_5 \xi \Re(i\hat{u}_2 \bar{\hat{u}}_4) + a_6 \xi \Re(i\hat{u}_2 \bar{\hat{u}}_6) = 0.$

Finally, multiplying (2.7) and (2.8) by a_5^2 and $-a_5(a_4^2 - 1)$, respectively, and combining the resultant equations, we have

$$(2.9) \quad \partial_t E_1 + a_5^2(|\hat{u}_1|^2 - |\hat{u}_4|^2)$$
$$+ a_5(a_4^2 + a_5^2 - 1)\xi\, \Re(i\hat{u}_1\bar{\hat{u}}_5) - a_5 a_6(a_4^2 - 1)\xi\, \Re(i\hat{u}_2\bar{\hat{u}}_6) = 0,$$

where we have defined that $E_1 := -\Re\{a_5^2(\hat{u}_1\bar{\hat{u}}_4 + a_4\hat{u}_2\bar{\hat{u}}_3) - a_5(a_4^2 - 1)\hat{u}_2\bar{\hat{u}}_5\}$.

Next, we multiply the first and second equations in (2.4) by $-i\xi\bar{\hat{u}}_2$ and $i\xi\bar{\hat{u}}_1$, respectively. Then, combining the resultant equations and taking the real part, we have

$$(2.10) \quad \xi\partial_t E_2 + \xi^2(|\hat{u}_2|^2 - |\hat{u}_1|^2) + \xi\, \Re(i\hat{u}_2\bar{\hat{u}}_4) = 0,$$

where $E_2 := -\Re(i\hat{u}_1\bar{\hat{u}}_2)$. Therefore, by Young inequality, the above equation becomes

$$(2.11) \quad \xi\partial_t E_2 + \frac{1}{2}\xi^2|\hat{u}_2|^2 \le \xi^2|\hat{u}_1|^2 + \frac{1}{2}|\hat{u}_4|^2.$$

We multiply the third and fourth equations in (2.4) by $i\xi a_4\bar{\hat{u}}_4$ and $-i\xi a_4\bar{\hat{u}}_3$, respectively. Then, combining the resultant equations and taking the real part, we have

$$a_4\xi\partial_t\Re(i\hat{u}_3\bar{\hat{u}}_4) + a_4^2\xi^2(|\hat{u}_3|^2 - |\hat{u}_4|^2) + a_4 a_5\xi^2\, \Re(\hat{u}_3\bar{\hat{u}}_5) + a_4\xi\, \Re(i\hat{u}_1\bar{\hat{u}}_3) = 0.$$

On the other hand, we multiply the second and third equations in (2.27) by $-a_4\bar{\hat{u}}_3$ and $-a_4\bar{\hat{u}}_2$, respectively. Then, combining the resultant equations and taking the real part, we have

$$-a_4\partial_t\Re(\hat{u}_2\bar{\hat{u}}_3) - a_4\xi\, \Re(i\hat{u}_1\bar{\hat{u}}_3) + a_4^2\xi\, \Re(i\hat{u}_2\bar{\hat{u}}_4) = 0.$$

Finally, combining the above two equations, we get

$$(2.12) \quad \partial_t\{\xi E_3 + F_1\} + a_4^2\xi^2(|\hat{u}_3|^2 - |\hat{u}_4|^2) + a_4 a_5\xi^2\, \Re(\hat{u}_3\bar{\hat{u}}_5) + a_4^2\xi\, \Re(i\hat{u}_2\bar{\hat{u}}_4) = 0.$$

where $E_3 := a_4\Re(i\hat{u}_3\bar{\hat{u}}_4)$ and $F_1 := -a_4\Re(\hat{u}_2\bar{\hat{u}}_3)$. By using Young inequality, we can obtain the following inequality:

$$(2.13) \quad \partial_t\{\xi E_3 + F_1\} + \frac{1}{2}a_4^2\xi^2|\hat{u}_3|^2 \le a_4^2\xi^2|\hat{u}_4|^2 + \frac{1}{2}a_5^2\xi^2|\hat{u}_5|^2 + a_4^2|\xi||\hat{u}_2||\hat{u}_4|.$$

Multiplying the fourth equation and fifth equation in (2.27) by $i\xi a_5\bar{\hat{u}}_5$ and $-i\xi a_5\bar{\hat{u}}_4$, respectively, combining the resultant equations, and taking the real part, then we have

$$(2.14) \quad \xi\partial_t E_4 + a_5^2\xi^2(|\hat{u}_4|^2 - |\hat{u}_5|^2)$$
$$- a_4 a_5\xi^2\, \Re(\hat{u}_3\bar{\hat{u}}_5) + a_5 a_6\xi^2\, \Re(\hat{u}_4\bar{\hat{u}}_6) - a_5\xi\, \Re(i\hat{u}_1\bar{\hat{u}}_5) = 0,$$

where $E_4 := a_5\Re(i\hat{u}_4\bar{\hat{u}}_5)$. Here, by using Young inequality, we obtain

$$(2.15) \quad \xi\partial_t E_4 + \frac{1}{2}a_5^2\xi^2|\hat{u}_4|^2$$
$$\le a_5^2\xi^2|\hat{u}_5|^2 + \frac{1}{2}a_6^2\xi^2|\hat{u}_6|^2 + a_4 a_5\xi^2\, \Re(\hat{u}_3\bar{\hat{u}}_5) + a_5\xi\, \Re(i\hat{u}_1\bar{\hat{u}}_5).$$

On the other hand, we multiply the fifth equation and the last equation in (2.4) by $i\xi a_6\bar{\hat{u}}_6$ and $-i\xi a_6\bar{\hat{u}}_5$, respectively. Then, combining the resultant equations and taking the real part, we obtain

$$a_6\xi\partial_t\Re(i\hat{u}_5\bar{\hat{u}}_6) + a_6^2\xi^2(|\hat{u}_5|^2 - |\hat{u}_6|^2) - a_5 a_6\xi^2\, \Re(\hat{u}_4\bar{\hat{u}}_6) + \gamma a_6\xi\, \Re(i\hat{u}_5\bar{\hat{u}}_6) = 0.$$

Using Young inequality, this yields

$$(2.16) \quad a_6\xi\partial_t\Re(i\hat{u}_5\bar{\hat{u}}_6) + \frac{1}{2}a_6^2\xi^2|\hat{u}_5|^2 \le a_6^2\xi^2|\hat{u}_6|^2 + \frac{1}{2}\gamma^2|\hat{u}_6|^2 + a_5a_6\xi^2\,\Re(\hat{u}_4\bar{\hat{u}}_6).$$

Step 3. In this step, we sum up the energy inequalities derived in the previous step, and then get the desired energy estimate. Throughout this step, β_j with $j \in \mathbb{N}$ denote the real numbers determined later. We first multiply (2.9) and (2.11) by ξ^2 and β_1, respectively. Then we combine the resultant equation, obtaining

$$\partial_t\{\xi^2 E_1 + \beta_1\xi E_2\} + (a_5^2 - \beta_1)\xi^2|\hat{u}_1|^2 + \frac{\beta_1}{2}\xi^2|\hat{u}_2|^2$$
$$\le \Big(\frac{\beta_1}{2} + a_5^2\xi^2\Big)|\hat{u}_4|^2 - a_5(a_4^2 + a_5^2 - 1)\xi^3\,\Re(i\hat{u}_1\bar{\hat{u}}_5) + a_5a_6(a_4^2 - 1)\xi^3\,\Re(i\hat{u}_2\bar{\hat{u}}_6).$$

Moreover, combining (2.9), (2.13) and the above inequality, we have

$$\partial_t\{(1+\xi^2)E_1 + \beta_1\xi E_2 + \xi E_3 + F_1\}$$
$$+ \{a_5^2 + (a_5^2 - \beta_1)\xi^2\}|\hat{u}_1|^2 + \frac{\beta_1}{2}\xi^2|\hat{u}_2|^2 + \frac{1}{2}a_4^2\xi^2|\hat{u}_3|^2$$
$$\le \Big\{a_5^2 + \frac{\beta_1}{2} + (a_4^2 + a_5^2)\xi^2\Big\}|\hat{u}_4|^2 + \frac{1}{2}a_5^2\xi^2|\hat{u}_5|^2 + a_4^2|\xi||\hat{u}_2||\hat{u}_4|$$
$$- a_5(a_4^2 + a_5^2 - 1)\xi(1+\xi^2)\,\Re(i\hat{u}_1\bar{\hat{u}}_5) + a_5a_6(a_4^2 - 1)\xi(1+\xi^2)\,\Re(i\hat{u}_2\bar{\hat{u}}_6).$$

For this inequality, letting β_1 suitably small and employing Young inequality, we can get

$$(2.17) \quad \partial_t\{(1+\xi^2)E_1 + c\xi E_2 + \xi E_3 + F_1\} + c(1+\xi^2)|\hat{u}_1|^2 + \beta_1\xi^2(|\hat{u}_2|^2 + |\hat{u}_3|^2)$$
$$\le C(1+\xi^2)|\hat{u}_4|^2 + C\xi^2|\hat{u}_5|^2$$
$$+ |a_4^2 + a_5^2 - 1|C|\xi|^3|\hat{u}_1||\hat{u}_5| + |a_4^2 - 1|C|\xi|(1+\xi^2)|\hat{u}_2||\hat{u}_6|.$$

Similarly, multiplying (2.15) and (2.17) by $1 + \xi^2$ and $\beta_2\xi^2$, respectively. Then we combine the resultant equation, obtainig

$$\partial_t\{\beta_2\xi^2((1+\xi^2)E_1 + \beta_1\xi E_2 + \xi E_3 + F_1) + \xi(1+\xi^2)E_4\}$$
$$+ \beta_2 c\xi^2(1+\xi^2)|\hat{u}_1|^2 + \beta_2 c\xi^4(|\hat{u}_2|^2 + |\hat{u}_3|^2) + \Big(\frac{1}{2}a_5^2 - \beta_2 C\Big)\xi^2(1+\xi^2)|\hat{u}_4|^2$$
$$\le \beta_2 C\xi^4|\hat{u}_5|^2 + a_5^2\xi^2(1+\xi^2)|\hat{u}_5|^2 + \frac{1}{2}a_6^2\xi^2(1+\xi^2)|\hat{u}_6|^2$$
$$+ a_4a_5\xi^2(1+\xi^2)\,\Re(\hat{u}_3\bar{\hat{u}}_5) + a_5\xi(1+\xi^2)\,\Re(i\hat{u}_1\bar{\hat{u}}_5)$$
$$+ \beta_2|a_4^2 + a_5^2 - 1|C|\xi|^5|\hat{u}_1||\hat{u}_5| + \beta_2|a_4^2 - 1|C|\xi|^3(1+\xi^2)|\hat{u}_2||\hat{u}_6|.$$

Letting β_2 suitably small and using Young inequality derive that

$$(2.18) \quad \partial_t\{\beta_2\xi^2((1+\xi^2)E_1 + \beta_1\xi E_2 + \xi E_3 + F_1) + \xi(1+\xi^2)E_4\}$$
$$+ c\xi^2(1+\xi^2)(|\hat{u}_1|^2 + |\hat{u}_4|^2) + c\xi^4(|\hat{u}_2|^2 + |\hat{u}_3|^2)$$
$$\le C(1+\xi^2)^2|\hat{u}_5|^2 + C\xi^2(1+\xi^2)|\hat{u}_6|^2$$
$$+ |a_4^2 + a_5^2 - 1|C\xi^6|\hat{u}_5|^2 + |a_4^2 - 1|C|\xi|^2(1+\xi^2)^2|\hat{u}_2||\hat{u}_6|.$$

If we assume that $a_4^2 - 1 = 0$, the estimate (2.18) can be rewritten as

$$
\begin{aligned}
(2.19) \quad & \partial_t \big\{ \beta_2 \xi^2 ((1+\xi^2)E_1 + \beta_1 \xi E_2 + \xi E_3 + F_1) + \xi(1+\xi^2)E_4 \big\} \\
& \quad + c\xi^2(1+\xi^2)(|\hat{u}_1|^2 + |\hat{u}_4|^2) + c\xi^4(|\hat{u}_2|^2 + |\hat{u}_3|^2) \\
& \qquad\qquad\qquad\qquad \leq C(1+\xi^2)^3 |\hat{u}_5|^2 + C\xi^2(1+\xi^2)|\hat{u}_6|^2 .
\end{aligned}
$$

Then, multiplying (2.16) and the above inequality by $(1+\xi^2)^3$ and $\beta_3 \xi^2$, respectively, and combining the resultant equation, we have

$$
\begin{aligned}
& \partial_t \big\{ \beta_3 \xi^2 (\beta_2 \xi^2((1+\xi^2)E_1 + \beta_1 \xi E_2 + \xi E_3 + F_1) + \xi(1+\xi^2)E_4) + \xi(1+\xi^2)^3 E_5 \big\} \\
& \quad + \beta_3 c \xi^4(1+\xi^2)(|\hat{u}_1|^2 + |\hat{u}_4|^2) + \beta_3 c \xi^6(|\hat{u}_2|^2 + |\hat{u}_3|^2) \\
& \quad + \Big(\frac{1}{2}a_6^2 - \beta_3 C\Big)\xi^2(1+\xi^2)^3 |\hat{u}_5|^2 \leq \beta_3 C \xi^4(1+\xi^2)|\hat{u}_6|^2 \\
& \qquad\qquad + \Big(a_6^2 \xi^2 + \frac{1}{2}\gamma^2\Big)(1+\xi^2)^3 |\hat{u}_6|^2 + a_5 a_6 \xi^2(1+\xi^2)^3 \Re(\hat{u}_4 \bar{\hat{u}}_6).
\end{aligned}
$$

Hence we arrive at

$$
\begin{aligned}
& \partial_t \big\{ \beta_3 \xi^2 (\beta_2 \xi^2((1+\xi^2)E_1 + \beta_1 \xi E_2 + \xi E_3 + F_1) \\
& \qquad\qquad + \xi(1+\xi^2)E_4) + \xi(1+\xi^2)^3 E_5 \big\} \\
& \quad + c\xi^4(1+\xi^2)(|\hat{u}_1|^2 + |\hat{u}_4|^2) + c\xi^6(|\hat{u}_2|^2 + |\hat{u}_3|^2) + c\xi^2(1+\xi^2)^3 |\hat{u}_5|^2 \\
& \quad \leq C(1+\xi^2)^4 |\hat{u}_6|^2 + C\xi^2(1+\xi^2)^3 |\hat{u}_4||\hat{u}_6|.
\end{aligned}
$$

Moreover, we multiply (2.13) and (2.15) by $\beta_4 \xi^6$ and $\beta_5 \xi^6$, respectively, and combining the resultant equations and the above inequality. Then, letting β_4 and β_5 suitably small, this yields

$$
\begin{aligned}
(2.20) \quad & \partial_t E + c\xi^4(1+\xi^2)|\hat{u}_1|^2 + c\xi^6 |\hat{u}_2|^2 + c\xi^6(1+\xi^2)|\hat{u}_3|^2 \\
& \qquad + c\xi^4(1+\xi^2)^2 |\hat{u}_4|^2 + c\xi^2(1+\xi^2)^3 |\hat{u}_5|^2 \leq C(1+\xi^2)^4 |\hat{u}_6|^2 ,
\end{aligned}
$$

where we have defined

$$
\begin{aligned}
(2.21) \quad E = {}& \beta_2 \beta_3 \xi^4 (1+\xi^2)E_1 + \beta_1 \beta_2 \beta_3 \xi^5 E_2 + \xi^4 (\beta_2 \beta_3 + \beta_4 \xi^2)(\xi E_3 + F_1) \\
& \qquad + \xi^3 (\beta_3(1+\xi^2) + \beta_5 \xi^4)E_4 + \xi(1+\xi^2)^3 E_5.
\end{aligned}
$$

Finally, combining the basic energy (2.5) with the above estimate, this yields

$$
\begin{aligned}
(2.22) \quad & \partial_t \Big\{ \frac{1}{2}(1+\xi^2)^4 |\hat{u}|^2 + \beta_7 E \Big\} + c\xi^4(1+\xi^2)|\hat{u}_1|^2 \\
& \qquad\qquad + c\xi^6 |\hat{u}_2|^2 + c\sum_{j=3}^{6} \xi^{2(6-j)}(1+\xi^2)^{j-2}|\hat{u}_j|^2 \leq 0.
\end{aligned}
$$

Thus, integrating the above estimate with respect to t, we obtain the following energy estimate

$$
\begin{aligned}
(2.23) \quad & |\hat{u}(t,\xi)|^2 + \int_0^t \Big\{ \frac{\xi^4}{(1+\xi^2)^3}|\hat{u}_1|^2 + \frac{\xi^6}{(1+\xi^2)^4}|\hat{u}_2|^2 \\
& \qquad\qquad + \sum_{j=3}^{6} \frac{\xi^{2(6-j)}}{(1+\xi^2)^{6-j}}|\hat{u}_j|^2 \Big\} d\tau \leq C|\hat{u}(0,\xi)|^2 .
\end{aligned}
$$

Here we used the following inequality

$$(2.24) \qquad c|\hat{u}|^2 \le \frac{1}{2}|\hat{u}|^2 + \frac{\beta_7}{(1+\xi^2)^4}E \le C|\hat{u}|^2$$

for suitably small β_7. Furthermore the estimate (2.22) with (2.24) gives us the following pointwise estimate

$$(2.25) \qquad |\hat{u}(t,\xi)| \le Ce^{-c\lambda(\xi)t}|\hat{u}(0,\xi)|, \qquad \lambda(\xi) = \frac{\xi^6}{(1+\xi^2)^4}.$$

On the other hand, if we assume that $a_4^2 + a_5^2 - 1 = 0$, the estimate (2.18) is rewritten as

$$\begin{aligned}(2.26) \qquad & \partial_t\big\{\beta_2\xi^2((1+\xi^2)E_1 + \beta_1\xi E_2 + \xi E_3 + F_1) + \xi(1+\xi^2)E_4\big\} \\ & \quad + c\xi^2(1+\xi^2)(|\hat{u}_1|^2 + |\hat{u}_4|^2) + c\xi^4(|\hat{u}_2|^2 + |\hat{u}_3|^2) \\ & \le C(1+\xi^2)^2|\hat{u}_5|^2 + C(1+\xi^2)^4|\hat{u}_6|^2.\end{aligned}$$

Then, multiplying (2.16) and the above inequality by $(1+\xi^2)^2$ and $\beta_3\xi^2$, respectively, and combining the resultant equation, we have

$$\partial_t\big\{\beta_3\xi^2(\beta_2\xi^2((1+\xi^2)E_1 + \beta_1\xi E_2 + \xi E_3 + F_1) + \xi(1+\xi^2)E_4) + \xi(1+\xi^2)^2E_5\big\}$$

$$+ \beta_3c\xi^4(1+\xi^2)(|\hat{u}_1|^2 + |\hat{u}_4|^2) + \beta_3c\xi^6(|\hat{u}_2|^2 + |\hat{u}_3|^2) + \Big(\frac{1}{2}a_6^2 - \beta_3C\Big)\xi^2(1+\xi^2)^2|\hat{u}_5|^2$$

$$\le \beta_3C(1+\xi^2)^4|\hat{u}_6|^2 + \Big(a_6^2\xi^2 + \frac{1}{2}\gamma^2\Big)(1+\xi^2)^2|\hat{u}_6|^2 + a_5a_6\xi^2(1+\xi^2)^2\,\Re(\hat{u}_4\bar{\hat{u}}_6).$$

Hence we arrive at

$$\begin{aligned} & \partial_t\big\{\beta_3\xi^2(\beta_2\xi^2((1+\xi^2)E_1 + \beta_1\xi E_2 + \xi E_3 + F_1) \\ & \qquad\qquad + \xi(1+\xi^2)E_4) + \xi(1+\xi^2)^2E_5\big\} \\ & \quad + c\xi^4(1+\xi^2)(|\hat{u}_1|^2 + |\hat{u}_4|^2) + c\xi^6(|\hat{u}_2|^2 + |\hat{u}_3|^2) + c\xi^2(1+\xi^2)^2|\hat{u}_5|^2 \\ & \le C(1+\xi^2)^4|\hat{u}_6|^2.\end{aligned}$$

Moreover, we multiply (2.13), (2.15) and (2.16) by $\beta_4\xi^6$, $\beta_5\xi^6$ and $\beta_6\xi^6$, respectively, and combine the resultant equations and the above inequality. Then, letting β_4 and β_5 suitably small, this yields

$$\begin{aligned} & \partial_t\big\{\beta_2\beta_3\xi^4(1+\xi^2)E_1 + \beta_1\beta_2\beta_3\xi^5E_2 + \xi^4(\beta_2\beta_3 + \beta_4\xi^2)(\xi E_3 + F_1) \\ & \qquad\qquad + \xi^3(\beta_3(1+\xi^2) + \beta_5\xi^4)E_4 + \xi((1+\xi^2)^2 + \beta_6\xi^6)E_5\big\} \\ & \quad + c\xi^4(1+\xi^2)|\hat{u}_1|^2 + c\xi^6|\hat{u}_2|^2 + c\xi^6(1+\xi^2)|\hat{u}_3|^2 \\ & \quad + c\xi^4(1+\xi^2)^2|\hat{u}_4|^2 + c\xi^2(1+\xi^2)^3|\hat{u}_5|^2 \le C(1+\xi^2)^4|\hat{u}_6|^2.\end{aligned}$$

We note that this estimate is essentially the same as (2.20). Hence we can obtain the energy estimate (2.23) and the pointwise estimate (2.25). Eventually, we arrive at the estimate for both cases $a_4^2 - 1 = 0$ and $a_4^2 + a_5^2 - 1 = 0$. Moreover, by using the similar argument, we can derive the same estimates in the case $a_4^2 - 1 \ne 0$, $a_4^2 + a_5^2 - 1 \ne 0$. Thus we complete the proof of Theorem 2.1 with $m = 6$.

2.3. **Energy method for model I.** Inspired by the concrete computation in Subsection 2.2, we consider the more general case $m \geq 6$. Now, our system (1.1) with (2.1) is described as

$$
\begin{aligned}
&\partial_t \hat{u}_1 + i\xi \hat{u}_2 + \hat{u}_4 = 0, \\
&\partial_t \hat{u}_2 + i\xi \hat{u}_1 = 0, \\
&\partial_t \hat{u}_3 + i\xi a_4 \hat{u}_4 = 0, \\
&\partial_t \hat{u}_4 + i\xi (a_4 \hat{u}_3 + a_5 \hat{u}_5) - \hat{u}_1 = 0, \\
&\partial_t \hat{u}_j + i\xi (a_j \hat{u}_{j-1} + a_{j+1} \hat{u}_{j+1}) = 0, \qquad j = 5, \cdots, m-1, \\
&\partial_t \hat{u}_m + i\xi a_m \hat{u}_{m-1} + \gamma \hat{u}_m = 0.
\end{aligned}
$$
(2.27)

We are going to apply the energy method to this system and derive Theorem 2.1. The proof is organized by the following three steps.

Step 1. We first derive the basic energy equality for the system (1.1) in the Fourier space. Taking the inner product of (1.1) with $\hat{u}$, we have

$$
\langle \hat{u}_t, \hat{u} \rangle + i\xi \langle A_m \hat{u}, \hat{u} \rangle + \langle L_m \hat{u}, \hat{u} \rangle = 0.
$$

Taking the real part, we get the basic energy equality

$$
\frac{1}{2} \frac{\partial}{\partial t} |\hat{u}|^2 + \langle L_m \hat{u}, \hat{u} \rangle = 0,
$$

and hence

$$
\frac{1}{2} \partial_t |\hat{u}|^2 + \gamma \hat{u}_m^2 = 0.
$$
(2.28)

Next we create the dissipation terms by the following two steps.

