ON THE CAUCHY PROBLEM FOR THE TWO-COMPONENT EULER-POINCARÉ EQUATIONS

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Abstract. In the paper, we first use the energy method to establish the local well-posedness as well as blow-up criteria for the Cauchy problem on the two-component Euler-Poincaré equations in multi-dimensional space. In the case of dimensions 2 and 3, we show that for a large class of smooth initial data with some concentration property, the corresponding solutions blow up in finite time by using Constantin-Escher Lemma and Littlewood-Paley decomposition theory. Then for the one-component case, a more precise blow-up estimate and a global existence result are also established by using similar methods. Next, we investigate the zero density limit and the zero dispersion limit. At the end, we also briefly demonstrate a Liouville type theorem for the stationary weak solution.

Keywords: two-component Euler-Poincaré equations; blow-up; global existence; zero density limit; zero dispersion limit

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Contents

1. Introduction

In this paper, we consider the Cauchy problem on the following two-component Euler-Poincaré equations in multi-dimensional space $\mathbb{R}^N$ ($N \geq 2$):

$$\begin{cases} m_t + u \cdot \nabla m + (\nabla u)^T m + m \nabla \cdot u = -\rho \nabla \rho, & \text{in} \quad \mathbb{R}^N \times (0,T), \\ \rho_t + \nabla \cdot (\rho u) = 0, & \text{in} \quad \mathbb{R}^N \times (0,T), \\ m = (1-\alpha^2 \Delta) u, & \text{in} \quad \mathbb{R}^N \times (0,T), \\ m(x,0) = m_0(x), \quad \rho(x,0) = \rho_0(x), & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, \cdots, u_N)$ represents the velocity of fluid, $m = (m_1, m_2, \cdots, m_N)$ denotes the momentum, and the scalar function ρ stands for the density or the total depth. The notation $(\nabla u)^T$ denotes the transpose of the matrix ∇u. The constant

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$\alpha > 0$ corresponds to the length scale and is called the dispersion parameter. Equations (1.1) were presented by [22, 25] as a framework for modeling and analyzing fluid dynamics, particularly for nonlinear shallow water waves, geophysical fluids and turbulence modeling, or recasting the geodesic flow on the diffeomorphism groups. In the case of $\alpha = 0$, equations (1.1) is called zero-dispersive Euler-Poincaré equations and can be written as

$$\begin{cases} u_t + u\cdot\nabla u + (\nabla u)^T u + u\nabla\cdot u = -\rho\nabla\rho, & \text{in} \quad \mathbb{R}^N \times (0,T), \\ \rho_t + \nabla\cdot(\rho u) = 0, & \text{in} \quad \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x), \quad \rho(x,0) = \rho_0(x), & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{1.2}$$

which is a symmetric hyperbolic system of conservation laws (see (2.5) below).

To motivate our study, we recall some related progresses on equations (1.1). When the system is decoupled (i.e., formally, $\rho \equiv 0$), equations (1.1) reduce to the classical mathematical model of the fully nonlinear shallow water waves or the one of the geodesic motion on diffeomorphism group:

$$\begin{cases} m_t + u\cdot\nabla m + (\nabla u)^T m + m\nabla\cdot u = 0, & \text{in} \quad \mathbb{R}^N \times (0,T), \\ m = (1-\alpha^2\Delta)u, & \text{in} \quad \mathbb{R}^N \times (0,T), \\ m(x,0) = m_0(x), & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{1.3}$$

(see [3, 5, 20, 21, 23]). In particular, equations (1.3) are the classical Camassa-Holm equations for $N = 1$, while it is also called the Euler-Poincaré equations in the higher dimensional case $N \geq 1$. The local well-posedness, blow-up criterion, existence of blow-up or global solutions, and simulation of Camassa-Holm equation (1.3) with $N = 1$ have been intensively studied (see [2, 3, 7, 9, 11, 30, 33, 34] and references therein). Recently, the rigorous analysis of the Euler-Poincaré equations (1.3) with $N \geq 1$ was initiated by Chae-Liu [5] who established a fairly complete well-posedness theory and obtained the local well-posedness, blow-up criterion, zero α limit and the Liouville type theorem. More recently, Li-Yu-Zhai [27] gave a further analysis and proved that for a large class of smooth initial data the corresponding solution to (1.3) blows up in finite time and that for some monotonous intial data the corresponding solution exists globally in time, which reveals the nonlinear depletion mechanism hidden in the Euler-Poincaré equation. The well-posedness of the Cauchy problem (1.3) posed on an arbitrary compact Riemannian manifold with boundary is also investigated by Gay-Balmaz [14]. We remark that for the non-dispersive case, i.e., $\alpha = 0$, the local well-posedness and existence of blow-up solutions to equations (1.3) are also studied by [5]. In this case, equation (1.1) will become a symmetric hyperbolic system of conservation laws

