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The relativistic Boltzmann equation for soft potentials

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The paper concerns the Cauchy problem on the relativistic Boltzmann equation for soft potentials in a periodic box. We show that the global-in-time solutions around relativistic Maxwellians exist in the weighted L^∞ perturbation framework and also approach equilibrium states in large time in the weighted L^2 framework at the rate of $\exp(-\lambda t^\beta)$ for some $\lambda > 0$ and $\beta \in (0, 1)$. The proof is based on the nonlinear L^2 energy method and nonlinear L^∞ pointwise estimate with appropriate exponential weights in momentum. The results extend those on the classical Boltzmann equation by Caflisch [2,3] and Strain and Guo [31] to the relativistic version, and also improve the recent result on almost exponential time-decay by Strain [28] to the exponential rate. Moreover, we study the propagation of spatial regularity for the obtained solutions and also the large time behavior in the corresponding regular Sobolev space, provided that the spatial derivatives of initial data are bounded, not necessarily small.

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1. Introduction

The relativistic Boltzmann equation, which is a fundamental model describing the motion of fast moving particles in kinetic theory, takes the form of

$$P \otimes \partial_X F = -\mathcal{C}(F, F). \tag{1.1}$$

Here $\otimes$ represents the Lorentz inner product $(+---)$ of 4-vector. As is customary we write $X = (x_0, x)$ with $x \in \mathbb{T}^3$ and $x_0 = -t$, and $P = (p_0, p)$ with momentum $p \in \mathbb{R}^3$ and energy $p_0 = \sqrt{c^2 + |p|^2}$, where c denotes the speed of light. For convenience of presentation, we rewrite (1.1) supplemented with initial data as

$$\partial_t F + \hat{p} \cdot \nabla_x F = \mathcal{Q}(F, F), \quad F(0, x, p) = F_0(x, p), \tag{1.2}$$

with $\mathcal{Q}(F, F) = \mathcal{C}(F, F)/p_0$, where the unknown $F = F(t, x, p)$ stands for the density distribution function of time $t \geq 0$, space $x \in \mathbb{T}^3$ and momentum $p \in \mathbb{R}^3$. Here the dot represents the standard Euclidean dot product, and the normalized velocity of a particle is denoted as

$$\hat{p} = c\frac{p}{p_0} = \frac{p}{\sqrt{1 + |p|^2/c^2}}.$$

It is known that the constant equilibrium state of (1.1) is the global relativistic Maxwellian, also called the Jütter solution, in the form of

$$J(p) = \frac{\exp\{-cp_0/(k_B T)\}}{4\pi c k_B T K_2(c^2/(k_B T)},$$

where $K_2(z) := \frac{z^2}{2}\int_1^\infty e^{-zt}(t^2 - 1)^{3/2} dt$ is the Bessel function, T is temperature and k_B is the Boltzmann's constant. For notational simplicity we normalize all the physical constants to be one. Then the normalized global relativistic Maxwellian takes the form of

$$J(p) = \frac{e^{-p_0}}{4\pi}, \quad p_0 = \sqrt{1 + |p|^2}. \tag{1.3}$$

Moreover, to describe the relativistic Boltzmann collision term, we introduce the relative momentum g as

$$g = g(p,q) = \sqrt{2(p_0 q_0 - p \cdot q - 1)}, \tag{1.4}$$

and also the quantity s as

$$s = s(p,q) = g^2 + 4 = 2(p_0 q_0 - p \cdot q + 1). \tag{1.5}$$

The Mϕller velocity is given by

$$v_\phi = v_\phi(p,q) = \sqrt{\left|\frac{p}{p_0} - \frac{q}{q_0}\right|^2 - \left|\frac{p}{p_0} \times \frac{q}{q_0}\right|^2} = \frac{g\sqrt{s}}{p_0 q_0}. \tag{1.6}$$

Then we may express the collision operator $\mathcal{Q}(F,G)$ in the form (see [7,12,14])

$$\mathcal{Q}(F,G) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} v_\phi \sigma(g,\theta)[F(p')G(q') - F(p)G(q)]\, dq d\omega, \tag{1.7}$$

where $d\omega$ is the surface measure on the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$, and $\sigma(g,\theta)$ is the scattering kernel. As is standard, we abbreviate $F(t,x,p)$ by $F(p)$, etc., and use primes to represent the results of collisions. The conservation of momentum and energy is

$$p' + q' = p + q, \tag{1.8}$$

$$\sqrt{1+|p'|^2} + \sqrt{1+|q'|^2} = \sqrt{1+|p|^2} + \sqrt{1+|q|^2}, \tag{1.9}$$

for any $p, q \in \mathbb{R}^3$. Finally, the scattering angle θ is defined as follows. Given 4-vectors $P = (p_0, p)$ and $Q = (q_0, q)$, with the Lorentz inner product, the angle θ is given by

$$\cos\theta = \frac{(P-Q)\otimes(P'-Q')}{(P-Q)\otimes(P-Q)}.$$

Here $q_0 = \sqrt{1+|q|^2}$ and $q_0' = \sqrt{1+|q'|^2}$. As in [28], by (1.8) and (1.9), the post-collisional momentum can be written:

$$\begin{cases} p' = \dfrac{p+q}{2} + \dfrac{g}{2}\Big(\omega + (\varrho - 1)(p+q)\dfrac{(p+q)\cdot\omega}{|p+q|^2}\Big), \\ q' = \dfrac{p+q}{2} - \dfrac{g}{2}\Big(\omega + (\varrho - 1)(p+q)\dfrac{(p+q)\cdot\omega}{|p+q|^2}\Big), \end{cases} \tag{1.10}$$

where $\varrho = (p_0 + q_0)/\sqrt{s}$. The energies are then

$$\begin{cases} p_0' = \dfrac{p_0+q_0}{2} + \dfrac{g}{2\sqrt{s}}\omega\cdot(p+q), \\ q_0' = \dfrac{p_0+q_0}{2} - \dfrac{g}{2\sqrt{s}}\omega\cdot(p+q). \end{cases}$$

For a smooth function $F(p)$ the collision operator satisfies

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ p \\ p_0 \end{pmatrix} \mathcal{Q}(F,F)(p)\, dp = 0.$$

With this identity in hand, by integrating the relativistic Boltzmann equation (1.2), we obtain the conservation of mass, momentum and energy for solutions as

$$\frac{d}{dt}\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ p \\ p_0 \end{pmatrix} F(t,x,p)\, dpdx = 0.$$

We define the standard perturbation $f(t,x,p)$ to the relativistic Maxwellian (1.3) as $F = J + \sqrt{J}f$. The Cauchy problem on the relativistic Boltzmann equation (1.2) for $f(t,x,p)$ is given by

$$\partial_t f + \hat{p}\cdot\nabla_x f + Lf = \Gamma(f,f), \qquad f(0,x,p) = f_0(x,p). \tag{1.11}$$

Here the standard linearized collision operator L is (see [7,12])

$$Lf = -\frac{1}{\sqrt{J}}\mathcal{Q}(J,\sqrt{J}f) - \frac{1}{\sqrt{J}}\mathcal{Q}(\sqrt{J}f,J) = \nu(p)f - Kf.$$

Above the multiplication operator takes the form

$$\nu(p) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} v_\phi \sigma(g,\theta) J(q)\, dqd\omega. \tag{1.12}$$

Notice that $K = K_2 - K_1$ is given by [7,12]:

$$K_1 f = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} v_\phi \sigma(g,\theta)\sqrt{J(q)J(p)}f(q)\, dqd\omega,$$

and

$$K_2 f = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} v_\phi \sigma(g,\theta)\sqrt{J(q)}\{\sqrt{J(q')}f(p') + \sqrt{J(p')}f(q')\}\, dqd\omega.$$

The nonlinear collision operator $\Gamma(f_1,f_2)$ is defined by

$$\Gamma(f_1,f_2) = \frac{1}{\sqrt{J}}\mathcal{Q}(\sqrt{J}f_1,\sqrt{J}f_2) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} v_\phi \sigma(g,\theta)\sqrt{J(q)}[f_1(p')f_2(q') - f_1(p)f_2(q)]\, dqd\omega. \tag{1.13}$$

We now state the conditions on the collisional cross-section as in [7,28]:

Hypothesis on the collision kernel: For soft potentials we assume the collision kernel $\sigma(g,\theta)$ in (1.7) satisfies the following growth/decay estimates:

$$c_1\Big(\frac{g}{\sqrt{s}}\Big)g^{-b}\sigma_0(\theta) \le \sigma(g,\theta) \le c_2 g^{-b}\sigma_0(\theta). \tag{1.14}$$

We consider $b \in (0,2)$ and angular factors $0 \le \sigma_0(\theta) \le C\sin^\gamma\theta$ with $\gamma \ge 0$. Additionally $\sigma_0(\theta)$ should be non-zero on a set of positive measure.

This hypothesis for soft potentials contains the general physical assumption on the kernel which was introduced in [7] (and we add the corresponding necessary lower bounds as in [28], where $0 < b < \min\{4, 4+\gamma\}$ is supposed). Due to some technical reason, we impose that the collision kernel satisfies the above assumptions, particularly $0 < b < 2$ for consideration of soft potentials.

It is well known that the linearized collision operator L is non-negative and self-adjoint. And for fixed (t,x), the null space of L is given by

$$\mathcal{N} = \mathrm{span}\{\sqrt{J}, p_1\sqrt{J}, p_2\sqrt{J}, p_3\sqrt{J}, p_0\sqrt{J}\}. \tag{1.15}$$

Define $\mathbf{P}$ as the orthogonal projection in $L^2(\mathbb{R}^3_p)$ to the null space $\mathcal{N}$ and $\mathcal{N}^\perp$ denotes the orthogonal complement of the null space $\mathcal{N}$. Any function $f(t,x,p)$ can be decomposed into

$$f(t,x,p) = \mathbf{P}f(t,x,p) + (\mathbf{I}-\mathbf{P})f(t,x,p),$$

where (t,x) is taken as a parameter. According to the basis in (1.15), it holds that

$$\mathbf{P}f(t,x,p) = \{a^f(t,x) + \sum_{j=1}^{3} b_j^f(t,x)p_j + c^f(t,x)p_0\}\sqrt{J}.$$

To present the results in this paper, the following notations are needed. Let $\alpha = [\alpha_1,\alpha_2,\alpha_3]$ and $\partial^\alpha \equiv \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$. If each component of α is not greater than the corresponding one of $\overline{\alpha}$, we use the standard notation $\alpha \le \overline{\alpha}$. And $\alpha < \overline{\alpha}$ means that $\alpha \le \overline{\alpha}$ and $|\alpha| < |\overline{\alpha}|$. $C_\alpha^{\bar{\alpha}}$ is the usual binomial coefficient. In addition, $\langle\cdot,\cdot\rangle$ is used to denote the standard L^2 inner product in $\mathbb{R}^3_p$, and $(\cdot,\cdot)$ for the one in $\mathbb{R}^3_x \times \mathbb{R}^3_p$. $|\cdot|_2$ denotes the L^2 norm in $\mathbb{R}^3_p$, and $\|\cdot\|$ denotes the L^2 norms in $\mathbb{R}^3_x$ or $\mathbb{R}^3_x \times \mathbb{R}^3_p$ without any ambiguity in the following discussion. And C denotes a generic positive constant which may vary from line to line. The notation $\mathcal{A} \sim \mathcal{B}$ means that there exist two generic positive constants $C_1 < C_2$ such that $C_1\mathcal{B} \le \mathcal{A} \le C_2\mathcal{B}$.

Define a weight function in p by

$$\varpi = \varpi(l,\vartheta)(p) \equiv p_0^{\frac{lb}{2}} \exp\Big(\tau p_0^{\vartheta}\Big).$$

Here $l \in \mathbb{R}$, $\tau > 0$ and $\vartheta \in [0,1]$. Denote the L^2 norms as

$$|f|_2^2 = \int_{\mathbb{R}^3} |f(p)|^2 \, dp, \quad \|f\|^2 = \int_{\mathbb{T}^3} |f|_2^2 \, dx,$$

and the weighted L^2 norms as

$$|f|_{l,\vartheta}^2 = \int_{\mathbb{R}^3} \varpi^2(l,\vartheta)|f(p)|^2 \, dp, \quad \|f\|_{l,\vartheta}^2 = \int_{\mathbb{T}^3} |f|_{l,\vartheta}^2 \, dx.$$

Define the weighted dissipation norm as

$$|f|_{\nu,l,\vartheta}^2 = \int_{\mathbb{R}^3} \varpi^2(l,\vartheta)\nu(p)|f(p)|^2 \, dp, \quad \|f\|_{\nu,l,\vartheta}^2 = \int_{\mathbb{T}^3} |f|_{\nu,l,\vartheta}^2 \, dx.$$

We also work with the L^∞ norms

$$|f|_\infty = \operatorname*{ess\,sup}_{p\in\mathbb{R}^3} |f(p)|, \quad \|f\|_\infty = \operatorname*{ess\,sup}_{x\in\mathbb{T}^3, p\in\mathbb{R}^3} |f(x,p)|,$$
$$|f|_{\infty,l,\vartheta} = \operatorname*{ess\,sup}_{p\in\mathbb{R}^3} |\varpi(l,\vartheta)f(p)|, \quad \|f\|_{\infty,l,\vartheta} = \operatorname*{ess\,sup}_{x\in\mathbb{T}^3, p\in\mathbb{R}^3} |\varpi(l,\vartheta)f(x,p)|.$$

If some index is zero, we drop the index, for example, $\|f\|_{\nu,l,0} = \|f\|_{\nu,l}$ if $\vartheta = 0$ and the same for the other norms.

Corresponding to the linearized operator L, it is shown in [28] or Remark 2.2 that there exists a constant $\delta > 0$ such that

$$\langle Lf, f\rangle \geq \delta|(\mathbf{I} - \mathbf{P})f|_\nu^2. \tag{1.16}$$

For any nonnegative integer N, we define the following instant functionals as

$$\mathfrak{E}_{N,l,\vartheta}(f)(t) = \sum_{|\alpha|=N} \|\partial^\alpha f(t)\|_{\infty,l,\vartheta}, \quad \mathcal{E}_{N,l,\vartheta}(f)(t) \sim \sum_{|\alpha|=N} \|\partial^\alpha f(t)\|_{l,\vartheta}^2. \tag{1.17}$$

Correspondingly, the energy dissipation functional $\mathcal{D}_{N,l,\vartheta}(f)(t)$ satisfies

$$\mathcal{D}_{N,l,\vartheta}(f)(t) \sim \sum_{|\alpha|=N} \|\partial^\alpha f(t)\|_{\nu,l,\vartheta}^2. \tag{1.18}$$

We will also write $\mathcal{E}_{0,l,\vartheta}(f)(t) = \mathcal{E}_{l,\vartheta}(f)(t)$, $\mathcal{E}_{N,l,0}(f)(t) = \mathcal{E}_{N,l}(f)(t)$ and also for the other functionals.

By assuming that initially $F_0(x,p)$ has the same mass, momentum and total energy as the relativistic Maxwellian $J(p)$, then for any $t \geq 0$,

$$\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ p \\ p_0 \end{pmatrix} \sqrt{J(p)} f(t,x,p)\, dp dx = 0. \tag{1.19}$$

We are now ready to state the main results of the paper. The first one concerns the exponential rate of convergence in terms of the L^2 energy functional $\mathcal{E}_l(f)(t)$ for global-in-time solutions which exist in L^∞ perturbation framework as proved in [28].

Theorem 1.1. *Let $l \geq 0$, $l_0 > 3/b$, $\vartheta \in [0,1]$ and $\tau > 0$. If $\vartheta = 1$, restrict $\tau > 0$ small enough. Choose initial data $F_0(x,p) = J(p) + \sqrt{J(p)} f_0(x,p) \geq 0$ such that $f_0(x,p)$ satisfies (1.19) and is continuous in $\mathbb{T}^3 \times \mathbb{R}^3$. There exists an instant L^∞ functional $\mathfrak{E}_{l+l_0,\vartheta}(f)(t)$ such that if $\mathfrak{E}_{l+l_0,\vartheta}(f_0)$ is sufficiently small, there exists a unique global solution $f(t,x,p)$ to the relativistic Boltzmann equation (1.11) with $F(t,x,p) = J(p) + \sqrt{J(p)} f(t,x,p) \geq 0$ and*

$$\sup_{0\leq t<\infty} \mathfrak{E}_{l+l_0,\vartheta}(f)(t) \leq C\mathfrak{E}_{l+l_0,\vartheta}(f_0).$$

Moreover, there exist instant L^2 functionals $\mathcal{E}_{l,\vartheta}(f)(t)$ and $\mathcal{D}_{l,\vartheta}(f)(t)$ as (1.17) and (1.18) such that

$$\frac{d}{dt}\mathcal{E}_{l,\vartheta}(f)(t) + \mathcal{D}_{l,\vartheta}(f)(t) \leq 0,$$

for all $t \geq 0$. In particular, if $\vartheta > 0$ and $\beta = \frac{\vartheta}{\vartheta + \frac{b}{2}}$, there exists a constant $\lambda_0 > 0$ such that for all $t \geq 0$,

$$\mathcal{E}_l(f)(t) \leq Ce^{-\lambda_0 t^\beta}\mathcal{E}_{l,\vartheta}(f_0). \tag{1.20}$$

The second result is further related to the exponential rate of convergence in terms of the higher order L^2 energy functional $\mathcal{E}_{N,l}(f)(t)$ under the additional assumption that the momentum-weighted L^∞ norms of the higher order spatial derivatives of the initial data are bounded, not necessarily small.

Theorem 1.2. *Let all assumptions of Theorem 1.1 hold true. Suppose further that $\mathfrak{E}_{N,l+l_0,\vartheta}(f_0) \leq C_N$ for some $N \in \{1,2,...\}$, where C_N is finite, not necessarily small. Then there exists a constant $C'_N > 0$ such that*

$$\sup_{0\leq t<\infty} \mathfrak{E}_{N,l+l_0,\vartheta}(f)(t) \leq C'_N.$$

Moreover, there exist instant L^2 functionals $\mathcal{E}_{N,l,\vartheta}(f)(t)$ and $\mathcal{D}_{N,l,\vartheta}(f)(t)$ as in (1.17) and (1.18) such that

$$\frac{d}{dt}\mathcal{E}_{N,l,\vartheta}(f)(t) + \mathcal{D}_{N,l,\vartheta}(f)(t)$$

$$\leq C \sum_{0\leq m\leq [\frac{N}{2}]} \{\mathfrak{E}_{m,l+l_0,\vartheta}(f)(t) + \mathfrak{E}^2_{m,l+l_0,\vartheta}(f)(t)\}\mathcal{D}_{N-m,l,\vartheta}(f)(t),$$

for all $t \geq 0$. In particular, if $\vartheta > 0$ and $\beta = \frac{\vartheta}{\vartheta+\frac{b}{2}}$, there exist constants $C''_N > 0$ and $\lambda_N > 0$ such that for all $t \geq 0$,

$$\mathcal{E}_{N,l}(f)(t) \leq C''_N e^{-\lambda_N t^\beta}.$$

In the spatially homogeneous setting, namely when $F = F(t,p)$, the equation (1.2) simplifies into the following form

$$\partial_t F = \mathcal{Q}(F,F), \qquad F(0,p) = F_0(p). \tag{1.21}$$

Correspondingly the linearized equation (1.14) simplifies into the following equation

$$\partial_t f + Lf = \Gamma(f,f), \qquad f(0,p) = f_0(p). \tag{1.22}$$

As before, we define the following instant functionals as

$$\overline{\mathfrak{E}}_{l,\vartheta}(f)(t) = |f(t)|_{\infty,l,\vartheta}, \quad \overline{\mathcal{E}}_{l,\vartheta}(f)(t) \sim |f(t)|^2_{l,\vartheta}, \tag{1.23}$$

and the energy dissipation functional $\mathcal{D}_{N,l,\vartheta}(f)(t)$ satisfies

$$\overline{\mathcal{D}}_{l,\vartheta}(f)(t) \sim |f(t)|^2_{\nu,l,\vartheta}. \tag{1.24}$$

By the similar arguments as for showing Theorem 1.1, we have the following convergence rate result for the spatially homogeneous equation (1.21).

Corollary 1.3. *Let $l \geq 0$, $l_0 > 3/b$, $\vartheta \in [0,1]$ and $\tau > 0$. If $\vartheta = 1$, restrict $\tau > 0$ small enough. Choose initial data $F_0(p) = J(p) + \sqrt{J(p)} f_0(p) \geq 0$ such that $f_0(p) \in \mathcal{N}^\perp$ and is continuous in $\mathbb{R}^3$. There exists an instant L^∞ functional $\overline{\mathfrak{E}}_{l+l_0,\vartheta}(f)(t)$ such that if $\overline{\mathfrak{E}}_{l+l_0,\vartheta}(f_0)$ is sufficiently small, there exists a unique global solution $f(t,p)$ to the relativistic Boltzmann equation (1.22) with $F(t,p) = J(p) + \sqrt{J(p)} f(t,p) \geq 0$ and*

$$\sup_{0\leq t<\infty} \overline{\mathfrak{E}}_{l+l_0,\vartheta}(f)(t) \leq C\overline{\mathfrak{E}}_{l+l_0,\vartheta}(f_0).$$

Moreover, there exists an instant L^2 functional $\overline{\mathcal{E}}_{l,\vartheta}(f)(t)$ as in (1.23) such that if $\vartheta > 0$ and $\beta = \frac{\vartheta}{\vartheta+\frac{b}{2}}$, then it holds that

$$\overline{\mathcal{E}}_l(f)(t) \leq Ce^{-\lambda_0 t^\beta}\overline{\mathcal{E}}_{l,\vartheta}(f_0),$$

for all $t \geq 0$, where $\lambda_0 > 0$ is a constant.

In what follows we would mention some mathematical results related to the topic of the paper. A brief history of relativistic kinetic theory (cf. [4,14]) was given in [28]; interested readers may refer to that paper and references therein. Here, as far as the relativistic Boltzmann equation is concerned, we only recall the local-in-time solution [1], solutions and hydrodynamics for the linearized equation [7,6], large-data solutions [8,24] by DiPerna-Lions' renormalized theory [5], small-data solutions near vacuum [11,27], asymptotic stability of the relativistic Maxwellian for hard potentials [12,13,21,23,36,37] and for soft potentials [28,32], and stability of solutions with respect to initial data [20, 18,19].

One of motivations in the paper is due to the work [28] mentioned above, where the unique global-in-time small-amplitude mild solution in the momentum weighted L^∞ framework (cf. [16]) was constructed for the relativistic Boltzmann equation (1.11) with soft potentials, and the polynomial rate of convergence towards the steady state was also obtained. However, the exponential time-decay rate of global solutions has remained unknown, even in the spatially homogeneous setting. On the other hand, for the classical Boltzmann equation with soft potentials, the existence and large-time behavior of solutions near Maxwellians were studied either in the whole space [22,26,29,33] or in the torus [2,3,15,30,31]. Particularly, for the torus case, any smooth perturbation approaches zero at the rate of $\exp(-\lambda t^\beta)$ for some $\lambda > 0$ and $0 < \beta < 1$.

From those results on the large-time behavior of solutions to the Boltzmann equation in the torus for soft potentials mentioned before, one may see that the polynomial time-decay rate obtained by [28] for the relativistic case could be improved to be exponential by using similar techniques as in [2,3,31] for the classical case. Note that the method in [2,3] can not be directly used to treat the case of very soft potentials. Moreover, as for the approach in [31], one may need to make a crucial use of the momentum derivatives and Sobolev imbedding in order to control singularity of the collision kernel. However, as pointed out in [28,17], in the relativistic case, high derivatives of the post-collisional variables (1.10) create additional high singularities which are hard to control. Also, derivatives of the post-collisional momentum exhibit enough momentum growth to make the method of [31] fail to be used. To overcome these difficulties, the main idea of [28] is to adapt the similar method from [16] in terms of the weighted $L^2 \cap L^\infty$ estimates. As only the polynomial momentum weight functions are involved, the time-decay rates are only polynomial.