Step 2. For $\ell = 6, \cdots, m-1$, we multiply the fifth equations with $j = \ell - 1$ and $j = \ell$ in (2.27) by $i\xi a_\ell \bar{\hat{u}}_\ell$ and $-i\xi a_\ell \bar{\hat{u}}_{\ell-1}$, respectively. Then, combining the resultant equations and taking the real part, we have

$$
\begin{aligned}
(2.29) \quad a_\ell \xi \partial_t \Re(i\hat{u}_{\ell-1} \bar{\hat{u}}_\ell) + a_\ell^2 \xi^2 (|\hat{u}_{\ell-1}|^2 - |\hat{u}_\ell|^2) \\
- a_\ell a_{\ell-1} \xi^2 \Re(\hat{u}_{\ell-2} \bar{\hat{u}}_\ell) + a_\ell a_{\ell+1} \xi^2 \Re(\hat{u}_{\ell-1} \bar{\hat{u}}_{\ell+1}) = 0.
\end{aligned}
$$

Here, by using Young inequality, we obtain

$$
(2.30) \quad \xi \partial_t E_{\ell-1} + \frac{1}{2} a_\ell^2 \xi^2 |\hat{u}_{\ell-1}|^2 \leq a_\ell^2 \xi^2 |\hat{u}_\ell|^2 + \frac{1}{2} a_{\ell+1}^2 \xi^2 |\hat{u}_{\ell+1}|^2 + a_\ell a_{\ell-1} \xi^2 \Re(\hat{u}_{\ell-2} \bar{\hat{u}}_\ell)
$$

for $\ell = 6, \cdots, m-1$, where we have defined $E_{\ell-1} = a_\ell \xi \Re(i\hat{u}_{\ell-1} \bar{\hat{u}}_\ell)$. On the other hand, we multiply the fifth equation with $j = m-1$ and the last equation in (2.27) by $i\xi a_m \bar{\hat{u}}_m$ and $-i\xi a_m \bar{\hat{u}}_{m-1}$, respectively. Then, combining the resultant equations and taking the real part, we obtain

$$
\begin{aligned}
(2.31) \quad a_m \xi \partial_t \Re(i\hat{u}_{m-1} \bar{\hat{u}}_m) + a_m^2 \xi^2 (|\hat{u}_{m-1}|^2 - |\hat{u}_m|^2) \\
- a_m a_{m-1} \xi^2 \Re(\hat{u}_{m-2} \bar{\hat{u}}_m) + \gamma a_m \xi \Re(i\hat{u}_{m-1} \bar{\hat{u}}_m) = 0.
\end{aligned}
$$

Using Young inequality, this yields

$$
\begin{aligned}
(2.32) \quad \xi \partial_t E_{m-1} + \frac{1}{2} a_m^2 \xi^2 |\hat{u}_{m-1}|^2 \\
\leq a_m^2 \xi^2 |\hat{u}_m|^2 + \frac{1}{2} \gamma^2 |\hat{u}_m|^2 + a_m a_{m-1} \xi^2 \Re(\hat{u}_{m-2} \bar{\hat{u}}_m),
\end{aligned}
$$

where we have defined $E_{m-1} = a_m \xi \Re(i\hat{u}_{m-1} \bar{\hat{u}}_m)$.

Step 3. We note that equations (2.27) with $1 \leq j \leq 5$ are the same as the five equations in (2.4). Thus we can adopt the useful estimates derived in Subsection 2.2. More precisely, we employ (2.13), (2.15) and (2.18) again.

For the estimate (2.18), if we assume that $a_4^2 - 1 = 0$, we can obtain (2.19). Then, multiplying (2.30) with $\ell = 6$ and (2.19) by $(1 + \xi^2)^3$ and $\beta_3 \xi^2$, respectively, and combining the resultant equation, we have

$$\partial_t \big\{ \beta_3 \xi^2 (\beta_2 \xi^2 ((1 + \xi^2) E_1 + \beta_1 \xi E_2 + \xi E_3 + F_1) + \xi(1 + \xi^2) E_4) + \xi(1 + \xi^2)^3 E_5 \big\}$$
$$+ \beta_3 c \xi^4 (1 + \xi^2)(|\hat{u}_1|^2 + |\hat{u}_4|^2) + \beta_3 c \xi^6 (|\hat{u}_2|^2 + |\hat{u}_3|^2)$$
$$+ \Big(\frac{1}{2} a_6^2 - \beta_3 C \Big) \xi^2 (1 + \xi^2)^3 |\hat{u}_5|^2 \leq \beta_3 C \xi^4 (1 + \xi^2) |\hat{u}_6|^2 + a_6^2 \xi^2 (1 + \xi^2)^3 |\hat{u}_6|^2$$
$$+ \frac{1}{2} a_7^2 \xi^2 (1 + \xi^2)^3 |\hat{u}_7|^2 + a_5 a_6 \xi^2 (1 + \xi^2)^3 \, \Re(\hat{u}_4 \bar{\hat{u}}_6).$$

Hence we arrive at

$$\partial_t \big\{ \beta_3 \xi^2 (\beta_2 \xi^2 ((1 + \xi^2) E_1 + \beta_1 \xi E_2 + \xi E_3 + F_1)$$
$$+ \xi(1 + \xi^2) E_4) + \xi(1 + \xi^2)^3 E_5 \big\}$$
$$+ c \xi^4 (1 + \xi^2)(|\hat{u}_1|^2 + |\hat{u}_4|^2) + c \xi^6 (|\hat{u}_2|^2 + |\hat{u}_3|^2) + c \xi^2 (1 + \xi^2)^3 |\hat{u}_5|^2$$
$$\leq C \xi^2 (1 + \xi^2)^3 (|\hat{u}_6|^2 + |\hat{u}_7|^2) + C \xi^2 (1 + \xi^2)^3 |\hat{u}_4| |\hat{u}_6|.$$

Moreover, we multiply (2.13) and (2.15) by $\beta_4 \xi^6$ and $\beta_5 \xi^6$, respectively, and combining the resultant equations and the above inequality. Then, letting β_4 and β_5 suitably small, this yields

$$(2.33) \quad \partial_t E + c \xi^4 (1 + \xi^2) |\hat{u}_1|^2 + c \xi^6 |\hat{u}_2|^2 + c \xi^6 (1 + \xi^2) |\hat{u}_3|^2 + c \xi^4 (1 + \xi^2)^2 |\hat{u}_4|^2$$
$$+ c \xi^2 (1 + \xi^2)^3 |\hat{u}_5|^2 \leq C (1 + \xi^2)^4 |\hat{u}_6|^2 + C \xi^2 (1 + \xi^2)^3 |\hat{u}_7|^2.$$

where E is defined in (2.21).

On the other hand, if we assume that $a_4^2 + a_5^2 - 1 = 0$, we employ (2.26). Then, multiplying (2.30) with $\ell = 6$ and (2.26) by $(1 + \xi^2)^2$ and $\beta_3 \xi^2$, respectively, and combining the resultant equation, we have

$$\partial_t \big\{ \beta_3 \xi^2 (\beta_2 \xi^2 ((1 + \xi^2) E_1 + \beta_1 \xi E_2 + \xi E_3 + F_1) + \xi(1 + \xi^2) E_4) + \xi(1 + \xi^2)^2 E_5 \big\}$$
$$+ \beta_3 c \xi^4 (1 + \xi^2)(|\hat{u}_1|^2 + |\hat{u}_4|^2) + \beta_3 c \xi^6 (|\hat{u}_2|^2 + |\hat{u}_3|^2)$$
$$+ \Big(\frac{1}{2} a_6^2 - \beta_3 C \Big) \xi^2 (1 + \xi^2)^2 |\hat{u}_5|^2 \leq \beta_3 C (1 + \xi^2)^4 |\hat{u}_6|^2 + a_6^2 \xi^2 (1 + \xi^2)^2 |\hat{u}_6|^2$$
$$+ \frac{1}{2} a_7^2 \xi^2 (1 + \xi^2)^2 |\hat{u}_7|^2 + a_5 a_6 \xi^2 (1 + \xi^2)^2 \, \Re(\hat{u}_4 \bar{\hat{u}}_6).$$

Hence we arrive at

$$\partial_t \big\{ \beta_3 \xi^2 (\beta_2 \xi^2 ((1 + \xi^2) E_1 + \beta_1 \xi E_2 + \xi E_3 + F_1)$$
$$+ \xi(1 + \xi^2) E_4) + \xi(1 + \xi^2)^2 E_5 \big\}$$
$$+ c \xi^4 (1 + \xi^2)(|\hat{u}_1|^2 + |\hat{u}_4|^2) + c \xi^6 (|\hat{u}_2|^2 + |\hat{u}_3|^2) + c \xi^2 (1 + \xi^2)^2 |\hat{u}_5|^2$$
$$\leq C (1 + \xi^2)^4 |\hat{u}_6|^2 + C \xi^2 (1 + \xi^2)^2 |\hat{u}_7|^2.$$

Moreover, we multiply (2.13), (2.15) and (2.30) with $\ell = 6$ by $\beta_4 \xi^6$, $\beta_5 \xi^6$ and $\beta_6 \xi^6$, respectively, and combine the resultant equations and the above inequality. Then,

letting β_4, β_5 and β_6 suitably small, this yields

$$\partial_t\big\{\beta_2\beta_3\xi^4(1+\xi^2)E_1 + \beta_1\beta_2\beta_3\xi^5 E_2 + \xi^4(\beta_2\beta_3 + \beta_4\xi^2)(\xi E_3 + F_1)$$
$$+ \xi^3(\beta_3(1+\xi^2) + \beta_5\xi^4)E_4 + \xi((1+\xi^2)^2 + \beta_6\xi^6)E_5\big\}$$
$$+ c\xi^4(1+\xi^2)|\hat{u}_1|^2 + c\xi^6|\hat{u}_2|^2 + c\xi^6(1+\xi^2)|\hat{u}_3|^2$$
$$+ c\xi^4(1+\xi^2)^2|\hat{u}_4|^2 + c\xi^2(1+\xi^2)^3|\hat{u}_5|^2 \leq C(1+\xi^2)^4|\hat{u}_6|^2 + C\xi^2(1+\xi^2)^3|\hat{u}_7|^2.$$

Consequently, this estimate is essentially the same as (2.33). Moreover, by using the similar argument, we can derive the same estimate in the case $a_4^2 - 1 \neq 1$ and $a_4^2 + a_5^2 - 1 \neq 0$.

By using the estimate (2.33), we construct the desired estimate. We multiply (2.30) with $\ell = 7$ and (2.33) by $(1+\xi^2)^4$ and $\beta_7\xi^2$, respectively, and combine the resultant equation. Moreover, letting β_7 suitably small and using Young inequality, we obtain

$$\partial_t\big\{\beta_7\xi^2 E + \xi(1+\xi^2)^4 E_6\big\} + c\xi^6(1+\xi^2)|\hat{u}_1|^2 + c\xi^8|\hat{u}_2|^2 + c\xi^8(1+\xi^2)|\hat{u}_3|^2$$
$$+ c\xi^6(1+\xi^2)^2|\hat{u}_4|^2 + c\xi^4(1+\xi^2)^3|\hat{u}_5|^2 + c\xi^2(1+\xi^2)^4|\hat{u}_6|^2$$
$$\leq C(1+\xi^2)^5|\hat{u}_7|^2 + C\xi^2(1+\xi^2)^4|\hat{u}_8|^2.$$

Eventually, by the induction argument with respect to j in (2.30), we can derive

$$(2.34) \quad \partial_t\mathcal{E}_{m-2} + c\xi^{2(m-5)}(1+\xi^2)|\hat{u}_1|^2 + c\xi^{2(m-4)}|\hat{u}_2|^2$$
$$+ c\sum_{j=3}^{m-2}\xi^{2(m-j-1)}(1+\xi^2)^{j-2}|\hat{u}_j|^2$$
$$\leq C(1+\xi^2)^{m-3}|\hat{u}_{m-1}|^2 + C\xi^2(1+\xi^2)^{m-4}|\hat{u}_m|^2.$$

for $m \geq 7$. Here we define $\mathcal{E}_{m-2}$ as $\mathcal{E}_5 = E$ and

$$\mathcal{E}_{m-2} = \beta_{m-1}\xi^2\mathcal{E}_{m-3} + \xi(1+\xi^2)^{m-4}E_{m-2}, \qquad m \geq 8.$$

Now, multiplying (2.32) and (2.34) by $(1+\xi^2)^{m-3}$ and $\beta_m\xi^2$, respectively, and making the appropriate combination, we get

$$(2.35) \quad \partial_t\mathcal{E}_{m-1} + c\xi^{2(m-4)}(1+\xi^2)|\hat{u}_1|^2 + c\xi^{2(m-3)}|\hat{u}_2|^2$$
$$+ c\sum_{j=3}^{m-1}\xi^{2(m-j)}(1+\xi^2)^{j-2}|\hat{u}_j|^2 \leq C(1+\xi^2)^{m-2}|\hat{u}_m|^2.$$

Finally, combining (2.28) with (2.35), this yields

$$(2.36) \quad \partial_t\Big\{\frac{1}{2}(1+\xi^2)^{m-2}|\hat{u}|^2 + \beta_{m+1}\mathcal{E}_{m-1}\Big\} + c\xi^{2(m-4)}(1+\xi^2)|\hat{u}_1|^2$$
$$+ c\xi^{2(m-3)}|\hat{u}_2|^2 + c\sum_{j=3}^{m}\xi^{2(m-j)}(1+\xi^2)^{j-2}|\hat{u}_j|^2 \leq 0.$$

Thus, integrating the above estimate with respect to t, we obtain the following energy estimate

$$(2.37) \quad |\hat{u}(t,\xi)|^2 + \int_0^t \Big\{ \frac{\xi^{2(m-4)}}{(1+\xi^2)^{m-3}}|\hat{u}_1|^2 + \frac{\xi^{2(m-3)}}{(1+\xi^2)^{m-2}}|\hat{u}_2|^2$$

$$+ \sum_{j=3}^m \frac{\xi^{2(m-j)}}{(1+\xi^2)^{m-j}}|\hat{u}_j|^2 \Big\}d\tau \leq C|\hat{u}(0,\xi)|^2$$

for $m \geq 7$. Here we have used the following inequality

$$c|\hat{u}|^2 \leq \frac{1}{2}|\hat{u}|^2 + \frac{\beta_{m+1}}{(1+\xi^2)^{m-2}}\mathcal{E}_{m-1} \leq C|\hat{u}|^2$$

for suitably small β_{m+1}. Furthermore the estimate (2.35) with (2.36) gives us the following pointwise estimate

$$|\hat{u}(t,\xi)| \leq Ce^{-c\lambda(\xi)t}|\hat{u}(0,\xi)|, \qquad \lambda(\xi) = \frac{\xi^{2(m-3)}}{(1+\xi^2)^{m-2}}$$

for $m \geq 7$. Therefore, together with the proof in Subsection 2.2, (2.2) is proved, and we then complete the proof of Theorem 2.1.

2.4. **Construction of the matrices K and S.** In this section, inspired by the energy method employed in Sections 2.2 and 2.3, we shall derive the matrices K and S.

Based on the energy method of Step 2 in Subsection 2.2, we introduce the following $m \times m$ matrices:

$$S_1 = \begin{pmatrix} \begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} & O \\ \hline O & O \end{pmatrix}, \quad S_2 = \begin{pmatrix} \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} & O \\ \hline O & O \end{pmatrix}, \quad S_3 = \begin{pmatrix} \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array} & O \\ \hline O & O \end{pmatrix},$$

and hence

$$\tilde{S} = -a_5\big\{a_5(S_1 + a_4 S_2) - a_5(a_4^2 - 1)S_3\big\}$$

$$= -a_5 \begin{pmatrix} \begin{array}{ccccc} 0 & 0 & 0 & a_5 & 0 \\ 0 & 0 & a_4 a_5 & 0 & 1-a_4^2 \\ 0 & a_4 a_5 & 0 & 0 & 0 \\ a_5 & 0 & 0 & 0 & 0 \\ 0 & 1-a_4^2 & 0 & 0 & 0 \end{array} & O \\ \hline O & O \end{pmatrix}.$$

Then, we multiply (1.4) by $\tilde{S}$ and take the inner product with $\hat{u}$. Furthermore, taking the real part of the resultant equation, we obtain

$$(2.38) \qquad \frac{1}{2}\partial_t\langle \tilde{S}\hat{u}, \hat{u}\rangle + \xi\langle i[\tilde{S}A_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle + \langle [\tilde{S}L_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle = 0,$$

where

$$\tilde{S}A_m = -a_5 \begin{pmatrix} \begin{array}{cccc|cc} 0 & 0 & a_4a_5 & 0 & a_5^2 & 0 \\ 0 & 0 & 0 & a_5 & 0 & a_6(1-a_4^2) \\ a_4a_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_5 & 0 & 0 & 0 & 0 \\ 1-a_4^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} & O \\ \hline & O & & & & O \end{array} \end{pmatrix},$$

$$\tilde{S}L_m = a_5^2 \begin{pmatrix} \begin{array}{cccc|c} 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & O \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & -1 & \\ \hline & O & & & O \end{array} \end{pmatrix}.$$

The equality (2.38) is equivalent to (2.9). We note that the symmetric matrix $S_1 + a_4 S_2$ is the key matrix for 4×4 Timoshenko system (see [13, 14]). The symmetric matrix $\tilde{S}$ is the one of the key matrix for the system (1.4).

On the other hand, we introduce the following $m \times m$ matrix:

$$K_1 = \begin{pmatrix} \begin{array}{cccc|c} 0 & -1 & 0 & 0 & \\ 1 & 0 & 0 & 0 & O \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ \hline & O & & & O \end{array} \end{pmatrix}.$$

Then, we multiply (1.4) by $-i\xi K_1$ and take the inner product with $\hat{u}$. Moreover, taking the real part of the resultant equation, we have

$$(2.39) \qquad -\frac{1}{2}\xi\partial_t\langle iK_1\hat{u}, \hat{u}\rangle + \xi^2\langle[K_1A_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle - \xi\langle i[K_1L_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle = 0,$$

where

$$K_1A_m = \begin{pmatrix} \begin{array}{cccc|c} -1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & O \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ \hline & O & & & O \end{array} \end{pmatrix}, \qquad K_1L_m = \begin{pmatrix} \begin{array}{cccc|c} 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & O \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ \hline & O & & & O \end{array} \end{pmatrix}.$$

The equality (2.39) is equivalent to (2.10).

We next introduce the following $m \times m$ matrices:

$$K_4 = a_4 \begin{pmatrix} \begin{array}{cccc|c} 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & O \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & -1 & 0 & \\ \hline & O & & & O \end{array} \end{pmatrix}, \qquad S_4 = -a_4 \begin{pmatrix} \begin{array}{cccc|c} 0 & 0 & 0 & 0 & \\ 0 & 0 & 1 & 0 & O \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ \hline & O & & & O \end{array} \end{pmatrix}.$$

Then, we multiply (1.4) by $-i\xi K_2$ and S_4, and take the inner product with $\hat{u}$, respectively. Moreover, taking the real part of the resultant equations and combining these, then we have

$$(2.40) \quad \frac{1}{2}\partial_t\langle(S_4 - i\xi K_4)\hat{u}, \hat{u}\rangle + \xi^2\langle[K_4 A_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle + \langle[S_4 L_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle$$
$$+ \xi\langle i[S_4 A_m - K_4 L_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle = 0,$$

where $S_4 L_m = O$ and

$$K_4 A_m = \begin{pmatrix} \begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_4^2 & 0 & a_4 a_5 \\ 0 & 0 & 0 & -a_4^2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array} & O \\ \hline O & & O \end{pmatrix}, \quad S_4 A_m - K_4 L_m = \begin{pmatrix} \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_4^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} & O \\ \hline O & O \end{pmatrix}.$$

The equality (2.40) is equivalent to (2.12).

Similarly we introduce the following $m \times m$ matrix:

$$K_5 = a_5 \begin{pmatrix} \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & -1 & 0 \end{array} & O \\ \hline O & O \end{pmatrix}.$$

Then, we multiply (1.4) by $-i\xi K_5$ and take the inner product with $\hat{u}$. Furthermore, taking the real part of the resultant equation, we obtain

$$(2.41) \qquad -\frac{1}{2}\xi\partial_t\langle iK_5\hat{u}, \hat{u}\rangle + \xi^2\langle[K_5 A_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle - \xi\langle i[K_5 L_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle = 0,$$

where

$$K_5 A_m = \begin{pmatrix} \begin{array}{cccc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_5^2 & 0 & a_5 a_6 \\ \hline 0 & 0 & -a_4 a_5 & 0 & -a_5^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} & O \\ \hline O & & O \end{pmatrix}, \quad K_5 L_m = \begin{pmatrix} \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline a_5 & 0 & 0 & 0 & 0 \end{array} & O \\ \hline O & O \end{pmatrix}.$$

The equality (2.41) is equivalent to (2.14).

Based on the energy method of Step 2 in Subsection 2.3, we introduce the following $m \times m$ matrices:

$$
K_\ell = a_\ell
\begin{pmatrix}
 & & & 0 & 0 & & & & \\
 & O & & \vdots & \vdots & & O & & \\
 & & & 0 & 0 & & & & \\
0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\
 & & & 0 & 0 & & & & \\
 & O & & \vdots & \vdots & & O & & \\
 & & & 0 & 0 & & & & \\
 & & & \ell-1 & \ell & & & &
\end{pmatrix}
\begin{matrix} \\ \\ \\ \ell-1 \\ \ell \\ \\ \\ \\ \end{matrix}
$$

for $\ell = 6, \cdots, m-1$. Then, we multiply (1.4) by $-iK_\ell$ and take the inner product with $\hat{u}$. Furthermore, taking the real part of the resultant equation, we obtain

(2.42) $$-\frac{1}{2}\xi\partial_t\langle iK_\ell\hat{u}, \hat{u}\rangle + \xi^2\langle [K_\ell A_m]^{\text{sy}}\hat{u}, \hat{u}\rangle = 0$$

for $\ell = 6, \cdots, m-1$, where

$$
K_\ell A_m =
\begin{pmatrix}
 & & & 0 & 0 & 0 & 0 & & & \\
 & O & & \vdots & \vdots & \vdots & \vdots & & O & \\
 & & & 0 & 0 & 0 & 0 & & & \\
0 & \cdots & 0 & 0 & a_\ell^2 & 0 & a_\ell a_{\ell+1} & 0 & \cdots & 0 \\
0 & \cdots & 0 & -a_{\ell-1}a_\ell & 0 & -a_\ell^2 & 0 & 0 & \cdots & 0 \\
 & & & 0 & 0 & 0 & 0 & & & \\
 & O & & \vdots & \vdots & \vdots & \vdots & & O & \\
 & & & 0 & 0 & 0 & 0 & & & \\
 & & & \ell-2 & \ell-1 & \ell & \ell+1 & & &
\end{pmatrix}
\begin{matrix} \\ \\ \\ \ell-1 \\ \ell \\ \\ \\ \\ \end{matrix}
$$

Moreover we have

(2.43) $$-\frac{1}{2}\xi\partial_t\langle iK_m\hat{u}, \hat{u}\rangle + \xi^2\langle [K_m A_m]^{\text{sy}}\hat{u}, \hat{u}\rangle - \xi\langle i[K_m L_m]^{\text{asy}}\hat{u}, \hat{u}\rangle = 0,$$

where

$$
K_m A_m =
\begin{pmatrix}
 & & 0 & 0 & 0 \\
 & O & \vdots & \vdots & \vdots \\
 & & 0 & 0 & 0 \\
0 & \cdots\ 0 & 0 & a_m^2 & 0 \\
0 & \cdots\ 0 & -a_{m-1}a_m & 0 & -a_m^2
\end{pmatrix}
, \quad
K_m L_m =
\begin{pmatrix}
 & & 0 \\
 & O & \vdots \\
 & & 0 \\
0 & \cdots\ 0 & a_m\gamma \\
0 & \cdots\ 0 & 0
\end{pmatrix}.
$$

The equalities (2.42) and (2.43) are equivalent to (2.29) and (2.31), respectively.

