BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 47(1), 307-321

ISSN: 1678-7544 (Print) / 1678-7714 (Online)

Large-time behavior for fluid and kinetic plasmas with collisions

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Abstract. The motion of collisional plasmas can be governed either by the Euler-Maxwell system with damping at the fluid level or by the Vlasov-Maxwell-Boltzmann system at the kinetic level. In the note, we present some recent results in [8] and [7] for the study of the non-trivial large-time behavior of solutions to the Cauchy problem on the related models in perturbation framework.

Keywords: collisional plasmas, damped Euler-Maxwell system, Vlasov-Poisson-Boltzmann system, time-asymptotic stability.

Mathematical subject classification: Primary: 35Q20, 76P05; Secondary: 35B35, 35B40.

1 Motivations

Consider the two-fluid kinetic system with collisions (cf. [3, 24])

$$\partial_t F_\alpha + \xi \cdot \nabla_x F_\alpha + \frac{q_\alpha}{m_\alpha}\Big(E + \frac{\xi}{c} \times B\Big) \cdot \nabla_\xi F_\alpha = C_\alpha, \quad \alpha = i, e,$$

coupling to the Maxwell system

$$\partial_t E - c\nabla \times B = -4\pi J, \quad \partial_t B + c\nabla \times E = 0,$$
$$\nabla \cdot E = 4\pi\rho, \quad \nabla \cdot B = 0,$$

with

$$J = \sum_\alpha q_\alpha \int_{\mathbb{R}^3} \xi F_\alpha \, d\xi,$$

$$\rho = \sum_\alpha q_\alpha \int_{\mathbb{R}^3} F_\alpha \, d\xi.$$

Received 12 January 2015.

Here $F_\alpha = F_\alpha(t, x, \xi)$ stands for the number distribution function of particles with position x and velocity ξ at time t, and E, B are the electric and magnetic fields. Constants q_α, m_α, c are the charge, mass, and speed of light, respectively. In general the term C_α depending on all F_α and particle masses denotes collisions of α-species particles with all like and unlike particles in plasma. For fully ionised plasma, the collision is grazing and the corresponding collision term C_α is usually described by the Landau collision operator of the Fokker-Planck type. Mathematically, in order to compute the macroscopic transport coefficients in terms of the kinetic model, it is convenient to use the Boltzmann collision operator which is of the integral form and acts only on the velocity variable. For weakly ionised plasma, in the case when collisions of charged particles ions or electrons with neutral particles are dominated, the collision term is determined by the relaxation operator conserving only the mass. In the spatially homogeneous case without any force, due to the relaxation property of C_α, the system tends in large time to the Maxwellian

$$\mathbf{M} = \mathbf{M}_{[n,u,T]}(\xi) = \frac{n}{(2\pi RT)^{3/2}} e^{-\frac{|\xi-u|^2}{2T}},$$

where n is the number density, u is the bulk velocity, and T is the temperature. One of the important goals of studying these models in mathematics is to establish stability and convergence rates of solutions around a global Maxwellian or some non-trivial profile (for instance, wave patterns and stationary solutions) in the spatially inhomogeneous case, cf. [41].

To study the problems on the kinetic equations above, particularly for investigating the structure of the complex coupling system, it could be better to first look at the corresponding fluid dynamic models with relaxations. Associated with $F_\alpha(t, x, \xi)$, one can introduce the macroscopic moments

$$n_\alpha(t,x) \equiv \int_{\mathbb{R}^3} F_\alpha(t,x,\xi)\,d\xi, \quad u_\alpha(t,x) \equiv \frac{1}{n_\alpha(t,x)} \int_{\mathbb{R}^3} \xi F_\alpha(t,x,\xi)\,d\xi,$$

$$\theta_\alpha(t,x) \equiv \frac{1}{3k_\alpha n_\alpha} \int_{\mathbb{R}^3} |\xi - u_\alpha(t,x)|^2 F_\alpha(t,x,\xi)\,d\xi, \quad k_\alpha = \frac{k_B}{m_\alpha},$$

and the high-order moments (thermal quantities)