In the present work, for the Cauchy problem (1.11) on the relativistic Boltzmann equation for soft potentials in the torus, we try to first obtain the global existence of solutions in the setting of $L^\infty_{x,p}$ with exponential weight functions of p. The method of the proof is a little different from those used in [2,3,31,28]. We shall directly work on the nonlinear equation in the space with the exponential weight function. Under the a priori assumption on the solution in the weighted $L^\infty_{x,p}$ space, we first use the compensation function of the relativistic equation with soft potentials and the property of the torus to get $L^2_{x,p}$ estimates with the exponential weight function of p for the nonlinear equation. Then, as in [16] (also cf. [35]), we use the iterations of the solution for the nonlinear equation so as to close the a priori assumption in L^∞ space. Note that we would not devote ourselves to getting the time-decay rate in the space $L^\infty_{x,p}$ as done in [28]. For completeness, we also give the proof of the local existence by the usual method and use the standard continuity argument to show the global-in-time existence. Once we obtain the global existence in the exponential p-weighted L^∞ framework, we are able to adopt the method of [2,3,31] to obtain the exponential time-decay rate for soft potentials in L^2 framework. Lastly we study the propagation of spatial regularity of the global solution under the assumption on the derivatives of the initial data being bounded, not necessarily small. And, we can also obtain the exponential time-decay rate of the derivatives of solutions as in [34], where the authors considered the similar problem of the classical Boltzmann equation for hard potentials.

The rest of the paper is organized as follows. In Section 2 we establish some basic estimates on linear and nonlinear terms. Then we obtain weighted L^2 estimates and L^∞ estimates on the nonlinear equation in Section 3 and Section 4, respectively. In Section 5, we first prove the local existence of solutions for the nonlinear equation and further obtain the global existence through uniform a priori estimates. Moreover, we prove the exponential time decay rate. Finally, in Section 6 we prove the propagation of the space regularity of the global solution and the exponential time decay of higher-order energy functionals.

2. Basic estimates

In this section, we will prove some basic estimates used to obtain global existence of solutions with an exponential weight in momentum p. We first start from the linearized operator K. We know from [7,12,28] that $Kf = K_2 f - K_1 f$, where K_1 and K_2 are integral operators defined by

$$K_i f(p) = \int_{\mathbb{R}^3} k_i(p,q) f(q)\, dq, \qquad i = 1, 2,$$

with the symmetric kernels

$$k_1(p,q) = \int_{\mathbb{S}^2} v_\phi \sigma(g,\theta) \sqrt{J(q)J(p)}\, d\omega, \tag{2.1}$$

$$k_2(p,q) = \frac{Cs^{3/2}}{g p_0 q_0} \int_0^\infty e^{-l\sqrt{y^2+1}} \sigma\Big(\frac{g}{\sin(\psi/2)}, \psi\Big) \frac{y\Big(1+\sqrt{y^2+1}\Big)}{\sqrt{y^2+1}} I_0(jy)\, dy. \tag{2.2}$$

Here $C > 0$ is a generic constant, the modified Bessel function of index zero is defined by

$$I_0(y) = \frac{1}{2\pi} \int_0^{2\pi} e^{y \cos \varphi} d\varphi,$$

and we are also using the simplified notation

$$\sin \frac{\psi}{2} = \frac{\sqrt{2} g}{[g^2 - 4 + (g^2 + 4)\sqrt{y^2 + 1}]^{1/2}},$$

and in addition

$$l = \frac{p_0 + q_0}{2}, \qquad j = \frac{|p \times q|}{g}.$$

Next we will deduce some basic estimates on the operator kernels $k_1(p, q)$ and $k_2(p, q)$, for instance (2.7) and (2.18). Some techniques in the proof are from [12,28]. We first consider the simple kernel $k_1(p, q)$. By [12,28], we recall the following estimates:

$$\frac{[|p \times q|^2 + |p - q|^2]^{1/2}}{\sqrt{p_0 q_0}} \le g \le |p - q|, \quad \text{and} \quad g \le 2\sqrt{p_0 q_0}. \tag{2.3}$$

With (1.4), (1.5) and the Mϕller velocity (1.6), we have

$$s = 4 + g^2 \le C p_0 q_0, \quad v_\phi = \frac{g\sqrt{s}}{p_0 q_0} \le C. \tag{2.4}$$

For $b \in (0, 1]$, it follows from (2.3) and (2.4) that

$$v_\phi g^{-b} = \frac{\sqrt{s}}{p_0 q_0} g^{1-b} \le C \frac{\sqrt{p_0 q_0}}{p_0 q_0} (p_0 q_0)^{(1-b)/2} \le C (p_0 q_0)^{-b/2}. \tag{2.5}$$

For $b \in (1, 2)$, similarly we have that

$$v_\phi g^{-b} = \frac{\sqrt{s}}{p_0 q_0} g^{1-b} \le C \frac{\sqrt{s}}{p_0 q_0} \frac{(p_0 q_0)^{(b-1)/2}}{|p - q|^{b-1}} \le C \frac{(p_0 q_0)^{(b-2)/2}}{|p - q|^{b-1}}. \tag{2.6}$$

It follows from (1.14), (2.1), (2.5) and (2.6) that

$$k_1(p, q) \le C \Big((p_0 q_0)^{-b/2} + \frac{(p_0 q_0)^{(b-2)/2}}{|p - q|^{b-1}} \Big) e^{-\frac{p_0}{2} - \frac{q_0}{2}}. \tag{2.7}$$

By using (1.12), (2.5) and (2.6), we can obtain that for any $a > 0$,

$$\nu_a(p) \equiv \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} v_\phi \sigma(g, \theta) J^a(q) dq d\omega \sim p_0^{-b/2} \sim \nu(p). \tag{2.8}$$

By symmetry the same estimate holds if the roles of p and q are reversed. The full proof of this estimate above was given in [28, Lemma 3.1].

Next we consider the estimates of the operator kernel $k_2(p,q)$. We recall the following estimates in [12,28]:

$$\frac{C_1}{\sqrt{s}(1+y^2)^{1/4}} \leq \frac{\sin(\psi/2)}{g} \leq \frac{C_2}{g(1+y^2)^{1/4}}, \tag{2.9}$$

and

$$\frac{y}{2(1+y^2)^{1/2}} \leq \cos(\psi/2) \leq 1. \tag{2.10}$$

By the assumption (1.14), one has that

$$\sigma\Big(\frac{g}{\sin(\psi/2)}, \psi\Big) \leq C\Big(\frac{\sin(\psi/2)}{g}\Big)^b \sin^\gamma \psi \leq Cg^{-b}\sin^{b+\gamma}(\psi/2)\cos^\gamma(\psi/2).$$

Since $\gamma \geq 0$ and $b \in (0,2)$, we have from (2.9) and (2.10) that

$$g^{-b}\sin^{b+\gamma}(\psi/2)\cos^\gamma(\psi/2) \leq g^{-b}(1+y^2)^{-(b+\gamma)/4}.$$

Thus we can deduce from (2.2) and the above estimate that

$$k_2(p,q) \leq \frac{Cs^{3/2}}{g^{1+b}p_0q_0}\int_0^\infty e^{-l\sqrt{y^2+1}}yI_0(jy)(1+y^2)^{-(b+\gamma)/4}dy. \tag{2.11}$$

To estimate the right hand side of (2.11), we define

$$\widetilde{K}_\alpha(i,j) = \int_0^\infty e^{-l\sqrt{y^2+1}}yI_0(jy)(1+y^2)^{\alpha/4}dy.$$

Then for $\alpha \in [-2,2]$, it is known from [12,28] that

$$\widetilde{K}_\alpha(i,j) \leq Cl^{1+\alpha/2}e^{-c|p-q|}. \tag{2.12}$$

We also define

$$\widetilde{I}_\beta(l,j) = \int_0^1 e^{-l\sqrt{y^2+1}}y^{1-\beta}I_0(jy)dy.$$

Then for $\beta \in [0,2)$, we have

$$\widetilde{I}_\beta(l,j) \leq Ce^{-c\sqrt{l^2-j^2}} \leq Ce^{-c|p-q|/2}. \tag{2.13}$$

By using (2.11), (2.12) and (2.13), for $\zeta = \min\{2, b+\gamma\}$, we can obtain

$$k_2(p,q) \leq \frac{Cs^{3/2}}{g^{1+b}p_0q_0}(p_0+q_0)^{1-\zeta/2}e^{-c|p-q|/2}. \tag{2.14}$$

As in [28], we can show that

$$(p_0+q_0)e^{-c|p-q|/8} \leq C(p_0q_0)^{1/2}. \tag{2.15}$$

We give the proof of (2.15) for completeness. If $\frac{|p|}{2} \leq |q| \leq 2|p|$, this inequality is obvious. If $\frac{|p|}{2} \geq |q|$, then $|p-q| \geq \frac{|p|}{2}$ and

$$(p_0+q_0)e^{-c|p-q|/8} \leq Cp_0e^{-c|p|/16} \leq C.$$

If $|q| \geq 2|p|$, then $|p-q| \geq \frac{|q|}{2}$ and

$$(p_0+q_0)e^{-c|p-q|/8} \leq Cq_0e^{-c|q|/16} \leq C.$$

Thus (2.15) holds. Noting that $b \in (0,2)$, $\gamma \geq 0$ and $\zeta = \min\{2, b+\gamma\}$, one has from (2.15) that

$$\begin{aligned}\frac{(p_0+q_0)^{1-\zeta/2}}{p_0q_0}e^{-c|p-q|/8} &\leq \frac{(p_0+q_0)^{-b/2}}{p_0q_0}(p_0+q_0)e^{-c|p-q|/8}\\ &\leq C(p_0q_0)^{-1/2}(p_0+q_0)^{-b/2}.\end{aligned} \tag{2.16}$$

Noticing that $s = 4+g^2 \leq 4+|p-q|^2$ from (2.3) and (2.4), we can obtain

$$s^{3/2}e^{-c|p-q|/8} \leq C. \tag{2.17}$$

By using (2.3), (2.14), (2.16) and (2.17) we can obtain

$$\begin{aligned}k_2(p,q) &\leq \frac{C}{g^{1+b}}(p_0q_0)^{-1/2}(p_0+q_0)^{-b/2}e^{-c|p-q|/4}\\ &\leq \frac{C(p_0q_0)^{b/2}}{[|p\times q|^2+|p-q|^2]^{(1+b)/2}}(p_0+q_0)^{-b/2}e^{-c|p-q|/4}.\end{aligned} \tag{2.18}$$

We remark that the estimates (2.7) and (2.18) of the operator K have the singularity near $p = q$. As the classical version [15], a cutoff function (2.19) is introduced to get rid of such a singularity.

Given a small $\epsilon > 0$, choose a smooth cut-off function $\chi = \chi(r) \in [0,1]$ satisfying

$$\chi(r) \equiv 1, \quad \text{for } r \geq 2\epsilon; \qquad \chi(r) \equiv 0, \quad \text{for } r \leq \epsilon. \tag{2.19}$$

We define by $K^\chi = K_2^\chi - K_1^\chi$ and

$$K_i^\chi f(p) = \int_{\mathbb{R}^3} k_i^\chi(p,q) f(q) dq = \int_{\mathbb{R}^3} \chi(|p-q|) k_i(p,q) f(q) dq, \qquad i = 1, 2. \tag{2.20}$$

We also use the splitting $K^{1-\chi} = K - K^\chi$.

In what follows we deduce the main estimates of the operator K, which will be used for the nonlinear L^2 energy analysis.

Lemma 2.1. *Let $l \geq 0$, $\vartheta \in [0,1]$ and $\tau > 0$. If $\vartheta = 1$, restrict $\tau > 0$ small enough. For any small $\eta > 0$, the operator K can be split into*

$$K = K_c + K_s,$$

where K_c is a compact operator in L^2_ν. In particular for some large constant $R > 0$, we have

$$|\langle \varpi^2(l,\vartheta) K_c h_1, h_2 \rangle| \leq C_\eta |\mathbf{1}_{\leq R} h_1|_2 |\mathbf{1}_{\leq R} h_2|_2. \tag{2.21}$$

Here $\mathbf{1}_{\leq R}$ is the indicator function of the ball of radius R. Furthermore,

$$|\langle \varpi^2(l,\vartheta) K_s h_1, h_2 \rangle| \leq C\eta |h_1|_{\nu,l,\vartheta} |h_2|_{\nu,l,\vartheta}. \tag{2.22}$$

Proof. Since $K = K_2 - K_1$, K_c and K_s are to be constructed separately for both K_1 and K_2 so that $K_c = K_{2c} - K_{1c}$ and $K_s = K_{2s} - K_{1s}$ accordingly. We define

$$K_{2c} h_1(p) = \int_{\mathbb{R}^3} \mathbf{1}_{|p|+|q| \leq R} k_2^\chi(p,q) h_1(q) dq.$$

By using (2.18) and (2.19) we can obtain

$$\varpi^2(l,\vartheta)(p) \mathbf{1}_{|p|+|q| \leq R} k_2^\chi(p,q) \leq C_{R,\epsilon} e^{-c'|p-q|} \mathbf{1}_{|p| \leq R} \mathbf{1}_{|q| \leq R}.$$

The Hilbert–Schmidt theorem clearly shows that K_{2c} is a compact operator in L^2_ν, for any given $\epsilon > 0$ and $R > 0$.

It follows from the above estimate and the Hölder's inequality that

$$\begin{aligned}
\left| \langle \varpi^2(l,\vartheta) K_{2c} h_1, h_2 \rangle \right| \leq\ & \left\{ \int_{\mathbb{R}^3 \times \mathbb{R}^3} dq dp \varpi^2(l,\vartheta)(p) \mathbf{1}_{|p|+|q| \leq R} k_2^\chi(p,q) |h_1(q)|^2 \right\}^{1/2} \\
& \times \left\{ \int_{\mathbb{R}^3 \times \mathbb{R}^3} dq dp \varpi^2(l,\vartheta)(p) \mathbf{1}_{|p|+|q| \leq R} k_2^\chi(p,q) |h_2(p)|^2 \right\}^{1/2} \\
\leq\ & C_R |\mathbf{1}_{\leq R} h_1|_2 |\mathbf{1}_{\leq R} h_2|_2.
\end{aligned}$$

We also define

$$K_{1c}h_1(p) = \int_{\mathbb{R}^3} \mathbf{1}_{|p|+|q|\le R} k_1^\chi(p,q)h_1(q)dq.$$

By using (2.20) and (2.7) we can obtain

$$\varpi^2(l,\vartheta)(p)\mathbf{1}_{|p|+|q|\le R} k_1^\chi(p,q) \le C_{R,\epsilon} e^{-c'p_0-c'q_0}\mathbf{1}_{|p|\le R}\mathbf{1}_{|q|\le R}.$$

Then we also obtain

$$\left|\langle \varpi^2(l,\vartheta)K_{1c}h_1, h_2\rangle\right| \le C_R|\mathbf{1}_{\le R}h_1|_2|\mathbf{1}_{\le R}h_2|_2.$$

The similar arguments as for K_{2c} show that K_{1c} is a compact operator in L^2_ν. Thus K_c is a compact operator in L^2_ν and the estimate (2.21) holds.

Next we will prove the estimate (2.22). First we consider the operator K_2. We define by $K_{2s} = K_{2s}^\chi + K_{2s}^{1-\chi}$, where

$$K_{2s}^\chi h_1(p) = \int_{\mathbb{R}^3} \mathbf{1}_{|p|+|q|\ge R} k_2^\chi(p,q)h_1(q)dq,$$

and

$$K_{2s}^{1-\chi} h_1(p) = \int_{\mathbb{R}^3} k_2^{1-\chi}(p,q)h_1(q)dq.$$

For the operator K_{2s}^χ, we have that

$$\begin{aligned}\left|\langle \varpi^2(l,\vartheta)K_{2s}^\chi h_1, h_2\rangle\right| \le& \left\{\int_{\mathbb{R}^3} |h_1(q)|^2 dq \int_{\mathbb{R}^3} \varpi^2(l,\vartheta)(p)\mathbf{1}_{|p|+|q|\ge R}|k_2^\chi(p,q)|dp\right\}^{1/2}\\ &\times\left\{\int_{\mathbb{R}^3} \varpi^2(l,\vartheta)(p)|h_2(p)|^2dp\int_{\mathbb{R}^3}\mathbf{1}_{|p|+|q|\ge R}|k_2^\chi(p,q)|dq\right\}^{1/2}.\end{aligned} \tag{2.23}$$

To estimate (2.23), we make some preparations. As in [12], we write $q\times p = p\times(p-q)$ and set $r=|p-q|$, $|p\times(p-q)| = |p|r\sin\theta$. Then for any small $\epsilon>0$, one has

$$\begin{aligned}\int_{\mathbb{R}^3}\frac{\chi(|p-q|)e^{-c|p-q|/32}}{[|p\times q|^2+|p-q|^2]^{(1+b)/2}}dq &\le C\int_\epsilon^\infty\int_0^\pi \frac{e^{-cr/32}r^{1-b}\sin\theta d\theta dr}{[|p|^2\sin^2\theta+1]^{(1+b)/2}}\\ &\le C_\epsilon\int_0^\pi\frac{\sin\theta d\theta}{[|p|^2\sin^2\theta+1]^{(1+b)/2}}.\end{aligned} \tag{2.24}$$

As in [12], by explicit computation, the angular integral is $O(p_0^{-b-\epsilon})$ for large $|p|$ and $b \in (0,2)$. Thus we can obtain that

$$\int\limits_{\mathbb{R}^3} \frac{\chi(|p-q|)e^{-c|p-q|/32}}{[|p\times q|^2+|p-q|^2]^{(1+b)/2}}dq \le C_\epsilon p_0^{-b-\epsilon}. \tag{2.25}$$

Notice that

$$\{(p,q) : |p|+|q| \ge R\} \subset \{(p,q) : |p| \ge \frac{R}{4}\} \cup \{(p,q) : |p| \le \frac{R}{4}, |q| \ge \frac{3R}{4}\}.$$

Thus we can obtain that

$$\mathbf{1}_{|p|+|q|\ge R}e^{-c|p-q|/32} \le (\mathbf{1}_{|p|\ge \frac{R}{4}} + \mathbf{1}_{|p|\le \frac{R}{4}}\mathbf{1}_{|q|\ge \frac{3R}{4}})e^{-c|p-q|/32} \le \mathbf{1}_{|p|\ge \frac{R}{4}} + e^{-cR/64}. \tag{2.26}$$

To estimate the second integral of (2.23), for any $\eta > 0$, we have from (2.8), (2.18), (2.25) and (2.26) that

$$\begin{aligned}
&\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \mathbf{1}_{|p|+|q|\ge R}|k_2^\chi(p,q)|dq \\
&\le \int\limits_{\mathbb{R}^3} \mathbf{1}_{|p|+|q|\ge R}\chi(|p-q|)\frac{c(p_0q_0)^{b/2}}{[|p\times q|^2+|p-q|^2]^{(1+b)/2}}(p_0+q_0)^{-b/2}e^{-c|p-q|/8}dq \\
&\le Cp_0^{b/2}(\mathbf{1}_{|p|\ge\frac{R}{4}} + e^{-cR/64}) \int\limits_{|p-q|\ge\epsilon} \frac{e^{-c|p-q|/32}}{[|p\times q|^2+|p-q|^2]^{(1+b)/2}}dq \\
&\le Cp_0^{-b/2-\epsilon}(\mathbf{1}_{|p|\ge\frac{R}{4}} + e^{-cR/64}) \le C\eta\nu(p),
\end{aligned} \tag{2.27}$$

where $\epsilon > 0$ is small enough and $R > 0$ large enough. For the second integral of (2.23), we have that

$$\int\limits_{\mathbb{R}^3} \varpi^2(l,\vartheta)(p)|h_2(p)|^2dp \int\limits_{\mathbb{R}^3} \mathbf{1}_{|p|+|q|\ge R}|k_2^\chi(p,q)|dq \le C\eta|h_2|^2_{\nu,l,\vartheta}. \tag{2.28}$$

Next we consider the first integral of (2.23). We first notice that

$$(\frac{p_0}{q_0})^{lb}e^{-c|p-q|/16} \le Ce^{-c|p-q|/32}, \tag{2.29}$$

which is an immediate consequence of

$$p_0 \le |p-q| + q_0. \tag{2.30}$$

By the assumptions of the weight $\varpi^2(l,\vartheta)(p)$, we see from (2.29) and (2.30) that

$$\begin{aligned}\varpi^2(l,\vartheta)(p)e^{-c|p-q|/8} &= \frac{\varpi^2(l,\vartheta)(p)}{\varpi^2(l,\vartheta)(q)}\varpi^2(l,\vartheta)(q)e^{-c|p-q|/8}\\ &= (\frac{p_0}{q_0})^{lb}e^{-c|p-q|/16}e^{2\tau(p_0^\vartheta-q_0^\vartheta-|p-q|^\vartheta)}\varpi^2(l,\vartheta)(q)e^{-c|p-q|/16+2\tau|p-q|^\vartheta}\\ &\le C\varpi^2(l,\vartheta)(q)e^{-c|p-q|/32}. \end{aligned}\tag{2.31}$$

It follows from (2.18), (2.20), (2.31) and (2.27) that

$$\begin{aligned}&\int_{\mathbb{R}^3}\varpi^2(l,\vartheta)(p)\mathbf{1}_{|p|+|q|\ge R}|k_2^\chi(p,q)|dp\\ &\le \int_{\mathbb{R}^3}\mathbf{1}_{|p|+|q|\ge R}\varpi^2(l,\vartheta)(p)\frac{c(p_0q_0)^{b/2}\chi(|p-q|)}{[|p\times q|^2+|p-q|^2]^{(1+b)/2}}(p_0+q_0)^{-b/2}e^{-c|p-q|/4}dp\\ &\le \varpi^2(l,\vartheta)(q)\int_{\mathbb{R}^3}\mathbf{1}_{|p|+|q|\ge R}\frac{c(p_0q_0)^{b/2}\chi(|p-q|)}{[|p\times q|^2+|p-q|^2]^{(1+b)/2}}(p_0+q_0)^{-b/2}e^{-c|p-q|/8}dp\\ &\le C\eta\nu(q)\varpi^2(l,\vartheta)(q).\end{aligned}\tag{2.32}$$

For the first integral of (2.23), we have from (2.32) that

$$\int_{\mathbb{R}^3}|h_1(q)|^2dq\int_{\mathbb{R}^3}\varpi^2(l,\vartheta)(p)\mathbf{1}_{|p|+|q|\ge R}|k_2^\chi(p,q)|dp\le C\eta|h_1|^2_{\nu,l,\vartheta}.\tag{2.33}$$

By using (2.28) and (2.33), for the operator K_{2s}, we have from (2.23) that

$$\left|\langle\varpi^2(l,\vartheta)K_{2s}^\chi h_1,h_2\rangle\right|\le C\eta|h_1|_{\nu,l,\vartheta}|h_2|_{\nu,l,\vartheta}.\tag{2.34}$$

For the operator $K_{2s}^{1-\chi}$, we have that

$$\begin{aligned}\left|\langle\varpi^2(l,\vartheta)K_{2s}^{1-\chi}h_1,h_2\rangle\right| &\le \Big\{\int_{\mathbb{R}^3}|h_1(q)|^2dq\int_{\mathbb{R}^3}\varpi^2(l,\vartheta)(p)|k_2^{1-\chi}(p,q)|dp\Big\}^{1/2}\\ &\quad\times\Big\{\int_{\mathbb{R}^3}\varpi^2(l,\vartheta)(p)|h_2(p)|^2dp\int_{\mathbb{R}^3}|k_2^{1-\chi}(p,q)|dq\Big\}^{1/2}.\end{aligned}\tag{2.35}$$

It follows from (2.24) and (2.25) that

$$\int_{\mathbb{R}^3}\frac{(1-\chi(|p-q|))e^{-c|p-q|/32}}{[|p\times q|^2+|p-q|^2]^{(1+b)/2}}dq\le C\int_0^{2\epsilon}\int_0^\pi\frac{e^{-cr/32}r^{1-b}\sin\theta d\theta dr}{[|p|^2\sin^2\theta+1]^{(1+b)/2}}$$

$$\leq C\epsilon^{2-b}\int_0^\pi \frac{\sin\theta d\theta}{[|p|^2\sin^2\theta+1]^{(1+b)/2}} \leq C\epsilon^{2-b}p_0^{-b-\epsilon}. \tag{2.36}$$

By using this, (2.8) and (2.18), we can obtain

$$\begin{aligned}\int_{\mathbb{R}^3}|k_2^{1-\chi}(p,q)|dq &\leq \int_{\mathbb{R}^3}\frac{c(p_0q_0)^{b/2}(1-\chi(|p-q|))}{[|p\times q|^2+|p-q|^2]^{(1+b)/2}}(p_0+q_0)^{-b/2}e^{-c|p-q|/8}dq\\ &\leq Cp_0^{b/2}\int_{\mathbb{R}^3}\frac{(1-\chi(|p-q|))}{[|p\times q|^2+|p-q|^2]^{(1+b)/2}}e^{-c|p-q|/32}dq \leq C\epsilon^{2-b}\nu(p).\end{aligned} \tag{2.37}$$

For the second integral of (2.35), we have from (2.37) that

$$\int_{\mathbb{R}^3}\varpi^2(l,\vartheta)(p)|h_2(p)|^2dp\int_{\mathbb{R}^3}|k_2^{1-\chi}(p,q)|dq \leq C\epsilon^{2-b}|h_2|^2_{\nu,l,\vartheta}. \tag{2.38}$$

It follows from (2.18), (2.31) and (2.37) that

$$\begin{aligned}&\int_{\mathbb{R}^3}\varpi^2(l,\vartheta)(p)|k_2^{1-\chi}(p,q)|dp\\ &\leq \int_{\mathbb{R}^3}\varpi^2(l,\vartheta)(p)\frac{c(p_0q_0)^{b/2}(1-\chi(|p-q|))}{[|p\times q|^2+|p-q|^2]^{(1+b)/2}}(p_0+q_0)^{-b/2}e^{-c|p-q|/4}dp\\ &\leq C\varpi^2(l,\vartheta)(q)\int_{\mathbb{R}^3}\frac{c(p_0q_0)^{b/2}(1-\chi(|p-q|))}{[|p\times q|^2+|p-q|^2]^{(1+b)/2}}(p_0+q_0)^{-b/2}e^{-c|p-q|/8}dp\\ &\leq C\epsilon^{2-b}\nu(q)\varpi^2(l,\vartheta)(q).\end{aligned} \tag{2.39}$$

We have from (2.35), (2.38) and (2.39) that

$$\left|\langle\varpi^2(l,\vartheta)K_{2s}^{1-\chi}h_1,h_2\rangle\right| \leq C\eta|h_1|_{\nu,l,\vartheta}|h_2|_{\nu,l,\vartheta}. \tag{2.40}$$

Here we used the fact that $b\in(0,2)$ and choose $\epsilon>0$ small enough.