For the rest of this subsection, we construct the desired matrices. According to the strategy of Step 3 in Subsection 2.2, we first combine (2.38) and (2.39). More precisely, multiplying (2.38), (2.40) and (2.39) by $(1+\xi^2)$, $(1+\xi^2)$ and δ_1,

respectively, and combining the resultant equations, we obtain

$$\frac{1}{2}\partial_t\big\langle\big\{(1+\xi^2)\mathcal{S} - i\xi(\delta_1 K_1 + (1+\xi^2)K_4)\big\}\hat{u}, \hat{u}\big\rangle$$
$$+ (1+\xi^2)\langle[\mathcal{S}L_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle + \xi^2\langle[(\delta_1 K_1 + (1+\xi^2)K_4)A_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle$$
$$+ \xi(1+\xi^2)\langle i[\mathcal{S}A_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle - \xi\langle i[(\delta_1 K_1 + (1+\xi^2)K_4)L_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle = 0.$$

Here we define $\mathcal{S} = \tilde{S} + S_4$. We next multiply (2.41) with $\ell = 6$ and the above equation by $(1+\xi^2)^2$ and $\delta_2\xi^2$, respectively, and combining the resultant equations, we obtain

$$\frac{1}{2}\partial_t\big\langle\big\{\delta_2\xi^2((1+\xi^2)\mathcal{S} - i\xi(\delta_1 K_1 + (1+\xi^2)K_4)) - \xi(1+\xi^2)^2 K_5\big\}\hat{u}, \hat{u}\big\rangle$$
$$+ \delta_2\xi^2(1+\xi^2)\langle[\mathcal{S}L_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle + \delta_2\xi^3(1+\xi^2)\langle i[\mathcal{S}A_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle$$
$$+ \xi^2\langle[(\delta_2\xi^2(\delta_1 K_1 + (1+\xi^2)K_4) + (1+\xi^2)^2 K_5)A_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle$$
$$- \xi\langle i[(\delta_2\xi^2(\delta_1 K_1 + (1+\xi^2)K_4) + (1+\xi^2)^2 K_5)L_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle = 0.$$

Moreover, multiplying (2.42) and the above equation by $(1+\xi^2)^3$ and $\delta_3\xi^2$, respectively, and combining the resultant equations, we get

$$\frac{1}{2}\partial_t\big\langle\big\{\delta_3\xi^2(\delta_2\xi^2((1+\xi^2)\mathcal{S} - i\xi(\delta_1 K_1 + (1+\xi^2)K_4))$$
$$- i\xi(1+\xi^2)^2 K_5) - i\xi(1+\xi^2)^3 K_6\big\}\hat{u}, \hat{u}\big\rangle$$
$$+ \delta_2\delta_3\xi^4(1+\xi^2)\langle[\mathcal{S}L_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle + \delta_2\delta_3\xi^5(1+\xi^2)\langle i[\mathcal{S}A_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle$$
$$+ \xi^2\langle[(\delta_3\xi^2(\delta_2\xi^2(\delta_1 K_1 + (1+\xi^2)K_4) + (1+\xi^2)^2 K_5) + (1+\xi^2)^3 K_6)A_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle$$
$$- \delta_3\xi^3\langle i[(\delta_2\xi^2(\delta_1 K_1 + (1+\xi^2)K_4) + (1+\xi^2)^2 K_5)L_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle = 0.$$

Consequently, by the induction argument with respect to ℓ in (2.42), we have

$$(2.44)\quad \frac{1}{2}\partial_t\Big\langle\Big\{\prod_{j=2}^{\ell-3}\delta_j\xi^{2(\ell-4)}(1+\xi^2)\mathcal{S} - i\xi\mathcal{K}_\ell\Big\}\hat{u}, \hat{u}\Big\rangle$$
$$+ \prod_{j=2}^{\ell-3}\delta_j\xi^{2(\ell-4)}(1+\xi^2)\langle[\mathcal{S}L_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle + \prod_{j=2}^{\ell-3}\delta_j\xi^{2(\ell-4)+1}(1+\xi^2)\langle i[\mathcal{S}A_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle$$
$$+ \xi^2\langle[\mathcal{K}_\ell A_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle - \prod_{j=3}^{\ell-3}\delta_j\xi^{2(\ell-5)+1}\langle i[\mathcal{K}_5 L_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle = 0.$$

for $5 \leq \ell \leq m-1$, where the last term of left hand side is replaced by $\xi\langle i[\mathcal{K}_5 L_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle$ for $\ell = 5$. Here we define $\mathcal{K}_\ell$ as $\mathcal{K}_4 = \delta_1 K_1 + (1+\xi^2)K_4$ and

$$\mathcal{K}_\ell = \delta_{\ell-3}\xi^2\mathcal{K}_{\ell-1} + (1+\xi^2)^{\ell-3}K_\ell$$

for $\ell \geq 5$. Therefore, we make the combination of (2.43) and (2.44) with $\ell = m-1$. Then we can obtain

$$(2.45) \quad \frac{1}{2}\partial_t\Big\langle\Big\{\prod_{j=2}^{m-4}\delta_j\xi^{2(m-4)}(1+\xi^2)\mathcal{S} - i\xi\mathcal{K}_m\Big\}\hat{u},\hat{u}\Big\rangle + \xi^2\langle[\mathcal{K}_m A_m]^{\mathrm{sy}}\hat{u},\hat{u}\rangle$$

$$+\prod_{j=2}^{m-4}\delta_j\xi^{2(m-4)}(1+\xi^2)\langle[\mathcal{S}L_m]^{\mathrm{sy}}\hat{u},\hat{u}\rangle + \prod_{j=2}^{m-4}\delta_j\xi^{2(m-4)+1}(1+\xi^2)\langle i[\mathcal{S}A_m]^{\mathrm{asy}}\hat{u},\hat{u}\rangle$$

$$-\prod_{j=3}^{m-4}\delta_j\xi^{2(m-5)+1}\langle i[\mathcal{K}_5 L_m]^{\mathrm{asy}}\hat{u},\hat{u}\rangle - \xi(1+\xi^2)^{m-3}\langle i[K_m L_m]^{\mathrm{asy}}\hat{u},\hat{u}\rangle = 0.$$

Finally, multiplying (2.45) by $\delta_{m-3}/(1+\xi^2)^{m-2}$, and combining (2.28) and the resultant equations, we can obtain

$$(2.46) \quad \frac{1}{2}\partial_t\Big\langle\Big[I + \frac{\delta_{m-3}}{(1+\xi^2)^{m-2}}\Big\{\prod_{j=2}^{m-4}\delta_j\xi^{2(m-4)}(1+\xi^2)\mathcal{S} - i\xi\mathcal{K}_m\Big\}\Big]\hat{u},\hat{u}\Big\rangle$$

$$+ \langle L_m\hat{u},\hat{u}\rangle + \prod_{j=2}^{m-3}\delta_j\frac{\xi^{2(m-4)}}{(1+\xi^2)^{m-3}}\langle[\mathcal{S}L_m]^{\mathrm{sy}}\hat{u},\hat{u}\rangle$$

$$+ \delta_{m-3}\frac{\xi^2}{(1+\xi^2)^{m-2}}\langle[\mathcal{K}_m A_m]^{\mathrm{sy}}\hat{u},\hat{u}\rangle + \prod_{j=2}^{m-3}\delta_j\frac{\xi^{2(m-4)+1}}{(1+\xi^2)^{m-3}}\langle i[\mathcal{S}A_m]^{\mathrm{asy}}\hat{u},\hat{u}\rangle$$

$$- \prod_{j=3}^{m-3}\delta_j\frac{\xi^{2(m-5)+1}}{(1+\xi^2)^{m-2}}\langle i[\mathcal{K}_5 L_m]^{\mathrm{asy}}\hat{u},\hat{u}\rangle - \delta_{m-3}\frac{\xi}{1+\xi^2}\langle i[K_m L_m]^{\mathrm{asy}}\hat{u},\hat{u}\rangle = 0.$$

where I denotes an identity matrix. Letting $\delta_1,\cdots,\delta_{m-3}$ suitably small, then (2.46) derives energy estimate (2.37). More precisely, noting that

$$\mathcal{K}_m = \prod_{j=2}^{m-3}\delta_j\xi^{2(m-4)}(\delta_1 K_1 + (1+\xi^2)K_4) + (1+\xi^2)^{m-3}K_m$$

$$+ \sum_{k=3}^{m-3}\prod_{j=k}^{m-3}\delta_j\xi^{2(m-k-2)}(1+\xi^2)^{k-1}K_{k+2}$$

for $m \geq 6$, we can estimate the dissipation terms as

$$(2.47) \quad \langle L_m\hat{u},\hat{u}\rangle + \prod_{j=2}^{m-3}\delta_j\frac{\xi^{2(m-4)}}{(1+\xi^2)^{m-3}}\langle[\mathcal{S}L_m]^{\mathrm{sy}}\hat{u},\hat{u}\rangle$$

$$+ \delta_{m-3}\frac{\xi^2}{(1+\xi^2)^{m-2}}\langle[\mathcal{K}_m A_m]^{\mathrm{sy}}\hat{u},\hat{u}\rangle$$

$$\geq c\Big\{\frac{\xi^{2(m-4)}}{(1+\xi^2)^{m-3}}|\hat{u}_1|^2 + \frac{\xi^{2(m-3)}}{(1+\xi^2)^{m-2}}|\hat{u}_2|^2 + \sum_{j=3}^{m}\frac{\xi^{2(m-j)}}{(1+\xi^2)^{m-j}}|\hat{u}_j|^2\Big\}$$

for suitably small $\delta_1,\cdots,\delta_{m-3}$. Consequently we conclude that our desired symmetric matrix S and skew-symmetric matrix K are described as

$$S = \frac{\xi^{2(m-4)}}{(1+\xi^2)^{m-3}}\mathcal{S}, \qquad K = \frac{\xi^2}{(1+\xi^2)^{m-2}}\mathcal{K}_m.$$

3. Model II

3.1. **Main result II.** In this section, we treat the Cauchy problem (1.1), (1.2) with

(3.1)

$$
A_m = \begin{pmatrix}
0 & 1 & 0 & 0 & & & & & & \\
1 & 0 & 0 & 0 & & & & & & \\
0 & 0 & 0 & a_4 & 0 & & & & & \\
0 & 0 & a_4 & 0 & 0 & 0 & & O & & \\
& & 0 & 0 & 0 & a_6 & & & & \\
& & 0 & a_6 & 0 & & & & & \\
& & & & & \ddots & & & & \\
& & & & & & 0 & a_{m-2} & & \\
& O & & & & a_{m-2} & 0 & 0 & 0 & \\
& & & & & & 0 & 0 & 0 & a_m \\
& & & & & & & 0 & a_m & 0
\end{pmatrix},
$$

$$
L_m = \begin{pmatrix}
0 & 0 & 0 & 0 & & & & & & \\
0 & \gamma & 1 & 0 & & & & & & \\
0 & -1 & 0 & 0 & 0 & & & O & & \\
0 & 0 & 0 & 0 & a_5 & & & & & \\
& & 0 & -a_5 & 0 & & & & & \\
& & & & \ddots & & & & & \\
& & & & & 0 & a_{m-3} & 0 & & \\
& & & & & -a_{m-3} & 0 & 0 & 0 & \\
& & & & & 0 & 0 & 0 & a_{m-1} & 0 \\
& O & & & & 0 & -a_{m-1} & 0 & 0 \\
& & & & & & 0 & 0 & 0
\end{pmatrix},
$$

where integer $m \geq 4$ is even, $\gamma > 0$, and all elements a_j $(4 \leq j \leq m)$ are nonzero. We note that the system (1.1) with (3.1) for $m = 4$ is the Timoshenko system (cf. [13, 14]). For this problem, we can derive the following decay structure.

Theorem 3.1. *The Fourier image $\hat{u}$ of the solution u to the Cauchy problem (1.1)-(1.2) with (3.1) satisfies the pointwise estimate:*

(3.2) $$|\hat{u}(t,\xi)| \leq Ce^{-c\lambda(\xi)t}|\hat{u}_0(\xi)|,$$

where $\lambda(\xi) := \xi^{3m-10}/(1 + \xi^2)^{2(m-3)}$. Furthermore, let $s \geq 0$ be an integer and suppose that the initial data u_0 belong to $H^s \cap L^1$. Then the solution u satisfies the decay estimate:

$$\|\partial_x^k u(t)\|_{L^2} \leq C(1 + t)^{-\frac{1}{3m-10}\left(\frac{1}{2}+k\right)}\|u_0\|_{L^1} + C(1 + t)^{-\frac{\ell}{m-2}}\|\partial_x^{k+\ell} u_0\|_{L^2}$$

for $k + \ell \leq s$. Here C and c are positive constants.

3.2. **Energy method in the case** $m = 6$. Ide-Hramoto-Kawashima [13] and Ide-Kawashima [14] had already obtained the desired estimates in the case $m = 4$. Thus we consider the case $m = 6$ in this subsection, which can shed light on the proof of the general case $m \geq 6$ to be given by Section 3.3. Then we rewrite the

system (1.1) with (3.1) as follows.

$$
\begin{aligned}
&\partial_t \hat{u}_1 + i\xi \hat{u}_2 = 0,\\
&\partial_t \hat{u}_2 + i\xi \hat{u}_1 + \gamma \hat{u}_2 + \hat{u}_3 = 0,\\
&\partial_t \hat{u}_3 + i\xi a_4 \hat{u}_4 - \hat{u}_2 = 0,\\
&\partial_t \hat{u}_4 + i\xi a_4 \hat{u}_3 + a_5 \hat{u}_5 = 0,\\
&\partial_t \hat{u}_5 + i\xi a_6 \hat{u}_6 - a_5 \hat{u}_4 = 0,\\
&\partial_t \hat{u}_6 + i\xi a_6 \hat{u}_5 = 0.
\end{aligned}
$$

(3.3)

Step 1. We first derive the basic energy equality for the system (3.3) in the Fourier space. We multiply the all equations of (3.3) by $\bar{\hat{u}} = (\bar{\hat{u}}_1, \bar{\hat{u}}_2, \bar{\hat{u}}_3, \bar{\hat{u}}_4, \bar{\hat{u}}_5, \bar{\hat{u}}_6)^T$, respectively, and combine the resultant equations. Furthermore, taking the real part for the resultant equality, we arrive at the basic energy equality

$$(3.4) \qquad \frac{1}{2}\partial_t |\hat{u}|^2 + \gamma |\hat{u}_2|^2 = 0.$$

Next we create the dissipation terms by the following two steps.

Step 2. We multiply the first and second equations in (3.3) by $i\xi \bar{\hat{u}}_2$ and $-i\xi \bar{\hat{u}}_1$, respectively. Then, combining the resultant equations and taking the real part, we have

$$(3.5) \qquad \xi \partial_t \Re(i\hat{u}_1 \bar{\hat{u}}_2) + \xi^2 (|\hat{u}_1|^2 - |\hat{u}_2|^2) + \gamma \xi \Re(i\hat{u}_1 \bar{\hat{u}}_2) + \xi \Re(i\hat{u}_1 \bar{\hat{u}}_3) = 0.$$

Next, we combine the fourth and sixth equations in (3.3), obtaining

$$\partial_t (\xi a_6 \hat{u}_4 + i a_5 \hat{u}_6) + i\xi^2 a_4 a_6 \hat{u}_3 = 0.$$

Then multiplying the first equation in (3.3) and the resultant equation by $\xi a_6 \bar{\hat{u}}_4 - i a_5 \bar{\hat{u}}_6$ and $\bar{\hat{u}}_1$, and combining the resultant equations and taking the real part, we obtain

$$(3.6) \quad \partial_t \big\{ a_6 \xi \Re(\hat{u}_1 \bar{\hat{u}}_4) - a_5 \Re(i\hat{u}_1 \bar{\hat{u}}_6) \big\} \\ - a_4 a_6 \xi^2 \Re(i\hat{u}_1 \bar{\hat{u}}_3) + a_6 \xi^2 \Re(i\hat{u}_2 \bar{\hat{u}}_4) + a_5 \xi \Re(\hat{u}_2 \bar{\hat{u}}_6) = 0.$$

To eliminate $\Re(i\hat{u}_1 \bar{\hat{u}}_3)$, we multiply (3.5) and (3.6) by $a_4^2 a_6^2 \xi^2$ and $a_4 a_6 \xi$, add the resultant equations. Then this yields

$$(3.7) \quad a_4 a_6 \xi \partial_t E_1^{(6)} + a_4^2 a_6^2 \xi^4 (|\hat{u}_1|^2 - |\hat{u}_2|^2) \\ + a_4 a_6^2 \xi^3 \Re(i\hat{u}_2 \bar{\hat{u}}_4) + a_4 a_5 a_6 \xi^2 \Re(\hat{u}_2 \bar{\hat{u}}_6) + \gamma a_4^2 a_6^2 \xi^3 \Re(i\hat{u}_1 \bar{\hat{u}}_2) = 0,$$

where $E_1^{(6)} = a_6 \xi \Re(\hat{u}_1 \bar{\hat{u}}_4) - a_5 \Re(i\hat{u}_1 \bar{\hat{u}}_6) + a_4 a_6 \xi^2 \Re(i\hat{u}_1 \bar{\hat{u}}_2)$.

On the other hand, we multiply the second and third equations in (3.3) by $\bar{\hat{u}}_3$ and $\bar{\hat{u}}_2$, respectively. Then, combining the resultant equations and taking the real part, we have

$$(3.8) \qquad \partial_t \Re(\hat{u}_2 \bar{\hat{u}}_3) + |\hat{u}_3|^2 - |\hat{u}_2|^2 + \xi \Re(i\hat{u}_1 \bar{\hat{u}}_3) - a_4 \xi \Re(i\hat{u}_2 \bar{\hat{u}}_4) + \gamma \Re(\hat{u}_2 \bar{\hat{u}}_3) = 0.$$

By the Young inequality, the equation (3.8) is estimated as

$$(3.9) \qquad \partial_t E_3 + \frac{1}{2}|\hat{u}_3|^2 \le \xi^2 |\hat{u}_1|^2 + (1 + \gamma^2)|\hat{u}_2|^2 + a_4 \xi \Re(i\hat{u}_2 \bar{\hat{u}}_4),$$

where $E_3 = \Re(\hat{u}_2 \bar{\hat{u}}_3)$.

Furthermore, we multiply the third equation and fourth equation of (3.3) by $-i\xi a_4\bar{\hat{u}}_4$ and $i\xi a_4\bar{\hat{u}}_3$, respectively. Then, combining the resultant equations and taking the real part, we have

$$(3.10) \quad -a_4\xi\partial_t\Re(i\hat{u}_3\bar{\hat{u}}_4) + a_4^2\xi^2(|\hat{u}_4|^2 - |\hat{u}_3|^2) + a_4\xi\,\Re(i\hat{u}_2\bar{\hat{u}}_4) - a_4a_5\xi\,\Re(i\hat{u}_3\bar{\hat{u}}_5) = 0.$$

By the Young inequality, the above equation is estimated as

$$(3.11) \qquad \xi\partial_t E_4 + \frac{1}{2}a_4^2\xi^2|\hat{u}_4|^2 \le \frac{1}{2}|\hat{u}_2|^2 + a_4^2\xi^2|\hat{u}_3|^2 + a_4a_5\xi\,\Re(i\hat{u}_3\bar{\hat{u}}_5),$$

where $E_4 = -a_4\Re(i\hat{u}_3\bar{\hat{u}}_4)$.

We multiply the fourth equation and fifth equation in (3.3) by $a_5\bar{\hat{u}}_5$ and $a_5\bar{\hat{u}}_4$, respectively. Then, combining the resultant equations and taking the real part, we have

$$a_5\partial_t\Re(\hat{u}_4\bar{\hat{u}}_5) + a_5^2(|\hat{u}_5|^2 - |\hat{u}_4|^2) + a_4a_5\xi\,\Re(i\hat{u}_3\bar{\hat{u}}_5) - a_5a_6\xi\,\Re(i\hat{u}_4\bar{\hat{u}}_6) = 0.$$

By using Young inequality, we obtain

$$(3.12) \qquad \partial_t E_5 + \frac{1}{2}a_5^2|\hat{u}_5|^2 \le a_5^2|\hat{u}_4|^2 + \frac{1}{2}a_5^2\xi^2|\hat{u}_3|^2 + a_5a_6\xi\,\Re(i\hat{u}_4\bar{\hat{u}}_6).$$

where $E_5 = a_5\partial_t\Re(\hat{u}_4\bar{\hat{u}}_5)$.

Moreover, we multiply the last equation and the fifth equation in (3.3) by $i\xi a_6\bar{\hat{u}}_5$ and $-i\xi a_6\bar{\hat{u}}_6$, respectively. Then, combining the resultant equations and taking the real part, we have

$$-a_6\xi\partial_t\Re(i\hat{u}_5\bar{\hat{u}}_6) + a_6^2\xi^2(|\hat{u}_6|^2 - |\hat{u}_5|^2) + a_5a_6\xi\,\Re(i\hat{u}_4\bar{\hat{u}}_6) = 0.$$

Using Young inequality, this yields

$$(3.13) \qquad\qquad \xi\partial_t E_6 + \frac{1}{2}a_6^2\xi^2|\hat{u}_6|^2 \le a_6^2\xi^2|\hat{u}_5|^2 + \frac{1}{2}a_5^2|\hat{u}_4|^2,$$

where $E_6 = -a_6\Re(i\hat{u}_5\bar{\hat{u}}_6)$.