$$\begin{aligned} P_\alpha(t,x) &\equiv m_\alpha \int_{\mathbb{R}^3} (\xi - u_\alpha) \otimes (\xi - u_\alpha) F_\alpha(t,x,\xi)\,d\xi \\ &= p_\alpha I + \Pi_\alpha, \quad p_\alpha = k_B n_\alpha \theta_\alpha, \end{aligned}$$

$$
\begin{aligned}
h_\alpha(t,x) &\equiv \frac{1}{2}m_\alpha \int_{\mathbb{R}^3} |\xi - u_\alpha|^2(\xi - u_\alpha)F_\alpha(t,x,\xi)\,d\xi,\\
\mathcal{R}_\alpha(t,x) &\equiv \int_{\mathbb{R}^3} m_\alpha(\xi - u_\alpha)C_\alpha\,d\xi,\\
\mathcal{Q}_\alpha(t,x) &\equiv \int_{\mathbb{R}^3} \frac{1}{2}m_\alpha|\xi - u_\alpha|^2 C_\alpha\,d\xi.
\end{aligned}
$$

At the formal level, one obtains the macroscopic fluid system which is unclosed and takes the form of the Euler-Maxwell system:

$$
\begin{aligned}
&(\partial_t + u_\alpha\cdot\nabla_x)n_\alpha + n_\alpha\nabla_x\cdot u_\alpha = 0,\\
&n_\alpha m_\alpha(\partial_t + u_\alpha\cdot\nabla_x)u_\alpha + \nabla_x(k_B n_\alpha\theta_\alpha)\\
&\qquad = n_\alpha q_\alpha\Big(E + \frac{u_\alpha}{c}\times B\Big) - \nabla_x\cdot\Pi_\alpha + \mathcal{R}_\alpha,\\
&\frac{3}{2}n_\alpha(\partial_t + u_\alpha\cdot\nabla_x)k_B\theta_\alpha + k_B n_\alpha\theta_\alpha\nabla_x\cdot u_\alpha\\
&\qquad = -\Pi_\alpha : \nabla_x u_\alpha - \nabla_x\cdot h_\alpha + \mathcal{Q}_\alpha,
\end{aligned}
$$

coupled to

$$
\partial_t E - c\nabla\times B = -4\pi\sum_\alpha q_\alpha n_\alpha u_\alpha,\qquad \partial_t B + c\nabla\times E = 0,
$$
$$
\nabla\cdot E = 4\pi\sum_\alpha q_\alpha n_\alpha,\qquad \nabla\cdot B = 0.
$$

To present the motivations for the study of collisional fluid and kinetic plasmas, we recall a few mathematical results on the Vlasov-Maxwell-Boltzmann (VMB) system near global Maxwellians in perturbation framework. In the case of the periodic box, the global existence of small-amplitude classical solutions to the Cauchy problem was firstly obtained by Guo [16] through the robust energy method, and the almost exponential time rate was later given by Jang [23] where the diffusive limit to the incompressible Navier-Stokes-Poisson system is also justified. In the case of the whole space, Strain [37] showed the global existence by using the two-species' cancelation property to control the electric field, and Duan-Strain [11] further provided explicit polynomial convergence rates of solutions to the constant steady state by carrying out the linearised analysis as well as the bootstrap argument to the nonlinear problem. In particular, it was shown in [11] by the Fourier energy method that for the linearised VMB system, there is an energy functional $\mathcal{E}(t,k)$ equivalent with the natural one $\|\hat{u}\|^2_{L^2_\xi} + |[\hat{E},\hat{B}]|^2$ such that

$$\partial_t \mathcal{E}(t,k) + \lambda \|\nu^{1/2}\{\mathbf{I}-\mathbf{P}\}\hat{u}\|^2_{L^2_\xi} + \frac{\lambda |k|^2}{1+|k|^2}\|\mathbf{P}\hat{u}\|^2_{L^2_\xi}$$