It follows from (2.34) and (2.40) that

$$\left|\langle\varpi^2(l,\vartheta)K_{2s}h_1,h_2\rangle\right| \leq C\eta|h_1|_{\nu,l,\vartheta}|h_2|_{\nu,l,\vartheta}. \tag{2.41}$$

As for treating the operator K_{2s}^{χ}, for the operator K_{1s}^{χ}, we have that

$$\Big|\langle\varpi^2(l,\vartheta)K_{1s}^{\chi}h_1,h_2\rangle\Big| \le \Big\{\int_{\mathbb{R}^3}|h_1(q)|^2dq\int_{\mathbb{R}^3}\varpi^2(l,\vartheta)(p)\mathbf{1}_{|p|+|q|\ge R}|k_1^{\chi}(p,q)|dp\Big\}^{1/2}$$
$$\times\Big\{\int_{\mathbb{R}^3}\varpi^2(l,\vartheta)(p)|h_2(p)|^2dp\int_{\mathbb{R}^3}\mathbf{1}_{|p|+|q|\ge R}|k_1^{\chi}(p,q)|dq\Big\}^{1/2}.$$

By using (2.7) and the assumptions of the weight $\varpi(l,\vartheta)(p)$, we can obtain

$$\mathbf{1}_{|p|+|q|\ge R}\varpi^2(l,\vartheta)(p)|k_1^{\chi}(p,q)| \le C_\epsilon(R^{(b-2)/2}+R^{-b/2}))e^{-\frac{p_0}{8}-\frac{q_0}{8}} \le C_\epsilon\eta e^{-\frac{p_0}{8}-\frac{q_0}{8}}.$$

Here we choose $R>0$ large enough. It follows from these estimates that

$$\Big|\langle\varpi^2(l,\vartheta)K_{1s}^{\chi}h_1,h_2\rangle\Big| \le C\eta|h_1|_{\nu,l,\vartheta}|h_2|_{\nu,l,\vartheta}. \tag{2.42}$$

By using (2.7) and the assumptions of the weight $\varpi(l,\vartheta)(p)$, we can obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3}\varpi^2(l,\vartheta)(p)|k_1^{1-\chi}(p,q)|dp\\
&\le \int_{\mathbb{R}^3}\varpi^2(l,\vartheta)(p)(1-\chi(|p-q|))\Big((p_0q_0)^{-b/2}+\frac{(p_0q_0)^{(b-2)/2}}{|p-q|^{b-1}}\Big)e^{-\frac{p_0}{2}-\frac{q_0}{2}}dp\\
&\le C\int_{|p-q|\le 2\epsilon}\Big(1+\frac{1}{|p-q|^{b-1}}\Big)e^{-\frac{p_0}{8}-\frac{q_0}{8}}dp \le C(\epsilon^{4-b}+\epsilon^3)e^{-\frac{q_0}{8}}.
\end{aligned} \tag{2.43}$$

For the operator $K_{1s}^{1-\chi}$, by using (2.43) and choosing $\epsilon>0$ small enough, one has that

$$\begin{aligned}
\Big|\langle\varpi^2(l,\vartheta)K_{1s}^{1-\chi}h_1,h_2\rangle\Big| &\le \Big\{\int_{\mathbb{R}^3}|h_1(q)|^2dq\int_{\mathbb{R}^3}\varpi^2(l,\vartheta)(p)|k_1^{1-\chi}(p,q)|dp\Big\}^{1/2}\\
&\quad\times\Big\{\int_{\mathbb{R}^3}\varpi^2(l,\vartheta)(p)|h_2(p)|^2dp\int_{\mathbb{R}^3}|k_1^{1-\chi}(p,q)|dq\Big\}^{1/2}\\
&\le C\eta|h_1|_{\nu,l,\vartheta}|h_2|_{\nu,l,\vartheta}.
\end{aligned} \tag{2.44}$$

It follows from (2.44) and (2.42) that

$$\Big|\langle\varpi^2(l,\vartheta)K_{1s}h_1,h_2\rangle\Big| \le C\eta|h_1|_{\nu,l,\vartheta}|h_2|_{\nu,l,\vartheta}.$$

This completes the estimate (2.22) for the operator K_1. This and (2.41) complete the proof of (2.22). $\square$

Remark 2.2. By using Lemma 2.1 and the similar arguments as in [15,28], one can prove the crucial coercive estimate (1.16) of the linearized Boltzmann operator.

Next we derive other estimate of the operator K, which will be used to perform the nonlinear L^∞ estimate of (1.11).

Lemma 2.3. *Let $l \geq 0$, $\vartheta \in [0,1]$ and $\tau > 0$. If $\vartheta = 1$, restrict $\tau > 0$ small enough. Denote $\varpi = \varpi(l,\vartheta)(p)$. Then for any $j \geq 0$, any small $\eta = \eta(\epsilon) > 0$ and some constant $c_0 > 0$ such that*

$$\left|\varpi K^{1-\chi}(\frac{h}{\varpi})(p)\right| \leq C\eta\nu(p)\|h\|_\infty.$$

Proof. It follows from (2.31) that

$$\frac{\varpi(l,\vartheta)(p)}{\varpi(l,\vartheta)(q)} e^{-c|p-q|/8} \leq C. \tag{2.45}$$

Recalling (2.18) and (2.20), for the operator K_2, we have from (2.37) that

$$\begin{aligned}
\left|\varpi K_2^{1-\chi}(\frac{h}{\varpi})(p)\right| &\leq C\int_{\mathbb{R}^3} \frac{\varpi(l,\vartheta)(p)}{\varpi(l,\vartheta)(q)} \frac{(p_0q_0)^{b/2}(1-\chi(|p-q|))}{[|p\times q|^2+|p-q|^2]^{(1+b)/2}}(p_0+q_0)^{-b/2} \\
&\quad \times e^{-c|p-q|/4}|h(q)|dq \\
&\leq C\|h\|_\infty \int_{\mathbb{R}^3} \frac{(p_0q_0)^{b/2}(1-\chi(|p-q|))}{[|p\times q|^2+|p-q|^2]^{(1+b)/2}}(p_0+q_0)^{-b/2}e^{-c|p-q|/8}dq \\
&\leq C\|h\|_\infty p_0^{b/2}\int_{\mathbb{R}^3} \frac{(1-\chi(|p-q|))}{[|p\times q|^2+|p-q|^2]^{(1+b)/2}}e^{-c|p-q|/32}dq \\
&\leq C\epsilon^{2-b}\nu(p)\|h\|_\infty.
\end{aligned}$$

For the operator K_1, by using (2.7) and (2.20), one has that

$$\begin{aligned}
\left|\varpi K_1^{1-\chi}(\frac{h}{\varpi})(p)\right| &\leq C\int_{\mathbb{R}^3}(1-\chi(|p-q|))\frac{\varpi(l,\vartheta)(p)}{\varpi(l,\vartheta)(q)}(\frac{(p_0q_0)^{(b-2)/2}}{|p-q|^{b-1}}+(p_0q_0)^{-b/2}) \\
&\quad \times e^{-\frac{p_0}{2}-\frac{q_0}{2}}|h(q)|dq \\
&\leq C\|h\|_\infty\int_{\mathbb{R}^3}(1-\chi(|p-q|))(1+\frac{1}{|p-q|^{b-1}})e^{-\frac{p_0}{4}-\frac{q_0}{4}}dq \\
&\leq Ce^{-\frac{p_0}{4}}\|h\|_\infty\int_0^{2\epsilon}(1+\frac{1}{r^{b-1}})r^2dr \leq C(\epsilon^3+\epsilon^{4-b})\nu(p)\|h\|_\infty.
\end{aligned}$$

By using $b \in (0,2)$ and choosing $\epsilon > 0$ small enough, we conclude the proof of the lemma. $\square$

Next we shall construct the L^2 estimates of the nonlinear collision term. Recalling (1.13), we have that

$$\partial^\alpha \Gamma(h_1, h_2) = \sum_{\alpha_1+\alpha_2=\alpha} C_\alpha^{\alpha_1} \Gamma(\partial^{\alpha_1} h_1, \partial^{\alpha_2} h_2).$$

Here $\Gamma(\partial^{\alpha_1} h_1, \partial^{\alpha_2} h_2)$ have the following expression

$$\begin{aligned}\Gamma(\partial^{\alpha_1} h_1, \partial^{\alpha_2} h_2) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} dq d\omega v_\phi \sigma(g,\theta) J^{1/2}(q) \partial^{\alpha_1} h_1(p') \partial^{\alpha_2} h_2(q') \\ &\qquad - \partial^{\alpha_1} h_1(p) \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} dq d\omega v_\phi \sigma(g,\theta) J^{1/2}(q) \partial^{\alpha_2} h_2(q) \\ &\equiv \Gamma_{gain}(\partial^{\alpha_1} h_1, \partial^{\alpha_2} h_2) - \Gamma_{loss}(\partial^{\alpha_1} h_1, \partial^{\alpha_2} h_2). \end{aligned} \tag{2.46}$$

The first L^2 estimate about the nonlinear collision term is as follows.

Lemma 2.4. *Assume that $e(p) = P(p)\sqrt{J(p)}$, where $P(p)$ is a polynomial of any order. Let $\alpha = \alpha_1 + \alpha_2$ with $|\alpha| \le N$. If $|\alpha_1| \le |\alpha|/2$, then*

$$\|\langle \Gamma(\partial^{\alpha_1} h_1, \partial^{\alpha_2} h_2), e(p)\rangle\| \le C \|J^{1/32} \partial^{\alpha_1} h_1\|_\infty \|J^{1/32}\partial^{\alpha_2} h_2\|.$$

Alternatively, if $|\alpha_2| \le |\alpha|/2$, then

$$\|\langle \Gamma(\partial^{\alpha_1} h_1, \partial^{\alpha_2} h_2), e(p)\rangle\| \le C \|J^{1/32} \partial^{\alpha_2} h_2\|_\infty \|J^{1/32}\partial^{\alpha_1} h_1\|.$$

Proof. We first consider the loss term. It follows from (2.46) and (2.8) that

$$\begin{aligned} &\Big|\langle \Gamma_{loss}(\partial^{\alpha_1} h_1, \partial^{\alpha_2} h_2), e(p)\rangle\Big|^2 \\ &= \Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} v_\phi \sigma(g,\theta) J^{1/2}(q) e(p) \partial^{\alpha_2} h_2(q) \partial^{\alpha_1} h_1(p) dp dq d\omega\Big|^2 \\ &\le C \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} v_\phi \sigma(g,\theta) J^{1/2}(q) J^{1/2}(p) |P(p)| |\partial^{\alpha_1} h_1(p)|^2 dp dq d\omega \\ &\qquad \times \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} v_\phi \sigma(g,\theta) J^{1/2}(q) J^{1/2}(p) |P(p)| |\partial^{\alpha_2} h_2(q)|^2 dp dq d\omega \\ &\le C \Big|J^{1/16}\partial^{\alpha_1} h_1\Big|_2^2 \Big|J^{1/16}\partial^{\alpha_2} h_2\Big|_2^2. \end{aligned} \tag{2.47}$$

By this and the fact that $s = 4 + g^2$, we see that g and s are invariant with respect to the pre–post collision change of variables. Notice that $dpdq = \frac{p_0 q_0}{p_0' q_0'} dp' dq'$, which is from [9]. For a function G: $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, it follows from (2.5) in [17] that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} v_\phi\sigma(g,\theta)G(p,q,p',q')d\omega dq dp$$
$$= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} v_\phi\sigma(g,\theta)G(p',q',p,q)d\omega dq dp. \tag{2.48}$$

Then we have from (2.46), (1.9), (2.48) and (2.8) that

$$\begin{aligned}
&\Big|\langle\Gamma_{gain}(\partial^{\alpha_1}h_1,\partial^{\alpha_2}h_2),e(p)\rangle\Big|^2\\
&=\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} v_\phi\sigma(g,\theta)J^{1/2}(q)e(p)\partial^{\alpha_1}h_1(p')\partial^{\alpha_2}h_2(q')dpdqd\omega\Big|^2\\
&\le C\Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} v_\phi\sigma(g,\theta)J^{1/4}(q')J^{1/4}(p')|\partial^{\alpha_1}h_1(p')\partial^{\alpha_2}h_2(q')|dpdqd\omega\Big)^2\\
&= C\Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} v_\phi\sigma(g,\theta)J^{1/4}(q)J^{1/4}(p)|\partial^{\alpha_1}h_1(p)\partial^{\alpha_2}h_2(q)|d\omega dqdp\Big)^2,
\end{aligned}$$

and hence

$$\begin{aligned}
&\Big|\langle\Gamma_{gain}(\partial^{\alpha_1}h_1,\partial^{\alpha_2}h_2),e(p)\rangle\Big|^2\\
&\le C\Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} v_\phi\sigma(g,\theta)J^{1/4}(q)J^{1/4}(p)|\partial^{\alpha_1}h_1(p)\partial^{\alpha_2}h_2(q)|d\omega dqdp\Big)^2\\
&\le C\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} v_\phi\sigma(g,\theta)J^{1/4}(q)J^{1/4}(p)|\partial^{\alpha_1}h_1(p)|^2d\omega dqdp\\
&\qquad\times\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} v_\phi\sigma(g,\theta)J^{1/4}(q)J^{1/4}(p)|\partial^{\alpha_2}h_2(q)|^2d\omega dqdp\\
&\le C\Big|J^{1/16}\partial^{\alpha_1}h_1\Big|_2^2\Big|J^{1/16}\partial^{\alpha_2}h_2\Big|_2^2.
\end{aligned}\tag{2.49}$$

Thus, by the above estimates, if $|\alpha_1|\le|\alpha|/2$, we can obtain

$$\begin{aligned}
\|\langle\Gamma(\partial^{\alpha_1}h_1,\partial^{\alpha_2}h_2),e(p)\rangle\|^2 &\le C\int_{\mathbb{T}^3}\Big|J^{1/16}\partial^{\alpha_1}h_1\Big|_2^2\Big|J^{1/16}\partial^{\alpha_2}h_2\Big|_2^2dx\\
&\le C\|J^{1/32}\partial^{\alpha_1}h_1\|_\infty^2\|J^{1/32}\partial^{\alpha_2}h_2\|^2.
\end{aligned}$$

The case that $|\alpha_2|\le|\alpha|/2$ can be handled in the same way. This concludes the proof of Lemma 2.4. □

The second L^2 estimate about the nonlinear collision term with the exponential weight is given in the following lemma. The proof of Lemma 2.5 is more or less similar as in [15,31]. However, some modifications are needed to facilitate the momentum exponentially growing weight in the relativistic case. And more care will be paid to conservations (1.8) and (1.9) and the momentum pre–post collision change of variables, which are different from the classical version.

Lemma 2.5. *Let $\alpha = \alpha_1 + \alpha_2$ with $|\alpha| \le N$, $l \ge 0$, $l_0 > 3/b$, $\vartheta \in [0,1]$ and $\tau > 0$. If $\vartheta = 1$, restrict $\tau \in (0,1)$. If $|\alpha_1| \le |\alpha|/2$, then*

$$|(\varpi^2(l,\vartheta)\Gamma(\partial^{\alpha_1}h_1, \partial^{\alpha_2}h_2), \partial^\alpha h_3)| \le C\mathfrak{E}_{|\alpha_1|,l+l_0,\vartheta}(h_1)\|\partial^{\alpha_2}h_2\|_{\nu,l,\vartheta}\|\partial^\alpha h_3\|_{\nu,l,\vartheta}.$$

Alternatively, if $|\alpha_2| \le |\alpha|/2$, then

$$|(\varpi^2(l,\vartheta)\Gamma(\partial^{\alpha_1}h_1, \partial^{\alpha_2}h_2), \partial^\alpha h_3)| \le C\mathfrak{E}_{|\alpha_2|,l+l_0,\vartheta}(h_2)\|\partial^{\alpha_1}h_1\|_{\nu,l,\vartheta}\|\partial^\alpha h_3\|_{\nu,l,\vartheta}.$$

Proof. We divide it into two cases for the gain term and loss term, respectively.
Case 1: The Loss Term Estimate. If $|\alpha_2| \le |\alpha|/2$, we have from (2.8) that

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} dqd\omega v_\phi \sigma(g,\theta) J^{1/2}(q)\partial^{\alpha_2}h_2(x,q) \\
&\le C\Big\{\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} dqd\omega v_\phi \sigma(g,\theta) J^{1/2}(q)|\partial^{\alpha_2}h_2(x,q)|^2\Big\}^{1/2}\Big\{\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} dqd\omega v_\phi \sigma(g,\theta) J^{1/2}(q)\Big\}^{1/2} \\
&\le C\sup_{x,q}\Big|J^{1/8}(q)\partial^{\alpha_2}h_2(x,q)\Big|\Big\{\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} dqd\omega v_\phi \sigma(g,\theta) J^{1/4}(q)\Big\} \\
&\le C\nu(p)\mathfrak{E}_{|\alpha_2|,l}(h_2).
\end{aligned}$$

Hence we see from (2.46) that

$$\begin{aligned}
&\Big|(\varpi^2(l,\vartheta)\Gamma_{loss}(\partial^{\alpha_1}h_1, \partial^{\alpha_2}h_2), \partial^\alpha h_3)\Big| \\
&\le C\mathfrak{E}_{|\alpha_2|,l}(h_2)\int_{\mathbb{T}^3\times\mathbb{R}^3} \nu(p)\varpi^2(l,\vartheta)(p)|\partial^{\alpha_1}h_1(x,p)\partial^\alpha h_3(x,p)|dpdx \\
&\le C\mathfrak{E}_{|\alpha_2|,l}(h_2)\|\partial^{\alpha_1}h_1\|_{\nu,l,\vartheta}\|\partial^\alpha h_3\|_{\nu,l,\vartheta}.
\end{aligned}$$

This completes the estimate for Γ_{loss} when $|\alpha_2| \le |\alpha|/2$. Next, to consider Γ_{loss} with $|\alpha_1| \le |\alpha|/2$, the integration domain in (p,q) is split into three parts

$$\{|q| \ge \frac{|p|}{2}\} \cup \{|q| \le \frac{|p|}{2}, |p| \ge 1\} \cup \{|q| \le \frac{|p|}{2}, |p| \le 1\}. \tag{2.50}$$

Case (1a): The Loss Term Estimate in the First Region $\{|q| \geq \frac{|p|}{2}\}$. On this region we have that

$$e^{-\frac{q_0}{2}} \leq e^{-\frac{q_0}{4}} e^{-\frac{p_0}{8}}. \tag{2.51}$$

Then the integral of $\varpi^2(l,\vartheta)\Gamma_{loss}(\partial^{\alpha_1} h_1, \partial^{\alpha_2} h_2)\partial^\alpha h_3$ over $\{|q| \geq \frac{|p|}{2}\}$ is bounded by

$$\begin{aligned}
&\int\limits_{\mathbb{T}^3\times\mathbb{S}^2} \int\limits_{\{|q|\geq\frac{|p|}{2}\}} v_\phi \sigma(g,\theta) e^{-\frac{q_0}{2}} \varpi^2(l,\vartheta)(p)\partial^{\alpha_1} h_1(p)\partial^{\alpha_2} h_2(q)\partial^\alpha h_3(p) dq dp d\omega dx \\
&\leq C \int\limits_{\mathbb{T}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} v_\phi \sigma(g,\theta) e^{-\frac{q_0}{4}} e^{-\frac{p_0}{8}} \varpi^2(l,\vartheta)(p)|\partial^{\alpha_1} h_1(p)\partial^{\alpha_2} h_2(q)\partial^\alpha h_3(p)| dq dp d\omega dx \\
&\leq C\Big\{ \int\limits_{\mathbb{T}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} v_\phi \sigma(g,\theta) e^{-\frac{q_0}{4}} e^{-\frac{p_0}{8}} |\partial^{\alpha_2} h_2(q)|^2 dq dp d\omega dx \Big\}^{1/2} \\
&\quad \times \Big\{ \int\limits_{\mathbb{T}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} v_\phi \sigma(g,\theta) e^{-\frac{q_0}{4}} e^{-\frac{p_0}{8}} \varpi^4(l,\vartheta)(p)|\partial^{\alpha_1} h_1(p)\partial^\alpha h_3(p)|^2 dq dp d\omega dx \Big\}^{1/2}.
\end{aligned}$$

Note that for $b \in (1,2)$, we have from (2.6) that

$$\int\limits_{\mathbb{S}^2} v_\phi \sigma(g,\theta) d\omega \leq C \int\limits_0^\pi \sin^{1+\gamma}\theta v_\phi g^{-b} d\theta \leq C \frac{(p_0 q_0)^{(b-2)/2}}{|p-q|^{b-1}}.$$

With this, the first integral in the previous estimate is bounded as

$$\begin{aligned}
&\int\limits_{\mathbb{T}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} v_\phi \sigma(g,\theta) e^{-\frac{q_0}{4}} e^{-\frac{p_0}{8}} |\partial^{\alpha_2} h_2(q)|^2 dq dp d\omega dx \\
&\leq C \int\limits_{\mathbb{T}^3\times\mathbb{R}^3} \Big\{ \int\limits_{\mathbb{R}^3} \frac{(p_0 q_0)^{(b-2)/2}}{|p-q|^{b-1}} e^{-\frac{p_0}{8}} dp \Big\} e^{-\frac{q_0}{4}} |\partial^{\alpha_2} h_2(q)|^2 dp dx \\
&\leq C \int\limits_{\mathbb{T}^3\times\mathbb{R}^3} \nu(q) e^{-\frac{q_0}{4}} |\partial^{\alpha_2} h_2(q)|^2 dp dx \leq C \|\partial^{\alpha_2} h_2\|_{\nu,l}^2,
\end{aligned}$$

and the second integral is bounded as

$$\begin{aligned}
&\int\limits_{\mathbb{T}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} v_\phi \sigma(g,\theta) e^{-\frac{q_0}{4}} e^{-\frac{p_0}{8}} \varpi^4(l,\vartheta)(p)|\partial^{\alpha_1} h_1(p)\partial^\alpha h_3(p)|^2 dq dp d\omega dx \\
&\leq C \int\limits_{\mathbb{T}^3\times\mathbb{R}^3} \Big\{ \int\limits_{\mathbb{R}^3} \frac{(p_0 q_0)^{(b-2)/2}}{|p-q|^{b-1}} e^{-\frac{q_0}{4}} dq \Big\} e^{-\frac{p_0}{8}} \varpi^4(l,\vartheta)(p)|\partial^{\alpha_1} h_1(p)\partial^\alpha h_3(p)|^2 dp dx \\
&\leq C \mathfrak{E}^2_{|\alpha_1|,l,\vartheta}(h_1) \|\partial^\alpha h_3\|^2_{\nu,l,\vartheta}.
\end{aligned}$$

Here we have used the facts that $|\alpha_1| \le |\alpha|/2$ and

$$\sup_{x,p}\left\{\varpi^2(l,\vartheta)(p)e^{-\frac{p_0}{16}}|\partial^{\alpha_1}h_1(p)|^2\right\} \le C\mathfrak{E}^2_{|\alpha_1|,l,\vartheta}(h_1). \tag{2.52}$$

For $b \in (0,1]$, by using (2.5), we handle it in the same way. We thus conclude the estimate over the first region.