Step 3. In this step, we sum up the energy inequalities and derive the desired energy inequality. For this purpose, we first multiply (3.12) and (3.13) by ξ^2 and β_1, respectively. Then we combine the resultant equation, obtaining

$$\partial_t\big\{\xi^2 E_5 + \beta_1\xi E_6\big\} + \frac{1}{2}\beta_1 a_6^2\xi^2|\hat{u}_6|^2 + \Big(\frac{1}{2}a_5^2 - \beta_1 a_6^2\Big)\xi^2|\hat{u}_5|^2$$
$$\le \Big(\frac{1}{2}\beta_1 + \xi^2\Big)a_5^2|\hat{u}_4|^2 + \frac{1}{2}a_5^2\xi^4|\hat{u}_3|^2 + |a_5||a_6||\xi|^3|\hat{u}_4||\hat{u}_6|.$$

Letting β_1 suitably small and using Young inequality, we get

$$\partial_t\big\{\xi^2 E_5 + \beta_1\xi E_6\big\} + c\xi^2(|\hat{u}_5|^2 + |\hat{u}_6|^2) \le C(1+\xi^2)^2|\hat{u}_4|^2 + \frac{1}{2}a_5^2\xi^4|\hat{u}_3|^2.$$

Moreover, combining the above estimate and (3.12), we get

$$(3.14) \quad \partial_t\big\{(1+\xi^2)E_5 + \beta_1\xi E_6\big\} + c(1+\xi^2)|\hat{u}_5|^2 + c\xi^2|\hat{u}_6|^2$$
$$\le C(1+\xi^2)^2|\hat{u}_4|^2 + \frac{1}{2}a_5^2\xi^2(1+\xi^2)|\hat{u}_3|^2.$$

Second, we multiply (3.11) and (3.14) by $(1 + \xi^2)^2$ and $\beta_2 \xi^2$, respectively, and combine the resultant equations. Then we obtain

$$\partial_t \big\{ \beta_2 \xi^2 ((1 + \xi^2) E_5 + \beta_1 \xi E_6) + \xi (1 + \xi^2)^2 E_4 \big\}$$
$$+ \beta_2 c \xi^2 (1 + \xi^2) |\hat{u}_5|^2 + \beta_2 c \xi^4 |\hat{u}_6|^2 + \Big(\frac{1}{2} a_4^2 - \beta_2 C \Big) \xi^2 (1 + \xi^2)^2 |\hat{u}_4|^2$$
$$\leq C (1 + \xi^2)^2 |\hat{u}_2|^2 + C \xi^2 (1 + \xi^2)^2 |\hat{u}_3|^2 + C \xi (1 + \xi^2)^2 \, \Re(i \hat{u}_3 \bar{\hat{u}}_5).$$

Letting β_2 suitably small and using Young inequality, we get

$$(3.15) \quad \partial_t \big\{ \beta_2 \xi^2 ((1 + \xi^2) E_5 + \beta_1 \xi E_6) + \xi (1 + \xi^2)^2 E_4 \big\} + c \xi^2 (1 + \xi^2) |\hat{u}_5|^2$$
$$+ c \xi^4 |\hat{u}_6|^2 + c \xi^2 (1 + \xi^2)^2 |\hat{u}_4|^2 \leq C (1 + \xi^2)^2 |\hat{u}_2|^2 + C (1 + \xi^2)^3 |\hat{u}_3|^2.$$

Third, we multiply (3.9) and (3.15) by $(1 + \xi^2)^3$ and β_3 and combine the resultant equations. Then we obtain

$$\partial_t \big\{ \beta_3 (\beta_2 \xi^2 ((1 + \xi^2) E_5 + \beta_1 \xi E_6) + \xi (1 + \xi^2)^2 E_4) + (1 + \xi^2)^3 E_3 \big\} + \beta_3 c \xi^4 |\hat{u}_6|^2$$
$$+ \beta_3 c \xi^2 (1 + \xi^2) |\hat{u}_5|^2 + \beta_3 c \xi^2 (1 + \xi^2)^2 |\hat{u}_4|^2 + \Big(\frac{1}{2} - \beta_3 C \Big) (1 + \xi^2)^3 |\hat{u}_3|^2$$
$$\leq C (1 + \xi^2)^3 |\hat{u}_2|^2 + \xi^2 (1 + \xi^2)^3 |\hat{u}_1|^2 + a_4 \xi (1 + \xi^2)^3 \, \Re(i \hat{u}_2 \bar{\hat{u}}_4).$$

Therefore, letting β_3 suitably small and using Young inequality, we get

$$(3.16) \quad \partial_t \big\{ \beta_3 (\beta_2 \xi^2 ((1 + \xi^2) E_5 + \beta_1 \xi E_6) + \xi (1 + \xi^2)^2 E_4) + (1 + \xi^2)^3 E_3 \big\}$$
$$+ c \xi^4 |\hat{u}_6|^2 + c \xi^2 (1 + \xi^2) |\hat{u}_5|^2 + c \xi^2 (1 + \xi^2)^2 |\hat{u}_4|^2 + c (1 + \xi^2)^3 |\hat{u}_3|^2$$
$$\leq C (1 + \xi^2)^4 |\hat{u}_2|^2 + \xi^2 (1 + \xi^2)^3 |\hat{u}_1|^2.$$

Fourth, we multiply (3.7) and (3.16) by $(1 + \xi^2)^3$ and $\beta_4 \xi^2$, respectively, and combine the resultant equalities. Moreover, letting β_4 suitably small and using Young inequality, then we obtain

$$(3.17) \quad \partial_t \tilde{E} + c \xi^6 |\hat{u}_6|^2 + c \xi^4 (1 + \xi^2) |\hat{u}_5|^2 + c \xi^4 (1 + \xi^2)^2 |\hat{u}_4|^2 + c \xi^2 (1 + \xi^2)^3 |\hat{u}_3|^2$$
$$+ c \xi^4 (1 + \xi^2)^3 |\hat{u}_1|^2 \leq C (1 + \xi^2)^5 |\hat{u}_2|^2 + a_4 a_5 a_6 \xi^2 (1 + \xi^2)^3 \, \Re(\hat{u}_2 \bar{\hat{u}}_6),$$

where we have defined

$$\tilde{E} = \beta_4 \xi^2 (\beta_3 (\beta_2 \xi^2 ((1 + \xi^2) E_5 + \beta_1 \xi E_6) + \xi (1 + \xi^2)^2 E_4) + (1 + \xi^2)^3 E_3)$$
$$+ a_4 a_6 \xi (1 + \xi^2)^3 E_1^{(6)}.$$

Moreover, to estimate $\Re(\hat{u}_2 \bar{\hat{u}}_6)$, we multiply (3.17) by ξ^2 and use Young inequality again. Then this yields

$$(3.18) \quad \xi^2 \partial_t \tilde{E} + c \xi^8 |\hat{u}_6|^2 + c \xi^6 (1 + \xi^2) |\hat{u}_5|^2 + c \xi^6 (1 + \xi^2)^2 |\hat{u}_4|^2$$
$$+ c \xi^4 (1 + \xi^2)^3 |\hat{u}_3|^2 + c \xi^6 (1 + \xi^2)^3 |\hat{u}_1|^2 \leq C (1 + \xi^2)^6 |\hat{u}_2|^2.$$

Finally, multiplying the basic energy (3.4) and (3.18) by $(1 + \xi^2)^6$ and β_5, respectively, combining the resultant equations and letting β_5 suitably small, then this yields

$$(3.19) \quad \partial_t \Big\{ \frac{1}{2} (1 + \xi^2)^6 |\hat{u}|^2 + \beta_5 \xi^2 \tilde{E} \Big\} + c \xi^6 (1 + \xi^2)^3 |\hat{u}_1|^2 + c (1 + \xi^2)^6 |\hat{u}_2|^2$$
$$+ c \xi^4 (1 + \xi^2)^3 |\hat{u}_3|^2 + c \xi^6 (1 + \xi^2)^2 |\hat{u}_4|^2 + c \xi^6 (1 + \xi^2) |\hat{u}_5|^2 + c \xi^8 |\hat{u}_6|^2 \leq 0.$$

Thus, integrating the above estimate with respect to t, we obtain the following energy estimate

$$|\hat{u}(t,\xi)|^2 + \int_0^t \Big\{ \frac{\xi^6}{(1+\xi^2)^3}|\hat{u}_1|^2 + |\hat{u}_2|^2 + \frac{\xi^4}{(1+\xi^2)^3}|\hat{u}_3|^2 + \frac{\xi^6}{(1+\xi^2)^4}|\hat{u}_4|^2$$
$$+ \frac{\xi^6}{(1+\xi^2)^5}|\hat{u}_5|^2 + \frac{\xi^8}{(1+\xi^2)^6}|\hat{u}_6|^2 \Big\} d\tau \le C|\hat{u}(0,\xi)|^2.$$

Here we have used the following inequality

$$(3.20) \qquad\qquad c|\hat{u}|^2 \le \frac{1}{2}|\hat{u}|^2 + \frac{\beta_5 \xi^2}{(1+\xi^2)^6}\tilde{E} \le C|\hat{u}|^2$$

for suitably small β_5. Furthermore the estimate (3.19) with (3.20) give us the following pointwise estimate

$$|\hat{u}(t,\xi)| \le Ce^{-c\lambda(\xi)t}|\hat{u}(0,\xi)|, \qquad \lambda(\xi) = \frac{\xi^8}{(1+\xi^2)^6}.$$

This therefore proves (3.2) in the case $m = 6$ for Theorem 3.1.

3.3. **Energy method for model II.** Inspired by the concrete calculation in Subsection 3.2, we consider the more general situation $m \ge 6$. Then we rewrite our system (1.4) with (3.1) as follows:

$$(3.21) \qquad \begin{aligned} &\partial_t \hat{u}_1 + i\xi \hat{u}_2 = 0, \\ &\partial_t \hat{u}_2 + i\xi \hat{u}_1 + \gamma \hat{u}_2 + \hat{u}_3 = 0, \\ &\partial_t \hat{u}_3 + i\xi a_4 \hat{u}_4 - \hat{u}_2 = 0, \\ &\partial_t \hat{u}_j + i\xi a_j \hat{u}_{j-1} + a_{j+1}\hat{u}_{j+1} = 0, \qquad j = 4, 6, \cdots, m-2, \text{ (for even)} \\ &\partial_t \hat{u}_j + i\xi a_{j+1}\hat{u}_{j+1} - a_j \hat{u}_{j-1} = 0, \qquad j = 5, 7, \cdots, m-1, \text{ (for odd)} \\ &\partial_t \hat{u}_m + i\xi a_m \hat{u}_{m-1} = 0. \end{aligned}$$

Step 1. We first derive the basic energy equality for the system (1.4) in the Fourier space. Taking the inner product of (1.4) with $\hat{u}$, we have

$$\langle \hat{u}_t, \hat{u} \rangle + i\xi \langle A_m \hat{u}, \hat{u} \rangle + \langle L_m \hat{u}, \hat{u} \rangle = 0.$$

Taking the real part, we get the basic energy equality

$$\frac{1}{2}\partial_t |\hat{u}|^2 + \langle L_m \hat{u}, \hat{u} \rangle = 0,$$

and hence

$$(3.22) \qquad\qquad \frac{1}{2}\partial_t |\hat{u}|^2 + \gamma |\hat{u}_2|^2 = 0.$$

Next we create the dissipation terms by the following three steps.

Step 2. We note that we had already derived some useful equations in Subsection 3.2. Indeed the equations (3.5), (3.9), (3.11) and (3.12) are valid for our general problem. Therefore we adopt these equations in this subsection.

To eliminate $\Re(i\hat{u}_1 \bar{\hat{u}}_3)$ in (3.5), we first prepare the useful equation. We combine the fourth equations with $j = 4, \cdots, 2\ell$ in (3.21) inductively. Then we obtain

$$(3.23) \qquad \partial_t \mathcal{U}_{2\ell} + i\xi(-i\xi)^{\ell-2} \prod_{j=2}^{\ell} a_{2j} \hat{u}_3 + \prod_{j=2}^{\ell} a_{2j+1}\hat{u}_{2\ell+1} = 0,$$

for $4 \leq 2\ell \leq m - 2$, where we have defined $\mathcal{U}_4 = \hat{u}_4$ and

$$\mathcal{U}_{2\ell} = -i\xi a_{2\ell}\mathcal{U}_{2\ell-2} + \prod_{j=2}^{\ell-1} a_{2j+1}\hat{u}_{2\ell}.$$

Moreover, combining the last equation in (3.21) and (3.23), this yields

$$(3.24) \qquad i^{m/2}\partial_t\mathcal{U}_m - i\xi^{m/2-1}\prod_{j=2}^{m/2} a_{2j}\hat{u}_3 = 0.$$

Multiplyig (3.24) by $-\bar{\hat{u}}_1$ and the first equation in (3.21) by $-\overline{i^{m/2}\mathcal{U}_m}$, combining the resultant equations and taking the real part, we obtain

$$(3.25) \qquad -\partial_t\Re(i^{m/2}\mathcal{U}_m\bar{\hat{u}}_1) - \prod_{j=2}^{m/2} a_{2j}\xi^{m/2-1}\Re(i\hat{u}_1\bar{\hat{u}}_3) + \xi\,\Re(i^{m/2+1}\mathcal{U}_m\bar{\hat{u}}_2) = 0.$$

In order to eliminate $\Re(i\hat{u}_1\bar{\hat{u}}_3)$, we multiply (3.5) by $\prod_{j=2}^{m/2} a_{2j}\xi^{m/2-2}$ and combine the resultant equation and (3.25). Then we obtain

$$(3.26) \quad \partial_t E_1^{(m)} + \prod_{j=2}^{m/2} a_{2j}\xi^{m/2}(|\hat{u}_1|^2 - |\hat{u}_2|^2)$$

$$+ \gamma\prod_{j=2}^{m/2} a_{2j}\xi^{m/2-1}\Re(i\hat{u}_1\bar{\hat{u}}_2) + \xi\,\Re(i^{m/2+1}\mathcal{U}_m\bar{\hat{u}}_2) = 0,$$

where we have defined

$$E_1^{(m)} = \prod_{j=2}^{m/2} a_{2j}\xi^{m/2-1}\Re(i\hat{u}_1\bar{\hat{u}}_2) - \Re(i^{m/2}\mathcal{U}_m\bar{\hat{u}}_1).$$

For $\ell = 4, 6, \cdots, m - 2$, we multiply the fourth equation and fifth equation with $j = \ell$ and $j = \ell+1$ in (3.21) by $a_{\ell+1}\bar{\hat{u}}_{\ell+1}$ and $a_{\ell+1}\bar{\hat{u}}_\ell$, respectively. Then, combining the resultant equations and taking the real part, we have

$$(3.27) \quad a_{\ell+1}\partial_t\Re(\hat{u}_\ell\bar{\hat{u}}_{\ell+1}) + a_{\ell+1}^2(|\hat{u}_{\ell+1}|^2 - |\hat{u}_\ell|^2)$$

$$+ a_\ell a_{\ell+1}\xi\,\Re(i\hat{u}_{\ell-1}\bar{\hat{u}}_{\ell+1}) - a_{\ell+1}a_{\ell+2}\xi\,\Re(i\hat{u}_\ell\bar{\hat{u}}_{\ell+2}) = 0.$$

By using Young inequality, we obtain

$$(3.28) \quad \partial_t E_{\ell+1} + \frac{1}{2}a_{\ell+1}^2|\hat{u}_{\ell+1}|^2 \leq a_{\ell+1}^2|\hat{u}_\ell|^2 + \frac{1}{2}a_{\ell+1}^2\xi^2|\hat{u}_{\ell-1}|^2 + a_{\ell+1}a_{\ell+2}\xi\,\Re(i\hat{u}_\ell\bar{\hat{u}}_{\ell+2}).$$

where $E_{\ell+1} = a_{\ell+1}\Re(\hat{u}_\ell\bar{\hat{u}}_{\ell+1})$.

On the other hand, for $\ell = 4, \cdots, m - 4$, we multiply the fourth and fifth equations with $j = \ell + 2$ and $j = \ell + 1$ in (3.21) by $i\xi a_{\ell+2}\bar{\hat{u}}_{\ell+1}$ and $-i\xi a_{\ell+2}\bar{\hat{u}}_{\ell+2}$, respectively. Then, combining the resultant equations and taking the real part, we have

$$(3.29) \quad \begin{aligned} & -a_{\ell+2}\xi\partial_t\Re(i\hat{u}_{\ell+1}\bar{\hat{u}}_{\ell+2}) + a_{\ell+2}^2\xi^2(|\hat{u}_{\ell+2}|^2 - |\hat{u}_{\ell+1}|^2) \\ & \quad + a_{\ell+1}a_{\ell+2}\xi\,\Re(i\hat{u}_\ell\bar{\hat{u}}_{\ell+2}) - a_{\ell+2}a_{\ell+3}\xi\,\Re(i\hat{u}_{\ell+1}\bar{\hat{u}}_{\ell+3}) = 0. \end{aligned}$$

Here, by using Young inequality, we obtain

$$(3.30) \quad \xi\partial_t E_{\ell+2} + \frac{1}{2}a_{\ell+2}^2\xi^2|\hat{u}_{\ell+2}|^2$$

$$\le a_{\ell+2}^2\xi^2|\hat{u}_{\ell+1}|^2 + \frac{1}{2}a_{\ell+1}^2|\hat{u}_\ell|^2 + a_{\ell+2}a_{\ell+3}\xi\,\Re(i\hat{u}_{\ell+1}\bar{\hat{u}}_{\ell+3}),$$

where $E_{\ell+2} = -a_{\ell+2}\Re(i\hat{u}_{\ell+1}\bar{\hat{u}}_{\ell+2})$.

Moreover, we multiply the last equation and the fifth equation with $j = m - 1$ in (3.21) by $i\xi a_m\bar{\hat{u}}_{m-1}$ and $-i\xi a_m\bar{\hat{u}}_m$, respectively. Then, combining the resultant equations and taking the real part, we have

$$(3.31) \quad -a_m\xi\partial_t\Re(i\hat{u}_{m-1}\bar{\hat{u}}_m) + a_m^2\xi^2(|\hat{u}_m|^2 - |\hat{u}_{m-1}|^2) + a_{m-1}a_m\xi\,\Re(i\hat{u}_{m-2}\bar{\hat{u}}_m) = 0.$$

Using Young inequality, this yields

$$(3.32) \quad \xi\partial_t E_m + \frac{1}{2}a_m^2\xi^2|\hat{u}_m|^2 \le a_m^2\xi^2|\hat{u}_{m-1}|^2 + \frac{1}{2}a_{m-1}^2|\hat{u}_{m-2}|^2.$$

where $E_m = -a_m\Re(i\hat{u}_{m-1}\bar{\hat{u}}_m)$.

Step 3. In this step, we sum up the energy inequalities constructed in the previous step and then make the desired energy inequality. The strategy is essentially the same as in Subsection 3.2.

For this purpose, we first multiply (3.28) with $\ell = m - 2$ and (3.32) by ξ^2 and β_1, respectively. Then we combine the resultant equation, obtaining

$$\partial_t\big\{\xi^2 E_{m-1} + \beta_1\xi E_m\big\} + \frac{1}{2}\beta_1 a_m^2\xi^2|\hat{u}_m|^2 + \Big(\frac{1}{2}a_{m-1}^2 - \beta_1 a_m^2\Big)\xi^2|\hat{u}_{m-1}|^2$$

$$\le \Big(\frac{1}{2}\beta_1 + \xi^2\Big)a_{m-1}^2|\hat{u}_{m-2}|^2 + \frac{1}{2}a_{m-1}^2\xi^4|\hat{u}_{m-3}|^2 + |a_{m-1}||a_m||\xi|^3|\hat{u}_{m-2}||\hat{u}_m|.$$

Letting β_1 suitably small and using Young inequality, we get

$$\partial_t\big\{\xi^2 E_{m-1} + \beta_1\xi E_m\big\} + c\xi^2(|\hat{u}_m|^2 + |\hat{u}_{m-1}|^2)$$

$$\le C(1 + \xi^2)^2|\hat{u}_{m-2}|^2 + \frac{1}{2}a_{m-1}^2\xi^4|\hat{u}_{m-3}|^2.$$

Moreover, combining the above estimate and (3.28) with $\ell = m - 2$, we get

$$(3.33) \quad \partial_t\big\{(1 + \xi^2)E_{m-1} + \beta_1\xi E_m\big\} + c\xi^2|\hat{u}_m|^2 + c(1 + \xi^2)|\hat{u}_{m-1}|^2$$

$$\le C(1 + \xi^2)^2|\hat{u}_{m-2}|^2 + \frac{1}{2}a_{m-1}^2\xi^2(1 + \xi^2)|\hat{u}_{m-3}|^2.$$

Second, we multiply (3.33) and (3.30) with $\ell = m - 4$ by $\beta_2\xi^2$ and $(1 + \xi^2)^2$ and combine the resultant equations. Then we obtain

$$\partial_t\big\{\beta_2\xi^2((1 + \xi^2)E_{m-1} + \beta_1\xi E_m) + \xi(1 + \xi^2)^2 E_{m-2}\big\}$$

$$+ \beta_2 c\xi^4|\hat{u}_m|^2 + \beta_2 c\xi^2(1 + \xi^2)|\hat{u}_{m-1}|^2 + \Big(\frac{1}{2}a_{m-2}^2 - \beta_2 C\Big)\xi^2(1 + \xi^2)^2|\hat{u}_{m-2}|^2$$

$$\le C\xi^2(1 + \xi^2)^2|\hat{u}_{m-3}|^2 + \frac{1}{2}a_{m-3}^2(1 + \xi^2)^2|\hat{u}_{m-4}|^2 + C|\xi|(1 + \xi^2)^2|\hat{u}_{m-3}||\hat{u}_{m-1}|,$$

Letting β_2 suitably small and using Young inequality, we get

$$(3.34) \quad \partial_t \big\{ \beta_2 \xi^2 ((1+\xi^2) E_{m-1} + \beta_1 \xi E_m) + \xi(1+\xi^2)^2 E_{m-2} \big\}$$
$$+ c\xi^4 |\hat{u}_m|^2 + c\xi^2(1+\xi^2)|\hat{u}_{m-1}|^2 + c\xi^2(1+\xi^2)^2|\hat{u}_{m-2}|^2$$
$$\leq C(1+\xi^2)^3 |\hat{u}_{m-3}|^2 + \frac{1}{2} a_{m-3}^2 (1+\xi^2)^2 |\hat{u}_{m-4}|^2.$$

Third, we multiply (3.34) and (3.28) with $\ell = m - 4$ by β_3 and $(1+\xi^2)^3$, respectively, and combine the resultant equations. Then we obtain

$$\partial_t \big\{ \beta_3 (\beta_2 \xi^2 ((1+\xi^2) E_{m-1} + \beta_1 \xi E_m) + \xi(1+\xi^2)^2 E_{m-2}) + (1+\xi^2)^3 E_{m-3} \big\}$$
$$+ \beta_3 c\xi^4 |\hat{u}_m|^2 + \beta_3 c\xi^2(1+\xi^2)|\hat{u}_{m-1}|^2 + \beta_3 c\xi^2(1+\xi^2)^2|\hat{u}_{m-2}|^2$$
$$+ \Big(\frac{1}{2} a_{m-3}^2 - \beta_3 C \Big)(1+\xi^2)^3 |\hat{u}_{m-3}|^2$$
$$\leq C(1+\xi^2)^3 |\hat{u}_{m-4}|^2 + \frac{1}{2} a_{m-3}^2 \xi^2 (1+\xi^2)^3 |\hat{u}_{m-5}|^2 + C|\xi|(1+\xi^2)^3 |\hat{u}_{m-4}||\hat{u}_{m-2}|.$$

Therefore, letting β_3 suitably small and using Young inequality, we get

$$(3.35) \quad \partial_t \big\{ \beta_3 (\beta_2 \xi^2 ((1+\xi^2) E_{m-1} + \beta_1 \xi E_m) + \xi(1+\xi^2)^2 E_{m-2})$$
$$+ (1+\xi^2)^3 E_{m-3} \big\} + c\xi^4 |\hat{u}_m|^2 + c\xi^2(1+\xi^2)|\hat{u}_{m-1}|^2 + c\xi^2(1+\xi^2)^2|\hat{u}_{m-2}|^2$$
$$+ c(1+\xi^2)^3 |\hat{u}_{m-3}|^2 \leq C(1+\xi^2)^4 |\hat{u}_{m-4}|^2 + \frac{1}{2} a_{m-3}^2 \xi^2 (1+\xi^2)^3 |\hat{u}_{m-5}|^2.$$

Inspired by the derivation of (3.33), (3.34) and (3.35), we can conclude that the following inequality

$$(3.36) \quad \partial_t \mathcal{E}_{m-5} + c \sum_{\ell=5}^{m} \xi^{2([\ell/2]-2)} (1+\xi^2)^{m-\ell} |\hat{u}_\ell|^2$$
$$\leq C(1+\xi^2)^{m-4} |\hat{u}_4|^2 + \frac{1}{2} a_5^2 \xi^2 (1+\xi^2)^{m-5} |\hat{u}_3|^2,$$

is derived by the induction argument. Here $[\,]$ denotes the greatest integer function, and $\mathcal{E}_1 = \beta_1 \xi E_m + (1+\xi^2) E_{m-1}$ and

$$(3.37) \qquad \begin{aligned} \mathcal{E}_\ell &= \beta_\ell \xi^2 \mathcal{E}_{\ell-1} + \xi(1+\xi^2)^\ell E_{m-\ell}, \\ \mathcal{E}_{\ell+1} &= \beta_{\ell+1} \mathcal{E}_\ell + (1+\xi^2)^{\ell+1} E_{m-(\ell+1)}, \end{aligned}$$

for ℓ are even integers with $\ell \geq 2$.