$$+\lambda|\hat{E}|^2 + \frac{\lambda|k|^2}{(1+|k|^2)^2}|\hat{B}|^2 \le 0,$$

for all $t \ge 0$ and $k \in \mathbb{R}^3$, where $\lambda > 0$ is a positive constant, u stands for the perturbation, and $\mathbf{P}$ is the projection to the null space of the linearised Boltzmann operator. Consider the case when $\nu = \nu(\xi)$ has a positive lower bound. The above energy inequality implies that the real part of eigenvalues of the linearised system in the Fourier space is less than or equal to $-\lambda|k|^2/(1+|k|^2)^2$. Therefore there is no eigenvalue on the imaginary axis in the complex plane except at $k=0$, and one eigenvalue could tend to zero with the rate $1/|k|^2$ as $|k|$ goes to infinity. Hence, both 0 and $\pm i\infty$ are singular points of the resolvent of the linear solution operator. Note that in the finite-dimensional case, there exists an abstract spectral theory to study such complex system of the regularity-loss type with many applications to the Bresse system for polynomial stability of the semigroup; see Liu-Rao [28], Batty-Duyckaerts [1], Muñoz Rivera-Racke [35], Ide-Kawashima [25], and reference therein.

The work [11] then motivates us to consider the following questions:

(a) Is the energy product rate optimal? To answer this question has to be based on the spectral analysis as in Ukai [38] for the pure Boltzmann equation without any force. The case with self-consistent forces has been recently done by Li-Yang-Zhong [26].

(b) Can the fluid-type system, that is the damped Euler-Maxwell, enjoy a similar property? In this direction, global existence and rates of convergence of solutions near constant steady states were considered in Ueda-Kawashima [40] and Duan [5]. Recently the analysis of eigenvalues for the two-fluid system with general physical parameters was made in Duan-Liu-Zhu [8], where the linear diffusion wave is also justified in the sense of the generalised Darcy's law in the context of plasma physics.

(c) What happens to the long-range potentials with/without angular cutoff assumptions? The previous work on the VMB system mentioned above is mainly for the hard sphere model. It is usually more difficult to generalise those results to the non hard sphere case, particularly for the very soft potentials. The main breakthrough in this direction was recently made by Guo [17] for the study of the Vlasov-Poisson-Landau system with the Coulomb potential. For the VMB system, we may refer to Duan-Liu-Yang-Zhao [10], Duan-Lei-Yang-Zhao [9] and references therein.

(d) Is there any abstract theory to treat such complex coupling system? At the fluid level, Ueda-Duan-Kawashima [39] indeed provided a structure condition to general hyperbolic conservation laws with degenerate dissipations. We also expect to generalise the result to the coupled kinetic-fluid system including the VMB system.

(e) What about the stability issue for some existing non-trivial profiles such as wave patterns? We may start from the simple case when only the potential force is present, and mainly focus on the interesting situation where the potential function is non-trivial, cf. Duan-Liu [6, 7].

In this note we consider the nontrivial large-time behaviour of fluid and kinetic plasmas only related to questions (b) and (e).

2 Linear diffusion waves of Euler-Maxwell with damping

Consider the two-fluid ($\alpha = i, e$) Euler-Maxwell system with damping in $\mathbb{R}^3$:

$$
\begin{aligned}
&\partial_t n_\alpha + \nabla \cdot (n_\alpha u_\alpha) = 0,\\
&m_\alpha n_\alpha (\partial_t u_\alpha + u_\alpha \cdot \nabla u_\alpha) + \nabla p_\alpha(n_\alpha)\\
&\qquad = q_\alpha n_\alpha \Big(E + \frac{u_\alpha}{c} \times B\Big) - \nu_\alpha m_\alpha n_\alpha u_\alpha,\\
&\partial_t E - c\nabla \times B = -4\pi J, \quad J = \sum_\alpha q_\alpha n_\alpha u_\alpha,\\
&\partial_t B + c\nabla \times E = 0,\\
&\nabla \cdot E = 4\pi\rho, \quad \nabla \cdot B = 0, \quad \rho = \sum_\alpha q_\alpha n_\alpha.
\end{aligned}
$$