Case (1b): The Loss Term Estimate in the Second Region $\{|q| \le \frac{|p|}{2}, |p| \ge 1\}$. When $|q| \le \frac{|p|}{2}$, it holds that

$$|p-q| \ge |p| - |q| \ge \frac{|p|}{2}. \tag{2.53}$$

It follows from this, (2.6) and (2.8) that for $b \in (1,2)$,

$$v_\phi g^{-b} \le C\frac{(p_0q_0)^{(b-2)/2}}{|p-q|^{b-1}} \le C(p_0q_0)^{(b-2)/2}p_0^{1-b} \le C\nu(p)q_0^{(b-2)/2}.$$

By using this, it follows from (2.46) that

$$\begin{aligned}
&\int_{\mathbb{T}^3\times\mathbb{S}^2}\int_{\{|q|\le\frac{|p|}{2},|p|\ge1\}} v_\phi\sigma(g,\theta)e^{-\frac{q_0}{2}}\varpi^2(l,\vartheta)(p)\partial^{\alpha_1}h_1(p)\partial^{\alpha_2}h_2(q)\partial^\alpha h_3(p)dqdpd\omega dx\\
&\le C\int_{\mathbb{T}^3}\int_{\{|q|\le\frac{|p|}{2},|p|\ge1\}} \nu(p)q_0^{(b-2)/2}e^{-\frac{q_0}{2}}\varpi^2(l,\vartheta)(p)|\partial^{\alpha_1}h_1(p)\partial^{\alpha_2}h_2(q)\partial^\alpha h_3(p)|dqdpdx\\
&\le C\int_{\mathbb{T}^3}\left\{\int_{\mathbb{R}^3} q_0^{(b-2)/2}e^{-\frac{q_0}{2}}|\partial^{\alpha_2}h_2(q)|dq\right\}\left\{\int_{\mathbb{R}^3}\nu(p)\varpi^2(l,\vartheta)(p)|\partial^{\alpha_1}h_1(p)\partial^\alpha h_3(p)|dp\right\}dx\\
&\le C\int_{\mathbb{T}^3}|\partial^{\alpha_2}h_2|_{\nu,l}|\partial^{\alpha_1}h_1|_{\nu,l,\vartheta}|\partial^\alpha h_3|_{\nu,l,\vartheta}dx\\
&\le C\sup_x|\partial^{\alpha_1}h_1|_{\nu,l,\vartheta}\|\partial^{\alpha_2}h_2\|_{\nu,l}\|\partial^\alpha h_3\|_{\nu,l,\vartheta}.
\end{aligned}$$

Since $\nu(p) \le C$ and $|\alpha_1| \le |\alpha|/2$, one has that for $l_0 > 3/b$

$$\sup_x|\partial^{\alpha_1}h_1|_{\nu,l,\vartheta} \le C\sup_x|\partial^{\alpha_1}h_1|_{l,\vartheta} \le C\mathfrak{E}_{|\alpha_1|,l+l_0,\vartheta}(h_1).$$

For $b \in (0,1]$, by the similar arguments, we use (2.5) and (2.8) to get the same estimate. Thus the term Γ_{loss} over the second region $\{|q| \le \frac{|p|}{2}, |p| \ge 1\}$ is bounded by $C\mathfrak{E}_{|\alpha_1|,l+l_0,\vartheta}(h_1)\|\partial^{\alpha_2}h_2\|_{\nu,l,\vartheta}\|\partial^\alpha h_3\|_{\nu,l,\vartheta}$.

Case (1c): The Loss Term Estimate in the Third Region $\{|q| \le \frac{|p|}{2}, |p| \le 1\}$. In this region, for $b \in (1,2)$, we have from (2.6) and (2.53) that

$$v_\phi g^{-b} \le C\frac{(p_0q_0)^{(b-2)/2}}{|p-q|^{b-1}} \le C\frac{(p_0q_0)^{(b-2)/2}|p|^{(1-b)/2}}{|p-q|^{(b-1)/2}}, \qquad \varpi^2(l,\vartheta)(p) \le C. \tag{2.54}$$

It follows from (2.54) and (2.46) that

$$\begin{aligned}
&\int\limits_{\mathbb{T}^3\times\mathbb{S}^2}\int\limits_{\{|q|\le\frac{|p|}{2},|p|\le 1\}} v_\phi\sigma(g,\theta)e^{-\frac{q_0}{2}}\varpi^2(l,\vartheta)(p)\partial^{\alpha_1}h_1(p)\partial^{\alpha_2}h_2(q)\partial^\alpha h_3(p)dqdpd\omega dx\\
&\le C\int\limits_{\mathbb{T}^3}\int\limits_{\{|q|\le\frac{|p|}{2},|p|\le 1\}} |p|^{\frac{1-b}{2}}|p-q|^{\frac{1-b}{2}}(p_0q_0)^{\frac{b-2}{2}}e^{-\frac{q_0}{2}}|\partial^{\alpha_1}h_1(p)\partial^{\alpha_2}h_2(q)\partial^\alpha h_3(p)|dqdpdx\\
&\le C\int\limits_{\mathbb{T}^3}\Big\{\int\limits_{\mathbb{R}^3}|p-q|^{\frac{1-b}{2}}e^{-\frac{q_0}{4}}|\partial^{\alpha_2}h_2(q)|dq\Big\}\Big\{\int\limits_{|p|\le 1}|p|^{\frac{1-b}{2}}|\partial^{\alpha_1}h_1(p)\partial^\alpha h_3(p)|dp\Big\}dx\\
&\le C\int\limits_{\mathbb{T}^3}\Big\{\int\limits_{\mathbb{R}^3}|p-q|^{1-b}e^{-\frac{q_0}{4}}dq\Big\}^{1/2}\Big\{\int\limits_{\mathbb{R}^3}e^{-\frac{q_0}{4}}|\partial^{\alpha_2}h_2(q)|^2dq\Big\}^{1/2}\\
&\qquad\times\Big\{\int\limits_{|p|\le 1}|p|^{\frac{1-b}{2}}|\partial^{\alpha_1}h_1(p)\partial^\alpha h_3(p)|dp\Big\}dx\\
&\le C\int\limits_{\mathbb{T}^3}|\partial^{\alpha_2}h_2(q)|_{\nu,l}\Big\{\int\limits_{|p|\le 1}|p|^{1-b}|\partial^{\alpha_1}h_1(p)|^2dp\Big\}^{1/2}\Big\{\int\limits_{|p|\le 1}|\partial^\alpha h_3(p)|^2dp\Big\}^{1/2}dx.
\end{aligned}$$

By the facts that $b\in(1,2)$ and $|\alpha_1|\le|\alpha|/2$, we have

$$\sup_x\Big\{\int\limits_{|p|\le 1}|p|^{1-b}|\partial^{\alpha_1}h_1(p)|^2dp\Big\}^{1/2} \le C\sup_{x,|p|\le 1}|\partial^{\alpha_1}h_1(p)| \le C\mathfrak{E}_{|\alpha_1|,l}(h_1).$$

Hence, if $|\alpha_1|\le|\alpha|/2$, the last part is bounded by $C\mathfrak{E}_{|\alpha_1|,l}(h_1)\|\partial^{\alpha_2}h_2\|_{\nu,l}\|\partial^\alpha h_3\|_{\nu,l}$. The case that $b\in(0,1]$ can be handled in the same way by using (2.5). This completes the proof of Case 1(c) and hence the whole Case 1.

Case 2: The Gain Term Estimate. Once again the integration domain in (p,q) is split into three parts as in (2.50).

Case (2a): The Gain Term Estimate in the First Region $\{|q|\ge\frac{|p|}{2}\}$. By using (2.51), the integral of $\varpi^2(l,\vartheta)\Gamma_{gain}(\partial^{\alpha_1}h_1,\partial^{\alpha_2}h_2)\partial^\alpha h_3$ over $\{|q|\ge\frac{|p|}{2}\}$ is bounded by

$$\begin{aligned}
&\int\limits_{\mathbb{T}^3\times\mathbb{S}^2}\int\limits_{\{|q|\ge\frac{|p|}{2}\}} v_\phi\sigma(g,\theta)e^{-\frac{q_0}{2}}\varpi^2(l,\vartheta)(p)\partial^{\alpha_1}h_1(p')\partial^{\alpha_2}h_2(q')\partial^\alpha h_3(p)dqdpd\omega dx\\
&\le C\int\limits_{\mathbb{T}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} v_\phi\sigma(g,\theta)e^{-\frac{q_0}{4}}e^{-\frac{p_0}{8}}\varpi^2(l,\vartheta)(p)|\partial^{\alpha_1}h_1(p')\partial^{\alpha_2}h_2(q')\partial^\alpha h_3(p)|dqdpd\omega dx
\end{aligned}$$

$$\leq C\Big\{ \int\limits_{\mathbb{T}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} v_\phi\sigma(g,\theta)e^{-\frac{p_0}{8}-\frac{q_0}{8}}\varpi^2(l,\vartheta)(p)|\partial^{\alpha_1}h_1(p')|^2|\partial^{\alpha_2}h_2(q')|^2 dqdpd\omega dx\Big\}^{1/2}$$

$$\times\Big\{ \int\limits_{\mathbb{T}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} v_\phi\sigma(g,\theta)e^{-\frac{p_0}{8}-\frac{q_0}{8}}\varpi^2(l,\vartheta)(p)|\partial^{\alpha}h_3(p)|^2 dqdpd\omega dx\Big\}^{1/2}.$$

By (2.8), the second factor can be bounded by $C\|\partial^\alpha h_3\|_{\nu,l,\vartheta}$. By (1.9) and the assumptions of $\varpi(l,\vartheta)(p)$, one has that

$$\begin{aligned}\varpi^2(l,\vartheta)(p) &= p_0^{bl}e^{2\tau p_0^\vartheta} \leq (p_0+q_0)^{bl}e^{2\tau(p_0+q_0)^\vartheta} = (p_0'+q_0')^{bl}e^{2\tau(p_0'+q_0')^\vartheta}\\ &\leq C(p_0')^{bl}(q_0')^{bl}e^{2\tau(p_0')^\vartheta}e^{2\tau(q_0')^\vartheta} = C\varpi^2(l,\vartheta)(p')\varpi^2(l,\vartheta)(q'). \end{aligned}\tag{2.55}$$

If $|\alpha_1|\leq|\alpha|/2$, as in (2.52),

$$\sup_{x,p}\varpi(l,\vartheta)(p)e^{-\frac{p_0}{32}}|\partial^{\alpha_1}h_1(p)|\leq C\mathfrak{E}_{|\alpha_1|,l,\vartheta}(h_1).\tag{2.56}$$

Then the integral in the first factor can be bounded by

$$\begin{aligned}&C\int\limits_{\mathbb{T}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} v_\phi\sigma(g,\theta)e^{-\frac{p_0'}{16}-\frac{q_0'}{16}}\varpi^2(l,\vartheta)(p')\varpi^2(l,\vartheta)(q')|\partial^{\alpha_1}h_1(p')|^2|\partial^{\alpha_2}h_2(q')|^2 dqdpd\omega dx\\ &= C\int\limits_{\mathbb{T}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} v_\phi\sigma(g,\theta)e^{-\frac{p_0}{16}-\frac{q_0}{16}}\varpi^2(l,\vartheta)(p)\varpi^2(l,\vartheta)(q)|\partial^{\alpha_1}h_1(p)|^2|\partial^{\alpha_2}h_2(q)|^2 dqdpd\omega dx\\ &\leq C\mathfrak{E}^2_{|\alpha_1|,l,\vartheta}(h_1)\int\limits_{\mathbb{T}^3\times\mathbb{R}^3}\Big\{\int\limits_{\mathbb{R}^3\times\mathbb{S}^2} v_\phi\sigma(g,\theta)e^{-\frac{p_0}{32}}dpd\omega\Big\}\varpi^2(l,\vartheta)(q)|\partial^{\alpha_2}h_2(q)|^2dqdx\\ &\leq C\mathfrak{E}^2_{|\alpha_1|,l,\vartheta}(h_1)\|\partial^{\alpha_2}h_2\|^2_{\nu,l,\vartheta}.\end{aligned}$$

Here we have used (2.8), (2.55), (2.56) and the pre–post collision change of variables as (2.48). Thus if $|\alpha_1|\leq|\alpha|/2$, the gain term estimate in this region is bounded by $C\mathfrak{E}_{|\alpha_1|,l,\vartheta}(h_1)\|\partial^{\alpha_2}h_2\|_{\nu,l,\vartheta}\|\partial^\alpha h_3\|_{\nu,l,\vartheta}$.

If $|\alpha_2|\leq|\alpha|/2$, we switch $\partial^{\alpha_2}h_2$ with $\partial^{\alpha_1}h_1$. This completes the estimate for the gain term over $\{|q|\geq\frac{|p|}{2}\}$.

Case (2b): The Gain Term Estimate in the Second Region $\{|q|\leq\frac{|p|}{2},|p|\geq1\}$. For $b\in(0,1]$, we have from (2.5) and (2.8) that

$$v_\phi g^{-b}e^{-\frac{q_0}{2}}\leq C(p_0q_0)^{-b/2}e^{-\frac{q_0}{2}}\leq C\nu(p)e^{-\frac{q_0}{4}}.$$

On this region $\{|q|\leq\frac{|p|}{2},|p|\geq1\}$, $|p-q|\geq|p|-|q|\geq\frac{|p|}{2}$ and $|p-q|^2\geq\frac{p_0^2}{8}$. By this and $b\in(1,2)$, it holds that

$$\frac{1}{|p-q|^{b-1}} \le (\frac{p_0^2}{8})^{\frac{1-b}{2}} \le C p_0^{1-b}.$$

For $b \in (1,2)$, it follows from this, (2.6) and (2.8) that

$$v_\phi g^{-b} e^{-\frac{q_0}{2}} \le C(p_0 q_0)^{(b-2)/2} p_0^{1-b} e^{-\frac{q_0}{2}} \le C\nu(p) e^{-\frac{q_0}{4}}.$$

By using these, it follows from (2.46) that

$$\begin{aligned}
&\int\limits_{\mathbb{T}^3\times\mathbb{S}^2}\int\limits_{\{|q|\le\frac{|p|}{2},|p|\ge1\}} v_\phi\sigma(g,\theta)e^{-\frac{q_0}{2}}\varpi^2(l,\vartheta)(p)\partial^{\alpha_1}h_1(p')\partial^{\alpha_2}h_2(q')\partial^\alpha h_3(p)dqdpd\omega dx \\
&\le C\int\limits_{\mathbb{T}^3}\int\limits_{\{|q|\le\frac{|p|}{2},|p|\ge1\}} \nu(p)e^{-\frac{q_0}{4}}\varpi^2(l,\vartheta)(p)|\partial^{\alpha_1}h_1(p')\partial^{\alpha_2}h_2(q')\partial^\alpha h_3(p)|dqdpdx \\
&\le C\Big\{\int\limits_{\mathbb{T}^3\times\mathbb{R}^3\times\mathbb{R}^3}\nu(p)e^{-\frac{q_0}{4}}\varpi^2(l,\vartheta)(p)|\partial^{\alpha_1}h_1(p')|^2|\partial^{\alpha_2}h_2(q')|^2dqdpdx\Big\}^{1/2} \\
&\qquad\times\Big\{\int\limits_{\mathbb{T}^3\times\mathbb{R}^3\times\mathbb{R}^3}\nu(p)e^{-\frac{q_0}{4}}\varpi^2(l,\vartheta)(p)|\partial^\alpha h_3(p)|^2dqdpdx\Big\}^{1/2}.
\end{aligned} \tag{2.57}$$

The last line is bounded by $\|\partial^\alpha h_3\|_{\nu,l,\vartheta}$. On this region $\{|q|\le\frac{|p|}{2},|p|\ge1\}$, we have from (1.9) and (2.8) that

$$\nu(p)\sim p_0^{-b/2}\le C(p_0'+q_0')^{-b/2}\le C\min\{(p_0')^{-b/2},(q_0')^{-b/2}\}\le C\min\{\nu(p'),\nu(q')\}. \tag{2.58}$$

It follows from (1.9) that

$$e^{-\frac{q_0}{4}}\frac{p_0q_0}{p_0'q_0'}\le C\frac{p_0}{p_0'q_0'}\le C\frac{p_0'+q_0'}{p_0'q_0'}\le C. \tag{2.59}$$

If $|\alpha_1|\le|\alpha|/2$, by using $dpdq=\frac{p_0q_0}{p_0'q_0'}dp'dq'$, the integral of the first factor in (2.57) is bounded by

$$\begin{aligned}
&C\int\limits_{\mathbb{T}^3\times\mathbb{R}^3\times\mathbb{R}^3}\nu(q')e^{-\frac{q_0}{4}}\varpi^2(l,\vartheta)(p')\varpi^2(l,\vartheta)(q')|\partial^{\alpha_1}h_1(p')|^2|\partial^{\alpha_2}h_2(q')|^2dqdpdx \\
&=C\int\limits_{\mathbb{T}^3\times\mathbb{R}^3\times\mathbb{R}^3}\nu(q')e^{-\frac{q_0}{4}}\varpi^2(l,\vartheta)(p')\varpi^2(l,\vartheta)(q')|\partial^{\alpha_1}h_1(p')|^2|\partial^{\alpha_2}h_2(q')|^2\frac{p_0q_0}{p_0'q_0'}dq'dp'dx \\
&\le C\int\limits_{\mathbb{T}^3\times\mathbb{R}^3\times\mathbb{R}^3}\nu(q')\varpi^2(l,\vartheta)(p')\varpi^2(l,\vartheta)(q')|\partial^{\alpha_1}h_1(p')|^2|\partial^{\alpha_2}h_2(q')|^2dq'dp'dx
\end{aligned}$$

$$\leq C \sup_x \int_{\mathbb{R}^3} \varpi^2(l,\vartheta)(p')|\partial^{\alpha_1} h_1(p')|^2 dp' \|\partial^{\alpha_2} h_2\|^2_{\nu,l,\vartheta}, \tag{2.60}$$

where we have used (2.55), (2.58) and (2.59). For $l_0 > 3/b$

$$\sup_x \left\{ \int_{\mathbb{R}^3} \varpi^2(l,\vartheta)(p')|\partial^{\alpha_1} h_1(p')|^2 dp' \right\}^{1/2} \leq C\mathfrak{E}_{|\alpha_1|,l+l_0,\vartheta}(h_1).$$

Thus if $|\alpha_1| \leq |\alpha|/2$, by this, (2.57) and (2.60), the gain term estimate in this region is bounded by $C\mathfrak{E}_{|\alpha_1|,l+l_0,\vartheta}(h_1)\|\partial^{\alpha_2} h_2\|_{\nu,l,\vartheta}\|\partial^{\alpha} h_3\|_{\nu,l,\vartheta}$.

If $|\alpha_2| \leq |\alpha|/2$, by the similar arguments, the gain term estimate is bounded by

$$C\mathfrak{E}_{|\alpha_2|,l+l_0,\vartheta}(h_2)\|\partial^{\alpha_1} h_1\|_{\nu,l,\vartheta}\|\partial^{\alpha} h_3\|_{\nu,l,\vartheta}.$$

This completes the estimate for the gain term over $\{|q| \leq \frac{|p|}{2}, |p| \geq 1\}$.