Furthermore, we multiply (3.36) and (3.11) by $\beta_{m-4} \xi^2$ and $(1+\xi^2)^{m-4}$, respectively, and combine the resultant equation. Then we obtain

$$\partial_t \mathcal{E}_{m-4} + \beta_{m-4} c \sum_{\ell=5}^{m} \xi^{2([\ell/2]-1)} (1+\xi^2)^{m-\ell} |\hat{u}_\ell|^2$$
$$+ \Big(\frac{1}{2} a_4^2 - \beta_{m-4} C \Big) \xi^2 (1+\xi^2)^{m-4} |\hat{u}_4|^2$$
$$\leq C\xi^2 (1+\xi^2)^{m-4} |\hat{u}_3|^2 + \frac{1}{2} (1+\xi^2)^{m-4} |\hat{u}_2|^2 + C|\xi|(1+\xi^2)^{m-4} |\hat{u}_3||\hat{u}_5|,$$

where $\mathcal{E}_{m-4}$ is defined by (3.37) with $\ell = m - 4$. Thus, letting β_{m-4} suitably small and using Young inequality, we obtain

$$(3.38) \quad \partial_t \mathcal{E}_{m-4} + c \sum_{\ell=4}^{m} \xi^{2([\ell/2]-1)}(1+\xi^2)^{m-\ell}|\hat{u}_\ell|^2$$

$$\leq C(1+\xi^2)^{m-3}|\hat{u}_3|^2 + \frac{1}{2}(1+\xi^2)^{m-4}|\hat{u}_2|^2.$$

Similarly, we multiply (3.38) and (3.9) by β_{m-3} and $(1+\xi^2)^{m-3}$, combine the resultant equalities, and take β_{m-3} suitably small. Then we have

$$(3.39) \quad \partial_t \mathcal{E}_{m-3} + c \sum_{\ell=3}^{m} \xi^{2([\ell/2]-1)}(1+\xi^2)^{m-\ell}|\hat{u}_\ell|^2$$

$$\leq C(1+\xi^2)^{m-2}|\hat{u}_2|^2 + \xi^2(1+\xi^2)^{m-3}|\hat{u}_1|^2,$$

where $\mathcal{E}_{m-3}$ is defined by (3.37) with $\ell = m - 3$.

To estimate $|\hat{u}_1|^2$ in (3.39), we next employ (3.26). Namely, we multiply (3.26) and (3.39) by $(1+\xi^2)^{m-3}$ and $\beta_{m-2}\alpha_m \xi^{m/2-2}$, respectively. Then we combine the resultant equation, obtaining

$$\partial_t \left\{ \beta_{m-2}\alpha_m \xi^{m/2-2}\mathcal{E}_{m-3} + (1+\xi^2)^{m-3}E_1^{(m)} \right\}$$

$$+\beta_{m-2}\alpha_m c\xi^{m/2-2} \sum_{\ell=3}^{m} \xi^{2([\ell/2]-1)}(1+\xi^2)^{m-\ell}|\hat{u}_\ell|^2 + \alpha_m(1-\beta_{m-2})\xi^{m/2}(1+\xi^2)^{m-3}|\hat{u}_1|^2$$

$$\leq C\xi^{m/2-2}(1+\xi^2)^{m-2}|\hat{u}_2|^2 + \gamma\alpha_m\xi^{m/2-1}(1+\xi^2)^{m-3}\Re(i\hat{u}_1\bar{\hat{u}}_2)$$

$$+ \xi(1+\xi^2)^{m-3}\Re(i^{m/2+1}\mathcal{U}_m\bar{\hat{u}}_2),$$

where we have defined $\alpha_m = \prod_{j=2}^{m/2} a_{2j}$. Here, taking β_{m-2} suitably small and using Young inequality, we get

$$(3.40) \quad \partial_t \left\{ \beta_{m-2}\alpha_m \xi^{m/2-2}\mathcal{E}_{m-3} + (1+\xi^2)^{m-3}E_1^{(m)} \right\}$$

$$+ c\xi^{m/2-2} \sum_{\ell=3}^{m} \xi^{2([\ell/2]-1)}(1+\xi^2)^{m-\ell}|\hat{u}_\ell|^2 + c\xi^{m/2}(1+\xi^2)^{m-3}|\hat{u}_1|^2$$

$$\leq C\xi^{m/2-2}(1+\xi^2)^{m-2}|\hat{u}_2|^2 + \xi(1+\xi^2)^{m-3}\Re(i^{m/2+1}\mathcal{U}_m\bar{\hat{u}}_2).$$

For the last term of the right hand side in (3.40), we note that

$$\mathcal{U}_m = \Big(\prod_{j=0}^{m/2-3} a_{m-2j} \Big)(-i\xi)^{m/2-2}\hat{u}_4 + \Big(\prod_{j=2}^{m/2-1} a_{2j+1} \Big)\hat{u}_m$$

$$+ \sum_{k=3}^{m/2-1} \Big(\prod_{j=2}^{k-1} a_{2j+1} \Big)\Big(\prod_{j=0}^{m/2-1-k} a_{m-2j} \Big)(-i\xi)^{m/2-k}\hat{u}_{2k},$$

for $m \geq 6$, where the last term of the right hand side is neglected in the case $m = 6$. Then, substituting the above equality into (3.40), we obtain

$$(3.41) \quad \partial_t \big\{ \beta_{m-2} \alpha_m \xi^{m/2-2} \mathcal{E}_{m-3} + (1+\xi^2)^{m-3} E_1^{(m)} \big\}$$

$$+ c\xi^{m/2-2} \sum_{\ell=3}^{m} \xi^{2([\ell/2]-1)} (1+\xi^2)^{m-\ell} |\hat{u}_\ell|^2 + c\xi^{m/2}(1+\xi^2)^{m-3} |\hat{u}_1|^2$$

$$\leq C\xi^{m/2-2}(1+\xi^2)^{m-2}|\hat{u}_2|^2 + C \sum_{k=2}^{m/2} |\xi|^{m/2+1-k}(1+\xi^2)^{m-3}|\hat{u}_2||\hat{u}_{2k}|.$$

In order to control the term of $|\hat{u}_m|$ on the right hand side of (3.41) we introduce the following inequality

$$|\xi|^{3m/2-5}(1+\xi^2)^{m-3}|\hat{u}_2||\hat{u}_m| \leq \varepsilon \xi^{3m-10}|\hat{u}_m|^2 + C_\varepsilon(1+\xi^2)^{2(m-3)}|\hat{u}_2|^2.$$

Inspired by the above inequality, we multiply (3.41) by $\xi^{3m/2-6}$ and employ this inequality. Then we obtain

$$\xi^{3m/2-6} \partial_t \big\{ \beta_{m-2}\alpha_m \xi^{m/2-2}\mathcal{E}_{m-3} + (1+\xi^2)^{m-3}E_1^{(m)} \big\} + (c-\varepsilon)\xi^{3m-10}|\hat{u}_m|^2$$

$$+ c\xi^{2m-10} \sum_{\ell=3}^{m-1} \xi^{2[\ell/2]}(1+\xi^2)^{m-\ell}|\hat{u}_\ell|^2 + c\xi^{2m-6}(1+\xi^2)^{m-3}|\hat{u}_1|^2$$

$$\leq \big\{ C\xi^{2m-8} + C_\varepsilon(1+\xi^2)^{m-3} \big\}(1+\xi^2)^{m-3}|\hat{u}_2|^2$$

$$+ C \sum_{k=2}^{m/2-1} |\xi|^{2m-5-k}(1+\xi^2)^{m-3}|\hat{u}_2||\hat{u}_{2k}|.$$

Therefore, letting ε suitably small, we have

$$(3.42) \quad \xi^{3m/2-6} \partial_t \big\{ \beta_{m-2}\alpha_m \xi^{m/2-2}\mathcal{E}_{m-3} + (1+\xi^2)^{m-3}E_1^{(m)} \big\}$$

$$+ c\xi^{2m-10} \sum_{\ell=3}^{m} \xi^{2[\ell/2]}(1+\xi^2)^{m-\ell}|\hat{u}_\ell|^2 + c\xi^{2m-6}(1+\xi^2)^{m-3}|\hat{u}_1|^2$$

$$\leq C(1+\xi^2)^{2(m-3)}|\hat{u}_2|^2 + C \sum_{k=2}^{m/2-1} |\xi|^{2m-5-k}(1+\xi^2)^{m-3}|\hat{u}_2||\hat{u}_{2k}|.$$

Moreover, applying the inequality

$$|\xi|^{2m-5-k}(1+\xi^2)^{m-3}|\hat{u}_2||\hat{u}_{2k}|$$

$$\leq \varepsilon \xi^{2m-10+2k}(1+\xi^2)^{m-2k}|\hat{u}_{2k}|^2 + C_\varepsilon \xi^{2m-4k}(1+\xi^2)^{m-6+2k}|\hat{u}_2|^2$$

to (3.42), we can get

$$(3.43) \quad \partial_t \mathcal{E}_{m-2} + c\xi^{2m-10} \sum_{\ell=3}^{m} \xi^{2[\ell/2]}(1+\xi^2)^{m-\ell}|\hat{u}_\ell|^2$$

$$+ c\xi^{2m-6}(1+\xi^2)^{m-3}|\hat{u}_1|^2 \leq C(1+\xi^2)^{2(m-3)}|\hat{u}_2|^2,$$

where we have defined $\mathcal{E}_{m-2} = \xi^{3m/2-6}(\beta_{m-2}\alpha_m\xi^{m/2-2}\mathcal{E}_{m-3} + (1+\xi^2)^{m-3}E_1^{(m)})$.

Finally, multiplying the basic energy (3.4) and (3.43) by $(1 + \xi^2)^{2(m-3)}$ and β_{m-1}, respectively, combining the resultant equations and letting β_{m-1} suitably small, then this yields

$$(3.44) \quad \partial_t \left\{ \frac{1}{2}(1+\xi^2)^{2(m-3)}|\hat{u}|^2 + \beta_{m-1}\mathcal{E}_{m-2} \right\} + c\xi^{2m-6}(1+\xi^2)^{m-3}|\hat{u}_1|^2$$

$$+ c(1+\xi^2)^{2(m-3)}|\hat{u}_2|^2 + c\xi^{2m-10}\sum_{\ell=3}^{m}\xi^{2[\ell/2]}(1+\xi^2)^{m-\ell}|\hat{u}_\ell|^2 \leq 0.$$

Thus, integrating the above estimate with respect to t, we obtain the following energy estimate

$$(3.45) \quad |\hat{u}(t,\xi)|^2 + \int_0^t \left\{ \frac{\xi^{2m-6}}{(1+\xi^2)^{m-3}}|\hat{u}_1|^2 + |\hat{u}_2|^2 \right.$$

$$\left. + \frac{\xi^{2m-10}}{(1+\xi^2)^{m-3}}\sum_{\ell=3}^{m}\frac{\xi^{2[\ell/2]}}{(1+\xi^2)^{\ell-3}}|\hat{u}_\ell|^2 \right\}d\tau \leq C|\hat{u}(0,\xi)|^2.$$

Here we have used the following inequality

$$(3.46) \qquad c|\hat{u}|^2 \leq \frac{1}{2}|\hat{u}|^2 + \frac{\beta_{m-1}}{(1+\xi^2)^{2(m-3)}}\mathcal{E}_{m-2} \leq C|\hat{u}|^2$$

for suitably small β_{m-1}. Furthermore the estimate (3.44) with (3.46) give us the following pointwise estimate

$$|\hat{u}(t,\xi)| \leq Ce^{-c\lambda(\xi)t}|\hat{u}(0,\xi)|, \qquad \lambda(\xi) = \frac{\xi^{3m-10}}{(1+\xi^2)^{2(m-3)}}.$$

This therefore proves (3.2) and completes the proof of Theorem 3.1.

3.4. **Construction of the matrices K and S.** In this section, inspired by the energy method stated in Sections 3.2 and 3.3, we derive the desired matrices K and S.

Based on the energy method of Step 2 in Subsection 3.2, we first introduce the following $m \times m$ matrices:

$$K_1 = \begin{pmatrix} \begin{array}{cccc|c} 0 & 1 & 0 & 0 & \\ -1 & 0 & 0 & 0 & O \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ \hline & O & & & O \end{array} \end{pmatrix}, \qquad K_4 = a_4 \begin{pmatrix} \begin{array}{cccc|c} 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & O \\ 0 & 0 & 0 & -1 & \\ 0 & 0 & 1 & 0 & \\ \hline & O & & & O \end{array} \end{pmatrix}.$$

Then, we multiply (1.4) by $-i\xi K_1$ and take the inner product with $\hat{u}$. Moreover, taking the real part of the resultant equation, we have

$$(3.47) \qquad -\frac{1}{2}\xi\partial_t\langle iK_1\hat{u}, \hat{u}\rangle + \xi^2\langle [K_1 A_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle - \xi\langle i[K_1 L_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle = 0,$$

where

$$
K_1 A_m = \begin{pmatrix} \begin{array}{cccc:c} 1 & 0 & 0 & 0 & \\ 0 & -1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & O \\ 0 & 0 & 0 & 0 & \\ \hdashline & & O & & O \end{array} \end{pmatrix}, \qquad
K_1 L_m = \begin{pmatrix} \begin{array}{cccc:c} 0 & \gamma & 1 & 0 & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & O \\ 0 & 0 & 0 & 0 & \\ \hdashline & & O & & O \end{array} \end{pmatrix}.
$$

The equality (3.47) is equivalent to (3.5). Similarly, by using the matrix K_4, we can obtain

(3.48) $\qquad -\dfrac{1}{2}\xi\partial_t\langle iK_4\hat{u}, \hat{u}\rangle + \xi^2\langle[K_4 A_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle - \xi\langle i[K_4 L_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle = 0,$

where

$$
K_4 A_m = a_4^2 \begin{pmatrix} \begin{array}{cccc:c} 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & -1 & 0 & O \\ 0 & 0 & 0 & 1 & \\ \hdashline & & O & & O \end{array} \end{pmatrix}, \qquad
K_4 L_m = -a_4 \begin{pmatrix} \begin{array}{cccc:c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & O \\ 0 & 1 & 0 & 0 & 0 \\ \hdashline & & O & & & O \end{array} \end{pmatrix}.
$$

The equality (3.48) is equivalent to (3.10).

We next introduce

$$
S_3 = \begin{pmatrix} \begin{array}{cccc:c} 0 & 0 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 1 & 0 & 0 & O \\ 0 & 0 & 0 & 0 & \\ \hdashline & & O & & O \end{array} \end{pmatrix}, \qquad
\tilde{S}_\ell = \begin{pmatrix} & & & & 1 & & \\ & O & & & 0 & O \\ & & & & \vdots & \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ & O & & & \vdots & O \\ & & & & 0 & \\ & & & & \ell & \end{pmatrix} \ell
$$

for $2 \le \ell \le m-1$. Then, by using the same argument, we can show that the equality

(3.49) $\qquad \dfrac{1}{2}\partial_t\langle S_3\hat{u}, \hat{u}\rangle + \xi\langle i[S_3 A_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle + \langle[S_3 L_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle = 0,$

which satisfies

$$
S_3 A_m = \begin{pmatrix} \begin{array}{cccc:c} 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & a_4 & \\ 1 & 0 & 0 & 0 & O \\ 0 & 0 & 0 & 0 & \\ \hdashline & & O & & O \end{array} \end{pmatrix}, \qquad
S_3 L_m = \begin{pmatrix} \begin{array}{cccc:c} 0 & 0 & 0 & 0 & \\ 0 & -1 & 0 & 0 & \\ 0 & \gamma & 1 & 0 & O \\ 0 & 0 & 0 & 0 & \\ \hdashline & & O & & O \end{array} \end{pmatrix}
$$

is equivalent to (3.8). Similarly, we derive that

(3.50) $\qquad \dfrac{1}{2}\partial_t\langle \tilde{S}_{2j}\hat{u}, \hat{u}\rangle + \xi\langle i[\tilde{S}_{2j} A_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle + \langle[\tilde{S}_{2j} L_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle = 0,$

which satisfies

$$\tilde{S}_{2j} A_m = \begin{pmatrix} & & & a_{2j} & & \\ & O & & 0 & O & \\ & & & \vdots & & \\ 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ & O & & \vdots & O & \\ & & & 0 & & \\ & & & 2j-1 & & \end{pmatrix} \!\! \begin{array}{c} \\ \\ 2j \\ \\ \\ \\ \end{array}$$

and

$$\tilde{S}_{2j} L_m = \begin{pmatrix} 0 & \cdots & 0 & a_{2j+1} & 0 & \cdots & 0 \\ & & & 0 & & & \\ & O & & \vdots & & O & \\ & & & 0 & & & \\ & & & 2j+1 & & & \end{pmatrix},$$

is equivalent to

$$\partial_t \Re(\hat{u}_1 \bar{\hat{u}}_{2j}) - a_{2j} \xi \, \Re(i\hat{u}_1 \bar{\hat{u}}_{2j-1}) + a_{2j+1} \, \Re(\hat{u}_1 \bar{\hat{u}}_{2j+1}) + \xi \, \Re(i\hat{u}_2 \bar{\hat{u}}_{2j}) = 0,$$

for $2 \leq j \leq (m-2)/2$. Therefore, to construct (3.25), we sum up (3.50) with respect to j with $2 \leq j \leq (m-2)/2$, and find that

$$(3.51) \qquad \frac{1}{2} \partial_t \langle \tilde{\mathcal{S}}_{m-2} \hat{u}, \hat{u} \rangle + \xi \langle i[\tilde{\mathcal{S}}_{m-2} A_m]^{\mathrm{asy}} \hat{u}, \hat{u} \rangle + \langle [\tilde{\mathcal{S}}_{m-2} L_m]^{\mathrm{sy}} \hat{u}, \hat{u} \rangle = 0$$

is equivalent to (3.25). Here we define $\tilde{\mathcal{S}}_{2\ell}$ as $\tilde{\mathcal{S}}_4 = \tilde{S}_4$ and

$$\tilde{\mathcal{S}}_{2\ell} = a_{2\ell} \xi \tilde{\mathcal{S}}_{2\ell-2} + \prod_{j=2}^{\ell-1} a_{2j+1} \tilde{S}_{2\ell}$$

for $\ell \geq 3$. Consequently, multiplying (3.47) by $\prod_{j=2}^{m/2} a_{2j} \xi^{m/2-2}$ and combining the resultant equality and (3.51), we obtain

$$\frac{1}{2} \partial_t \Big\langle \Big(\tilde{\mathcal{S}}_{m-2} - i \prod_{j=2}^{m/2} a_{2j} \xi^{m/2-1} K_1 \Big) \hat{u}, \hat{u} \Big\rangle$$

$$(3.52) \qquad + \Big\langle \Big[\tilde{\mathcal{S}}_{m-2} L_m + \prod_{j=2}^{m/2} a_{2j} \xi^{m/2} K_1 A_m \Big]^{\mathrm{sy}} \hat{u}, \hat{u} \Big\rangle$$

$$+ \xi \langle i[\tilde{\mathcal{S}}_{m-2} A_m]^{\mathrm{asy}} \hat{u}, \hat{u} \rangle - \prod_{j=2}^{m/2} a_{2j} \xi^{m/2-1} \langle i[K_1 L_m]^{\mathrm{asy}} \hat{u}, \hat{u} \rangle = 0.$$

This equality is the same as (3.26).

Based on the energy method of Step 3 in Subsection 3.3, we next introduce the following $m \times m$ matrices:

$$
S_{\ell+1} = a_{\ell+1}
\left(
\begin{array}{ccccccccc}
 & & & 0 & 0 & & & & \\
 & O & & \vdots & \vdots & & & O & \\
 & & & 0 & 0 & & & & \\
0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\
 & & & 0 & 0 & & & & \\
 & O & & \vdots & \vdots & & & O & \\
 & & & 0 & 0 & & & & \\
 & & & \ell & \ell+1 & & & &
\end{array}
\right)
\begin{array}{c}
\\ \\ \\ \ell \\ \ell+1 \\ \\ \\ \\
\end{array}
$$

for $\ell = 4, 6, \cdots, m-2$. Then, we multiply (1.4) by $S_{\ell+1}$ and take the inner product with $\hat{u}$. Furthermore, taking the real part of the resultant equation, we obtain

(3.53)
$$
\frac{1}{2}\partial_t \langle S_{\ell+1}\hat{u}, \hat{u}\rangle + \xi\langle i[S_{\ell+1}A_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle + \langle [S_{\ell+1}L_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle = 0
$$

for $\ell = 4, 6, \cdots, m-2$, where

$$
S_{\ell+1}A_m = a_\ell
\left(
\begin{array}{ccccccccccc}
 & & & 0 & 0 & 0 & 0 & & & & \\
 & O & & \vdots & \vdots & \vdots & \vdots & & O & & \\
 & & & 0 & 0 & 0 & 0 & & & & \\
0 & \cdots & 0 & 0 & 0 & 0 & a_{\ell+2} & 0 & \cdots & 0 \\
0 & \cdots & 0 & a_\ell & 0 & 0 & 0 & 0 & \cdots & 0 \\
 & & & 0 & 0 & 0 & 0 & & & & \\
 & O & & \vdots & \vdots & \vdots & \vdots & & O & & \\
 & & & 0 & 0 & 0 & 0 & & & & \\
 & & & \ell-1 & \ell & \ell+1 & \ell+2 & & & &
\end{array}
\right)
\begin{array}{c}
\\ \\ \\ \ell \\ \ell+1 \\ \\ \\ \\
\end{array}
$$

and

$$
S_{\ell+1}L_m = a_{\ell+1}^2
\left(
\begin{array}{ccccccccc}
 & & & 0 & 0 & & & & \\
 & O & & \vdots & \vdots & & & O & \\
 & & & 0 & 0 & & & & \\
0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\
 & & & 0 & 0 & & & & \\
 & O & & \vdots & \vdots & & & O & \\
 & & & 0 & 0 & & & & \\
 & & & \ell & \ell+1 & & & &
\end{array}
\right)
\begin{array}{c}
\\ \\ \\ \ell \\ \ell+1 \\ \\ \\ \\
\end{array}.
$$

We note that the equalities (3.53) is equivalent to (3.27).