We first mention that in the non damped case, the dispersive Euler-Maxwell system near constant steady states has been studied by Germain-Masmoudi [13] and Guo-Ionescu-Pausader [18] by using the tools in the harmonic analysis. In the damped situation, it is easy to obtain the global existence of small amplitude classical solutions near the constant steady state only basing on the energy method. We mainly focus on the convergence of the obtained solutions to the linear diffusion waves, that is to show that the first-order approximation of $n_{i,e} - 1$

and B in large time are linear diffusion waves and the first-order approximation of $u_{i,e}$ and E are defined in terms of a generalised Darcy's law. For that purpose, as the extra time-decay rate is needed, it seems necessary to make a complete analysis of the eigenvalue problem on the linearised system. In general it is difficult to do so due to the fact that the size of the system is so large that it is an impossible task to find out the explicit formulas of all eigenvalues in the different frequency region. In what follows, we will state the main results obtained in Duan-Liu-Zhu [8] for the study of the problem. Note that for the damped Euler system without any force, there are a lot of works in different aspects, for instance, Lions [27], Hsiao-Liu [20], Huang-Marcati-Pan [21] and references therein. It would be interesting to extend those results to the damped Euler-Maxwell system.

Let us give a heuristic derivation of diffusion waves. Assume that the desired long-term asymptotic profile satisfies the quasi-neutral condition: $n_i = n_e = n(t,x), u_i = u_e = u(t,x)$, and the background magnetic field is zero. From the momentum equations $\nabla p_\alpha(n) = q_\alpha n E - \nu_\alpha m_\alpha n u$ $(\alpha = i, e)$, along the direction parallel to the frequency mode we define $u_\parallel$, $E_\parallel$ by

$$nu_\parallel = -\frac{T_i + T_e}{m_i\nu_i + m_e\nu_e}\nabla n, \quad nE_\parallel = \frac{T_i m_e \nu_e - T_e m_i \nu_i}{m_i\nu_i + m_e\nu_e}\nabla n,$$

where we have assumed $p_\alpha(n) = T_\alpha n$ without loss of generality. Recalling the conservation of mass $\partial_t n + \nabla\cdot(nu_\parallel) = 0$, then n satisfies the diffusion equation

$$\partial_t n - \mu_1 \Delta n = 0, \quad \mu_1 := \frac{T_i + T_e}{m_i\nu_i + m_e\nu_e}.$$

Note that whenever the pressure function $p_\alpha(\cdot)$ takes the general form, for instance, of the form of γ-law $(\gamma > 1)$, the corresponding diffusion equation is nonlinear. Moreover, we point out that the situation is completely different when the background magnetic field is nonzero, and in that case, the diffusion coefficients are anisotropic, namely, along the direction parallel to B diffusion is independent of B and takes the similar form as above, but along the direction perpendicular to B diffusion is inversely proportional to $|B|$ that is the magnitude of B, cf. [14]. Finally, the expected asymptotic equations for the electromagnetic part are determined in the sense of the generalised Darcy's Law:

$$\begin{cases} -eE_\perp + m_i\nu_i u_{i,\perp} = 0, \\ eE_\perp + m_e\nu_e u_{e,\perp} = 0, \\ -c\nabla\times B + 4\pi n(eu_{i,\perp} - eu_{e,\perp}) = 0, \\ \partial_t B + c\nabla\times E_\perp = 0, \end{cases}$$

where $\perp$ stands for the direction perpendicular to the frequency mode. Letting $n = 1$, the above gives

$$\begin{cases} \partial_t B - \mu_2 \Delta B = 0, \quad \mu_2 = \dfrac{c^2 m_i \nu_i m_e \nu_e}{4\pi e^2 (m_i \nu_i + m_e \nu_e)}, \\ u_{i,\perp} = \dfrac{e}{m_i \nu_i} E_\perp = \dfrac{c}{4\pi e} \dfrac{m_e \nu_e}{m_i \nu_i + m_e \nu_e} \nabla \times B, \\ u_{e,\perp} = -\dfrac{e}{m_e \nu_e} E_\perp = -\dfrac{c}{4\pi e} \dfrac{m_i \nu_i}{m_i \nu_i + m_e \nu_e} \nabla \times B, \\ E_\perp = \dfrac{c}{4\pi e^2} \dfrac{m_i \nu_i m_e \nu_e}{m_i \nu_i + m_e \nu_e} \nabla \times B. \end{cases}$$