Case (2c): The Gain Term Estimate in the Third Region $\{|q| \leq \frac{|p|}{2}, |p| \leq 1\}$.
For the last region $\{|q| \leq \frac{|p|}{2}, |p| \leq 1\}$, it holds that $|q| \leq 1/2$. For $b \in (1,2)$, we have from (2.46) and (2.54) that

$$\begin{aligned}
&C \int_{\mathbb{T}^3\times\mathbb{S}^2} \int_{\{|q|\leq\frac{|p|}{2},|p|\leq 1\}} v_\phi \sigma(g,\theta) e^{-\frac{q_0}{2}} \varpi^2(l,\vartheta)(p)\partial^{\alpha_1} h_1(p')\partial^{\alpha_2} h_2(q')\partial^{\alpha} h_3(p) dq dp d\omega dx \\
&\leq C \int_{\mathbb{T}^3} \int_{\{|q|\leq\frac{|p|}{2},|p|\leq 1\}} |p|^{\frac{1-b}{2}} |p-q|^{\frac{1-b}{2}} (p_0 q_0)^{\frac{b-2}{2}} e^{-\frac{q_0}{2}} |\partial^{\alpha_1} h_1(p')\partial^{\alpha_2} h_2(q')\partial^{\alpha} h_3(p)| dq dp dx \\
&\leq C \int_{\mathbb{T}^3} \int_{\{|p|\leq 1\}} \left\{ |p|^{\frac{1-b}{2}} \int_{\{|q|\leq\frac{|p|}{2}\}} |p-q|^{\frac{1-b}{2}} e^{-\frac{q_0}{4}} |\partial^{\alpha_1} h_1(p')\partial^{\alpha_2} h_2(q')| dq \right\} |\partial^{\alpha} h_3(p)| dp dx \\
&\leq C \int_{\mathbb{T}^3} \int_{\{|p|\leq 1\}} \left\{ \int_{\{|q|\leq\frac{|p|}{2}\}} |p|^{1-b} |\partial^{\alpha_1} h_1(p')|^2 |\partial^{\alpha_2} h_2(q')|^2 dq \right\}^{1/2} \\
&\qquad \times \left\{ \int_{\{|q|\leq\frac{|p|}{2}\}} |p-q|^{1-b} e^{-\frac{q_0}{2}} dq \right\}^{1/2} |\partial^{\alpha} h_3(p)| dp dx \\
&\leq C \int_{\mathbb{T}^3} \int_{\{|p|\leq 1\}} \left\{ \int_{\{|q|\leq\frac{|p|}{2}\}} |p|^{1-b} |\partial^{\alpha_1} h_1(p')|^2 |\partial^{\alpha_2} h_2(q')|^2 dq \right\}^{1/2} |\partial^{\alpha} h_3(p)| dp dx \\
&\leq C \left\{ \int_{\mathbb{T}^3} \int_{\{|p|\leq 1,|q|\leq\frac{|p|}{2}\}} |p|^{1-b} |\partial^{\alpha_1} h_1(p')|^2 |\partial^{\alpha_2} h_2(q')|^2 dq dp dx \right\}^{1/2} \|\partial^{\alpha} h_3\|_{\nu,l}.
\end{aligned} \tag{2.61}$$

We now estimate the first factor. Since $|q| \leq |p|/2$ and $|p| \leq 1$, by using (1.4), (1.5) and (2.3), we have from (1.10) that

$$|p'|+|q'|\leq C|p|+g(2+\frac{p_0+q_0}{\sqrt{4+g^2}})\leq C|p|+Cp_0|p-q|\leq C|p|+C(|p|+|q|)\leq C|p|.$$

Since $1-b<0$, this implies

$$|p|^{1-b}\leq C'|p'|^{1-b},\qquad |p|^{1-b}\leq C'|q'|^{1-b}.$$

Thus we have

$$\begin{aligned}
&\int_{\mathbb{T}^3}\int_{\{|p|\leq 1,|q|\leq\frac{|p|}{2}\}}|p|^{1-b}|\partial^{\alpha_1}h_1(p')|^2|\partial^{\alpha_2}h_2(q')|^2dqdpdx\\
&\leq C\int_{\mathbb{T}^3}\int_{\{|p|\leq 1,|q|\leq\frac{|p|}{2}\}}|p|^{1-b}e^{-\frac{q_0}{4}}|\partial^{\alpha_1}h_1(p')|^2|\partial^{\alpha_2}h_2(q')|^2dqdpdx\\
&\leq C\int_{\mathbb{T}^3}\int_{\{|p'|\leq C,|q'|\leq C\}}e^{-\frac{q_0}{4}}\min\{|p'|^{1-b},|q'|^{1-b}\}|\partial^{\alpha_1}h_1(p')|^2|\partial^{\alpha_2}h_2(q')|^2\frac{p_0q_0}{p_0'q_0'}dq'dp'dx\\
&\leq C\int_{\mathbb{T}^3}\int_{\{|p'|\leq C,|q'|\leq C\}}\min\{|p'|^{1-b},|q'|^{1-b}\}|\partial^{\alpha_1}h_1(p')|^2|\partial^{\alpha_2}h_2(q')|^2dq'dp'dx.
\end{aligned}$$

Here we used the fact that $e^{\frac{q_0}{4}}\mathbf{I}_{|q|\leq\frac{1}{2}}\leq C$. Assume $|\alpha_1|\leq|\alpha|/2$ and majorize the above by

$$\begin{aligned}
&C\int_{\mathbb{T}^3}\Big\{\int_{\{|p'|\leq C\}}|p'|^{1-b}|\partial^{\alpha_1}h_1(p')|^2dp'\Big\}\Big\{\int_{\{|q'|\leq C\}}|\partial^{\alpha_2}h_2(q')|^2dq'\Big\}dx\\
&\qquad\leq C\sup_{x,|p'|\leq C}|\partial^{\alpha_1}h_1(p')|^2\|\partial^{\alpha_2}h_2\|^2_{\nu,l}\leq C\mathfrak{E}_{|\alpha_1|,l,\vartheta}(h_1)\|\partial^{\alpha_2}h_2\|^2_{\nu,l}.
\end{aligned}$$

If $|\alpha_2|\leq|\alpha|/2$, we have

$$\begin{aligned}
&C\int_{\mathbb{T}^3}\Big\{\int_{\{|p'|\leq C\}}|\partial^{\alpha_1}h_1(p')|^2dp'\Big\}\Big\{\int_{\{|q'|\leq C\}}|q'|^{1-b}|\partial^{\alpha_2}h_2(q')|^2dq'\Big\}dx\\
&\qquad\leq C\sup_{x,|p'|\leq C}|\partial^{\alpha_2}h_2(p')|^2\|\partial^{\alpha_1}h_1\|^2_{\nu,l}\leq C\mathfrak{E}_{|\alpha_2|,l,\vartheta}(h_2)\|\partial^{\alpha_1}h_1\|^2_{\nu,l}.
\end{aligned}$$

Then for $b\in(1,2)$, we combine this upper bound with (2.61) to complete the estimate for the gain term over the last region. The case that $b\in(0,1]$ can be handled in the same way by using (2.5). Thus we complete the proof of Lemma 2.5. □

The following corollary is used to prove existence of global solutions to the homogeneous equation (1.22), which can be shown by the similar arguments as for obtaining Lemma 2.5.

Corollary 2.6. *Let $l \geq 0$, $l_0 > 3/b$, $\vartheta \in [0,1]$ and $\tau > 0$. If $\vartheta = 1$, restrict $\tau \in (0,1)$. It holds that*

$$|\langle \varpi^2(l,\vartheta)\Gamma(h_1,h_2),h_3\rangle| \leq C\overline{\mathfrak{E}}_{l+l_0,\vartheta}(h_1)|h_2|_{\nu,l,\vartheta}|h_3|_{\nu,l,\vartheta}.$$

Alternatively, we have that

$$|\langle \varpi^2(l,\vartheta)\Gamma(h_1,h_2),h_3\rangle| \leq C\overline{\mathfrak{E}}_{l+l_0,\vartheta}(h_2)|h_1|_{\nu,l,\vartheta}|h_3|_{\nu,l,\vartheta}.$$

The next lemma concerns the L^∞ estimates of the nonlinear collision operator $\Gamma(h_1,h_2)$ with the exponential weight function of p.

Lemma 2.7. *Under the assumptions of Lemma 2.5, we have the following estimates:*

$$\left|\varpi\Gamma(\frac{h_1}{\varpi},\frac{h_2}{\varpi})(p)\right| \leq C\nu(p)\|h_1\|_\infty\|h_2\|_\infty.$$

Proof. We have from (1.13), (2.55) and (2.8) that

$$\begin{aligned}\left|\varpi\Gamma(\frac{h_1}{\varpi},\frac{h_2}{\varpi})(p)\right| \leq& \int\limits_{\mathbb{R}^3\times\mathbb{S}^2} d\omega dq v_\phi \sigma(g,\theta)e^{-q_0/2}\frac{1}{\varpi(l,\vartheta)(q)}|h_1(p)h_2(q)| \\ &+\int\limits_{\mathbb{R}^3\times\mathbb{S}^2} d\omega dq v_\phi \sigma(g,\theta)e^{-q_0/2}\frac{\varpi(l,\vartheta)(p)}{\varpi(l,\vartheta)(p')\varpi(l,\vartheta)(q')}|h_1(p')h_2(q')| \\ &\leq C\|h_1\|_\infty\|h_2\|_\infty \int\limits_{\mathbb{R}^3\times\mathbb{S}^2} d\omega dq v_\phi \sigma(g,\theta)e^{-q_0/2} \leq C\nu(p)\|h_1\|_\infty\|h_2\|_\infty.\end{aligned}$$

This completes the proof of Lemma 2.7. □

3. Nonlinear L^2 estimates

In this section we will deduce the nonlinear L^2 estimates for the relativistic Boltzmann equation (1.11) with soft potentials by using the Fourier transform and the compensation function method (cf. [25]). For the case of hard potentials the compensation functions of the relativistic Boltzmann equation (1.10) have been derived in [13,37]. Here we will represent the details of the proof in order to take into account the effect of soft potentials and the periodic domain.

For the purpose mentioned above, we start from the following linear inhomogeneous relativistic Boltzmann equation with a source term G:

$$\{\partial_t + \hat{p}\cdot\nabla_x + L\}f = G. \tag{3.1}$$

Let us first recall the definition of a compensation function for the relativistic Boltzmann equation (3.1), which has been introduced in [25] as well as in [13,37].

Definition 3.1. A bounded linear operator $S(\omega)$ with $\omega \in \mathbb{S}^2$ on $L^2(\mathbb{R}^3)$ is called a compensation function for (3.1) if

(i) $S(\cdot)$ is C^∞ on $\mathbb{S}^2$ with values in the space of bounded linear operators on $L^2(\mathbb{R}^3)$, and $S(-\omega) = -S(\omega)$ for all $\omega \in \mathbb{S}^2$.

(ii) $iS(\omega)$ is self-adjoint on $L^2(\mathbb{R}^3)$ for all $\omega \in \mathbb{S}^2$.

(iii) There exists $c_0 > 0$ such that for all $f \in L^2(\mathbb{R}^3)$ and $\omega \in \mathbb{S}^2$,

$$\mathcal{R}\langle S(\omega)(\hat{p}\cdot\omega)f, f\rangle + \langle Lf, f\rangle \geq c_0\left(|\mathbf{P}f|_2^2 + |(\mathbf{I}-\mathbf{P})f|_\nu^2\right). \tag{3.2}$$

Here $\mathcal{R}z$ is the real part of $z \in \mathbb{C}$.

To construct the compensation function $S(\omega)$, we first consider the fourteen moments in the relativistic case as in [13,10]. The subspace $\widetilde{W}$ for the 14 moments is defined as the space generated by $\mathcal{N}$ and the images of $\mathcal{N}$ under the mappings $f(p) \mapsto \hat{p}_j f(p)$ $(j = 1, 2, 3)$. That is,

$$\widetilde{W} = \text{span}\{\sqrt{J}\varphi_j | j = 1, ..., 14\},$$

where

$$\varphi_1 = 1, \quad \varphi_{j+1} = p_j, \quad \varphi_5 = p_0, \qquad \varphi_{j+5} = p_j\hat{p}_j,$$
$$\varphi_9 = p_1\hat{p}_2, \quad \varphi_{10} = p_2\hat{p}_3, \quad \varphi_{11} = p_3\hat{p}_1, \quad \varphi_{j+11} = \hat{p}_j \quad (j = 1, 2, 3).$$

Here, $\mathcal{N} \subset \widetilde{W}$ and the operator of multiplication by $\hat{p}\cdot$ maps $\mathcal{N}$ into $\widetilde{W}$. Denote an orthogonal basis for this 14 dimensional space spanned by e_j, $1 \leq j \leq 14$ as in [13,10].

Let P_0 be the orthogonal projection from $L^2(\mathbb{R}^3_p)$ onto $\widetilde{W}$:

$$P_0 f = \sum_{k=1}^{14}\langle f, e_k\rangle e_k.$$

Set $W_k = \langle f, e_k\rangle$. Then we have by using (3.1) that

$$\partial_t W + \sum_j V^j \partial_{x_j} W + \overline{L}W = \overline{G} + R,$$

where V^j $(j = 1, 2, 3)$ and $\overline{L}$ are the symmetric matrices given by

$$\overline{L} = \{\langle L[e_l], e_k\rangle\}_{k,l=1}^{14}, \quad V(\xi) = \sum_{j=1}^{3} V^j\xi_j = \{\langle(\hat{p}\cdot\xi)e_k, e_l\rangle\}_{k,l=1}^{14},$$

and $\overline{G}$ is the vector component $\langle G, e_j\rangle$. Here R is the remaining term which has the factor $(\mathbf{I} - P_0)f$. We denote

$$W = [W_I, W_{II}]^T, \quad W_I = [W_1, ..., W_5]^T, \quad W_{II} = [W_6, ..., W_{14}]^T.$$

To construct the compensation function of the relativistic equation (3.1), the following lemma was proved in [13,10].

Lemma 3.2. *There exist three 14×14 real constant skew-symmetric matrices R^j ($j = 1, 2, 3$) and positive constants c_1 and c_2 such that*

$$R(\omega) \equiv \sum_{j=1}^{3} R^j \omega_j, \tag{3.3}$$

satisfies

$$\mathcal{R}\langle\langle R(\omega)V(\omega)W, W\rangle\rangle \geq c_1|W_I|^2 - c_2|W_{II}|^2$$

for all $W \in \mathbb{C}^{14}$. Here $\langle\langle\cdot,\cdot\rangle\rangle$ represents the inner product on $\mathbb{C}^{14}$.

Now a compensation function for the relativistic equation (3.1) can be defined as follows. Given $\omega \in \mathbf{S}^2$, set $R(\omega) \equiv \{r_{ij}(\omega)\}_{i,j=1}^{14}$ as in (3.3), and let

$$S(\omega)f \equiv \sum_{k,\ell=1}^{14} \lambda r_{k\ell}(\omega)\langle f, e_\ell\rangle e_k, \quad f \in L^2(\mathbb{R}^3), \tag{3.4}$$

where $\lambda > 0$ is a constant to be chosen later.

Lemma 3.3. *There exists $\lambda > 0$ such that $S(\omega) : L^2(\mathbb{R}^3) \to \widetilde{W}$ is a compensation function for the relativistic equation (3.1).*

Proof. The first two properties can be verified straightforwardly by using (3.4). It suffices to verify (3.2). Note

$$\langle S(\omega)(\hat{p}\cdot\omega)f, f\rangle \equiv \sum_{k,\ell=1}^{14} \lambda r_{k\ell}(\omega)\langle(\hat{p}\cdot\omega)f, e_\ell\rangle\overline{\langle f, e_k\rangle}.$$

One can compute that

$$\langle(\hat{p}\cdot\omega)P_0 f, e_\ell\rangle = \sum_{j=1}^{14} W_j(V(\omega))_{\ell j}.$$

It follows that

$$\mathcal{R}\langle S(\omega)(\hat{p}\cdot\omega)f, f\rangle = \mathcal{R}\langle S(\omega)(\hat{p}\cdot\omega)[P_0 f + (\mathbf{I}-P_0)f], f\rangle$$
$$= \mathcal{R}\lambda \sum_{k,\ell=1}^{14} r_{k\ell}(\omega)\Big[\sum_{j=1}^{14} V_{\ell j}(\omega)W_j + \langle(\hat{p}\cdot\omega)(\mathbf{I}-P_0)h, e_\ell\rangle\Big]\overline{W}_k.$$

The first term in this expression is $\mathcal{R}\lambda\langle\langle R(\omega)V(\omega)W, W\rangle\rangle$. Notice that

$$|W_I|^2 = \sum_{j=1}^{5} |\langle f, e_j\rangle|^2 = |\mathbf{P}f|_2^2,$$

and

$$|W_{II}|^2 = \sum_{j=6}^{14} |\langle f, e_j\rangle|^2 = \sum_{j=6}^{14} |\langle(\mathbf{I}-\mathbf{P})f, e_j\rangle|^2 \le C|(\mathbf{I}-\mathbf{P})f|_\nu^2.$$

Here we have used the fast decay in p for e_j. It follows from Lemma 3.2 that

$$\mathcal{R}\lambda\langle\langle R(\omega)V(\omega)W, W\rangle\rangle \ge \lambda[c_3|\mathbf{P}f|_2^2 - c_4|(\mathbf{I}-\mathbf{P})f|_\nu^2].$$

The second term is dominated by

$$C\lambda \max_{k,\ell} |\langle(\hat{p}\cdot\omega)(\mathbf{I}-P_0)f, e_\ell\rangle| \cdot |\langle f, e_k\rangle| \le C\lambda|(\mathbf{I}-P_0)f|_\nu |f|_\nu$$
$$\le C_\epsilon\lambda|(\mathbf{I}-\mathbf{P})f|_\nu^2 + C_\epsilon\lambda|(P_0-\mathbf{P})f|_\nu^2 + C\epsilon\lambda|f|_\nu^2$$
$$\le C'_\epsilon\lambda|(\mathbf{I}-\mathbf{P})f|_\nu^2 + C\lambda\epsilon|\mathbf{P}f|_2^2.$$

Here we have used the fact that

$$|(P_0-\mathbf{P})f|_\nu^2 \le C|(P_0-\mathbf{P})f|_2^2 \le C\sum_{j=6}^{14} |\langle f, e_j\rangle|^2 \le C|(\mathbf{I}-\mathbf{P})f|_\nu^2.$$

By choosing $\epsilon > 0$ small enough we have

$$\mathcal{R}\langle S(\omega)(\hat{p}\cdot\omega)f, f\rangle \ge C_1\lambda|\mathbf{P}f|_2^2 - C_2\lambda|(\mathbf{I}-\mathbf{P})f|_\nu^2.$$

By choosing $\lambda > 0$ small enough, (3.2) then follows from (1.16). □

We now use the compensation function $S(\omega)$ to derive an energy estimate. Set $\omega = \xi/|\xi|$ and take the Fourier transform in x of (3.1). We have

$$\partial_t \widehat{f} + i|\xi|(\hat{p}\cdot\omega)\widehat{f} + L\widehat{f} = \hat{G}. \tag{3.5}$$

By multiplying (3.5) by the conjugate of $\widehat{f}$, we have

$$\frac{1}{2}\partial_t|\widehat{f}|_2^2 + \langle L\widehat{f}, \widehat{f}\rangle = \mathcal{R}\langle \widehat{f}, \hat{G}\rangle. \tag{3.6}$$

Then applying $-i|\xi|S(\omega)$ to (3.5) gives

$$-i|\xi|S(\omega)\partial_t\widehat{f} + |\xi|^2 S(\omega)((\hat{p}\cdot\omega)\widehat{f}) - i|\xi|S(\omega)L\widehat{f} = -i|\xi|S(\omega)\hat{G}.$$

The inner product of the above equation with $\widehat{f}$ yields

$$\mathcal{R}\langle -i|\xi|S(\omega)\partial_t\widehat{f}, \widehat{f}\rangle + |\xi|^2\mathcal{R}\langle S(\omega)(\hat{p}\cdot\omega)\widehat{f}, \widehat{f}\rangle = |\xi|\mathcal{R}\Big\{\langle iS(\omega)L\widehat{f}, \widehat{f}\rangle - \langle iS(\omega)\hat{G}, \widehat{f}\rangle\Big\}. \tag{3.7}$$

Since $iS(\omega)$ is self-adjoint, the first term is just $-\frac{1}{2}\partial_t[|\xi|\langle iS(\omega)\widehat{f}, \widehat{f}\rangle]$. By multiplying $(1+|\xi|^2)$ by (3.6), and adding κ times (3.7), we have

$$\begin{aligned}
&\partial_t\Big[\frac{(1+|\xi|^2)}{2}|\widehat{f}|_2^2 - \frac{\kappa|\xi|}{2}\langle iS(\omega)\widehat{f}, \widehat{f}\rangle\Big] \\
&\quad + (1+|\xi|^2 - \kappa|\xi|^2)\langle L\widehat{f}, \widehat{f}\rangle + \kappa|\xi|^2\{\mathcal{R}\langle S(\omega)(\hat{p}\cdot\omega)\widehat{f}, \widehat{f}\rangle + \langle L\widehat{f}, \widehat{f}\rangle\} \\
&= (1+|\xi|^2)\mathcal{R}\langle \widehat{f}, \hat{G}\rangle + \kappa|\xi|\mathcal{R}\Big\{\langle iS(\omega)L\widehat{f}, \widehat{f}\rangle - \langle iS(\omega)\hat{G}, \widehat{f}\rangle\Big\}.
\end{aligned} \tag{3.8}$$

For the second term on the left hand side of (3.8), when $0 < \kappa < 1$, we have

$$(1+|\xi|^2 - \kappa|\xi|^2)\langle L\widehat{f}, \widehat{f}\rangle \geq (1-\kappa)(1+|\xi|^2)\cdot\delta_0|(\mathbf{I}-\mathbf{P})\widehat{f}|_\nu^2.$$

And by (3.2), the third term on the left hand side of (3.8) is bounded by

$$\kappa|\xi|^2\{\mathcal{R}\langle S(\omega)(\hat{p}\cdot\omega)\widehat{f}, \widehat{f}\rangle + \langle L\widehat{f}, \widehat{f}\rangle\} \geq \kappa|\xi|^2\cdot c_0(|\mathbf{P}\widehat{f}|_2^2 + |(\mathbf{I}-\mathbf{P})\widehat{f}|_\nu^2).$$

Now we estimate the last term in (3.8). By using (3.4), we see that

$$\kappa|\xi||\langle iS(\omega)\hat{G}, \widehat{f}\rangle| \leq C\kappa|\xi|\sum_{k,\ell=1}^{14}|\langle \hat{G}, e_\ell\rangle|\cdot|\langle \widehat{f}, e_k\rangle| \leq c\kappa\varepsilon|\xi|^2|\widehat{f}|_\nu^2 + c_\varepsilon\kappa\sum_{\ell=1}^{14}|\langle \widehat{G}, e_\ell\rangle|^2.$$

Recalling $Lf = \Gamma[\sqrt{J}, f] + \Gamma[f, \sqrt{J}]$, we have from (2.47) and (2.49) that

$$|\langle Lf, e_\ell\rangle| = |\langle L[(\mathbf{I}-\mathbf{P})f], e_\ell\rangle| \leq C|(\mathbf{I}-\mathbf{P})f|_\nu.$$

By using (3.4), we see that

$$\kappa|\xi||\langle iS(\omega)L\widehat{f}, \widehat{f}\rangle| \leq c_\varepsilon\kappa|(\mathbf{I}-\mathbf{P})\widehat{f}|_\nu^2 + \kappa\varepsilon|\xi|^2|\widehat{f}|_\nu^2.$$

The last term on the right hand side of (3.8) is dominated by

$$\kappa|\xi|\Big\{|\langle iS(\omega)L\widehat{f},\widehat{f}\rangle|+|\langle iS(\omega)\hat{G},\widehat{f}\rangle|\Big\}$$
$$\leq c_\varepsilon\kappa|(\mathbf{I}-\mathbf{P})\widehat{f}|_\nu^2+C\kappa\varepsilon|\xi|^2|(\mathbf{I}-\mathbf{P})\widehat{f}|_\nu^2+C\kappa\varepsilon|\xi|^2|\mathbf{P}\widehat{f}|_2^2+c_\varepsilon\sum_{\ell=1}^{14}|\langle\widehat{G},e_\ell\rangle|^2.$$

If we choose κ, $\varepsilon>0$ small enough and combine the above estimates, we know that there exist δ_1, $\delta_2>0$ such that

$$\partial_t\Big[(1+|\xi|^2)|\widehat{f}|_2^2-\kappa|\xi|\langle iS(\omega)\widehat{f},\widehat{f}\rangle\Big]+\delta_1(1+|\xi|^2)|(\mathbf{I}-\mathbf{P})\widehat{f}|_\nu^2+\delta_2|\xi|^2|\mathbf{P}\widehat{f}|_2^2$$
$$\leq C(1+|\xi|^2)\mathcal{R}\langle\widehat{f},\widehat{G}\rangle+c_\varepsilon\sum_{\ell=1}^{14}|\langle\widehat{G},e_\ell\rangle|^2. \quad (3.9)$$

By (1.19), it holds that $|\mathbf{P}\widehat{f}(t,0,\cdot)|_2=0$. Since the domain considered is the torus, ξ here is a vector with integer components. Thus there exists a small constant $\delta_2'>0$ such that

$$\delta_2|\xi|^2|\mathbf{P}\widehat{f}|_2^2\geq\delta_2'(1+|\xi|^2)|\mathbf{P}\widehat{f}|_2^2.$$

With this and (3.9), we can obtain

$$\partial_t\Big[(1+|\xi|^2)|\widehat{f}|_2^2-\kappa|\xi|\langle iS(\omega)\widehat{f},\widehat{f}\rangle\Big]+\delta_1(1+|\xi|^2)|\widehat{f}|_\nu^2\leq C(1+|\xi|^2)\mathcal{R}\langle\widehat{f},\widehat{G}\rangle+C\sum_{\ell=1}^{14}|\langle\widehat{G},e_\ell\rangle|^2.$$

This implies that

$$\partial_t\Big[|\widehat{f}|_2^2-\kappa\frac{|\xi|}{1+|\xi|^2}\langle iS(\omega)\widehat{f},\widehat{f}\rangle\Big]+\delta|\widehat{f}|_\nu^2\leq C\mathcal{R}\langle\widehat{f},\hat{G}\rangle+C\sum_{\ell=1}^{14}|\langle\widehat{G},e_\ell\rangle|^2. \quad (3.10)$$

We will use the crucial estimate (3.10) to deduce the desired L^2 estimates.