On the other hand, we introduce the following $m \times m$ matrices:

$$
K_{\ell+2} = a_{\ell+2}
\begin{pmatrix}
 & & 0 & 0 & & & \\
O & & \vdots & \vdots & & O & \\
 & & 0 & 0 & & & \\
0 & \cdots & 0 & 0 & -1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\
 & & 0 & 0 & & & \\
O & & \vdots & \vdots & & O & \\
 & & 0 & 0 & & & \\
 & & \ell+1 & \ell+2 & & &
\end{pmatrix}
\begin{matrix} \\ \\ \\ \ell+1 \\ \ell+2 \\ \\ \\ \\ \end{matrix}
$$

for $\ell = 4, 6, \cdots, m - 2$. Then, we multiply (1.4) by $-iK_{\ell+2}$ and take the inner product with $\hat{u}$. Furthermore, taking the real part of the resultant equation, we obtain

$$
(3.54) \qquad -\frac{1}{2}\xi\partial_t\langle iK_{\ell+2}\hat{u}, \hat{u}\rangle + \xi^2\langle [K_{\ell+2}A_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle - \xi\langle i[K_{\ell+2}L_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle = 0,
$$

for $\ell = 4, 6, \cdots, m - 4$, where

$$
K_{\ell+2}A_m = a_{\ell+2}^2
\begin{pmatrix}
 & & 0 & 0 & & & \\
O & & \vdots & \vdots & & O & \\
 & & 0 & 0 & & & \\
0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\
 & & 0 & 0 & & & \\
O & & \vdots & \vdots & & O & \\
 & & 0 & 0 & & & \\
 & & \ell+1 & \ell+2 & & &
\end{pmatrix}
\begin{matrix} \\ \\ \\ \ell+1 \\ \ell+2 \\ \\ \\ \\ \end{matrix}
$$

and

$$
K_{\ell+2}L_m = a_{\ell+2}
\begin{pmatrix}
 & & 0 & 0 & 0 & 0 & & & \\
O & & \vdots & \vdots & \vdots & \vdots & & O & \\
 & & 0 & 0 & 0 & 0 & & & \\
0 & \cdots & 0 & 0 & 0 & 0 & -a_{\ell+3} & 0 & \cdots & 0 \\
0 & \cdots & 0 & -a_{\ell+1} & 0 & 0 & 0 & 0 & \cdots & 0 \\
 & & 0 & 0 & 0 & 0 & & & \\
O & & \vdots & \vdots & \vdots & \vdots & & O & \\
 & & 0 & 0 & 0 & 0 & & & \\
 & & \ell-1 & \ell & \ell+1 & \ell+3 & & &
\end{pmatrix}
\begin{matrix} \\ \\ \\ \ell+1 \\ \ell+2 \\ \\ \\ \\ \end{matrix}.
$$

Moreover we have

$$
(3.55) \qquad -\frac{1}{2}\xi\partial_t\langle iK_m\hat{u}, \hat{u}\rangle + \xi^2\langle [K_mA_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle - \xi\langle i[K_mL_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle = 0,
$$

where

$$K_m A_m = a_m^2 \begin{pmatrix} & & 0 & 0 \\ O & \vdots & \vdots \\ & & 0 & 0 \\ 0 & \cdots & 0 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}, \quad K_m L_m = a_{m-1} a_m \begin{pmatrix} & & 0 & 0 & 0 \\ O & \vdots & \vdots & \vdots \\ & & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & -1 & 0 & 0 \end{pmatrix}.$$

The equalities (3.54) and (3.55) are equivalent to (3.29) and (3.31), respectively.

For the rest of this subsection, we construct the desired matrices. According to the strategy of Step 3 in Subsection 3.2, we first combine (3.53) and (3.55). More precisely, multiplying (3.53) with $\ell = m - 2$ and (3.55) by $(1 + \xi^2)$ and δ_1, respectively, and combining the resultant equations, we obtain

$$\frac{1}{2}\partial_t \langle \{(1+\xi^2)S_{m-1} - \delta_1 i\xi K_m\}\hat{u}, \hat{u}\rangle$$
$$+ (1+\xi^2)\langle [S_{m-1}L_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle + \delta_1\xi^2\langle [K_m A_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle$$
$$+ \xi(1+\xi^2)\langle i[S_{m-1}A_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle - \delta_1\xi\langle i[K_1 L_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle = 0.$$

We next multiply (3.54) with $\ell = m - 4$ and the above equation by $(1 + \xi^2)^2$ and $\delta_2\xi^2$, respectively, and combining the resultant equations, we obtain

$$\frac{1}{2}\partial_t \langle \{\delta_2\xi^2((1+\xi^2)S_{m-1} - \delta_1 i\xi K_m) - i\xi(1+\xi^2)^2 K_{m-2}\}\hat{u}, \hat{u}\rangle$$
$$+ \delta_2\xi^2(1+\xi^2)\langle [S_{m-1}L_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle + \xi^2\langle [(\delta_1\delta_2\xi^2 K_m + (1+\xi^2)^2 K_{m-2})A_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle$$
$$+ \delta_2\xi^3(1+\xi^2)\langle i[S_{m-1}A_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle$$
$$- \xi\langle i[(\delta_1\delta_2\xi^2 K_m + (1+\xi^2)^2 K_{m-2})L_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle = 0.$$

Furthermore, multiplying (3.53) with $\ell = m - 4$ and the above equation by $(1+\xi^2)^3$ and δ_3, respectively, and combining the resultant equations, we get

$$(3.56) \quad \frac{1}{2}\partial_t \langle \{\delta_3(\delta_2\xi^2((1+\xi^2)S_{m-1} - \delta_1 i\xi K_m)$$
$$- i\xi(1+\xi^2)^2 K_{m-2}) + (1+\xi^2)^3 S_{m-3}\}\hat{u}, \hat{u}\rangle$$
$$+ (1+\xi^2)\langle [(\delta_2\delta_3\xi^2 S_{m-1} + (1+\xi^2)^2 S_{m-3})L_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle$$
$$+ \delta_3\xi^2\langle [(\delta_1\delta_2\xi^2 K_m + (1+\xi^2)^2 K_{m-2})A_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle$$
$$+ \xi(1+\xi^2)\langle i[(\delta_2\delta_3\xi^2 S_{m-1} + (1+\xi^2)^2 S_{m-3})A_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle$$
$$- \delta_3\xi\langle i[(\delta_1\delta_2\xi^2 K_m + (1+\xi^2)^2 K_{m-2})L_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle = 0.$$

Now, we introduce the new matrices $\mathcal{K}_\ell$ and $\mathcal{S}_\ell$ as $\mathcal{K}_0 = K_m$ and

$$\mathcal{K}_\ell = \delta_{\ell-1}\delta_\ell\xi^2\mathcal{K}_{\ell-2} + (1+\xi^2)^\ell K_{m-\ell}$$

for $\ell \geq 2$, and $\mathcal{S}_1 = S_{m-1}$ and

$$\mathcal{S}_\ell = \delta_{\ell-1}\delta_\ell\xi^2\mathcal{S}_{\ell-2} + (1+\xi^2)^{\ell-1}S_{m-\ell}$$

for $\ell \geq 3$. Then the equation (3.56) is rewritten as

$$\frac{1}{2}\partial_t \langle \{(1+\xi^2)\mathcal{S}_3 - \delta_3 i\xi\mathcal{K}_2\}\hat{u}, \hat{u}\rangle + (1+\xi^2)\langle [\mathcal{S}_3 L_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle + \delta_3\xi^2\langle [\mathcal{K}_2 A_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle$$
$$+ \xi(1+\xi^2)\langle i[\mathcal{S}_3 A_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle - \delta_3\xi\langle i[\mathcal{K}_2 L_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle = 0.$$

Consequently, by the induction argument with respect to ℓ in (3.53) and (3.54), we arrive at

$$
\begin{aligned}
(3.57) \quad \frac{1}{2}\partial_t \big\langle \big\{(1+\xi^2)\mathcal{S}_{m-5} - \delta_{m-5}i\xi\mathcal{K}_{m-6}\big\}\hat{u}, \hat{u}\big\rangle &+ (1+\xi^2)\langle[\mathcal{S}_{m-5}L_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle \\
+ \delta_{m-5}\xi^2\langle[\mathcal{K}_{m-6}A_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle + \xi(1+\xi^2)\langle i[\mathcal{S}_{m-5}A_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle & \\
- \delta_{m-5}\xi\langle i[\mathcal{K}_{m-6}L_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle &= 0.
\end{aligned}
$$

Applying Young inequality to (3.57), we can obtain (3.36).

Moreover, we multiply (3.48) and (3.57) by $(1+\xi^2)^{m-4}$ and $\delta_{m-4}\xi^2$, respectively, and combine the resultant equations. Then this yields

$$
\begin{aligned}
\frac{1}{2}\partial_t \big\langle \big\{\delta_{m-4}\xi^2(1+\xi^2)\mathcal{S}_{m-5} - i\xi\mathcal{K}_{m-4}\big\}\hat{u}, \hat{u}\big\rangle & \\
+ \delta_{m-4}\xi^2(1+\xi^2)\langle[\mathcal{S}_{m-5}L_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle + \xi^2\langle[\mathcal{K}_{m-4}A_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle & \\
+ \delta_{m-4}\xi^3(1+\xi^2)\langle i[\mathcal{S}_{m-5}A_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle - \xi\langle i[\mathcal{K}_{m-4}L_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle &= 0.
\end{aligned}
$$

Similarly, Moreover, we multiply (3.49) and the above equation by $(1+\xi^2)^{m-3}$ and δ_{m-3}, respectively, and combine the resultant equations. Then we get

$$
\begin{aligned}
(3.58) \quad \frac{1}{2}\partial_t \big\langle \big\{(1+\xi^2)\mathcal{S}_{m-3} - \delta_{m-3}i\xi\mathcal{K}_{m-4}\big\}\hat{u}, \hat{u}\big\rangle &+ (1+\xi^2)\langle[\mathcal{S}_{m-3}L_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle \\
+ \delta_{m-3}\xi^2\langle[\mathcal{K}_{m-4}A_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle + \xi(1+\xi^2)\langle i[\mathcal{S}_{m-3}A_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle & \\
- \delta_{m-3}\xi\langle i[\mathcal{K}_{m-4}L_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle &= 0.
\end{aligned}
$$

By Young inequality to (3.58), we can derive (3.39).

We next employ (3.52) constructed before. Multiplying (3.52) and (3.58) by $(1+\xi^2)^{m-3}$ and $\delta_{m-2}\alpha_m\xi^{m/2-2}$, respectively, and combining the resultant equations, we get

$$
\begin{aligned}
(3.59) \quad \frac{1}{2}\partial_t \big\langle \big\{(1+\xi^2)\mathcal{S}' - \alpha_m i\xi^{m/2-1}\mathcal{K}'\big\}\hat{u}, \hat{u}\big\rangle &+ (1+\xi^2)\langle[\mathcal{S}'L_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle \\
+ \alpha_m\xi^{m/2}\langle[\mathcal{K}'A_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle + \xi(1+\xi^2)\langle i[\mathcal{S}'A_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle & \\
- \alpha_m\xi^{m/2-1}\langle i[\mathcal{K}'L_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle &= 0,
\end{aligned}
$$

where we have defined

$$
\begin{aligned}
\mathcal{S}' &= \delta_{m-2}\alpha_m\xi^{m/2-2}\mathcal{S}_{m-3} + (1+\xi^2)^{m-4}\tilde{\mathcal{S}}_{m-2}, \\
\mathcal{K}' &= \delta_{m-2}\delta_{m-3}\mathcal{K}_{m-4} + (1+\xi^2)^{m-3}K_1,
\end{aligned}
$$

and had already defined $\alpha_m = \prod_{j=2}^{m/2} a_{2j}$. By (3.59), we can get (3.43).

Finally, multiplying (3.59) by $\delta_{m-1}\xi^{3m/2-6}/(1+\xi^2)^{2(m-3)}$, and combining (3.22) and the resultant equations, we can obtain

$$(3.60) \quad \frac{1}{2}\partial_t\Big\langle\Big[I + \frac{\delta_{m-1}}{(1+\xi^2)^{2(m-3)}}\big\{\xi^{3m/2-6}(1+\xi^2)\mathcal{S}' - \alpha_m i\xi^{2m-7}\mathcal{K}'\big\}\Big]\hat{u}, \hat{u}\Big\rangle$$

$$+ \langle L_m\hat{u}, \hat{u}\rangle + \delta_{m-1}\frac{\xi^{3m/2-6}}{(1+\xi^2)^{2m-7}}\langle[\mathcal{S}'L_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle$$

$$+ \alpha_m\delta_{m-1}\frac{\xi^{2(m-3)}}{(1+\xi^2)^{2(m-3)}}\langle[\mathcal{K}'A_m]^{\mathrm{sy}}\hat{u}, \hat{u}\rangle - \alpha_m\delta_{m-1}\frac{\xi^{2m-7}}{(1+\xi^2)^{2(m-3)}}\langle i[\mathcal{K}'L_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle$$

$$+ \delta_{m-1}\frac{\xi^{3m/2-5}}{(1+\xi^2)^{2m-7}}\langle i[\mathcal{S}'A_m]^{\mathrm{asy}}\hat{u}, \hat{u}\rangle = 0,$$

where I denotes an identity matrix. Letting $\delta_1, \cdots, \delta_{m-1}$ suitably small, then (3.60) derives the energy estimate (3.45). To be more precise, we introduce

$$\mathcal{K}_{m-4} = (1+\xi^2)^{m-4}K_4 + \sum_{k=3}^{m/2}\prod_{j=2}^{k-1}\delta_{m-2j}\delta_{m-2j-1}\xi^{2(k-2)}(1+\xi^2)^{m-2k}K_{2k}$$

for $m \geq 6$, and hence

$$\mathcal{K}' = (1+\xi^2)^{m-3}K_1 + \delta_{m-2}\delta_{m-3}(1+\xi^2)^{m-4}K_4$$

$$(3.61) \qquad + \delta_{m-2}\delta_{m-3}\sum_{k=3}^{m/2}\prod_{j=2}^{k-1}\delta_{m-2j}\delta_{m-2j-1}\xi^{2(k-2)}(1+\xi^2)^{m-2k}K_{2k}.$$

Moreover, we find that

$$\mathcal{S}_{m-3} = (1+\xi^2)^{m-4}S_3 + \sum_{k=3}^{m/2}\prod_{j=2}^{k-1}\delta_{m-2j}\delta_{m-2j+1}\xi^{2(k-2)}(1+\xi^2)^{m-2k}S_{2k-1}$$

for $m \geq 6$, and $\tilde{\mathcal{S}}_4 = \tilde{S}_4$, $\tilde{\mathcal{S}}_6 = a_5\tilde{S}_6 + a_6\xi\tilde{S}_4$ and

$$\tilde{\mathcal{S}}_{m-2} = \prod_{j=2}^{m/2-2}a_{2j+1}\tilde{S}_{m-2} + \prod_{j=1}^{m/2-3}a_{m-2j}\xi^{m/2-3}\tilde{S}_4$$

$$+ \sum_{k=2}^{m/2-3}\Big(\prod_{j=2}^{m/2-k-1}a_{2j+1}\Big)\Big(\prod_{j=1}^{k-1}a_{m-2j}\Big)\xi^{k-1}\tilde{S}_{m-2k}$$

for $m \geq 10$, and also

$$\mathcal{S}' = \delta_{m-2}\alpha_m\xi^{m/2-2}(1+\xi^2)^{m-4}S_3$$

$$+ \alpha_m\sum_{k=3}^{m/2}\prod_{j=1}^{k-1}\delta_{m-2j}\delta_{m-2j+1}\xi^{m/2+2(k-3)}(1+\xi^2)^{m-2k}S_{2k-1}$$

$$(3.62) \qquad + \prod_{j=2}^{m/2-2}a_{2j+1}(1+\xi^2)^{m-4}\tilde{S}_{m-2} + \prod_{j=1}^{m/2-3}a_{m-2j}\xi^{m/2-3}(1+\xi^2)^{m-4}\tilde{S}_4$$

$$+ \sum_{k=2}^{m/2-3}\Big(\prod_{j=2}^{m/2-k-1}a_{2j+1}\Big)\Big(\prod_{j=1}^{k-1}a_{m-2j}\Big)\xi^{k-1}(1+\xi^2)^{m-4}\tilde{S}_{m-2k}$$

Therefore, by using (3.61) and (3.62), we can estimate the dissipation terms as

$$(3.63) \quad \langle L_m \hat{u}, \hat{u} \rangle + \delta_{m-1} \frac{\xi^{3(m-4)/2}}{(1+\xi^2)^{2m-7}} \langle [\mathcal{S}' L_m]^{\text{sy}} \hat{u}, \hat{u} \rangle$$

$$+ \delta_{m-1} \frac{\xi^{2(m-3)}}{(1+\xi^2)^{2(m-3)}} \langle [\mathcal{K}' A_m]^{\text{asy}} \hat{u}, \hat{u} \rangle$$

$$\geq c \Big\{ \frac{\xi^{2(m-3)}}{(1+\xi^2)^{m-3}} |\hat{u}_1|^2 + |\hat{u}_2|^2 + \sum_{j=2}^{m/2} \frac{\xi^{2(m+j-6)}}{(1+\xi^2)^{m+2j-7}} |\hat{u}_{2j-1}|^2$$

$$+ \sum_{j=2}^{m/2} \frac{\xi^{2(m+j-5)}}{(1+\xi^2)^{m+2j-6}} |\hat{u}_{2j}|^2 \Big\},$$

for suitably small $\delta_1, \cdots, \delta_{m-1}$. We note that this estimate is the same as the dissipation part of (3.45). Consequently we conclude that our desired symmetric matrix S and skew-symmetric matrix K are described as

$$S = \frac{\xi^{3(m-4)/2}}{(1+\xi^2)^{2m-7}} \mathcal{S}', \qquad K = \frac{\xi^{2(m-3)}}{(1+\xi^2)^{2(m-3)}} \mathcal{K}'.$$

4. ALTERNATIVE APPROACH

4.1. **General strategy.** In this section, by using the Fourier energy method, we provide an alternative way to justify the dissipative structure of the linear symmetric hyperbolic system with relaxation (1.1). The key point of the approach is to derive from the above system a new system of m number of equations or inequalities

$$(I_1), (I_2), \cdots, (I_j), \cdots, (I_m),$$

in the Fourier space, such that their appropriate linear combination can capture the dissipation rate of all the degenerate components only over the frequency domain far from $|\xi| = 0$ and $|\xi| = \infty$. Precisely, for any $0 < \epsilon < M < \infty$, by considering

$$(4.1) \qquad \sum_{j=1}^{m} c_j I_j$$

for an appropriate choice of constants $c_j > 0$ $(1 \leq j \leq m)$ which may depend on ϵ and M, we expect to obtain that for $\epsilon \leq |\xi| \leq M$,

$$(4.2) \qquad \partial_t \{ |\hat{u}|^2 + \Re E_1^{int}(\hat{u}) \} + c_{\epsilon,M} |\hat{u}|^2 \leq 0,$$

where $c_{\epsilon,M} > 0$ depending on ϵ and M is a constant, and $E_1^{int}(\hat{u})$ is an interactive functional such that $|\hat{u}|^2 + \Re E_1^{int}(\hat{u}) \sim |\hat{u}|^2$ over $\epsilon \leq |\xi| \leq M$. To deal with the dissipation rate around $|\xi| = 0$ or $|\xi| = \infty$, instead of (4.1), we re-consider the frequency weighted linear combination in the form of

$$(4.3) \qquad \sum_{j=1}^{m} c_j \frac{|\xi|^{\alpha_j}}{(1+|\xi|)^{\alpha_j + \beta_j}} I_j.$$

Here $\alpha_j \geq 0$ and $\beta_j \geq 0$ $(1 \leq j \leq m)$ are constants to be chosen such that the similar computations for deriving (4.2) can be applied so as to obtain a Lyapunov

inequality taking the form

$$(4.4) \qquad \partial_t\{|\hat{u}|^2 + \Re E^{int}(\hat{u})\} + c\sum_{j=1}^{m}\lambda_j(\xi)|\hat{u}_j|^2 \leq 0,$$

for all $t \geq 0$ and all $\xi \in \mathbb{R}$, where $c > 0$ is a constant, $\lambda_j(\xi)$ $(j = 1, 2, \cdots, m)$ are nonnegative rational functions of $|\xi|$, and $E^{int}(\hat{u})$ is an interactive functional such that $|\hat{u}|^2 + \Re E^{int}(\hat{u}) \sim |\hat{u}|^2$ for all $\xi \in \mathbb{R}$. If (4.4) was proved then by defining

$$\lambda_{min}(\xi) = \min_{1 \leq j \leq m}\lambda_j(\xi), \quad \xi \in \mathbb{R},$$

it follows that

$$|\hat{u}(t,\xi)|^2 \leq Ce^{-c\lambda_{min}(\xi)t}|\hat{u}(0,\xi)|^2,$$

for all $t \geq 0$ and all $\xi \in \mathbb{R}$, which thus implies the dissipative structure of the considered system (1.1). Observe that $\lambda_j(\xi)$ $(1 \leq j \leq m)$ and hence $\lambda_{min}(\xi)$ may depend on $\alpha_j \geq 0$ and $\beta_j \geq 0$ $(1 \leq j \leq m)$. In general, α_j and β_j are required to satisfy a series of inequalities such that (4.3) indeed can be applied to deduce (4.4) by using the Cauchy-Schwarz inequalities. Therefore we always expect to choose constants α_j and β_j such that $\lambda_{min}(\xi)$ is optimal in the sense that $\lambda_{min}(\xi)$ may tend to zero when $|\xi| \to 0$ or $|\xi| \to \infty$ in the slowest rate. Finally, we remark that due to (4.2) which holds over $\epsilon \leq |\xi| \leq M$, considering (4.3) is equivalent to considering $\sum_{j=1}^{m} c_j|\xi|^{\alpha_j}I_j$ over $|\xi| \leq \epsilon$ with $0 < \epsilon \leq 1$, and $\sum_{j=1}^{m} c_j|\xi|^{-\beta_j}I_j$ over $|\xi| \geq M$ with $M \geq 1$. In such way, it is more convenient to derive those inequalities satisfied by $\lambda_j(\xi)$ $(1 \leq j \leq m)$.