The main results in [8] can be divided into two parts. For the first part, indeed one can show

$$|\widehat{U}(t,k) - \widehat{\overline{U}}(t,k)| \lesssim \chi_{|k|\leq 1} |k| e^{-\lambda |k|^2 t} |\widehat{U}_0(k)| + \chi_{|k|\geq 1} e^{-\frac{\lambda}{|k|^2} t} |\widehat{U}_0(k)|,$$

where U is the solution to the linearized Cauchy problem with initial data U_0, and $\overline{U}$ is the solution to the constructed-above diffusion equations with the same initial data. The estimate above implies that at the linear level, as long as initial data are smooth enough, U converges to $\overline{U}$ in L^2 norm with the extra time-rate $(1+t)^{-1/2}$, and hence the diffusion equations are justified to be good approximations in large time for the more complex Euler-Maxwell system. For the proof, the upper bound over the high-frequency domain is obtained directly in terms of the Fourier energy estimates on U as $\overline{U}$ itself has a much better bound, and over the low-frequency domain it is based on solving the eigenvalue problem which is a key issue of our work.

In the second part, one can further prove that solutions to the nonlinear Cauchy problem tend time-asymtotically toward the diffusion waves with a faster time-rate than the one in which solutions themselves decay. Precisely, let $U = [n_\alpha - 1, u_\alpha, E, B]$ be the solution to the perturbed Cauchy problem on the Euler-Maxwell system with initial data U_0. The asymptotic profile $\overline{U} = [\overline{n}, \overline{u}_\alpha, \overline{E}, \overline{B}]$ is defined by

$$\overline{n} = \sum_{\alpha=i,e} \frac{m_\alpha \nu_\alpha}{m_i \nu_i + m_e \nu_e} G_{\mu_1}(t,\cdot) * (n_{\alpha 0} - 1),$$

$$\overline{B} = G_{\mu_2}(t,\cdot) * B_0,$$

and

$$\overline{u}_\alpha(t,x) = -\frac{T_i + T_e}{m_i\nu_i + m_e\nu_e}\nabla\overline{n}(t,x) + \frac{c}{4\pi q_\alpha}\frac{m_e\nu_e}{m_i\nu_i + m_e\nu_e}\nabla\times\overline{B}(t,x),$$
$$\alpha = i, e,$$

$$\overline{E}(t,x) = \frac{T_i m_e\nu_e - T_e m_i\nu_i}{e(m_i\nu_i + m_e\nu_e)}\nabla\overline{n}(t,x) + \frac{c}{4\pi e^2}\frac{m_i\nu_i m_e\nu_e}{m_i\nu_i + m_e\nu_e}\nabla\times\overline{B}(t,x).$$

Theorem 2.1. *One has*

$$\|(U - \overline{U})(t)\|_{L^2} \lesssim (1+t)^{-\frac{5}{4}},$$

for all $t \geq 0$*, provided that*

$$\sum_{\alpha=i,e} \|[n_{\alpha 0} - 1, u_{\alpha 0}]\|_{H^{11}\cap L^1} + \|[E_0, B_0]\|_{H^{11}\cap L^1},$$

is sufficiently small.

The complete proof of the above theorem is given in [8]. We remark that H^{11} is a technical condition on initial data, and the optimal Sobolev regularity is not pursued yet. The final remark is concerned with the nonlinear diffusion of the two fluid Euler-Maxwell system with damping. In fact, the current work is done at the linearized level. Even for the general pressure functions P_α ($\alpha = i, e$), the density in large time satisfies the nonlinear heat equation. It would be interesting and challenging to further investigate the asymptotic stability of the nonlinear diffusion waves.