$$u_t + u\cdot\nabla u + (\nabla u)^T u + u\nabla\cdot u = 0. \tag{1.4}$$

When the system is coupled and ρ is a non-zero constant, equations (1.1) rose from work on the imcompressible shallow water equations and are derived by considering the variational principles and Largrangian averaging (see [3, 20, 29]). The existence, uniqueness and simulation have been investigated by many scholars (see Holm-Titi [24] and references therein). For $N = 2, 3, 4$, in particular, Bjorland-Schonbek [1] established the existence and decay estimates for the viscous version.

When the system is coupled and ρ is a non-constant function, which plays a role in the equation of u, equations (1.1) are called the two-component Euler-Poincaré equations (or the two-component Camassa-Holm equations), which was presented by [6, 13]. In the case of $N = 1$, Constantin-Ivanov [10] gave a rigorous justification of the derivation of equations (1.1), which is a valid approximation to the governing equations for water waves in the shallow water regime, and investigated conditions for wave-breaking and global small solutions to the system. Then Guan-Yin [18, 19] and Gui-Liu [18, 19] further studied the local well-posedness and uniqueness, established several improved wave breaking results, and investigated the global existence. Mathematical properties of the related system have been also studied further in many works (see, e.g. [12, 35] and references therein). In the case of $N \geq 2$, Kohlmann [25] obtained some well-posedness, conservation laws or stability results for equations (1.1) posed on the torus. Thus, counter to the large amount of papers referring to the case $N = 1$, the two-component Euler-Poincaré equations in higher dimensions have rarely been studied. However, multi-variable extensions of these equations are of interest from both the physical and the mathematical point of view as explained in, e.g., [14, 25, 26].

Motivated by the above works, the main aim of this paper is to give a complete well-posedness analysis for the Cauchy problem (1.1). Precisely, we will establish the local well-posedness in the Sobolev space framework as well as blow-up criteria, show the existence of solutions blowing up in finite time and of solutions existing globally in time, and investigate the zero density limit and the zero dispersion ($\alpha = 0$) limit.

Since equations (1.1) are a system with *two* components in multidimensional space, there are more difficulties in analyzing it than a single equation or the equations in one-dimensional space. The main difficulties are the mutual effect between two components ρ and u and the estimates of ∇u and ρ. One cannot follow directly the same argument as in [5, 27] or [16, 17, 18, 19] to deal with this problem.

Before stating our results, we would like to remark that the boundary conditions are usually taken as $u \to 0$ and $\rho \to \rho_0 = constant$ as $|x| \to \infty$ (see e.g. [20]). In particular, [10, 16, 17, 18, 19] posed the boundary assumption $\rho_0 = 1$. Since our main purpose is to show the effcet of the non-constant ρ on the velocity u, we follow [4] and take the boundary condition as $u \to 0$ and $\rho \to 0$ as $|x| \to \infty$ in this paper, that is, we are assuming that the spatial infinity is vacuum. However, we can obtain the corresponding results for the case $\rho_0 = 1$ by some nonessential modifications.

We now state our main results. The first one is to deal with the local well-posedness. To the end, for brevity we denote the solution space by

$$\begin{aligned} X_k(0,T) = C([0,T]; H^{k+1}(\mathbb{R}^N)) \cap C^1([0,T]; H^k(\mathbb{R}^N)) \\ \times C([0,T]; H^k(\mathbb{R}^N)) \cap C^1([0,T]; H^{k-1}(\mathbb{R}^N)). \end{aligned}$$

Theorem 1.1. *(i) Let $(u_0, \rho_0) \in H^{k+1}(\mathbb{R}^N) \times H^k(\mathbb{R}^N)$ with $k > \frac{N}{2} + 2$. Then there exists a unique classical solution $(u, \rho) \in X_k(0,T)$ to equations (1.1) for some $T > 0$, depending only on $\|u_0\|_{H^{k+1}}$ and $\|\rho_0\|_{H^k}$.*

(ii) Let $(u_0, \rho_0) \in H^k(\mathbb{R}^N) \times H^k(\mathbb{R}^N)$ with $k > \frac{N}{2} + 1$. Then there exists a unique classical solution $(u, \rho) \in \Big(C([0,T]; H^k(\mathbb{R}^N)) \cap C^1([0,T]; H^{k-1}(\mathbb{R}^N))\Big)^2$ to equations (1.2) for some $T > 0$, depending only on $\|u_0\|_{H^k}$ and $\|\rho_0\|_{H^k}$.