4.2. **Revisit Model I.** By using the same strategy as in Subsection 2.2 and 2.3, one can obtain m number of identities (I_j) with $j = 1, 2, \cdots, m$ as follows:

$(I_1):$ $\partial_t\langle i\xi\hat{u}_2, \hat{u}_1\rangle + |\xi|^2|\hat{u}_2|^2 = -\langle i\xi\hat{u}_2, \hat{u}_4\rangle + |\xi|^2|\hat{u}_1|^2.$

$(I_2):$ $\partial_t\langle -\hat{u}_1, \hat{u}_4\rangle + |\hat{u}_1|^2 = |\hat{u}_4|^2 + \langle i\xi\hat{u}_2, \hat{u}_4\rangle + \langle\hat{u}_1, i\xi a_4\hat{u}_3 + i\xi a_5\hat{u}_5\rangle.$

$(I_3):$ $\partial_t\{\langle i\xi a_4\hat{u}_3, \hat{u}_4\rangle - \langle a_4\hat{u}_3, \hat{u}_2\rangle\} + a_4^2|\xi|^2|\hat{u}_3|^2 =$
$\qquad\qquad + a_4^2|\xi|^2|\hat{u}_4|^2 + \langle i\xi a_4\hat{u}_3, -i\xi a_5\hat{u}_5\rangle + a_4^2\langle i\xi\hat{u}_4, \hat{u}_2\rangle.$

$(I_4):$ $\partial_t\langle i\xi a_5\hat{u}_4, \hat{u}_5\rangle + a_5^2|\xi|^2|\hat{u}_4|^2 = \langle i\xi a_5\hat{u}_4, -i\xi a_6\hat{u}_6\rangle$
$\qquad\qquad + a_5^2|\xi|^2|\hat{u}_5|^2 + a_5 a_4|\xi|^2\langle\hat{u}_3, \hat{u}_5\rangle + \langle i\xi a_5\hat{u}_1, \hat{u}_5\rangle.$

$(I_{j-1}):$ $\partial_t\langle i\xi a_j\hat{u}_{j-1}, \hat{u}_j\rangle + a_j^2|\xi|^2|\hat{u}_{j-1}|^2 = \langle i\xi a_j\hat{u}_{j-1}, -i\xi a_{j+1}\hat{u}_{j+1}\rangle$
$\qquad\qquad + a_j^2|\xi|^2|\hat{u}_j|^2 + a_j a_{j-1}|\xi|^2\langle\hat{u}_{j-2}, \hat{u}_j\rangle, \quad j = 6, 7, \cdots, m-1.$

$(I_{m-1}):$ $\partial_t\langle i\xi a_m\hat{u}_{m-1}, \hat{u}_m\rangle + a_m^2|\xi|^2|\hat{u}_{m-1}|^2 = \langle i\xi a_m\hat{u}_{m-1}, -\gamma\hat{u}_m\rangle$
$\qquad\qquad + a_m^2|\xi|^2|\hat{u}_m|^2 + a_{m-1}a_m|\xi|^2\langle\hat{u}_{m-2}, \hat{u}_m\rangle.$

$(I_m):$ $\dfrac{1}{2}\partial_t|\hat{u}|^2 + \gamma|\hat{u}_m|^2 = 0.$

We note that the equations $(I_1), (I_2), (I_3), (I_4), (I_{j-1}), (I_{m-1}), (I_m)$ are parallel to (2.10), (2.6), (2.12), (2.14), (2.29), (2.29), (2.28), respectively. Hence we omit the proof for the derivation of these equations.

Step 1. We claim that for any $0 < \epsilon < M < \infty$, there is $c_{\epsilon,M} > 0$ such that for all $\epsilon \leq |\xi| \leq M$,

$$(4.5) \qquad \partial_t\{|\hat{u}|^2 + \Re E_1^{int}(\hat{u})\} + c_{\epsilon,M}|\hat{u}|^2 \leq 0,$$

where $E_1^{int}(\hat{u})$ is an interactive functional chosen such that

(4.6) $|\hat{u}|^2 + \Re E_1^{int}(\hat{u}) \sim |\hat{u}|^2.$

Proof of claim: The key observation is that all the right-hand terms of identities (I_j) $(1 \le j \le m)$ can be absorbed by the left-hand dissipative terms after taking an appropriate linear combination of all identities. In fact, let us define

$$E_1^{int}(\hat{u}) = c_1 \langle i\xi \hat{u}_2, \hat{u}_1 \rangle + c_2 \langle -\hat{u}_1, \hat{u}_4 \rangle$$

$$+c_3\{\langle i\xi a_4 \hat{u}_3, \hat{u}_4 \rangle - \langle a_4 \hat{u}_3, \hat{u}_2 \rangle\} + \sum_{j=4}^{m-1} c_j \langle i\xi a_j \hat{u}_{j-1}, \hat{u}_j \rangle.$$

By taking the real part of each identity (I_j), taking the sum $\sum_{j=1}^m c_j I_j$ with an appropriate choice of constants c_j $(1 \le j \le m)$, and applying the Cauchy-Schwarz inequality to the right-hand product terms, one can obtain (4.5), where constants c_j $(1 \le j \le m)$ depending on ϵ and M are chosen such that

$$0 < c_1 \ll c_2 \ll \cdots \ll c_{m-2} \ll c_{m-1} \ll 1 = c_m.$$

The detailed representation of the proof is omitted for brevity. (4.6) holds true due to $|E^{int}(\hat{u})| \le C_M c_{m-1} |\hat{u}|^2$ for some constant C_M depending on M and also due to smallness of c_{m-1}. $\square$

Step 2. Let $|\xi| \ge M$ for $M \ge 1$. We consider the weighted linear combination of identities (I_j) $(1 \le j \le m)$ in the form of

$$I_m + \sum_{j=1}^{m-1} c_j |\xi|^{-\beta_j} I_j,$$

where c_j $(1 \le j \le m-1)$ are chosen in terms of step 2, and $\beta_j \ge 0$ are chosen such that all the right-hand product terms can be absorbed after using the Cauchy-Schwarz inequality. In fact, multiplying (I_j) by $|\xi|^{-\beta_j}$, one has

$(I_{\beta_1}):$ $\partial_t \langle i\xi |\xi|^{-\beta_1} \hat{u}_2, \hat{u}_1 \rangle + |\xi|^{2-\beta_1} |\hat{u}_2|^2 = -\langle i\xi |\xi|^{-\beta_1} \hat{u}_2, \hat{u}_4 \rangle + |\xi|^{2-\beta_1} |\hat{u}_1|^2.$

$(I_{\beta_2}):$ $\partial_t \langle -|\xi|^{-\beta_2} \hat{u}_1, \hat{u}_4 \rangle + |\xi|^{-\beta_2} |\hat{u}_1|^2 = |\xi|^{-\beta_2} |\hat{u}_4|^2 + \langle i\xi |\xi|^{-\beta_2} \hat{u}_2, \hat{u}_4 \rangle$
$\qquad + \langle \hat{u}_1, i\xi |\xi|^{-\beta_2} a_4 \hat{u}_3 + i\xi |\xi|^{-\beta_2} a_5 \hat{u}_5 \rangle.$

$(I_{\beta_3}):$ $\partial_t \{\langle i\xi |\xi|^{-\beta_3} a_4 \hat{u}_3, \hat{u}_4 \rangle - \langle a_4 |\xi|^{-\beta_3} \hat{u}_3, \hat{u}_2 \rangle\} + a_4^2 |\xi|^{2-\beta_3} |\hat{u}_3|^2 =$
$\qquad + a_4^2 |\xi|^{2-\beta_3} |\hat{u}_4|^2 + \langle i\xi |\xi|^{-\beta_3} a_4 \hat{u}_3, -i\xi a_5 \hat{u}_5 \rangle + a_4^2 \langle i\xi |\xi|^{-\beta_3} \hat{u}_4, \hat{u}_3 \rangle.$

$(I_{\beta_4}):$ $\partial_t \langle i\xi |\xi|^{-\beta_4} a_5 \hat{u}_4, \hat{u}_5 \rangle + a_5^2 |\xi|^{2-\beta_4} |\hat{u}_4|^2 = \langle i\xi |\xi|^{-\beta_4} a_5 \hat{u}_4, -i\xi a_6 \hat{u}_6 \rangle$
$\qquad + a_5^2 |\xi|^{2-\beta_4} |\hat{u}_5|^2 + a_5 a_4 |\xi|^{2-\beta_4} \langle \hat{u}_3, \hat{u}_5 \rangle + \langle i\xi |\xi|^{-\beta_4} a_5 \hat{u}_1, \hat{u}_5 \rangle.$

$(I_{\beta_{j-1}}):$ $\partial_t \langle i\xi |\xi|^{-\beta_{j-1}} a_j \hat{u}_{j-1}, \hat{u}_j \rangle + a_j^2 |\xi|^{2-\beta_{j-1}} |\hat{u}_{j-1}|^2$
$\qquad = \langle i\xi |\xi|^{-\beta_{j-1}} a_j \hat{u}_{j-1}, -i\xi a_{j+1} \hat{u}_{j+1} \rangle + a_j^2 |\xi|^{2-\beta_{j-1}} |\hat{u}_j|^2$
$\qquad\qquad + a_j a_{j-1} |\xi|^{2-\beta_{j-1}} \langle \hat{u}_{j-2}, \hat{u}_j \rangle, \quad j = 6, 7, \cdots, m-1.$

$(I_{\beta_{m-1}}):$ $\partial_t \langle i\xi |\xi|^{-\beta_{m-1}} a_m \hat{u}_{m-1}, \hat{u}_m \rangle + a_m^2 |\xi|^{2-\beta_{m-1}} |\hat{u}_{m-1}|^2$
$\qquad = \langle i\xi |\xi|^{-\beta_{m-1}} a_m \hat{u}_{m-1}, -\gamma \hat{u}_m \rangle + a_m^2 |\xi|^{2-\beta_{m-1}} |\hat{u}_m|^2$
$\qquad\qquad + a_{m-1} a_m |\xi|^{2-\beta_{m-1}} \langle \hat{u}_{m-2}, \hat{u}_m \rangle.$

We then require β_j $(1 \le j \le m-1)$ to satisfy the following relations. From (I_{β_1}),

$$\beta_1 - 1 \ge 0, \quad \beta_1 - 2 \ge 0,$$
$$2(\beta_1 - 1) \ge (\beta_1 - 2) + (\beta_4 - 2), \quad \beta_1 - 2 \ge \beta_2,$$

where since $|\xi| \ge M$, $\beta_1 - 1 \ge 0$ is such that $\xi|\xi|^{-\beta_1}$ in the left first product term of (I_{β_1}) is bounded, $\beta_1 - 2 \ge 0$ is such that $|\xi|^{2-\beta_1}$ in the left second product term of (I_{β_1}) is bounded, $2(\beta_1 - 1) \ge (\beta_1 - 2) + (\beta_4 - 2)$ is such that the product term $\langle i\xi|\xi|^{-\beta_1}\hat{u}_2, \hat{u}_4 \rangle$ on the right first term of (I_{β_1}) can be bounded by the linear combination of the dissipative term $|\xi|^{2-\beta_1}|\hat{u}_2|^2$ in (I_{β_1}) and $|\xi|^{2-\beta_4}|\hat{u}_4|^2$ in (I_{β_4}), $\beta_1 - 2 \ge \beta_2$ is such that the term $|\xi|^{2-\beta_1}|\hat{u}_1|^2$ on the right second term of (I_{β_1}) can be bounded by the dissipative term $|\xi|^{-\beta_2}|\hat{u}_1|^2$ of (I_{β_2}). In terms of the completely same way, from (I_{β_j}) for $j = 2, 3, \cdots, m-1$, respectively, we require

$$\beta_2 \ge 0,$$
$$\beta_2 \ge \beta_4 - 2, \quad 2(\beta_2 - 1) \ge (\beta_1 - 2) + (\beta_4 - 2), \quad \beta_2 \ge \beta_3, \quad \beta_2 \ge \beta_5,$$

and

$$\beta_3 - 1 \ge 0, \quad \beta_3 \ge 0, \quad \beta_3 - 2 \ge 0,$$
$$\beta_3 - 2 \ge \beta_4 - 2, \quad \beta_3 \ge \beta_5, \quad 2(\beta_3 - 1) \ge (\beta_3 - 2) + (\beta_4 - 2),$$

and

$$\beta_4 \ge 1, \quad \beta_4 \ge 2, \quad \beta_4 \ge \beta_6, \quad \beta_4 \ge \beta_5,$$
$$2(\beta_4 - 2) \ge (\beta_3 - 2) + (\beta_5 - 2), \quad 2(\beta_4 - 1) \ge \beta_2 + (\beta_5 - 2),$$

and for $j = 6, \cdots, m-1$,

$$\beta_{j-1} \ge 1, \quad \beta_{j-1} \ge 2,$$
$$\beta_{j-1} \ge \beta_{j+1}, \quad \beta_{j-1} \ge \beta_j, \quad 2(\beta_{j-1} - 2) \ge (\beta_{j-2} - 2) + (\beta_j - 2),$$

and

$$\beta_{m-1} \ge 1, \quad \beta_{m-1} \ge 2,$$
$$2(\beta_{m-1} - 1) \ge \beta_{m-1} - 2, \quad \beta_{m-1} - 2 \ge 0, \quad 2(\beta_{m-1} - 2) \ge \beta_{m-2} - 2.$$

Let us choose

$$\beta_1 = 4, \quad \beta_2 = \beta_3 = \cdots = \beta_{m-1} = 2,$$

which satisfy all the above inequalities of β_j $(1 \le j \le m-1)$.

We now define

$$E_\infty^{int}(\hat{u}) = c_1 \langle i\xi|\xi|^{-4}\hat{u}_2, \hat{u}_1 \rangle + c_2 \langle -|\xi|^{-2}\hat{u}_1, \hat{u}_4 \rangle$$
$$+ c_3 \{ \langle i\xi|\xi|^{-2}a_4\hat{u}_3, \hat{u}_4 \rangle - \langle a_4|\xi|^{-2}\hat{u}_3, \hat{u}_2 \rangle \}$$
$$+ \sum_{j=4}^{m-1} c_j \langle i\xi|\xi|^{-2}a_j\hat{u}_{j-1}, \hat{u}_j \rangle.$$

Then, as in Step 1, one can show that for any $M \ge 1$, there is $c_M > 0$ such that for all $|\xi| \ge M$,

$$\partial_t \{|\hat{u}|^2 + \Re E_\infty^{int}(\hat{u})\} + c_M \{|\xi|^{-2}(|\hat{u}_1|^2 + |\hat{u}_2|^2) + \sum_{j=3}^{m} |\hat{u}_j|^2\} \le 0.$$

Step 3. Let $|\xi| \leq \epsilon$ for $0 < \epsilon \leq 1$. As in Step 2, we consider the weighted linear combination of identities (I_j) $(1 \leq j \leq m)$ in the form of

$$I_m + \sum_{j=1}^{m-1} c_j |\xi|^{\alpha_j} I_j,$$

where c_j $(1 \leq j \leq m-1)$ are chosen in terms of step 1, and $\alpha_j \geq 0$ are chosen such that all the right-hand product terms can be absorbed after using the Cauchy-Schwarz inequality. In fact, as in Step 2, multiplying (I_j) by $|\xi|^{\alpha_j}$, one has

$(I_{\alpha_1}):$ $\partial_t \langle i\xi|\xi|^{\alpha_1}\hat{u}_2, \hat{u}_1 \rangle + |\xi|^{2+\alpha_1}|\hat{u}_2|^2 = -\langle i\xi|\xi|^{\alpha_1}\hat{u}_2, \hat{u}_4 \rangle + |\xi|^{2+\alpha_1}|\hat{u}_1|^2.$

$(I_{\alpha_2}):$ $\partial_t \langle -|\xi|^{\alpha_2}\hat{u}_1, \hat{u}_4 \rangle + |\xi|^{\alpha_2}|\hat{u}_1|^2 = |\xi|^{\alpha_2}|\hat{u}_4|^2 + \langle i\xi|\xi|^{\alpha_2}\hat{u}_2, \hat{u}_4 \rangle$
$\qquad\qquad + \langle \hat{u}_1, i\xi|\xi|^{\alpha_2}a_4\hat{u}_3 + i\xi|\xi|^{\alpha_2}a_5\hat{u}_5 \rangle.$

$(I_{\alpha_3}):$ $\partial_t \{ \langle i\xi|\xi|^{\alpha_3}a_4\hat{u}_3, \hat{u}_4 \rangle - \langle a_4|\xi|^{\alpha_3}\hat{u}_3, \hat{u}_2 \rangle \} + a_4^2|\xi|^{2+\alpha_3}|\hat{u}_3|^2 =$
$\qquad\qquad + a_4^2|\xi|^{2+\alpha_3}|\hat{u}_4|^2 + \langle i\xi|\xi|^{\alpha_3}a_4\hat{u}_3, -i\xi a_5\hat{u}_5 \rangle + a_4^2 \langle i\xi|\xi|^{\alpha_3}\hat{u}_4, \hat{u}_3 \rangle.$

$(I_{\alpha_4}):$ $\partial_t \langle i\xi|\xi|^{\alpha_4}a_5\hat{u}_4, \hat{u}_5 \rangle + a_5^2|\xi|^{2+\alpha_4}|\hat{u}_4|^2 = \langle i\xi|\xi|^{\alpha_4}a_5\hat{u}_4, -i\xi a_6\hat{u}_6 \rangle$
$\qquad\qquad + a_5^2|\xi|^{2+\alpha_4}|\hat{u}_5|^2 + a_5a_4|\xi|^{2+\alpha_4}\langle \hat{u}_3, \hat{u}_5 \rangle + \langle i\xi|\xi|^{\alpha_4}a_5\hat{u}_1, \hat{u}_5 \rangle.$

$(I_{\alpha_{j-1}}):$ $\partial_t \langle i\xi|\xi|^{\alpha_{j-1}}a_j\hat{u}_{j-1}, \hat{u}_j \rangle + a_j^2|\xi|^{2+\alpha_{j-1}}|\hat{u}_{j-1}|^2$
$\qquad = \langle i\xi|\xi|^{\alpha_{j-1}}a_j\hat{u}_{j-1}, -i\xi a_{j+1}\hat{u}_{j+1} \rangle + a_j^2|\xi|^{2+\alpha_{j-1}}|\hat{u}_j|^2$
$\qquad\qquad + a_ja_{j-1}|\xi|^{2+\alpha_{j-1}}\langle \hat{u}_{j-2}, \hat{u}_j \rangle, \quad j = 6,7,\cdots,m-1.$

$(I_{\alpha_{m-1}}):$ $\partial_t \langle i\xi|\xi|^{\alpha_{m-1}}a_m\hat{u}_{m-1}, \hat{u}_m \rangle + a_m^2|\xi|^{2+\alpha_{m-1}}|\hat{u}_{m-1}|^2$
$\qquad = \langle i\xi|\xi|^{\alpha_{m-1}}a_m\hat{u}_{m-1}, -\gamma\hat{u}_m \rangle + a_m^2|\xi|^{2+\alpha_{m-1}}|\hat{u}_m|^2$
$\qquad\qquad + a_{m-1}a_m|\xi|^{2+\alpha_{m-1}}\langle \hat{u}_{m-2}, \hat{u}_m \rangle.$

As in the case of the large frequency domain, for $|\xi| \leq \epsilon$ with $\epsilon > 0$, in order for all the right product terms to be bounded, from equations (I_{α_j}) $(j = 1, 2, \cdots, m-1)$ above, respectively, we have to require

$$\alpha_1 + 1 \geq 0, \ 2(\alpha_1 + 1) \geq (\alpha_1 + 2) + (\alpha_4 + 2), \ \alpha_1 + 2 \geq \alpha_2,$$

and

$$\alpha_2 \geq \alpha_4 + 2, \ 2(\alpha_2 + 1) \geq (\alpha_1 + 2) + (\alpha_4 + 2), \ \alpha_2 \geq \alpha_3, \ \alpha_2 \geq \alpha_5,$$

and

$$\alpha_3 \geq \alpha_4, \ \alpha_3 \geq \alpha_5, \ 2(\alpha_3 + 1) \geq (\alpha_4 + 2) + (\alpha_1 + 2),$$

and

$$\alpha_4 \geq \alpha_6, \ \alpha_4 \geq \alpha_5,$$
$$2(\alpha_4 + 2) \geq (\alpha_3 + 2) + (\alpha_5 + 2), \ 2(\alpha_4 + 1) \geq \alpha_2 + (\alpha_5 + 2),$$

and for $j = 6, \cdots, m-1$,

$$\alpha_{j-1} \geq \alpha_{j+1}, \ \alpha_{j-1} \geq \alpha_j, \ 2(\alpha_{j-1} + 2) \geq (\alpha_{j-2} + 2) + (\alpha_j + 2),$$

and

$$\alpha_{m-1} \geq 0, \ \alpha_{m-1} + 2 \geq 0, \ 2(\alpha_{m-1} + 2) \geq \alpha_{m-2} + 2.$$

To consider the best choice of $\{\alpha_j\}_{j=1}^{m-1}$, one can see

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_j \geq \alpha_{j+1} \geq \cdots \geq \alpha_{m-2} \geq \alpha_{m-1} \geq 0 := \alpha_m,$$

with

$$\alpha_1 - \alpha_4 \geq 2,$$
$$\alpha_2 - \alpha_4 \geq 2,$$
$$\alpha_3 - \alpha_4 \geq 2,$$
$$\alpha_{j-1} - \alpha_j \leq \alpha_j - \alpha_{j+1}, \quad 4 \leq j \leq m - 1.$$

Therefore, the possible best choice satisfies

$$\alpha_1 - \alpha_4 = 2,$$
$$\alpha_2 - \alpha_4 = 2,$$
$$\alpha_3 - \alpha_4 = 2,$$
$$2 = \alpha_3 - \alpha_4 \leq \alpha_4 - \alpha_5 \leq \cdots \leq \alpha_{m-1} - \alpha_m = \alpha_{m-1} = 2,$$

which implies

$$\alpha_1 = \alpha_2 = \alpha_3 = 2(m - 4),$$
$$\alpha_j = 2(m - j - 1), \quad 4 \leq j \leq m - 1.$$

We now define

$$E_0^{int}(\hat{u}) = c_1 \langle i\xi|\xi|^{2(m-4)}\hat{u}_2, \hat{u}_1 \rangle + c_2 \langle -|\xi|^{2(m-4)}\hat{u}_1, \hat{u}_4 \rangle$$
$$+ c_3 \{ \langle i\xi|\xi|^{2(m-4)}a_4\hat{u}_3, \hat{u}_4 \rangle - \langle a_4|\xi|^{2(m-4)}\hat{u}_3, \hat{u}_2 \rangle \}$$
$$+ \sum_{j=4}^{m-1} c_j \langle i\xi|\xi|^{2(m-j-1)}a_j\hat{u}_{j-1}, \hat{u}_j \rangle.$$

Then, as in Step 1, one can show that for any $0 < \epsilon \leq 1$, there is $c_\epsilon > 0$ such that for all $|\xi| \leq \epsilon$,

$$\partial_t \{|\hat{u}|^2 + \Re E_0^{int}(\hat{u})\} + c_\epsilon \{|\xi|^{2m-8}|\hat{u}_1|^2 + |\xi|^{2m-6}|\hat{u}_2|^2 + \sum_{j=3}^{m} |\xi|^{2(m-j)}|\hat{u}_j|^2\} \leq 0,$$

which further implies that for $|\xi| \leq \epsilon$,

$$\partial_t \{|\hat{u}|^2 + \Re E_0^{int}(\hat{u})\} + c_\epsilon |\xi|^{2m-6}|\hat{u}|^2 \leq 0.$$

Step 4. For $\xi \in \mathbb{R}$ let us define

$$E^{int}(\hat{u}) = c_1 \frac{|\xi|^{2(m-4)}}{(1+|\xi|)^{2m-4}} \langle i\xi\hat{u}_2, \hat{u}_1 \rangle + c_2 \frac{|\xi|^{2(m-4)}}{(1+|\xi|)^{2m-6}} \langle -\hat{u}_1, \hat{u}_4 \rangle$$
$$+ c_3 \frac{|\xi|^{2(m-4)}}{(1+|\xi|)^{2m-6}} \{ \langle i\xi a_4\hat{u}_3, \hat{u}_4 \rangle - \langle a_4\hat{u}_3, \hat{u}_2 \rangle \}$$
$$+ \sum_{j=4}^{m-1} c_j \frac{|\xi|^{2(m-j-1)}}{(1+|\xi|)^{2(m-j)}} \langle i\xi a_j\hat{u}_{j-1}, \hat{u}_j \rangle.$$

As in Step 2 and Step 3, we consider the weighted linear combination of identities (I_j) $(1 \leq j \leq m)$ in the form of

$$
I_m + c_1 \frac{|\xi|^{2(m-4)}}{(1+|\xi|)^{2m-4}} I_1 + c_2 \frac{|\xi|^{2(m-4)}}{(1+|\xi|)^{2m-6}} I_2
$$
$$
+ c_3 \frac{|\xi|^{2(m-4)}}{(1+|\xi|)^{2m-6}} I_3 + \sum_{j=4}^{m-1} c_j \frac{|\xi|^{2(m-j-1)}}{(1+|\xi|)^{2(m-j)}} I_j,
$$

where c_j $(1 \leq j \leq m-1)$ are chosen in terms of Step 1. Thanks to computations in Step 1, Step 2, and Step 3, in the completely same way, one can deduce that for $\xi \in \mathbb{R}$,

$$
\partial_t \{|\hat{u}|^2 + \Re E^{int}(\hat{u})\} + c\{ \frac{|\xi|^{2m-8}}{(1+|\xi|)^{2m-6}} |\hat{u}_1|^2
$$
$$
+ \frac{|\xi|^{2m-6}}{(1+|\xi|)^{2m-4}} |\hat{u}_2|^2 + \sum_{j=3}^{m} \frac{|\xi|^{2(m-j)}}{(1+|\xi|)^{2(m-j)}} |\hat{u}_j|^2 \} \leq 0,
$$

which further gives

$$
\partial_t \{|\hat{u}|^2 + \Re E^{int}(\hat{u})\} + c\frac{|\xi|^{2m-6}}{(1+|\xi|)^{2m-4}} |\hat{u}|^2 \leq 0.
$$

Noticing $|\hat{u}|^2 + \Re E^{int}(\hat{u}) \sim |\hat{u}|^2$, it follows that

$$
|\hat{u}(t,\xi)| \leq C e^{-c\eta(\xi)t} |\hat{u}(0,\xi)|, \quad \eta(\xi) = \frac{|\xi|^{2m-6}}{(1+|\xi|)^{2m-4}},
$$

for all $t \geq 0$ and all $\xi \in \mathbb{R}$. Notice that the result here is consistent with (2.2) proved in Section 2.3.