3 Rarefaction waves of Vlasov-Poisson-Boltzmann system

In this section we consider the large-time behaviour of the kinetic plasma with collisions which is governed by the following Vlasov-Poisson-Boltzmann system:

$$\begin{cases} \partial_t F + \xi_1\partial_x F - \partial_x\phi\partial_{\xi_1}F = Q(F,F), \\ -\partial_x^2\phi = \rho - \rho_e(\phi), \quad \rho = \displaystyle\int_{\mathbb{R}^3} F\,d\xi, \end{cases}$$

with $F(0,x,\xi) = F_0(x,\xi) \geq 0$. We assume

$$\lim_{x\to\pm\infty} F_0(x,\xi) = \frac{\rho_\pm}{(2\pi\theta_\pm)^{3/2}}e^{-\frac{|\xi-u_\pm|^2}{2\theta_\pm}}, \; u_\pm = [u_{1\pm}, 0, 0],$$

$$\lim_{x\to\pm\infty} \phi(t,x) = \phi_\pm, \; \rho_\pm = \rho_e(\phi_\pm).$$

Also, the following assumption on $\rho_e(\cdot)$ holds:

$(\mathcal{A})$ $\rho_e(\phi) : (\phi_m, \phi_M) \to (\rho_m, \rho_M)$ is a positive smooth function with

$$\rho_m = \inf_{\phi_m<\phi<\phi_M} \rho_e(\phi), \qquad \rho_M = \sup_{\phi_m<\phi<\phi_M} \rho_e(\phi),$$

and
$(\mathcal{A}_1)$ $\rho_e(0) = 1$ with $0 \in (\phi_m, \phi_M)$;
$(\mathcal{A}_2)$ $\rho_e(\phi) > 0$, $\rho_e'(\phi) > 0$ for each $\phi \in (\phi_m, \phi_M)$;
$(\mathcal{A}_3)$ $\rho_e(\phi)\rho_e''(\phi) \leq [\rho_e'(\phi)]^2$ for each $\phi \in (\phi_m, \phi_M)$.

A typical example takes the form of

$$\rho_e(\phi) = \left[1 + \frac{\gamma_e - 1}{\gamma_e}\frac{\phi}{A_e}\right]^{\frac{1}{\gamma_e-1}},$$

with $\gamma_e > 1$. This is motivated by the momentum equation of electrons under the assumption that the electron mass is sufficiently small. Note that $\rho_e(\phi) = e^{\frac{\phi}{A_e}}$ as $\gamma_e \to 1$, which corresponds to the isothermal case for electrons.

There recently has been some progress on the nonlinear stability of three basic wave patterns for the Boltzmann equation with slab symmetry for the shock, rarefaction wave and contact discontinuity, respectively; see [2, 31, 32, 42], for instance. Here we also mention the fundamental work [15, 22, 33, 34] in the context of gas dynamic equations. However, for the Boltzmann equation with forces, to the best of our knowledge, there are few results on the same issue. In [7], we have showed the time-asymptotic stability of the rarefaction wave for the model mentioned above. In particular, the potential function may take the distinct states at both far fields $x = \pm\infty$. In what follows, we will present the main result of [7].

Recall that

$$\psi_0 = 1, \quad \psi_i = \xi_i \ (i = 1, 2, 3), \quad \psi_4 = \frac{1}{2}|\xi|^2$$

are five collision invariants. As in [30], one can introduce the maro-micro decomposition:

$$F(t, x, \xi) = \mathbf{M}(t, x, \xi) + \mathbf{G}(t, x, \xi),$$

with

$$\mathbf{M}_{[\rho(t,x),u(t,x),\theta(t,x)]}(\xi) \equiv \frac{\rho(t, x)}{(2\pi R\theta(t, x))^{\frac{3}{2}}} \exp\left(-\frac{|\xi - u(t, x)|^2}{2R\theta(t, x)}\right),$$

through

$$
\begin{aligned}
\rho(t,x) &\equiv \int_{\mathbb{R}^3} F(t,x,\xi)\,d\xi,\\
\rho(t,x)u_i(t,x) &\equiv \int_{\mathbb{R}^3} \psi_i(\xi)F(t,x,\xi)\,d\xi, \quad i=1,2,3,\\
\left[\rho\left(\frac{3}{2}R\theta(t,x)+\frac{1}{2}|u(t,x)|^2\right)\right] &\equiv \int_{\mathbb{R}^3} \psi_4(\xi)F(t,x,\xi)\,d\xi.
\end{aligned}
$$