The proof of Theorems 1.1 is based on the standard energy estimates as the argument of [5] in the study of one component equation (1.3). However, one problematic issue is that we here deal with a coupled system with these two components of the solution in different Sobolev spaces, making the proof of several required nonlinear estimates somewhat delicate. It is noted that the second equation of (1.1) is a transport equation with the component ρ and no more regularity can be obtained from this equation. With the help of invariant properties of the transport equation, these difficulties are nevertheless overcome by carefully estimating each component of solutions.

We next will consider the existence of finite time blow-up solutions and of global solutions to the two-component Euler-Poincaré equation (1.1). For this purpose, the first step usually consists of deriving a blow-up criterion. To state our results, we introduce the Besov space as follows. Define the Littlewood-Paley operators Δ_j and $\dot{\Delta}_j$ by the Fourier transform

$$\mathcal{F}\left(\Delta_j f\right)(\xi) = \varphi\left(\frac{\xi}{2^j}\right)\mathcal{F}(f)(\xi)$$

for any integers $j \geq 0$, and

$$\mathcal{F}\left(\dot{\Delta}_j f\right)(\xi) = \left(\varphi\left(\frac{\xi}{2^j}\right) - \varphi\left(\frac{\xi}{2^{j-1}}\right)\right)\mathcal{F}(f)(\xi)$$

for any $j \in \mathbb{Z}$, where $\varphi(x) \in C_0^\infty(\mathbb{R}^N)$ is a nonnegative radial bump function supported in the ball $|x| \leq 2$ and equal to one on the ball $|x| \leq 1$. Then the homogeneous Besov space $\dot{B}^0_{\infty,\infty}(\mathbb{R}^N)$ can be defined as

$$\dot{B}^0_{\infty,\infty} := \left\{ f \in \mathcal{S}'(\mathbb{R}^N) \,\middle|\, \|f\|_{\dot{B}^0_{\infty,\infty}} < \infty \right\}$$

with $\|f\|_{\dot{B}^0_{\infty,\infty}} := \sup_{j\in\mathbb{Z}} \|\dot{\Delta}_j f\|_{L^\infty}$.

Then we have the following blow-up criteria.

Theorem 1.2. *Let $(u, \rho) \in X_k(0,T)$ be a classical solution to (1.1) with initial data $(u_0, \rho_0) \in H^{k+1}(\mathbb{R}^N) \times H^k(\mathbb{R}^N)$ for $k > \frac{N}{2} + 2$. Then*

$$\lim_{t\to T} \left(\|u(t)\|_{H^{k+1}} + \|\rho(t)\|_{H^k}\right) = \infty$$

if and only if

$$\int_0^T \|\nabla u(\tau)\|_{L^\infty} d\tau = \infty,$$

or if and only if

$$\int_0^T \left(\|\nabla u(\tau)\|_{\dot{B}^0_{\infty,\infty}} + \|\rho(\tau)\|_{\dot{B}^0_{\infty,\infty}}\right) d\tau = \infty.$$

Remark 1.1. *Notice that $L^\infty(\mathbb{R}^N) \hookrightarrow \dot{B}^0_{\infty,\infty}(\mathbb{R}^N)$. Theorem 1.2 shows that in the sense of L^∞-norm, the blow-up criterion can be completely determined by u only. It is unclear for the case of $\dot{B}^0_{\infty,\infty}$-norm.*

With the aid of Theorem 1.2, we can show that for a large class of smooth initial data with some concentration property, the corresponding solution to (1.1) will blow up in finite time for the case $N = 2, 3$. These solutions belong to the class of radial functions. Thus, for brevity, we will slightly abuse the notation $f(x) = f(|x|) = f(r)$ for radial function f.