4.3. **Revisit Model II.** In this section we revisit the Model II (1.1) with coefficients matrices A_m and L_m defined in (3.1). For simplicity of representation, we rewrite A_m with $m = 2n$ as

$$
A_{2n} = \begin{pmatrix} 0 & a_{12} & & & & \\ a_{21} & 0 & & & & \\ & & 0 & a_{34} & & \\ & & a_{43} & 0 & & \\ & & & & \ddots & \\ & & & & & 0 & a_{2n-1,2n} \\ & & & & & a_{2n,2n-1} & 0 \end{pmatrix},
$$

with $a_{2j-1,2j} = a_{2j,2j-1} = a_j$ for $1 \leq j \leq n$, and also choose L_m with $m = 2n$ as

$$
L_{2n} = \begin{pmatrix} 0 & & & & & & \\ & 1 & 1 & & & & \\ & -1 & 0 & & & & \\ & & & 0 & 1 & & \\ & & & -1 & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 & 1 \\ & & & & & & -1 & 0 \\ & & & & & & & 0 \end{pmatrix}.
$$

With notations above, system (1.1) can read

$$\partial_t \hat{u}_{2j-1} + i\xi a_j \hat{u}_{2j} - \hat{u}_{2j-2} = 0,$$
$$\partial_t \hat{u}_{2j} + i\xi a_j \hat{u}_{2j-1} + \hat{u}_{2j+1} + \delta_{2,2j}\hat{u}_2 = 0, \quad j = 1, 2, \cdots, n,$$

with the convention that $\hat{u}_{2n+1} \equiv 0$ and $\hat{u}_0 \equiv 0$. As for the model I, we can obtain the following estimates

$$\text{(4.7)} \qquad \frac{1}{2}\partial_t |\hat{u}|^2 + |\hat{u}_2|^2 = 0,$$

and

$$\text{(4.8)} \quad \partial_t \Re \langle i\xi a_1 \hat{u}_1, \sum_{j=1}^{n} (-i\xi)^{1-j} (\prod_{k=2}^{j} a_k)^{-1} \hat{u}_{2j} \rangle + c a_1^2 \xi^2 |\hat{u}_1|^2$$

$$\lesssim (1 + |\xi|)^2 |\hat{u}_2|^2 + \Re \langle \xi^2 a_1^2 \hat{u}_2, \sum_{j=2}^{n} (-i\xi)^{1-j} (\prod_{k=2}^{j} a_k)^{-1} \hat{u}_{2j} \rangle,$$

and

$$\text{(4.9)} \ \ \partial_t \Re \langle \hat{u}_{2j-1}, u_{2j-2} \rangle + c|\hat{u}_{2j-1}|^2 \lesssim |\hat{u}_{2j-2}|^2 + \xi^2 |\hat{u}_{2j-3}|^2 + \Re \langle -i\xi a_j \hat{u}_{2j}, \hat{u}_{2j-2} \rangle,$$

and

$$\text{(4.10)} \quad \partial_t \Re \langle i\xi a_j \hat{u}_{2j}, \hat{u}_{2j-1} \rangle + c a_j^2 \xi^2 |\hat{u}_{2j}|^2$$

$$\lesssim |\hat{u}_{2j-2}|^2 + a_j^2 \xi^2 |\hat{u}_{2j-1}|^2 + \Re \langle -i\xi a_j \hat{u}_{2j+1}, \hat{u}_{2j-1} \rangle,$$

for $j = 2, 3, \cdots, n$. Indeed, by using the equations (3.22), (3.26), (3.28), (3.30) derived in Subsection 3.3, we can get (4.7), (4.8), (4.9), (4.10), immediately.

Let us denote (4.7), (4.8), (4.9), (4.10) by (I_1), (I_2), (I_{2j-1}) and (I_{2j}), respectively, where $j = 2, 3, \cdots, n$. Consider the linear combination of all $2n$ number of equations

$$\sum_{j=1}^{n} (c_{2j-1} I_{2j-1} + c_{2j} I_{2j}),$$

where $c_1 = 1$, and $c_k > 0$ $(k = 2, 3, \cdots, 2n)$ are constants to be properly chosen. It is straightforward to verify that for any $0 < \epsilon < M < \infty$, one can choose constants c_k $(1 \leq k \leq 2n)$ depending on ϵ and M, with

$$0 < c_{2n} \ll c_{2n-1} \ll \cdots \ll c_{2j} \ll c_{2j-1} \ll \cdots \ll c_3 \ll c_2 \ll 1 = c_1,$$

such that there is $c_{\epsilon,M} > 0$ such that for all $\epsilon \leq |\xi| \leq M$,

$$\partial_t \{|\hat{u}|^2 + \Re E_1^{int}(\hat{u})\} + c_{\epsilon,M} |\hat{u}|^2 \leq 0,$$

where $E_1^{int}(\hat{u})$ is an interactive functional given by

$$E_1^{int}(\hat{u}) = c_2 \langle i\xi a_1 \hat{u}_1, \sum_{j=1}^{n} (-i\xi)^{1-j} (\prod_{k=2}^{j} a_k)^{-1} \hat{u}_{2j} \rangle$$

$$+ \sum_{j=2}^{n} \{c_{2j-1} \langle \hat{u}_{2j-1}, u_{2j-2} \rangle + c_{2j} \langle i\xi a_j \hat{u}_{2j}, \hat{u}_{2j-1} \rangle\},$$

satisfying

$$|\hat{u}|^2 + \Re E_1^{int}(\hat{u}) \sim |\hat{u}|^2, \quad \text{for } \epsilon \leq |\xi| \leq M.$$

Furthermore, we can consider the frequency weighted linear combination in the form of

$$(4.11) \qquad \sum_{j=1}^{n}\{c_{2j-1}\frac{|\xi|^{\alpha_{2j-1}}}{(1+|\xi|)^{\alpha_{2j-1}+\beta_{2j-1}}}I_{2j-1}+c_{2j}\frac{|\xi|^{\alpha_{2j}}}{(1+|\xi|)^{\alpha_{2j}+\beta_{2j}}}I_{2j}\},$$

where $\alpha_1 = \beta_1 = 0$. As for the model I, we use the same strategy to determine the choice of constants

$$\alpha_2, \alpha_3, \cdots, \alpha_{2n}; \quad \beta_2, \beta_3, \cdots, \beta_{2n}.$$

In fact, by considering the low frequency domain $|\xi| \leq \epsilon$ with $\epsilon \leq 1$, $\alpha_2, \alpha_3, \cdots, \alpha_{2n}$ are required to satisfy inequalities

$$2 - j + \alpha_2 \geq 0, j = 2, 3, \cdots, n,$$
$$\alpha_2 \geq 0,$$
$$2 + \alpha_2 \geq 0,$$
$$\alpha_3 \geq 0, 2 + \alpha_3 \geq 2 + \alpha_2,$$
$$\alpha_4 \geq 0, 2 + \alpha_4 \geq \alpha_3,$$
$$\alpha_{2j} \geq 2 + \alpha_{2j-2}, 2 + \alpha_{2j} \geq \alpha_{2j-1},$$
$$\alpha_{2j-1} \geq 2 + \alpha_{2j-2}, 2 + \alpha_{2j-1} \geq \alpha_{2j-3}, \ j = 3, 4, \cdots, n,$$

and

$$2(3 - j + \alpha_2) \geq \alpha_{2j} + 2, j = 2, \cdots, n,$$
$$1 + \alpha_3 \geq \frac{2 + \alpha_4}{2},$$
$$\alpha_{2j} \geq \frac{1}{2}(\alpha_{2j+1} + \alpha_{2j-1}) - 1,$$
$$\alpha_{2j+1} \geq \frac{1}{2}(\alpha_{2j+2} + \alpha_{2j}) - 1, j = 2, \cdots, n-1.$$

One can take the best choice

$$\alpha_2 = 4(n - 2),$$
$$\alpha_{2j-1} = \alpha_{2j} = 4(n - 2) + 2(j - 2), \ j = 2, 3, \cdots, n.$$

Similarly, by considering the high frequency domain $|\xi| \geq M$ with $M \geq 1$, constants $\beta_2, \beta_3, \cdots, \beta_{2n}$ are required to satisfy inequalities

$$\beta_2 - 2 \geq 0,$$
$$\beta_3 \geq 0, \beta_3 - 2 \geq \beta_2 - 2,$$
$$\beta_4 \geq 0, \beta_4 - 2 \geq \beta_3,$$
$$\beta_{2j} \geq \beta_{2j-2}, \beta_{2j} - 2 \geq \beta_{2j-1},$$
$$\beta_{2j-1} \geq \beta_{2j-2} - 2, \beta_{2j-1} - 2 \geq \beta_{2j-3}, \ j = 3, \cdots, n,$$

and

$$2(\beta_3 - 1) \geq \beta_4 - 2,$$
$$\beta_2 + j - 2 \geq 0, 2(\beta_2 + j - 3) \geq \beta_{2j} - 2, j = 2, \cdots, n,$$
$$2(\beta_{2j} - 1) \geq \beta_{2j+1} + \beta_{2j-1},$$
$$2(\beta_{2j+1} - 1) \geq (\beta_{2j+2} - 2) + (\beta_{2j} - 2), j = 2, \cdots, n-1.$$

One can take the best choice

$$\beta_{2j} = \beta_{2j+1} = 2j, \quad j = 1, 2, \cdots, n.$$

Now, by (4.11), let us define the interactive functional

$$E^{int}(\hat{u}) = c_2 \frac{|\xi|^{\alpha_2}}{(1+|\xi|)^{\alpha_2+\beta_2}} \langle i\xi a_1 \hat{u}_1, \sum_{j=1}^{n}(-i\xi)^{1-j}(\prod_{k=2}^{j} a_k)^{-1}\hat{u}_{2j}\rangle$$

$$+ \sum_{j=2}^{n}\{c_{2j-1}\frac{|\xi|^{\alpha_{2j-1}}}{(1+|\xi|)^{\alpha_{2j-1}+\beta_{2j-1}}} \langle \hat{u}_{2j-1}, \hat{u}_{2j-2}\rangle$$

$$+ c_{2j}\frac{|\xi|^{\alpha_{2j}}}{(1+|\xi|)^{\alpha_{2j}+\beta_{2j}}} \langle i\xi a_j \hat{u}_{2j}, \hat{u}_{2j-1}\rangle\},$$

that is,

$$E^{int}(\hat{u}) = c_2 \frac{|\xi|^{4n-8}}{(1+|\xi|)^{4n-6}} \langle i\xi a_1 \hat{u}_1, \sum_{j=1}^{n}(-i\xi)^{1-j}(\prod_{k=2}^{j} a_k)^{-1}\hat{u}_{2j}\rangle$$

$$+ \sum_{j=2}^{n}\{c_{2j-1}\frac{|\xi|^{4n+2j-12}}{(1+|\xi|)^{4n+4j-14}} \langle \hat{u}_{2j-1}, \hat{u}_{2j-2}\rangle$$

$$+ c_{2j}\frac{|\xi|^{4n+2j-12}}{(1+|\xi|)^{4n+4j-12}} \langle i\xi a_j \hat{u}_{2j}, \hat{u}_{2j-1}\rangle\},$$

and also define the energy dissipation rate

$$D(\hat{u}) = |\hat{u}_2|^2 + \frac{|\xi|^{2+\alpha_2}}{(1+|\xi|)^{\alpha_2+\beta_2}} |\hat{u}_1|^2$$

$$+ \sum_{j=2}^{n}\{\frac{|\xi|^{\alpha_{2j-1}}}{(1+|\xi|)^{\alpha_{2j-1}+\beta_{2j-1}}} |\hat{u}_{2j-1}|^2 + \frac{|\xi|^{2+\alpha_{2j}}}{(1+|\xi|)^{\alpha_{2j}+\beta_{2j}}} |\hat{u}_{2j}|^2\},$$

that is,

$$D(\hat{u}) = |\hat{u}_2|^2 + \frac{|\xi|^{4n-6}}{(1+|\xi|)^{4n-6}} |\hat{u}_1|^2$$

$$+ \sum_{j=2}^{n}\{\frac{|\xi|^{4n+2j-12}}{(1+|\xi|)^{4n+4j-14}} |\hat{u}_{2j-1}|^2 + \frac{|\xi|^{4n+2j-10}}{(1+|\xi|)^{4n+4j-12}} |\hat{u}_{2j}|^2\}.$$

Then it follows that

$$\partial_t\{|\hat{u}|^2 + \Re E^{int}(\hat{u})\} + cD(\hat{u}) \leq 0,$$

for all $t \geq 0$ and all $\xi \in \mathbb{R}$. Noticing

$$|\hat{u}|^2 + \Re E^{int}(\hat{u}) \sim |\hat{u}|^2,$$

and

$$D(\hat{u}) \gtrsim \frac{|\xi|^{6n-10}}{(1+|\xi|)^{8n-12}} |\hat{u}|^2,$$

one can see that the Model II (1.1), where coefficients matrices A_m and L_m are defined in (3.1) with $m = 2n$, enjoys the dissipative structure

$$|\hat{u}(t,\xi)|^2 \leq Ce^{-c\eta(\xi)t}|\hat{u}(0,\xi)|,$$

with

$$\eta(\xi) = \frac{|\xi|^{6n-10}}{(1+|\xi|)^{8n-12}} = \frac{|\xi|^{3m-10}}{(1+|\xi|)^{4m-12}}.$$

Hence the derived result here is consistent with (3.2) proved in Theorem 3.1.

References

[1] F. Ammar Khodja, A. Benabdallah, J.E. Muñoz Rivera and R. Racke, *Energy decay for Timoshenko systems of memory type*, J. Differential Equations, **194** (2003), 82–115.

[2] K. Beauchard and E. Zuazua, *Large time asymptotics for partially dissipative hyperbolic systems*, Arch. Rational Mech. Anal., **199** (2011), 177–227.

[3] S. Bianchini, B. Hanouzet and R. Natalini, Asymptotic behavior of smooth solutions for partially dissipative hyperbolic systems with a convex entropy, *Comm. Pure Appl. Math.* **60** (2007), 1559–1622.

[4] G.Q. Chen, C. Levermore and T.-P. Liu, Hyperbolic conservation laws with stiff relaxation terms and entropy, *Comm. Pure Appl. Math.* **47** (1994), 787–830.

[5] C.M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*. Third edition. Springer-Verlag, Berlin, 2010. xxxvi+708 pp.

[6] R.-J. Duan, Dissipative property of the Vlasov-Maxwell-Boltzmann System with a uniform ionic background, *SIAM J. Math. Anal.* **43** (2011), 2732–2757.

[7] R.-J. Duan, *Global smooth flows for the compressible Euler-Maxwell system: Relaxation case*, J. Hyperbolic Differ. Equ. **8** (2011), 375–413.

[8] R.-J. Duan, Green's function and large time behavior of the Navier-Stokes-Maxwell system, *Anal. Appl. (Singap.)* **10** (2012), 133–197.

[9] R.-J. Duan and R.M. Strain, *Optimal large-time behavior of the Vlasov-Maxwell-Boltzmann system in the whole space*, Comm. Pure Appl. Math. **64** (2011), 1497–1546.

[10] H.D. Fernández Sare and R. Racke, *On the stability of damped Timoshenko systems: Cattaneo versus Fourier law*, Arch. Rational Mech. Anal. **194** (2009), 221–251.

[11] B. Hanouzet and R. Natalini, Global existence of smooth solutions for partially dissipative hyperbolic systems with a convex entropy, *Arch. Rational Mech. Anal.* **169** (2003), 89–117.

[12] T. Hosono and S. Kawashima, *Decay property of regularity-loss type and application to some nonlinear hyperbolic-elliptic system*, Math. Models Meth. Appl. Sci., **16** (2006), 1839–1859.

[13] K. Ide, K. Haramoto and S. Kawashima, *Decay property of regularity-loss type for dissipative Timoshenko system*, Math. Models Meth. Appl. Sci., **18** (2008), 647–667.

[14] K. Ide and S. Kawashima, *Decay property of regularity-loss type and nonlinear effects for dissipative Timoshenko system*, Math. Models Meth. Appl. Sci., **18** (2008), 1001–1025.

[15] S. Kawashima and M. Okada, Smooth global solutions for the one-dimensional equations in magnetohydrodynamics, Proc. Japan Acad., **58** (1982), 384-387.

[16] S. Kawashima, W.-A. Yong, Dissipative structure and entropy for hyperbolic systems of balance laws, *Arch. Rat. Mech. Anal.* **174** (2004), 345–364.

[17] S. Kawashima and W.-A. Yong, Decay estimates for hyperbolic balance laws, ZAA (J. Anal. Appl.), **28** (2009), 1-33.

[18] T. Kubo and S. Kawashima, *Decay property of regularity-loss type and nonlinear effects for some hyperbolic-elliptic system*, Kyushu J. Math., **63** (2009), 1–21.

[19] T.-P. Liu, Hyperbolic conservation laws with relaxation. *Comm. Math. Phys.* **108** (1987), 153–175.

[20] Y. Liu and S. Kawashima, *Global existence and asymptotic behavior of solutions for quasilinear dissipative plate equation*, Discrete Contin. Dyn. Syst., **29** (2011), 1113–1139.

[21] Y. Liu and S. Kawashima, *Global existence and decay of solutions for a quasi-linear dissipative plate equation*, J. Hyperbolic Differential Equations, to appear.

[22] Y. Liu and S. Kawashima, *Decay property for a plate equation with memory-type dissipation*, Kinetic and Related Models, **4** (2011), 531–547.

[23] Y. Liu and S. Kawashima, *Decay property for the Timoshenko system with memory-type dissipation*, Math. Models Meth. Appl. Sci., to appear.

[24] C.R. da Luz and R.C. Charão, *Asymptotic properties for a semilinear plate equation in unbounded domains*, J. Hyperbolic Differential Equations, **6** (2009), 269–294.

[25] C. Mascia and R. Natalini, On relaxation hyperbolic systems violating the ShizutaKawashima condition, *Arch. Rational Mech. Anal.* **195** (2010), 729–762.

[26] N. Mori and S. Kawashima, *Decay property for the Timoshenko system with Fourier's type heat conduction*, J. Hyperbolic Differential Equations, **11** (2014), 135–157.

[27] N. Mori and S. Kawashima, Decay property of the Timoshenko-Cattaneo system, preprint.

[28] J.E. Muñoz Rivera and R. Racke, *Global stability for damped Timoshenko systems*, Discrete Contin. Dyn. Syst., **9** (2003), 1625–1639.

[29] T. Ruggeri and D. Serre, Stability of constant equilibrium state for dissipative balance laws system with a convex entropy, *Quart. Appl. Math.* **62** (2004), 163–179.

[30] M.L. Santos, D.S. Almeida Júnior and J.E. Muñoz Rivera, *The stability number of the Timoshenko system with second sound*, *J. Differential Equations* **253** (2012), 2715–2733.

[31] Y. Shizuta and S. Kawashima, *Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation*, Hokkaido Math. J., **14** (1985), 249–275.

[32] Y. Sugitani and S. Kawashima, *Decay estimates of solutions to a semi-linear dissipative plate equation*, J. Hyperbolic Differerential Equations, **7** (2010), 471–501.

[33] Y. Ueda, R. Duan and S. Kawashima, Decay structure for symmetric hyperbolic systems with non-symmetric relaxation and its application. Arch. Ration. Mech. Anal. **205** (2012), no. 1, 239–266.

[34] Y. Ueda and S. Kawashima, *Decay property of regularity-loss type for the Euler-Maxwell system*, Methods and Applications of Analysis, **205** (2011), 239–266.

[35] Y. Ueda, S. Wang and S. Kawashima, *Dissipative structure of the regularity-loss type and time asymptotic decay of solutions for the Euler-Maxwell system*, SIAM Journal on Mathematical Analysis, **44** (2012), no. 3, 2002–2017.

[36] T. Umeda, S. Kawashima and Y. Shizuta, *On the devay of solutions to the linearized equations of electro-magneto-fluid dynamics*, Japan J. Appl. Math., **1** (1984), 435–457.

[37] C. Villani, Hypocoercivity, *Memoirs Amer. Math. Soc.* **202** (2009), no. 950, iv+141 pp.

[38] W.-A. Yong, Entropy and global existence for hyperbolic balance laws, *Arch. Rational Mech. Anal.* **172** (2004), 247–266.

(YU) Faculty of Maritime Sciences, Kobe University, Kobe 658-0022, Japan
E-mail address: `ueda@maritime.kobe-u.ac.jp`

(RJD) Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong
E-mail address: `rjduan@math.cuhk.edu.hk`

(SK) Faculty of Mathematics, Kyushu University, Fukuoka 819-0395, Japan
E-mail address: `kawashim@math.kyushu-u.ac.jp`