Then, the VPB system can be rewritten in the form of the Euler-Poisson system with unknown high moments:

$$
\begin{cases}
\partial_t\rho+\partial_x(\rho u_1)=0,\\
\partial_t(\rho u_1)+\partial_x(\rho u_1^2)+\partial_x P+\rho\partial_x\phi=-\displaystyle\int_{\mathbb{R}^3}\xi_1^2\partial_x\mathbf{G}\,d\xi,\\
\partial_t(\rho u_i)+\partial_x(\rho u_1 u_i)=-\displaystyle\int_{\mathbb{R}^3}\xi_i\xi_1\partial_x\mathbf{G}\,d\xi,\quad i=2,3,\\
\partial_t\left[\rho\left(\frac{3}{2}R\theta+\frac{1}{2}|u|^2\right)\right]+\partial_x\left[u_1\left(\rho\left(\frac{3}{2}R\theta+\frac{1}{2}|u|^2\right)+P\right)\right]\\
\qquad +\rho u_1\partial_x\phi=-\dfrac{1}{2}\displaystyle\int_{\mathbb{R}^3}|\xi|^2\xi_1\partial_x\mathbf{G}\,d\xi,\\
-\partial_x^2\phi=\rho-\rho_e(\phi).
\end{cases}
$$

To capture viscosity and heat-conductivity, one can plug into the above system

$$
\mathbf{G}=L_{\mathbf{M}}^{-1}\Big(\mathbf{P}_1^{\mathbf{M}}(\xi_1\partial_x\mathbf{M})\Big)+\Theta,
$$

with

$$
\Theta=L_{\mathbf{M}}^{-1}\left[\partial_t\mathbf{G}+\mathbf{P}_1^{\mathbf{M}}(\xi_1\partial_x\mathbf{G})-\partial_x\phi\partial_{\xi_1}\mathbf{G}\right]-L_{\mathbf{M}}^{-1}[Q(\mathbf{G},\mathbf{G})],
$$

so as to obtain the Navier-Stokes-Poisson type system with unknown higher-order moments.

To construct the large-time rarefaction wave, it is natural to make use of the quasineutral Euler system by ignoring all high-order terms:

$$
\begin{cases}
\partial_t\rho+\rho\partial_x u_1+u_1\partial_x\rho=0,\\
\partial_t u_1+u_1\partial_x u_1+\dfrac{\partial_x[P+P^{\phi}(\rho)]}{\rho}=0,\\
\partial_t\theta+u_1\partial_x\theta+\dfrac{P\partial_x u_1}{\rho}=0.
\end{cases}
$$

As in [29], we let $[\rho^R, u_1^R, \theta^R](x/t)$ denote the 3-rarefaction solution with Riemann data consistent with far-field data given before, and let $[\rho^r, u_1^r, \theta^r](t,x)$ denote the usual smooth approximation in terms of the Burgers' equation with initial data of the form

$$w(0,x) = \frac{1}{2}(w_+ + w_-) + \frac{1}{2}(w_+ - w_-)\tanh(\epsilon x),$$

where $w_\pm$ are values of the 3rd eigenfunction at both far-field data, and $\epsilon > 0$ is a small constant to be chosen later on.

To state the result, we need to introduce a reference weight function $\mathbf{M}_* = \mathbf{M}_*(\xi) = \mathbf{M}_{[\rho_*, u_*, \theta_*]}(\xi)$ which is a global Maxwellian such that the constant state $[\rho_*, u_*, \theta_*]$ with $u_* = [u_{1*}, 0, 0]$ satisfies

$$\begin{cases} \dfrac{1}{2} \sup\limits_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} \theta^r(t,x) < \theta_* < \inf\limits_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} \theta^r(t,x), \\ \sup\limits_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} \{|\rho^r(t,x) - \rho_*| + |u^r(t,x) - u_*| + |\theta^r(t,x) - \theta_*|\} < \eta_0, \end{cases}$$

for a constant $\eta_0 > 0$ which is not necessarily small.