Lemma 3.4. *Assume that $l\geq 0$, $l_0>3/b$, $\vartheta\in[0,1]$ and $\tau>0$. If $\vartheta=1$, restrict $\tau>0$ small enough. Let $f(t,x,p)$ be the solution to (1.11) satisfying (1.19). There exists $M>0$ small enough such that if*

$$\mathfrak{E}_{l+l_0,\vartheta}(f)(t)\leq M, \quad (3.11)$$

we have

$$\frac{d}{dt}\mathcal{E}_{l,\vartheta}(f)(t)+\mathcal{D}_{l,\vartheta}(f)(t)\leq 0. \quad (3.12)$$

Proof. Let $G = \Gamma(f, f)$ and integrate (3.10) over $\mathbb{Z}^3$ to give

$$\frac{d}{dt}\Big[\|f\|^2 - \kappa \int_{\mathbb{Z}^3} \frac{|\xi|}{1+|\xi|^2}\langle iS(\omega)\widehat{f}, \widehat{f}\rangle d\xi\Big] + \delta\|f\|_\nu^2$$

$$\leq C\int_{\mathbb{Z}^3} \mathcal{R}\langle\widehat{\Gamma(f,f)}, \widehat{f}\rangle d\xi + C\sum_{\ell=1}^{14}\int_{\mathbb{Z}^3} |\langle\widehat{\Gamma(f,f)}, e_\ell\rangle|^2 d\xi.$$

It follows from Lemma 2.5 and Lemma 2.4 that

$$\Big|\int_{\mathbb{Z}^3}\langle\widehat{\Gamma(f,f)}, \widehat{f}\rangle d\xi\Big| + \int_{\mathbb{Z}^3}|\langle\widehat{\Gamma(f,f)}, e_\ell\rangle|^2 d\xi \leq C(\mathfrak{E}_{l_0,\vartheta}(f)(t) + \mathfrak{E}^2_{l_0,\vartheta}(f)(t))\|f\|_\nu^2.$$

By choosing $M > 0$ small enough we can have from the properties of the Fourier transform that

$$\frac{d}{dt}\Big[\|f\|^2 - \kappa \int_{\mathbb{Z}^3} \frac{|\xi|}{1+|\xi|^2}\langle iS(\omega)\widehat{f}, \widehat{f}\rangle d\xi\Big] + \delta\|f\|_\nu^2 \leq CM\|f\|_\nu^2. \tag{3.13}$$

Multiply $\varpi^2(l,\vartheta)(p)f$ by (1.11) and then integrate over $\mathbb{T}^3 \times \mathbb{R}^3$ to get

$$\frac{1}{2}\frac{d}{dt}\|f\|_{l,\vartheta}^2 + \|f\|_{\nu,l,\vartheta}^2 - (\varpi^2(l,\vartheta)Kf, f) \leq (\varpi^2(l,\vartheta)\Gamma(f,f), f). \tag{3.14}$$

It follows from Lemma 2.1 that, for any small $\eta > 0$

$$|(\varpi^2(l,\vartheta)Kf, f)| \leq C\eta\|f\|_{\nu,l,\vartheta}^2 + C\|f\|_\nu^2.$$

Notice that Lemma 2.5 implies

$$|(\varpi^2(l,\vartheta)\Gamma(f,f), f)| \leq C\mathfrak{E}_{l+l_0,\vartheta}(f)(t)\|f\|_{\nu,l,\vartheta}^2.$$

By plugging these estimates into (3.14), we have from (3.11) that

$$\frac{d}{dt}\|f\|_{l,\vartheta}^2 + \delta_1\|f\|_{\nu,l,\vartheta}^2 \leq CM\|f\|_{\nu,l,\vartheta}^2 + C\|f\|_\nu^2. \tag{3.15}$$

By using a suitable linear combination of (3.13) and (3.15) and choosing $M > 0$ small enough, we have

$$\frac{d}{dt}\mathcal{E}_{l,\vartheta}(f)(t) + \mathcal{D}_{l,\vartheta}(f)(t) \leq 0.$$

Here $\mathcal{D}_{l,\vartheta}(f)(t)$ is as in (1.18) and $\mathcal{E}_{l,\vartheta}(f)(t)$ is defined as

$$\mathcal{E}_{l,\vartheta}(f)(t) = \frac{C}{\delta}\|f\|^2 - \frac{C\kappa}{\delta}\int_{\mathbb{Z}^3} \frac{|\xi|}{1+|\xi|^2}\langle iS(\omega)\widehat{f},\widehat{f}\rangle d\xi + \|f\|^2_{l,\vartheta}.$$

Since $\kappa > 0$ is small enough and $S(\omega)$ is bounded, we see that

$$\kappa\Big|\int_{\mathbb{Z}^3} \frac{|\xi|}{1+|\xi|^2}\langle iS(\omega)\widehat{f},\widehat{f}\rangle d\xi\Big| \le C\kappa\|f\|^2.$$

By this we know that $\mathcal{E}_{l,\vartheta}(f)(t) \sim \|f\|^2_{l,\vartheta}$ and this completes the proof of (3.12) and hence Lemma 3.4. □

4. Nonlinear L^∞ estimates

In this section we will prove the L^∞ estimates of (1.11) with the exponential p-weighted function in order to close the a priori estimate (3.11) in Lemma 3.4. Namely we will show the following lemma.

Lemma 4.1. *Assume that $l \ge 0$, $l_0 > 3/b$, $\vartheta \in [0,1]$ and $\tau > 0$. If $\vartheta = 1$, restrict $\tau > 0$ small enough. Let $f(t,x,p)$ be the solution to (1.11) satisfying (1.19). Then for any $T > 0$, we have*

$$\sup_{0\le s\le T} \mathfrak{E}_{l+l_0,\vartheta}(f)(t) \le C\mathfrak{E}_{l+l_0,\vartheta}(f_0) + C\sup_{0\le s\le T}\{\mathfrak{E}_{l+l_0,\vartheta}(f)(t)\}^2 + C\sup_{0\le s\le T}\|f(s)\|. \tag{4.1}$$

Proof. Let $\varpi = \varpi(l+l_0,\vartheta)(p)$ with $\ell_0 > 3/b$ and $K_\varpi g = \varpi K(\frac{g}{\varpi})$. Let $h = \varpi f$. Noticing that

$$\mathfrak{E}_{l+l_0,\vartheta}(f)(t) = \|\varpi(l+l_0,\vartheta)(p)f(t,x,p)\|_\infty = \|h(t)\|_\infty, \tag{4.2}$$

we have from (1.11) that

$$\partial_t h + \hat{p}\cdot\nabla_x h + \nu(p)h - K_\varpi h = \varpi\Gamma(\frac{h}{\varpi},\frac{h}{\varpi}). \tag{4.3}$$

Note that $K_\varpi = K_\varpi^\chi + K_\varpi^{1-\chi}$. By Duhamel's principle, we then expand out

$$\begin{aligned} h(t,x,p) &= e^{-\nu(p)t}h_0(x-\hat{p}t,p) + \int_0^t e^{-\nu(p)(t-s_1)}K_\varpi^{1-\chi}h(s_1,y_1,p)ds_1 \\ &\quad + \int_0^t e^{-\nu(p)(t-s_1)}K_\varpi^{\chi}h(s_1,y_1,p)ds_1 + \int_0^t e^{-\nu(p)(t-s_1)}\varpi\Gamma(\frac{h}{\varpi},\frac{h}{\varpi})(s_1,y_1,p)ds_1, \end{aligned} \tag{4.4}$$

with $y_1 = x - \hat{p}(t - s_1)$. We now estimate (4.4) term by term. It is direct to see that

$$\|e^{-\nu(p)t} h_0(x - \hat{p}t, p)\|_\infty \le \|h_0\|_\infty.$$

By Lemma 2.3, for any small $\eta > 0$, the second term on the right of (4.4) is bounded by

$$C\eta \sup_{0\le s\le t} \|h(s)\|_\infty \int_0^t e^{-\nu(p)(t-s_1)}\nu(p) ds_1 \le C\eta \sup_{0\le s\le t} \|h(s)\|_\infty.$$

By Lemma 2.7, the fourth term on the right of (4.4) is bounded by

$$C\{\sup_{0\le s\le t} \|h(s)\|_\infty\}^2 \int_0^t e^{-\nu(p)(t-s_1)}\nu(p) ds_1 \le C\{\sup_{0\le s\le t} \|h(s)\|_\infty\}^2.$$

It remains to estimate the third term on the right of (4.4). By (4.4) we have

$$\begin{aligned} h(s_1, y_1, q_1) = {} & e^{-\nu(q_1)s_1} h_0(y_1 - \hat{q}_1 s_1, q_1) + \int_0^{s_1} e^{-\nu(q_1)(s_1-s_2)} K_\varpi^{1-\chi} h(s_2, y_2, q_1) ds_2 \\ & + \int_0^{s_1} e^{-\nu(q_1)(s_1-s_2)} K_\varpi^{\chi} h(s_2, y_2, q_1) ds_2 \\ & + \int_0^{s_1} e^{-\nu(q_1)(s_1-s_2)} \varpi\Gamma\left(\frac{h}{\varpi}, \frac{h}{\varpi}\right)(s_2, y_2, q_1) ds_2, \end{aligned} \tag{4.5}$$

with $y_2 = y_1 - \hat{q}_1(s_1 - s_2)$. Let $k_\varpi^\chi(p,q)$ be the integral kernel of the operator K_ϖ^χ, i.e.,

$$k_\varpi^\chi(p,q) = k_\varpi^{\chi,2}(p,q) - k_\varpi^{\chi,1}(p,q), \qquad k_\varpi^{\chi,i}(p,q) = k_i^\chi(p,q)\frac{\varpi(l,\vartheta)(p)}{\varpi(l,\vartheta)(q)}, \quad i = 1, 2.$$

By using (2.18), (2.19), (2.20), (2.45) and the assumption of the weight function, one has

$$\begin{aligned} |k_\varpi^{\chi,2}(p,q_1)| &= |k_2^\chi(p,q_1)| \frac{\varpi(l,\vartheta)(p)}{\varpi(l,\vartheta)(q_1)} \\ &\le \frac{C(p_0 q_{10})^{b/2}\chi(|p-q_1|)}{[|p\times q_1|^2 + |p-q_1|^2]^{(1+b)/2}} (p_0 + q_{10})^{-b/2} e^{-\frac{c}{8}|p-q_1|} e^{-\frac{c}{8}|p-q_1|} \frac{\varpi(l,\vartheta)(p)}{\varpi(l,\vartheta)(q_1)} \\ &\le \frac{C(p_0 q_{10})^{b/2}\chi(|p-q_1|)}{[|p\times q_1|^2 + |p-q_1|^2]^{(1+b)/2}} (p_0 + q_{10})^{-b/2} e^{-\frac{c}{8}|p-q_1|}. \end{aligned} \tag{4.6}$$

By using (4.6) and (2.25), for any $\epsilon > 0$ small enough, we can obtain

$$\int_{\mathbb{R}^3} |k_{\varpi}^{\chi,2}(p,q_1)|dq_1 \le \int_{\mathbb{R}^3} \frac{C(p_0q_{10})^{b/2}\chi(|p-q_1|)}{[|p\times q_1|^2+|p-q_1|^2]^{(1+b)/2}}(p_0+q_{10})^{-b/2}e^{-\frac{c}{8}|p-q_1|}dq_1$$
$$\le Cp_0^{b/2}\int_{\mathbb{R}^3} \frac{\chi(|p-q_1|)e^{-c|p-q_1|/32}}{[|p\times q_1|^2+|p-q_1|^2]^{(1+b)/2}}dq_1 \le C\nu(p)p_0^{-\epsilon}.$$

By using (2.7), (2.19), and (2.20), one has

$$|k_{\varpi}^{\chi,1}(p,q_1)| = |k_1^{\chi}(p,q_1)|\frac{\varpi(l,\vartheta)(p)}{\varpi(l,\vartheta)(q_1)} \le C_\chi e^{-\frac{p_0}{8}-\frac{q_0}{8}}. \tag{4.7}$$

This implies that

$$\int_{\mathbb{R}^3} |k_{\varpi}^{\chi,1}(p,q_1)|dq_1 \le Ce^{-\frac{p_0}{8}}.$$

Thus, for any $\epsilon > 0$ small enough, we can obtain

$$\int_{\mathbb{R}^3} |k_{\varpi}^{\chi}(p,q_1)|dq_1 \le C\nu(p)p_0^{-\epsilon}. \tag{4.8}$$

By using (4.5), we can expand out the third term on the right of (4.4) as

$$\int_0^t e^{-\nu(p)(t-s_1)}K_{\varpi}^{\chi}h(s_1,y_1,p)ds_1 = H_1(t,x,p)+H_2(t,x,p)+H_3(t,x,p)+H_4(t,x,p), \tag{4.9}$$

where $H_i(t,x,p)$ $(i=1,2,3,4)$ are defined as

$$H_1(t,x,p) = \int_0^t\int_{\mathbb{R}^3} dq_1ds_1e^{-\nu(p)(t-s_1)}k_{\varpi}^{\chi}(p,q_1)e^{-\nu(q_1)s_1}h_0(y_1-\hat{q}_1s_1,q_1),$$

$$H_2(t,x,p) = \int_0^t\int_0^{s_1}\int_{\mathbb{R}^3} dq_1ds_2ds_1e^{-\nu(p)(t-s_1)}k_{\varpi}^{\chi}(p,q_1)e^{-\nu(q_1)(s_1-s_2)}K_{\varpi}^{1-\chi}h(s_2,y_2,q_1),$$

$$H_3(t,x,p) = \int_0^t\int_0^{s_1}\int_{\mathbb{R}^3} dq_1ds_2ds_1e^{-\nu(p)(t-s_1)}k_{\varpi}^{\chi}(p,q_1)e^{-\nu(q_1)(s_1-s_2)}\varpi\Gamma(\frac{h}{\varpi},\frac{h}{\varpi})(s_2,y_2,q_1),$$

$$H_4(t,x,p) = \int_0^t\int_0^{s_1}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} dq_2dq_1ds_2ds_1e^{-\nu(p)(t-s_1)}k_{\varpi}^{\chi}(p,q_1)$$
$$\times e^{-\nu(q_1)(s_1-s_2)}k_{\varpi}^{\chi}(q_1,q_2)h(s_2,y_2,q_2).$$

We now estimate (4.9) term by term. It follows from (4.8) that

$$
\begin{aligned}
|H_1(t,x,p)| &\leq \|h_0\|_\infty \int_{\mathbb{R}^3} |k_\varpi^\chi(p,q_1)|dq_1 \int_0^t e^{-\nu(p)(t-s_1)} e^{-\nu(q_1)s_1} ds_1 \\
&\leq \|h_0\|_\infty \nu^{-1}(p) \int_{\mathbb{R}^3} |k_\varpi^\chi(p,q_1)|dq_1 \leq C\|h_0\|_\infty.
\end{aligned}
$$

For the term $H_2(t,x,p)$, we use Lemma 2.3, for any $\eta > 0$, to obtain

$$
\begin{aligned}
|H_2(t,x,p)| &\leq \int_0^t \int_0^{s_1} \int_{\mathbb{R}^3} dq_1 ds_2 ds_1 |k_\varpi^\chi(p,q_1)| e^{-\nu(p)(t-s_1)} e^{-\nu(q_1)(s_1-s_2)} |K_\varpi^{1-\chi} h(s_2,y_2,q_1)| \\
&\leq C\eta \sup_{0\leq s\leq t} \|h(s)\|_\infty \int_0^t \int_0^{s_1} \int_{\mathbb{R}^3} dq_1 ds_2 ds_1 |k_\varpi^\chi(p,q_1)| e^{-\nu(p)(t-s_1)} \\
&\quad \times e^{-\nu(q_1)(s_1-s_2)} \nu(q_1) \\
&\leq C\eta \sup_{0\leq s\leq t} \|h(s)\|_\infty \int_{\mathbb{R}^3} dq_1 |k_\varpi^\chi(p,q_1)| \nu^{-1}(p).
\end{aligned}
$$

Thus we can obtain from (4.8) that

$$
|H_2(t,x,p)| \leq C\eta \sup_{0\leq s\leq t} \|h(s)\|_\infty.
$$

By (4.8) and Lemma 2.7 it holds that

$$
\begin{aligned}
|H_3(t,x,p)| &\leq \sup_{0\leq s\leq t} \|\nu^{-1}\varpi\Gamma(\frac{h}{\varpi},\frac{h}{\varpi})(s)\|_\infty \int_{\mathbb{R}^3} |k_\varpi^\chi(p,q_1)|dq_1 \\
&\quad \times \int_0^t \int_0^{s_1} e^{-\nu(p)(t-s_1)} e^{-\nu(q_1)(s_1-s_2)} \nu(q_1) ds_2 ds_1 \\
&\leq C \sup_{0\leq s\leq t} \|h(s)\|_\infty^2 \nu^{-1}(p) \int_{\mathbb{R}^3} |k_\varpi^\chi(p,q_1)|dq_1 \\
&\leq C \sup_{0\leq s\leq t} \|h(s)\|_\infty^2.
\end{aligned}
$$

We now concentrate on the last term in (4.9), which will be estimated as in [16, Theorem 6]. We first consider the case $|p| \geq N$.

Case 1: For $|p| \geq N$, we have from (4.8) that

$$\mathbf{1}_{|p|\geq N}\int_{\mathbb{R}^3}|k_{\varpi}^{\chi}(p,q_1)|dq_1 \leq C\nu(p)p_0^{-\epsilon}\mathbf{1}_{|p|\geq N}\leq CN^{-\epsilon}\nu(p).$$

Similarly it holds that

$$\int_{\mathbb{R}^3}|k_{\varpi}^{\chi}(q_1,q_2)|dq_2 \leq C\nu(q_1).$$

We thus have that

$$\begin{aligned}|H_4(t,x,p)\mathbf{1}_{|p|\geq N}| &\leq \sup_{0\leq s\leq t}\|h(s)\|_\infty \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathbf{1}_{|p|\geq N}|k_{\varpi}^{\chi}(p,q_1)k_{\varpi}^{\chi}(q_1,q_2)|dq_2dq_1\\ &\qquad\times\int_0^t\int_0^{s_1}e^{-\nu(p)(t-s_1)}e^{-\nu(q_1)(s_1-s_2)}ds_2ds_1\\ &\leq \sup_{0\leq s\leq t}\|h(s)\|_\infty \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathbf{1}_{|p|\geq N}|k_{\varpi}^{\chi}(p,q_1)k_{\varpi}^{\chi}(q_1,q_2)|\nu^{-1}(p)\nu^{-1}(q_1)dq_2dq_1\\ &\leq CN^{-\epsilon}\sup_{0\leq s\leq t}\|h(s)\|_\infty.\end{aligned}$$

Case 2: For either $|p|\leq N$, $|q_1|\geq 2N$ or $|q_1|\leq 2N$, $|q_2|\geq 3N$. Notice that we have either $|p-q_1|\geq N$ or $|q_1-q_2|\geq N$, and either one of the following is valid for some small $\eta>0$:

$$\begin{aligned}|k_{\varpi}^{\chi,2}(p,q_1)| &\leq e^{-\eta N}|k_{\varpi}^{\chi,2}(p,q_1)|e^{\eta|p-q_1|}\\ &\leq \frac{Ce^{-\eta N}(p_0q_{10})^{b/2}\chi(|p-q_1|)}{[|p\times q_1|^2+|p-q_1|^2]^{(1+b)/2}}(p_0+q_{10})^{-b/2}e^{-(\frac{c}{8}-\eta)|p-q_1|},\\ |k_{\varpi}^{\chi,2}(q_1,q_2)| &\leq e^{-\eta N}|k_{\varpi}^{\chi,2}(q_1,q_2)|e^{\eta|q_1-q_2|}\\ &\leq \frac{Ce^{-\eta N}(q_{10}q_{20})^{b/2}\chi(|q_1-q_2|)}{[|q_1\times q_2|^2+|q_1-q_2|^2]^{(1+b)/2}}(q_{10}+q_{20})^{-b/2}e^{-(\frac{c}{8}-\eta)|q_1-q_2|}.\end{aligned}$$

Here we have used (4.6). It follows from the arguments of (4.8) and the above inequalities that

$$\int_{\mathbb{R}^3}|k_{\varpi}^{\chi}(p,q_1)|e^{\eta|p-q_1|}dq_2\leq C\nu(p),\quad \int_{\mathbb{R}^3}|k_{\varpi}^{\chi}(q_1,q_2)|e^{\eta|q_1-q_2|}dq_2\leq C\nu(q_1). \tag{4.10}$$

Thus we use these to obtain

$$\int_0^t\int_0^{s_1}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}dq_2dq_1ds_2ds_1\mathbf{1}_{|p|\leq N,|q_1|\geq 2N}e^{-\nu(p)(t-s_1)}k_{\varpi}^{\chi}(p,q_1)$$

$$
\begin{aligned}
&\quad \times e^{-\nu(q_1)(s_1-s_2)} k_{\varpi}^{\chi}(q_1,q_2) h(s_2,y_2,q_2) \\
&\le \sup_{0\le s\le t} \|h(s)\|_\infty \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathbf{1}_{|p|\le N, |q_1|\ge 2N} |k_{\varpi}^{\chi}(p,q_1) k_{\varpi}^{\chi}(q_1,q_2)| dq_2 dq_1 \\
&\quad \times \int_0^t \int_0^{s_1} e^{-\nu(p)(t-s_1)} e^{-\nu(q_1)(s_1-s_2)} ds_2 ds_1 \\
&\le e^{-\eta N} \sup_{0\le s\le t} \|h(s)\|_\infty \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |k_{\varpi}^{\chi}(p,q_1)| e^{\eta|p-q_1|} |k_{\varpi}^{\chi}(q_1,q_2)| \nu^{-1}(p)\nu^{-1}(q_1) dq_2 dq_1 \\
&\le C e^{-\eta N} \sup_{0\le s\le t} \|h(s)\|_\infty.
\end{aligned}
$$

Similarly we also have that

$$
\begin{aligned}
&\int_0^t \int_0^{s_1} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} dq_2 dq_1 ds_2 ds_1 \mathbf{1}_{|q_1|\le 2N, |q_2|\ge 3N} e^{-\nu(p)(t-s_1)} k_{\varpi}^{\chi}(p,q_1) \\
&\qquad \times e^{-\nu(q_1)(s_1-s_2)} k_{\varpi}^{\chi}(q_1,q_2) h(s_2,y_2,q_2) \le C e^{-\eta N} \sup_{0\le s\le t} \|h(s)\|_\infty.
\end{aligned}
$$

Case 3: $|p| \le N$, $|q_1| \le 2N$ and $|q_2| \le 3N$. This is the last remaining case because if $|q_1| \ge 2N$, it is included in Case 2; while if $|q_2| \ge 3N$, either $|q_1| \le 2N$ or $|q_1| \ge 2N$ is also included in Case 2. We further assume that $s_1 - s_2 \le \kappa$ for $\kappa > 0$ small. Notice that

$$
\mathbf{1}_{|p|\le N, |q_1|\le 2N} e^{-\nu(p)(t-s_1)} e^{-\nu(q_1)(s_1-s_2)} \le e^{-C(t-s_2)/N^{b/2}}. \tag{4.11}
$$

Noticing that $\nu(p) \le C$, we have from (4.10) and (4.11) that

$$
\begin{aligned}
&\int_0^t \int_{s_1-\kappa}^{s_1} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} dq_2 dq_1 ds_2 ds_1 \mathbf{1}_{|p|\le N, |q_1|\le 2N, |q_2|\le 3N} \\
&\quad \times e^{-\nu(p)(t-s_1)} k_{\varpi}^{\chi}(p,q_1) e^{-\nu(q_1)(s_1-s_2)} k_{\varpi}^{\chi}(q_1,q_2) h(s_2,y_2,q_2) \\
&\le \sup_{0\le s\le t} \|h(s)\|_\infty \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathbf{1}_{|p|\le N, |q_1|\le 2N, |q_2|\le 3N} |k_{\varpi}^{\chi}(p,q_1) k_{\varpi}^{\chi}(q_1,q_2)| dq_2 dq_1 \\
&\quad \times \int_0^t \int_{s_1-\kappa}^{s_1} e^{-C(t-s_2)/N^{b/2}} ds_2 ds_1 \\
&\le C\kappa \sup_{0\le s\le t} \|h(s)\|_\infty.
\end{aligned}
$$

Case 4: $|p| \le N$, $|q_1| \le 2N$, $|q_2| \le 3N$ and $s_1 - s_2 \ge \kappa$. It follows from (4.6) and (4.7) that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathbf{1}_{|p|\le N,|q_1|\le 2N,|q_2|\le 3N}|k_\varpi^\chi(p,q_1)k_\varpi^\chi(q_1,q_2)|^2 dq_2dq_1 \le C_N.$$

We use this to obtain that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} dq_2dq_1\mathbf{1}_{|p|\le N,|q_1|\le 2N,|q_2|\le 3N}k_\varpi^\chi(p,q_1)k_\varpi^\chi(q_1,q_2)h(s_2,y_2,q_2)\Big|\\
&\le \Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathbf{1}_{|p|\le N,|q_1|\le 2N,|q_2|\le 3N}|k_\varpi^\chi(p,q_1)k_\varpi^\chi(q_1,q_2)|^2 dq_2dq_1\Big)^{1/2}\\
&\qquad\qquad\times\Big(\int_{|q_1|\le 2N}\int_{|q_2|\le 3N}|h(s_2,y_2,q_2)|^2dq_2dq_1\Big)^{1/2}\\
&\le C\Big(\int_{|q_1|\le 2N}\int_{|q_2|\le 3N}|h(s_2,y_2,q_2)|^2dq_2dq_1\Big)^{1/2}.
\end{aligned}$$