Theorem 3.1. *Assume that $[\rho_+, u_{1+}, \theta_+]$ is connected to $[\rho_-, u_{1-}, \theta_-]$ by the third rarefaction wave, $\rho_\pm = \rho_e(\phi_\pm)$ with $\phi_\pm \in (\phi_m, \phi_M)$, and $\rho_e(\cdot)$ satisfies $(\mathcal{A})$. Denote δ_r to be the wave strength which is not necessarily small. There are $\epsilon_0 > 0$, $0 < \sigma_0 < 1/3$ and $C_0 > 0$, which may depend on δ_r and η_0, such that if $F_0(x,\xi) \geq 0$ and*

$$\sum_{|\alpha|+|\beta|\leq 2} \left\| \partial_x^\alpha \partial_\xi^\beta \left(F_0(x,\xi) - \mathbf{M}_{[\rho^r, u^r, \theta^r](0,x)}(\xi) \right) \right\|^2_{L_x^2\left(L_\xi^2\left(\frac{1}{\sqrt{\mathbf{M}_*(\xi)}}\right)\right)} + \epsilon \leq \epsilon_0^2,$$

then the Cauchy problem on the Vlasov-Poisson-Boltzmann system admits a unique global solution $[F(t,x,\xi), \phi(t,x)]$, satisfying $F(t,x,\xi) \geq 0$ and

$$\sup_{t\to+\infty} \sup_{x\in\mathbb{R}} \left\{ \left\| F(t,x,\xi) - \mathbf{M}_{[\rho^R, u^R, \theta^R](x/t)}(\xi) \right\|_{L_\xi^2\left(\frac{1}{\sqrt{\mathbf{M}_*(\xi)}}\right)} + \left| \phi(t,x) - \rho_e^{-1}\left(\rho^R(x/t)\right) \right| \right\} = 0.$$

Readers may refer to [7] for the complete proof of the above theorem. The main idea is based on the study of the same problem on the Navier-Stokes-Poisson system; see [6]. Here we remark that the similar results should also hold true for the two-fluid models. Indeed, we also considered in [6] the two-fluid Navier-Stokes-Poisson system:

$$\begin{cases} \partial_t n_i + \partial_x (n_i u_i) = 0, \\ m_i n_i (\partial_t u_i + u_i \partial_x u_i) + T_i \partial_x n_i - n_i \partial_x \phi = \mu_i \partial_x^2 u_i, \\ \partial_t n_e + \partial_x (n_e u_e) = 0, \\ m_e n_e (\partial_t u_e + u_e \partial_x u_e) + T_e \partial_x n_e + n_e \partial_x \phi = \mu_e \partial_x^2 u_e, \\ \partial_x^2 \phi = n_i - n_e, \quad t > 0,\ x \in \mathbb{R}. \end{cases}$$

The large-time behavior for rarefaction waves can be determined by the quasi-neutral Euler system

$$\begin{cases} \partial_t n + \partial_x (nu) = 0, \\ n(\partial_t u + u \partial_x u) + \dfrac{T_i + T_e}{m_i + m_e} \partial_x n = 0, \end{cases}$$

with the potential function ϕ in large time determined by

$$\phi = \frac{T_i m_e - T_e m_i}{m_i + m_e} \ln n.$$

Finally we conclude the note with discussions on several closely relative problems arising from the current work [7]. First of all, it is of course an interesting problem to justify the fluid dynamic limit of the VPB system to the one-fluid Euler-Poisson system for ions, cf. [19]. In the mean time, motivated by [12] and [36], we point out that it should be an even more interesting and challenging problem to study the current model on the half space, which is related to the justification of the kinetic Bohm criterion (cf. [4]). In the end, we note that in the context of plasma, collisions between particles are usually described by the Boltzmann operator for long-range potentials or more physically by the classical Landau operator for the Coulomb potential taking into account the grazing effect of plasma. Thus, it is a problem to extend the result in [7] to those interesting cases, cf. [17].

Acknowledgements. This work was partially supported by the General Research Fund (Project No. 400912) from RGC of Hong Kong.

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