Since $y_2 = y_1 - \hat{q}_1(s_1-s_2)$, we make a change of variable $q_1 \to y_2$. In [28],

$$\Big(\frac{dy_2}{dq_1}\Big)_{mn} = -(s_1-s_2)\Big(\frac{\delta_{mn}q_{10}^2 - q_{1m}q_{1nn}}{q_{10}^3}\Big).$$

Thus the Jacobian is

$$\Big|\frac{dy_2}{dq_1}\Big| = \frac{|(s_1-s_2)|^3}{q_{10}^5} \ge C\frac{\kappa^3}{N^5}.$$

Recalling that $h = \varpi f$, we can obtain

$$\begin{aligned}
&\Big(\int_{|q_1|\le 2N}\int_{|q_2|\le 3N}|h(s_2,y_2,q_2)|^2dq_2dq_1\Big)^{1/2}\\
&\le C\Big(\frac{\kappa^3}{N^5}\Big)^{1/2}\Big(\int_{|y_2-x|\le c(t-s_2)}\int_{|q_2|\le 3N}|h(s_2,y_2,q_2)|^2dq_2dy_2\Big)^{1/2}\\
&\le C_{N,\kappa}\Big(\int_{|y_2-x|\le c(t-s_2)}\int_{|q_2|\le 3N}|f(s_2,y_2,q_2)|^2dq_2dy_2\Big)^{1/2}\\
&\le C_{N,\kappa}\{1+(t-s_2)^{3/2}\}\Big(\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}|f(s_2,y_2,q_2)|^2dq_2dy_2\Big)^{1/2}.
\end{aligned}$$

Thus we have from the above estimates that

$$\int_0^t\int_0^{s_1-\kappa}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} dq_2dq_1ds_2ds_1\mathbf{1}_{|p|\le N,|q_1|\le 2N,|q_2|\le 3N}$$

$$
\begin{aligned}
&\qquad\qquad \times e^{-\nu(p)(t-s_1)} k_{\varpi}^{\chi}(p,q_1) e^{-\nu(q_1)(s_1-s_2)} k_{\varpi}^{\chi}(q_1,q_2) h(s_2,y_2,q_2) \\
&\leq C \int_0^t \int_0^{s_1-\kappa} ds_2 ds_1 e^{-C(t-s_1)/N^{b/2}} e^{-C(s_1-s_2)/N^{b/2}} \\
&\qquad\qquad \times \left| \int_{|q_1|\leq 2N} \int_{|q_2|\leq 3N} dq_2 dq_1 k_{\varpi}^{\chi}(p,q_1) k_{\varpi}^{\chi}(q_1,q_2) h(s_2,y_2,q_2) \right| \\
&\leq C_{N,\kappa} \sup_{0\leq s\leq t} \|f(s)\| \int_0^t \int_0^{s_1} e^{-C(t-s_2)/(2N^{b/2})} e^{-C(t-s_2)/(2N^{b/2})} \{1+(t-s_2)^{3/2}\} ds_2 ds_1 \\
&\leq C_{N,\kappa} \sup_{0\leq s\leq t} \|f(s)\|.
\end{aligned}
$$

To summarize, if we take any small $\kappa > 0$, any small $\epsilon > 0$ and $\eta > 0$, and large $N > 0$, we have established, for any $T > 0$

$$
\begin{aligned}
\sup_{0\leq s\leq T} \{\|h(s)\|_\infty\} \leq C\|h_0\|_\infty + C(\eta + N^{-\epsilon} + e^{-\eta N} + \kappa) \sup_{0\leq s\leq T} \{\|h(s)\|_\infty\} \\
+ C \sup_{0\leq s\leq T} \{\|h(s)\|_\infty^2\} + C_{N,\kappa} \sup_{0\leq s\leq T} \|f(s)\|.
\end{aligned}
$$

Thus we can obtain that

$$
\sup_{0\leq s\leq T} \{\|h(s)\|_\infty\} \leq C\|h_0\|_\infty + C \sup_{0\leq s\leq T} \{\|h(s)\|_\infty\}^2 + C \sup_{0\leq s\leq T} \|f(s)\|.
$$

This implies that (4.1) holds by (4.2). The proof of Lemma 4.1 is complete. □

5. Existence and time-decay

In this section, we will first construct local-in-time solutions to the relativistic Boltzmann equation and then give the proofs of Theorem 1.1 and Corollary 1.3.

The construction of local-in-time solutions is based on the uniform estimate for a sequence of iterative approximate solutions.

Theorem 5.1. *Let any $l \geq 0$, $l_0 > 3/b$, $\vartheta \in [0,1]$ and $\tau > 0$. If $\vartheta = 1$, restrict $\tau > 0$ small enough. There exist both $\epsilon > 0$ and $T^* > 0$ small enough such that if*

$$
\mathfrak{E}_{l+l_0,\vartheta}(f_0) \leq \epsilon,
$$

then there exists a unique solution $f(t,x,p)$ to the relativistic Boltzmann equation (1.11) in $[0,T^*] \times \mathbb{T}^3 \times \mathbb{R}^3$ such that

$$
\sup_{0\leq s\leq T^*} \mathfrak{E}_{l+l_0,\vartheta}(f)(s) \leq C\mathfrak{E}_{l+l_0,\vartheta}(f_0).
$$

These solutions are continuous provided that it is so initially. We further have the positivity, i.e., $F = J + \sqrt{J}f \geq 0$ *if* $F_0 = J + \sqrt{J}f_0 \geq 0$*. Furthermore, the conservation laws* (1.19) *hold for all* $t \in (0, T^*]$ *if they are valid initially at* $t = 0$.

Proof. We consider the following iterative sequence $\{F^n(t, x, p)\}$ by solving the original relativistic Boltzmann equation (1.2):

$$\{\partial_t + \hat{p} \cdot \nabla_x\}F^{n+1} + R(F^n)F^{n+1} = Q_{gain}(F^n, F^n), \tag{5.1}$$

with the initial data $F^{n+1}(0, x, p) = F_0(x, p)$ and starting with $F^0(t, x, p) \equiv F_0(x, p)$. Here we have used the notations

$$R(F^n)F^{n+1} = F^{n+1}(p) \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} v_\phi \sigma(g, \theta) F^n(q) dq d\omega,$$

and

$$Q_{gain}(F^n, F^n) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} v_\phi \sigma(g, \theta) F^n(p') F^n(q') dq d\omega.$$

Since $F^{n+1}(t, x, p) = J + \sqrt{J} f^{n+1}(t, x, p)$, equivalently $f^{n+1}(t, x, p)$ satisfies

$$\{\partial_t + \hat{p} \cdot \nabla_x + \nu(p)\}f^{n+1} = K(f^n) + \Gamma_{gain}(f^n, f^n) - \Gamma_{loss}(f^n, f^{n+1}), \tag{5.2}$$

with the initial data $f^{n+1}(0, x, p) = f_0(x, p)$ for all $n \geq 0$ and with $f^0(t, x, p) \equiv f_0(x, p)$. The first goal of the proof is to get a uniform-in-n estimate for $\mathfrak{E}_{l+l_0,\vartheta}(f^{n+1})(t)$. The crucial estimate is given as follows.

Lemma 5.2. *Let any* $l \geq 0$, $l_0 > 3/b$, $\vartheta \in [0, 1]$ *and* $\tau > 0$*. If* $\vartheta = 1$*, restrict* $\tau > 0$ *small enough. There exist both* $\epsilon > 0$ *and* $T^* = T^*(\epsilon)$ *small enough such that*

$$\mathfrak{E}_{l+l_0,\vartheta}(f_0) \leq \epsilon, \quad \text{and} \quad \sup_{0 \leq t \leq T^*} \mathfrak{E}_{l+l_0,\vartheta}(f^n)(t) \leq C\mathfrak{E}_{l+l_0,\vartheta}(f_0), \tag{5.3}$$

we have

$$\sup_{0 \leq t \leq T^*} \mathfrak{E}_{l+l_0,\vartheta}(f^{n+1})(t) \leq C\mathfrak{E}_{l+l_0,\vartheta}(f_0). \tag{5.4}$$

Suppose that $\widetilde{f}^{n+1}(t, x, p) = f^{n+1}(t, x, p) - f^n(t, x, p)$*. It holds that*

$$\sup_{0 \leq t \leq T^*} \mathfrak{E}_{l+l_0,\vartheta}(\widetilde{f}^{n+1})(t) \leq \frac{1}{2} \sup_{0 \leq t \leq T^*} \mathfrak{E}_{l+l_0,\vartheta}(\widetilde{f}^n)(t). \tag{5.5}$$

Proof. Let $\varpi = \varpi(l + l_0, \vartheta)(v)$ with $l_0 > 3/b$ and $K_\varpi g = \varpi K(\frac{g}{\varpi})$. If we suppose that $h^n = \varpi f^n$ and $h^{n+1} = \varpi f^{n+1}$, we have from (5.2) that

$$\{\partial_t + \hat{p} \cdot \nabla_x + \nu(p)\}h^{n+1} = K_\varpi(h^n) + \varpi\Gamma_{gain}(\frac{h^n}{\varpi}, \frac{h^n}{\varpi}) - \varpi\Gamma_{loss}(\frac{h^n}{\varpi}, \frac{h^{n+1}}{\varpi}).$$

By Duhamel's principle, we then expand out

$$\begin{aligned} h^{n+1}(t, x, p) = e^{-\nu(p)t} h_0(x - \hat{p}t, p) &+ \int_0^t e^{-\nu(p)(t-s_1)} K_\varpi h^n(s_1, y_1, p) ds_1 \\ &+ \int_0^t e^{-\nu(p)(t-s_1)} \varpi\Gamma_{gain}(\frac{h^n}{\varpi}, \frac{h^n}{\varpi})(s_1, y_1, p) ds_1 \\ &- \int_0^t e^{-\nu(p)(t-s_1)} \varpi\Gamma_{loss}(\frac{h^n}{\varpi}, \frac{h^{n+1}}{\varpi})(s_1, y_1, p) ds_1. \end{aligned} \tag{5.6}$$

Here $y_1 = x - \hat{p}(t - s_1)$. By Lemma 2.3 and (4.8) we have for any $t \in (0, T^*)$

$$\int_0^t e^{-\nu(p)(t-s_1)} K_\varpi h^n(s_1, y_1, p) ds_1 \leq Ct \sup_{0 \leq s \leq t} \|K_\varpi h^n(s)\|_\infty \leq CT^* \sup_{0 \leq s \leq T^*} \|h^n(s)\|_\infty.$$

By Lemma 2.7 we can obtain for any $t \in (0, T^*)$

$$\begin{aligned} &\int_0^t e^{-\nu(p)(t-s_1)} \varpi\Gamma_{gain}(\frac{h^n}{\varpi}, \frac{h^n}{\varpi})(s_1, y_1, p) ds_1 \\ &\quad + \int_0^t e^{-\nu(p)(t-s_1)} \varpi\Gamma_{loss}(\frac{h^n}{\varpi}, \frac{h^{n+1}}{\varpi})(s_1, y_1, p) ds_1 \\ &\leq C \sup_{0 \leq s \leq t} \|\nu^{-1}\varpi\Gamma_{gain}(\frac{h^n}{\varpi}, \frac{h^n}{\varpi})(s)\|_\infty + C \sup_{0 \leq s \leq t} \|\nu^{-1}\varpi\Gamma_{loss}(\frac{h^n}{\varpi}, \frac{h^{n+1}}{\varpi})(s)\|_\infty \\ &\leq C\{\sup_{0 \leq s \leq T^*} \|h^n(s)\|_\infty\}^2 + C \sup_{0 \leq s \leq T^*} \|h^n(s)\|_\infty \sup_{0 \leq s \leq T^*} \|h^{n+1}(s)\|_\infty. \end{aligned}$$

By these estimates we have from (5.6) that for any $t \in (0, T^*)$

$$\begin{aligned} \|h^{n+1}(t)\|_\infty \leq \|h_0\|_\infty &+ CT^* \sup_{0 \leq s \leq T^*} \|h^n(s)\|_\infty + C\{\sup_{0 \leq s \leq T^*} \|h^n(s)\|_\infty\}^2 \\ &+ C \sup_{0 \leq s \leq T^*} \|h^n(s)\|_\infty \sup_{0 \leq s \leq T^*} \|h^{n+1}(s)\|_\infty. \end{aligned} \tag{5.7}$$

If we choose $\epsilon > 0$ and $T^* > 0$ small enough, (5.4) follows from (5.3) and (5.7).

Suppose that $\widetilde{f}^{n+1} = f^{n+1} - f^n$ and $\widetilde{h}^{n+1} = h^{n+1} - h^n$. Then $\widetilde{h}^{n+1}$ satisfies

$$\{\partial_t + \hat{p}\cdot\nabla_x + \nu(p)\}\widetilde{h}^{n+1} = K_\varpi(\widetilde{h}^n) + \varpi\Gamma_{gain}(\frac{h^{n+1}}{\varpi}, \frac{\widetilde{h}^{n+1}}{\varpi}) + \varpi\Gamma_{gain}(\frac{\widetilde{h}^{n+1}}{\varpi}, \frac{h^n}{\varpi}) - \varpi\Gamma_{loss}(\frac{\widetilde{h}^{n}}{\varpi}, \frac{h^{n+1}}{\varpi}) - \varpi\Gamma_{loss}(\frac{h^{n-1}}{\varpi}, \frac{\widetilde{h}^{n+1}}{\varpi}),$$

with $\widetilde{h}^{n+1}(0,x,p) = 0$. The similar arguments as for (5.7) imply that for any $t \in (0, T^*)$

$$\begin{aligned} \|\widetilde{h}^{n+1}(t)\|_\infty \le CT^* \sup_{0\le s\le T^*}\|\widetilde{h}^n(s)\|_\infty + C\sup_{0\le s\le T^*}\|\widetilde{h}^{n+1}(s)\|_\infty \\ \times\Big(\sup_{0\le s\le T^*}\|h^{n+1}(s)\|_\infty + \sup_{0\le s\le T^*}\|h^{n}(s)\|_\infty + \sup_{0\le s\le T^*}\|h^{n-1}(s)\|_\infty\Big) \\ + C\sup_{0\le s\le T^*}\|\widetilde{h}^n(s)\|_\infty \sup_{0\le s\le T^*}\|h^{n+1}(s)\|_\infty. \end{aligned} \tag{5.8}$$

By using (5.3), (5.4) and (5.8), we have

$$\|\widetilde{h}^{n+1}(t)\|_\infty \le C(T^*+\epsilon)\sup_{0\le s\le T^*}\|\widetilde{h}^n(s)\|_\infty \le \frac{1}{2}\sup_{0\le s\le T^*}\|\widetilde{h}^n(s)\|_\infty.$$

Here we choose both $\epsilon > 0$ and $T^* > 0$ small enough. This concludes the proof of (5.5) and hence Lemma 5.2. □

By Lemma 5.2, $\{f^n\}$ is a Cauchy sequence and the limit f is a desired solution. Now for uniqueness, there is another solution $\bar{f}$ to the relativistic Boltzmann equation with the same initial condition as f. Assume that $\sup_{0\le t\le T^*}\mathfrak{E}_{l+l_0,\vartheta}(\bar{f})(t)$ is also sufficiently small. Then the difference between $h = \varpi f$ and $\bar{h} = \varpi\bar{f}$ satisfies

$$\{\partial_t + \hat{p}\cdot\nabla_x + \nu(p)\}\{h-\bar{h}\} = K_\varpi\{h-\bar{h}\} + \varpi\Gamma(\frac{h-\bar{h}}{\varpi}, \frac{h}{\varpi}) + \varpi\Gamma(\frac{\bar{h}}{\varpi}, \frac{h-\bar{h}}{\varpi}).$$

By the similar arguments as for (5.7), we can obtain

$$\|h-\bar{h}\|_\infty \le CT^*\|h-\bar{h}\|_\infty + C\Big(\sup_{0\le t\le T^*}\|h\|_\infty + \sup_{0\le t\le T^*}\|\bar{h}\|_\infty\Big)\sup_{0\le t\le T^*}\|h-\bar{h}\|_\infty.$$

Since $T^* > 0$, $\sup_{0\le t\le T^*}\|h\|_\infty$ and $\sup_{0\le t\le T^*}\|\bar{h}\|_\infty$ are small enough, we deduce $h = \bar{h}$. This completes the proof of the uniqueness.

Next we prove that the solution $h(t,x,p)$ is continuous in $[0,T^*]\times\mathbb{T}^3\times\mathbb{R}^3$ by the similar arguments as in [16]. We claim that $h^{n+1}(t,x,p)$ is continuous in $[0,T^*]\times\mathbb{T}^3\times\mathbb{R}^3$ inductively. To prove this claim for any given fixed n, we can use another iteration to solve the linear problem for h^{n+1} in (5.6) as the limit of $n' \to \infty$:

$$\{\partial_t + \hat{p}\cdot\nabla_x + \nu(p)\}h^{n+1,n'+1} = K_\varpi h^{n+1,n'} + \varpi\Gamma(\frac{h^n}{\varpi}, \frac{h^n}{\varpi}).$$

By induction in n', $h^{n+1,n'+1}$ is continuous in $[0,T^*]\times\mathbb{T}^3\times\mathbb{R}^3$, and by Corollary 2.6, it is standard to verify that $K_\varpi^{1-\chi}h^{n+1,n'}$ is continuous in $[0,T^*]\times\mathbb{T}^3\times\mathbb{R}^3$. It follows from (4.7), (4.6) and (2.27) that

$$\int_{\mathbb{R}^3} \mathbf{1}_{|p|+|q|\geq N}|k_\varpi^\chi(p,q)|dq \leq C\eta\nu(p).$$

By (4.7), (4.6) and (2.19), we can choose $k_N(p,q)$ smooth with compact support such that for some $\zeta>0$ and some large $N>0$,

$$\sup_{|p|\leq N}\int_{|q|\leq N} |k_\varpi^\chi(p,q) - k_N(p,q)|dq \leq \frac{C}{N^\zeta}.$$

By the above two estimates and the induction hypothesis on continuity of $h^{n+1,n'}$ in $[0,T^*]\times\mathbb{T}^3\times\mathbb{R}^3$, it is also routine to verify that $K_\varpi^\chi h^{n+1,n'}$ is continuous in $[0,T^*]\times\mathbb{T}^3\times\mathbb{R}^3$. From Lemma 2.5 and the induction hypothesis on continuity of h^n in $[0,T^*]\times\mathbb{T}^3\times\mathbb{R}^3$, it is also straightforward to verify that $\varpi\Gamma[\frac{h^n}{\varpi},\frac{h^n}{\varpi}]$ is continuous in $[0,T^*]\times\mathbb{T}^3\times\mathbb{R}^3$. By the similar equation as (5.6), we deduce that $h^{n+1,n'+1}$ is continuous in $[0,T^*]\times\mathbb{T}^3\times\mathbb{R}^3$. Finally we have

$$\{\partial_t + \hat{p}\cdot\nabla_x + \nu(p)\}\{h^{n+1,n'+1} - h^{n+1,n'}\} = K_\varpi\{h^{n+1,n'} - h^{n+1,n'-1}\}.$$

It follows from (5.6), (5.7), (2.21), Corollary 2.6 and the above equation that

$$\begin{aligned}\sup_{0\leq t\leq T^*}\|h^{n+1,n'+1}(t) - h^{n+1,n'}(t)\|_\infty &\leq C\int_0^{T^*}\|h^{n+1,n'}(s) - h^{n+1,n'-1}(s)\|_\infty ds\\ &\leq \dots \leq \frac{(CT^*)^{n'}}{n'!}.\end{aligned}$$

Therefore, $\{h^{n+1,n'}\}$ is Cauchy in L^∞, and its limit h^{n+1} is continuous in $[0,T^*]\times\mathbb{T}^3\times\mathbb{R}^3$. We conclude our claim. We can then deduce that as the limit of h^{n+1}, h preserves the continuity in $[0,T^*]\times\mathbb{T}^3\times\mathbb{R}^3$ from uniform convergence.

Finally we show the positivity of $F(t,x,p) = J(p) + \sqrt{J(p)}f(t,x,p)$. As for obtaining (5.6), for (5.1) we may also write it as the mild form

$$\begin{aligned}F^{n+1}(t,x,p) = {}& e^{\int_0^t R(F^n)(s_1,x-\hat{p}s_1,p)ds_1}F_0(t,x-\hat{p}t,p)\\ &+\int_0^t e^{\int_{s_1}^t R(F^n)(s_2,x-\hat{p}s_2,p)ds_2}Q_{gain}(s_1,x-\hat{p}(t-s_1),p)ds_1.\end{aligned}$$

By this and induction on n, $F^{n+1} \geq 0$ if $F^n \geq 0$. This implies that the limit $F \geq 0$ if the initial data $F_0 \geq 0$. Since $\sup_{0\leq s\leq T^*} \mathfrak{E}_{l+l_0,\vartheta}(f)(s) \leq C\epsilon$, $f(t,x,p)$ is bounded and continuous $[0,T^*] \times \mathbb{T}^3 \times \mathbb{R}^3$. By these it is straightforward to verify that classical mass, total momentum and total energy conservations hold for such solutions constructed. We thus conclude Theorem 5.1. □

Proof of Theorem 1.1. There exists $\epsilon > 0$ small enough such that Theorem 5.1 is valid. We choose a constant $C_1 > 1$ such that for any $t \geq 0$,

$$\frac{1}{C_1}\|f(t)\|^2_{l,\vartheta} \leq \mathcal{E}_{l,\vartheta}(f)(t) \leq C_1\|f(t)\|^2_{l,\vartheta}.$$

From Theorem 5.1, we may denote $T > 0$ so that for some constant $C_0 > 0$,

$$T = \sup_{t>0}\{t : \mathfrak{E}_{l+l_0,\vartheta}(f)(t) \leq C_0\epsilon\} > 0.$$

Set $M = C_0\epsilon$ where M is as in Lemma 3.4. By Lemma 3.4 we can obtain that

$$\sup_{0\leq t\leq T}\|f(t)\|^2_{l,\vartheta} \leq C_1 \sup_{0\leq t\leq T} \mathcal{E}_{l,\vartheta}(f)(t) \leq C_1\mathcal{E}_{l,\vartheta}(f_0) \leq C_1^2\|f_0\|^2_{l,\vartheta} \leq C_1^2\mathfrak{E}^2_{l+l_0,\vartheta}(f_0).$$

By this and Lemma 4.1 we have

$$\begin{aligned}\sup_{0\leq t\leq T} \mathfrak{E}_{l+l_0,\vartheta}(f)(t) &\leq C\mathfrak{E}_{l+l_0,\vartheta}(f_0) + \sup_{0\leq t\leq T}\|f(t)\|_{l,\vartheta} + C\{\sup_{0\leq t\leq T} \mathfrak{E}_{l+l_0,\vartheta}(f)(t)\}^2 \\ &\leq (C + C_1)\mathfrak{E}_{l+l_0,\vartheta}(f_0) + CC_0\epsilon \sup_{0\leq t\leq T} \mathfrak{E}_{l+l_0,\vartheta}(f)(t).\end{aligned}$$

If we choose both $\epsilon > 0$ and $\mathfrak{E}_{l+l_0,\vartheta}(f_0)$ small enough, we have that

$$\sup_{0\leq t\leq T} \mathfrak{E}_{l+l_0,\vartheta}(f)(t) \leq \frac{C + C_1}{1 - CC_0\epsilon}\mathfrak{E}_{l+l_0,\vartheta}(f_0) \leq \frac{C_0\epsilon}{2} < C_0\epsilon.$$

We thus deduce $T = \infty$ from the continuity of $\mathfrak{E}_{l+l_0,\vartheta}(f)(t)$, and the existence of global solution follows.

Next we will prove the exponential decay of the global solution to (1.11) by using the similar idea as [31,2,3]. The key point is to split $\mathcal{E}_l(f)(t)$ into a time dependent low momentum

$$E_0 = \{p_0 \leq \rho t^{\beta'}\}, \quad \text{and} \quad E_0^c = \{p_0 > \rho t^{\beta'}\}. \tag{5.9}$$

Here $\beta' > 0$ and $\rho > 0$ will be chosen later.

Let $\mathcal{E}_l^{low}(f)(t)$ be the instant energy $\mathcal{E}_l(f)(t)$ restricted to E_0. Then we have that

$$\mathcal{D}_l(f)(t) \geq C\rho^{-b/2}t^{-b\beta'/2}\mathcal{E}_l^{low}(f)(t).$$

By this and (3.12) with $\vartheta = 0$ we can obtain

$$\frac{d}{dt}\mathcal{E}_l(f)(t) + C\rho^{-b/2}t^{-b\beta'/2}\mathcal{E}_l^{low}(f)(t) \le 0.$$

Letting $\mathcal{E}_l^{high}(f)(t) = \mathcal{E}_l(f)(t) - \mathcal{E}_l^{low}(f)(t)$, we have

$$\frac{d}{dt}\mathcal{E}_l(f)(t) + C\rho^{-b/2}t^{-b\beta'/2}\mathcal{E}_l(f)(t) \le C\rho^{-b/2}t^{-b\beta'/2}\mathcal{E}_l^{high}(f)(t).$$

Define $\lambda_0\beta = C\rho^{-b/2}$ and $\beta - 1 = -b\beta'/2$ where $\beta' > 0$ will be chosen later. Then

$$\frac{d}{dt}\mathcal{E}_l(f)(t) + \lambda_0\beta t^{\beta-1}\mathcal{E}_l(f)(t) \le \lambda_0\beta t^{\beta-1}\mathcal{E}_l^{high}(f)(t).$$

This implies that

$$\frac{d}{dt}\Big(e^{\lambda_0 t^\beta}\mathcal{E}_l(f)(t)\Big) \le \lambda_0\beta t^{\beta-1}e^{\lambda_0 t^\beta}\mathcal{E}_l^{high}(f)(t).$$

It follows from this that

$$\mathcal{E}_l(f)(t) \le e^{-\lambda_0 t^\beta}\Big(\mathcal{E}_l(f_0) + \lambda_0\beta\int_0^t s^{\beta-1}e^{\lambda_0 s^\beta}\mathcal{E}_l^{high}(f)(s)ds\Big). \tag{5.10}$$

Notice that $\beta - 1 = -b\beta'/2$. Letting $\vartheta\beta' = \beta$, then $\beta = \frac{\vartheta}{\vartheta+\frac{b}{2}}$. It follows from (3.12) that $\mathcal{E}_{l,\vartheta}(f)(s) \le \mathcal{E}_{l,\vartheta}(f_0)$. By these facts we can obtain

$$\begin{aligned}\mathcal{E}_l^{high}(f)(s) = \int_{\mathbb{T}^3}\int_{E_0^c} p_0^{bl}|f(s,x,p)|^2dpdx &\le \int_{\mathbb{T}^3}\int_{E_0^c}\frac{e^{2\tau p_0^\vartheta}}{e^{2\tau\rho^\vartheta s^{\vartheta\beta'}}}p_0^{bl}|f(s,x,p)|^2dpdx\\ &= e^{-2\tau\rho^\vartheta s^{\vartheta\beta'}}\mathcal{E}_{l,\vartheta}(f)(s) \le e^{-2\tau\rho^\vartheta s^\beta}\mathcal{E}_{l,\vartheta}(f_0).\end{aligned}$$

Further choose $\rho > 0$ large enough so that $\lambda_0 = \frac{C}{\beta}\rho^{-b/2} < 2\tau\rho^\vartheta$. With this we have from the above inequality that

$$\int_0^t s^{\beta-1}e^{\lambda_0 s^\beta}\mathcal{E}_l^{high}(f)(s)ds \le \mathcal{E}_{l,\vartheta}(f_0)\int_0^t s^{\beta-1}e^{\lambda_0 s^\beta}e^{-2\tau\rho^\vartheta s^\beta}ds \le C\mathcal{E}_{l,\vartheta}(f_0). \tag{5.11}$$

By (5.10) and (5.11) we see that (1.20) holds. This then completes the proof of Theorem 1.1. □

Proof of Corollary 1.3. We only deduce the a priori estimate. Multiply f by (1.22), integrate over $\mathbb{R}^3$ to get

$$\frac{1}{2}\frac{d}{dt}|f|_2^2 + \delta|(\mathbf{I}-\mathbf{P})f|_\nu^2 \le \langle \Gamma(f,f), f\rangle. \tag{5.12}$$

Due to the fact that $f_0 \in \mathcal{N}^\perp$, it holds that $f \in \mathcal{N}^\perp$ and $\mathbf{P}f = 0$. By Corollary 2.6 it follows that

$$\langle \Gamma(f,f), f\rangle \le C\overline{\mathfrak{E}}_{l_0}(f)(t)|f|_\nu^2.$$

By these facts it follows from (5.12) that

$$\frac{d}{dt}|f|_2^2 + \delta_0|f|_\nu^2 \le C\overline{\mathfrak{E}}_{l_0}(f)(t)|f|_\nu^2. \tag{5.13}$$

Multiply $\varpi^2(l,\vartheta)(p)f$ by (1.22), integrate over $\mathbb{R}^3$ to get

$$\frac{1}{2}\frac{d}{dt}|f|_{l,\vartheta}^2 + |f|_{\nu,l,\vartheta}^2 - \langle \varpi^2(l,\vartheta)Kf, f\rangle \le \langle \varpi^2(l,\vartheta)\Gamma(f,f), f\rangle. \tag{5.14}$$

It follows from Lemma 2.1 that, for any $\eta > 0$

$$|\langle \varpi^2(l,\vartheta)Kf, f\rangle| \le C\eta|f|_{\nu,l,\vartheta}^2 + C|f|_\nu^2.$$

It follows from Corollary 2.6 that

$$|\langle \varpi^2(l,\vartheta)\Gamma(f,f), f\rangle| \le C\overline{\mathfrak{E}}_{l+l_0,\vartheta}(f)(t)|f|_{\nu,l,\vartheta}^2.$$

By plugging these estimates into (5.14), we have that

$$\frac{d}{dt}|f|_{l,\vartheta}^2 + \delta_1|f|_{\nu,l,\vartheta}^2 \le C\overline{\mathfrak{E}}_{l+l_0,\vartheta}(f)(t)|f|_{\nu,l,\vartheta}^2 + C|f|_\nu^2. \tag{5.15}$$

By using a suitable linear combination of (5.13) and (5.15) and assuming that $\overline{\mathfrak{E}}_{l+l_0,\vartheta}(f)(t)$ is small enough, we have

$$\frac{d}{dt}\overline{\mathcal{E}}_{l,\vartheta}(f)(t) + \overline{\mathcal{D}}_{l,\vartheta}(f)(t) \le 0. \tag{5.16}$$

Here $\overline{\mathcal{D}}_{l,\vartheta}(f)(t)$ is (1.24) and $\overline{\mathcal{E}}_{l,\vartheta}(f)(t)$ is defined as

$$\overline{\mathcal{E}}_{l,\vartheta}(f)(t) = C|f|_2^2 + |f|_{l,\vartheta}^2 \sim |f|_{l,\vartheta}^2.$$

To close the a priori estimate, the similar arguments as for Lemma 4.1 imply that for any $T > 0$, if $f_0 \in \mathcal{N}^\perp$ and $\overline{\mathfrak{E}}_{l+l_0,\vartheta}(f_0)$ is small enough, the solution $f(t,p)$ to the equation (1.22) satisfies

$$\sup_{0\le s\le T}\overline{\mathfrak{E}}_{l+l_0,\vartheta}(f)(t) \le C\overline{\mathfrak{E}}_{l+l_0,\vartheta}(f_0) + C\sup_{0\le s\le T}\{\overline{\mathfrak{E}}_{l+l_0,\vartheta}(f)(t)\}^2 + C\sup_{0\le s\le T}|f(s)|_2. \tag{5.17}$$

Once we obtain (5.16) and (5.17), the similar arguments as for Theorem 1.1 conclude the proof of Corollary 1.3. □

6. Propagation of spatial regularity

In this section we will show the propagation of space regularity and also the large time behavior of the higher-order energy functional of global solutions obtained in Theorem 1.1.

Proof of Theorem 1.2. Since the local solution obtained in Theorem 5.1 is unique, we can use the proof of Theorem 5.1 and the assumption of Theorem 1.2 to prove the propagation of space regularity. We omit the proof for simplicity and we focus on the uniform bounds and the large time behavior of the higher-order energy functional of global solutions. We shall use induction in the nonnegative integer N. The case $N = 0$ is just a direct consequence of Theorem 1.1. Suppose that Theorem 1.2 is true up to the case $k \leq N-1$, that is, it holds that for all nonnegative integers $k \leq N-1$,

$$\begin{aligned}&\frac{d}{dt}\mathcal{E}_{k,l,\vartheta}(f)(t) + \mathcal{D}_{k,l,\vartheta}(f)(t)\\&\quad\leq C\sum_{0\leq m\leq[\frac{k}{2}]}\left\{\mathfrak{E}_{m,l+l_0,\vartheta}(f)(t) + \mathfrak{E}^2_{m,l+l_0,\vartheta}(f)(t)\right\}\mathcal{D}_{k-m,l,\vartheta}(f)(t),\end{aligned}\tag{6.1}$$

and there exist positive constants C_k', C_k'' and a decreasing number sequence $\lambda_k > 0$ such that

$$\sup_{0\leq t<\infty}\mathfrak{E}_{k,l+l_0,\vartheta}(f)(t) \leq C_k' \quad \text{and} \quad \mathcal{E}_{k,l}(f)(t) \leq C_k'' e^{-\lambda_k t^\beta},\ t\geq 0. \tag{6.2}$$

We need to show the theorem in the case $k = N$. Letting $G = \Gamma(f, f)$ in (3.10), multiplying (3.10) by $|\xi|^{2N}$ and integrating over ξ yield

$$\begin{aligned}&\frac{d}{dt}\left[\sum_{|\alpha|=N}\|\partial^\alpha f\|^2 - \kappa\int_{\mathbb{Z}^3}\frac{|\xi|^{2N+1}}{1+|\xi|}\langle iS(\omega)\hat{f},\hat{f}\rangle d\xi\right] + \delta_1\sum_{|\alpha|=N}\|\partial^\alpha f\|^2_\nu\\&\qquad\leq\int_{\mathbb{Z}^3}|\xi|^{2N}\mathcal{R}\langle\widehat{f},\widehat{\Gamma(f,f)}\rangle d\xi + C\sum_{\ell=1}^{14}\int_{\mathbb{Z}^3}|\xi|^{2N}|\langle\widehat{\Gamma(f,f)},e_\ell\rangle|^2 d\xi.\end{aligned}\tag{6.3}$$

By Lemma 2.5 and the properties of the Fourier transform, it follows that

$$\begin{aligned}\left|\int_{\mathbb{Z}^3}|\xi|^{2N}\mathcal{R}\langle\widehat{f},\widehat{\Gamma(f,f)}\rangle d\xi\right| &\leq C\sum_{|\alpha|=N}|(\partial^\alpha\Gamma(f,f),\partial^\alpha f)|\\&\leq C\sum_{0\leq m\leq[\frac{N}{2}]}\mathfrak{E}_{m,l+l_0,\vartheta}(f)(t)\mathcal{D}_{N-m,l,\vartheta}(f)(t).\end{aligned}$$

Similarly one has from Lemma 2.4 that

$$\int_{\mathbb{Z}^3} |\xi|^{2N} |\langle \widehat{\Gamma(f,f)}, e_\ell\rangle|^2 d\xi \le C \sum_{|\alpha|=N} \int_{\mathbb{Z}^3} |\langle \partial^\alpha \Gamma(f,f), e_\ell\rangle|^2 d\xi$$
$$\le C \sum_{0\le m\le [\frac{N}{2}]} \mathfrak{E}^2_{m,l+l_0,\vartheta}(f)(t) \mathcal{D}_{N-m,l,\vartheta}(f)(t).$$

Plugging the above two estimates into (6.3) gives

$$\frac{d}{dt}\left[\sum_{|\alpha|=N} \|\partial^\alpha f\|^2 - \kappa \int_{\mathbb{Z}^3} \frac{|\xi|^{2N+1}}{1+|\xi|} \langle iS(\omega)\hat{f}, \hat{f}\rangle d\xi\right] + \delta_1' \sum_{|\alpha|=N} \|\partial^\alpha f\|_\nu^2$$
$$\le C \sum_{0\le m\le [\frac{N}{2}]} \left\{\mathfrak{E}_{m,l+l_0,\vartheta}(f)(t) + \mathfrak{E}^2_{m,l+l_0,\vartheta}(f)(t)\right\} \mathcal{D}_{N-m,l,\vartheta}(f)(t). \tag{6.4}$$

We take the ∂^α of (1.11), multiply $\varpi^2(l,\vartheta)(p)\partial^\alpha f$ by the resulting equation, and integrate it over $\mathbb{T}^3 \times \mathbb{R}^3$ to deduce the following estimate

$$\sum_{|\alpha|=N} \left\{\frac{1}{2}\frac{d}{dt}\|\partial^\alpha f\|^2_{l,\vartheta} + (\varpi^2(l,\vartheta) L\partial^\alpha f, \partial^\alpha f)\right\} = (\varpi^2(l,\vartheta)\partial^\alpha \Gamma(f,f), \partial^\alpha f). \tag{6.5}$$

Notice that Lemma 2.1 implies

$$(\varpi^2(l,\vartheta) L\partial^\alpha f, \partial^\alpha f) = \|\partial^\alpha f\|^2_{\nu,l,\vartheta} - (\varpi^2(l,\vartheta) K\partial^\alpha f, \partial^\alpha f) \ge \frac{1}{2}\|\partial^\alpha f\|^2_{\nu,l,\vartheta} - C\|\partial^\alpha f\|^2_\nu,$$

and also Lemma 2.5 implies

$$(\varpi^2(l,\vartheta)\partial^\alpha \Gamma(f,f), \partial^\alpha f) \le C \sum_{0\le m\le [\frac{N}{2}]} \mathfrak{E}_{m,l+l_0,\vartheta}(f)(t) \mathcal{D}_{N-m,l,\vartheta}(f)(t).$$

Plugging the above estimates into (6.5), one has

$$\sum_{|\alpha|=N} \left\{\frac{d}{dt}\|\partial^\alpha f\|^2_{l,\vartheta} + \|\partial^\alpha f\|^2_{\nu,l,\vartheta} - C\|\partial^\alpha f\|^2_\nu\right\}$$
$$\le C \sum_{0\le m\le [\frac{N}{2}]} \mathfrak{E}_{m,l+l_0,\vartheta}(f)(t) \mathcal{D}_{N-m,l,\vartheta}(f)(t). \tag{6.6}$$

A suitable linear combination of (6.4) and (6.6) yields

$$\frac{d}{dt}\mathcal{E}_{N,l,\vartheta}(f)(t) + \mathcal{D}_{N,l,\vartheta}(f)(t)$$
$$\le C \sum_{0\le m\le [\frac{N}{2}]} \{\mathfrak{E}_{m,l+l_0,\vartheta}(f)(t) + \mathfrak{E}^2_{m,l+l_0,\vartheta}(f)(t)\} \mathcal{D}_{N-m,l,\vartheta}(f)(t), \tag{6.7}$$

where $\mathcal{D}_{N,l,\vartheta}(f)(t)$ is given in (1.18) and $\mathcal{E}_{N,l,\vartheta}(f)(t)$ is defined as

$$\mathcal{E}_{N,l,\vartheta}(f)(t) = \sum_{|\alpha|=N} \|\partial^\alpha f\|_{l,\vartheta}^2 + 2C \sum_{|\alpha|=N} \|\partial^\alpha f\|^2 - 2C\kappa \int_{\mathbb{Z}^3} \frac{|\xi|^{2N+1}}{1+|\xi|^2} \langle iS(\omega)\hat{f}, \hat{f}\rangle d\xi.$$

Since $S(\omega)$ is bounded, it holds that

$$\Big| \int_{\mathbb{Z}^3} \frac{|\xi|^{2N+1}}{1+|\xi|^2} \langle iS(\omega)\hat{f}, \hat{f}\rangle d\xi \Big| \le C \sum_{|\alpha|=N} \|\partial^\alpha f\|^2.$$

Further due to the fact that $\kappa > 0$ is small enough, one can see $\mathcal{E}_{N,l,\vartheta}(f)(t) \sim \sum_{|\alpha|=N} \|\partial^\alpha f\|_{l,\vartheta}^2$. This concludes the proof of (6.1) for $k = N$.

Recall by Theorem 1.1 that $\mathfrak{E}_{l+l_0,\vartheta}(f)(t)$ is small enough. It follows from (6.7) that

$$\frac{d}{dt}\mathcal{E}_{1,l,\vartheta}(f)(t) + \mathcal{D}_{1,l,\vartheta}(f)(t) \le 0. \tag{6.8}$$

For $N \ge 2$, one has

$$\begin{aligned} &\frac{d}{dt}\mathcal{E}_{N,l,\vartheta}(f)(t) + \mathcal{D}_{N,l,\vartheta}(f)(t) \\ &\quad \le C \sum_{1\le m\le [\frac{N}{2}]} \{\mathfrak{E}_{m,l+l_0,\vartheta}(f)(t) + \mathfrak{E}^2_{m,l+l_0,\vartheta}(f)(t)\}\mathcal{D}_{N-m,l,\vartheta}(f)(t). \end{aligned} \tag{6.9}$$

As $\mathfrak{E}_{l+l_0,\vartheta}(f_0)$ is small enough, it follows from (3.12) that

$$\mathcal{E}_{l,\vartheta}(f)(t) + \int_0^t \mathcal{D}_{l,\vartheta}(f)(s)ds \le \mathcal{E}_{l,\vartheta}(f_0) \le C\mathfrak{E}^2_{l+l_0,\vartheta}(f_0), \tag{6.10}$$

and hence by (6.8), (6.9) and the induction assumption, we have from (6.1) that for any $|k| \le N-1$,

$$\mathcal{E}_{k,l,\vartheta}(f)(t) + \int_0^t \mathcal{D}_{k,l,\vartheta}(f)(s)ds \le C. \tag{6.11}$$

By using (6.10), (6.11) and the induction assumption again, we have from (6.8) and (6.9) that

$$\mathcal{E}_{N,l,\vartheta}(f)(t) + \int_0^t \mathcal{D}_{N,l,\vartheta}(f)(s)ds \le C. \tag{6.12}$$

In terms of (4.2), (4.3), (6.10), Lemma 2.3 and Lemma 2.7, the similar arguments as in the proof of Lemma 4.1 imply that for any $T > 0$,

$$\sup_{0\le s\le T} \mathfrak{E}_{N,l+l_0,\vartheta}(f)(t) \le C\mathfrak{E}_{N,l+l_0,\vartheta}(f_0) + C \sup_{0\le s\le T} \sum_{|\alpha|=N} \|\partial^\alpha f(s)\|$$

$$+ C\{\sup_{0\le s\le T} \mathfrak{E}_{l+l_0,\vartheta}(f)(t)\}\{\sup_{0\le s\le T} \mathfrak{E}_{N,l+l_0,\vartheta}(f)(t)\} + C\{\sum_{1\le k\le N-1} \sup_{0\le s\le T} \mathfrak{E}_{k,l+l_0,\vartheta}(f)(t)\}^2.$$

Notice that $\mathfrak{E}_{l+l_0,\vartheta}(f_0)$ is small enough. By using (6.2) and (6.12) we have from the above inequality that for some constant $C'_N > 0$,

$$\sup_{0\le t\le\infty} \mathfrak{E}_{N,l+l_0,\vartheta}(f)(t) \le C'_N.$$

Next we shall prove the exponential time decay for the Nth-order spatial derivative of the solution. Let $\vartheta = 0$ in (6.9). We have from (6.2) that for $N \ge 2$,

$$\frac{d}{dt}\mathcal{E}_{N,l}(f)(t) + \mathcal{D}_{N,l}(f)(t) \le C \sum_{[\frac{N}{2}]\le m\le N-1} \mathcal{D}_{m,l}(f)(t) \le C \sum_{[\frac{N}{2}]\le m\le N-1} \mathcal{E}_{m,l}(f)(t), \tag{6.13}$$

where we have used (1.17), (1.18) and the fact that $\nu(p) \le C$. For any $m \in \{1, 2, ..., N\}$, as (5.9), we define

$$E_m = \{p_0 \le (2^m\rho)t^{\beta'}\} \text{ and } E_m^c = \{p_0 > (2^m\rho)t^{\beta'}\}.$$

Let $\mathcal{E}^{low}_{N,l}(f)(t)$ be the instant energy restricted to E_N. Then we have that

$$\mathcal{D}_{N,l}(f)(t) \ge C(2^N\rho)^{-b/2}t^{-b\beta'/2}\mathcal{E}^{low}_{N,l}(f)(t).$$

Define $\lambda_N\beta = C(2^N\rho)^{-b/2}$. Recalling $\beta - 1 = -b\beta'/2$, we have from this and (6.13) that

$$\frac{d}{dt}\mathcal{E}_{N,l}(f)(t) + \lambda_N\beta t^{\beta-1}\mathcal{E}_{N,l}(f)(t) \le \lambda_N\beta t^{\beta-1}\mathcal{E}^{high}_{l}(f)(t) + C \sum_{[\frac{N}{2}]\le m\le N-1} \mathcal{E}_{m,l}(f)(t),$$

where $\mathcal{E}^{high}_{N,l}(f)(s)$ is the instant energy functional $\mathcal{E}_{N,l}(f)(s)$ restricted to the set E_N^c. It follows from the above inequality that

$$\mathcal{E}_{N,l}(f)(t) \le e^{-\lambda_N t^\beta}\Big(\mathcal{E}_{N,l}(f_0) + \lambda_N\beta \int_0^t s^{\beta-1}e^{\lambda_N s^\beta}\mathcal{E}^{high}_{N,l}(f)(s)ds$$

$$+C \sum_{[\frac{N}{2}]\le m\le N-1} \int_0^t e^{\lambda_N s^\beta}\mathcal{E}_{m,l}(f)(s)ds\Big). \quad (6.14)$$

Choose $\rho > 0$ large enough so that for $m \in \{1, 2, ..., N\}$,

$$\lambda_m = \frac{C}{\beta}(2^m\rho)^{-b/2} < 2\tau(2^m\rho)^\vartheta. \quad (6.15)$$

Then λ_m is decreasing in m. By (6.2) we have that for $m < N$,

$$\int_0^t e^{\lambda_N s^\beta}\mathcal{E}_{m,l}(f)(s)ds \le C''_m \int_0^t e^{\lambda_N s^\beta} e^{-\lambda_m s^\beta} ds \le C''_m. \quad (6.16)$$

Notice that

$$\mathcal{E}^{high}_{N,l}(f)(s) \le e^{-2\tau(2^N\rho)^\vartheta s^{\vartheta\beta'}}\mathcal{E}_{N,l,\vartheta}(f)(s) \le Ce^{-2\tau(2^N\rho)^\vartheta s^\beta}\mathcal{E}_{N,l,\vartheta}(f_0).$$

By this we can obtain from (6.15) that

$$\int_0^t s^{\beta-1}e^{\lambda_N s^\beta}\mathcal{E}^{high}_{N,l}(f)(s)ds \le C\mathcal{E}_{N,l,\vartheta}(f_0)\int_0^t s^{\beta-1}e^{\lambda_N s^\beta}e^{-2\tau(2^N\rho)^\vartheta s^\beta}ds \le C\mathcal{E}_{N,l,\vartheta}(f_0). \quad (6.17)$$

It follows from (6.12), (6.14), (6.16) and (6.17) that

$$\mathcal{E}_{N,l}(f)(t) \le C''_N e^{-\lambda_N t^\beta}.$$

This completes the proof of Theorem 1.2 for the case $N \ge 2$. In the case $N = 1$, one can use (6.8) and the similar proof as (5.9) and (5.10) to get the desired results. This then completes the proof of Theorem 1.2. □

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