Theorem 1.3. *Suppose that $(\psi_0, \rho_0) \in H^k(\mathbb{R}^N) \times H^k(\mathbb{R}^N)$ is a pair of radial functions with $k > \frac{N}{2}+4$ and $N = 2, 3$. Assume that $u_0 = (1-\alpha^2\Delta)^{-1}\nabla\psi_0$, $\psi_0(0) = \sup_{r\geq 0}\psi(r)$ and $\psi_0(0) \geq C\left(\|\psi_0\|_{L^2} + \|\rho_0\|_{L^2}\right)$ for some $C > 0$ large enough. Then* *the solution (u,ρ) to equation (1.1) with initial data (u_0,ρ_0) will blow up at some finite time T^*.*

For the decoupled system (1.3), the blow-up solution was obtained by [27] without concentration restriction. Here we establish the more precise blow-up estimate under some concentration assumption. For a large class of initial data with some non-positive property at the origin, we also show that the corresponding solution exists globally in time, which improves the gobal existence result of [27] in the sense that we don't postulate any monotony assumption on the initial data ψ.

Proposition 1.1. *Suppose that $\psi_0 \in H^k(\mathbb{R}^N)$ is a radial function with $k > \frac{N}{2}+4$ and $N = 2,3$. Assume that $u_0 = (1-\alpha^2\Delta)^{-1}\nabla\psi_0$.*

(i) If $\psi_0(0) = \sup_{r\geq 0}\psi(r)$ and $\psi_0(0) \geq C\|\psi_0\|_{L^2}$ for some $C>0$ large enough, then the solution u to equation (1.3) with initial data u_0 will blow up at some finite time T^* in the sense that

$$c(T^*-t)^{-1} \leq \|\operatorname{div} u(t)\|_\infty \leq C(T^*-t)^{-1} \quad \text{as} \quad t \to T^*$$

for some $C > c > 0$.

(ii) If $\psi_0(0) = \inf_{r\geq 0}\psi(r)$ and $\psi_0(r) < 0$ for any $r \geq 0$, then the solution u to equation (1.3) with initial data u_0 exists globally in time.

To prove Theorem 1.3 and Proposition 1.1, our main idea is to transfer the higher dimensional problem to a one-dimensional problem. This process will result in a nonlocal integral, and no monotony is available due to the appearance of the component ρ. To overcome these difficulties, we will use the Constantin-Escher Lemma and Littlewood-Paley decomposition theory.

Now we turn to the limit problem. In [15], Grunert-Holden-Raynaud showed that by taking the limit of vanishing density ρ in system (1.1) with $N=1$, one can obtain the global conservative solution of the corresponding Camassa-Holm equation (1.3), which provides a novel way to define and obtain these solutions. On the other hand, Chae-Liu [5] proved that as the dispersion parameter α vanishes, the weak solution to the Euler-Poincarée equations (1.3) converges to the solution of the zero dispersion equation (1.4), provided that the limiting solution is classical. Our next theorem is motivated by these two works.

Theorem 1.4. *(i) Let $k > \frac{N}{2}+2$. Assume that $(u_n,\rho_n) \in L^\infty((0,T);H^1(\mathbb{R}^N)) \times L^\infty((0,T);L^2(\mathbb{R}^N))$ is a weak solution of equations (1.1) with initial data (u_{0n},ρ_{0n}) and that $u \in C\left([0,T),H^k(\mathbb{R}^N)\right)$ is a classical solution of equations (1.3) with initial data u_0. Then*

$$\begin{aligned}
&\|u_n-u\|_{L^2} + \alpha\|\nabla(u_n-u)\|_{L^2} + \|\rho_n\|_{L^2}\\
&\leq C\left(\|u_{0n}-u_0\|_{L^2} + \alpha\|\nabla(u_{0n}-u_0)\|_{L^2} + \|\rho_{0n}\|_{L^2}\right),
\end{aligned}$$

where C is a constant depending only on $\|u\|_{C([0,T),H^k)}$. The corresponding conclusion holds true for the case $\alpha = 0$.

(ii) Let $k > \frac{N}{2}+2$. Assume that

$$(u^\alpha,\rho^\alpha) \in L^\infty((0,T);H^1(\mathbb{R}^N)) \times L^\infty((0,T);L^2(\mathbb{R}^N))$$

is a weak solution of equations (1.1) *with initial data* $(u_0^\alpha, \rho_0^\alpha)$ *and that* $(u, \rho) \in C([0,T); H^{k+1}(\mathbb{R}^N)) \cap C^1([0,T); H^2(\mathbb{R}^N)) \times C([0,T); H^{k-1}(\mathbb{R}^N))$ *is a classical solution of equations* (1.2) *with initial data* (u_0, ρ_0)*. Then*

$$\begin{aligned}&\|u^\alpha - u\|_{L^2} + \|\rho^\alpha - \rho\|_{L^2} + \alpha\|\nabla(u^\alpha - u)\|_{L^2}\\ &\leq C\left(\alpha^2 + \|u_0^\alpha - u_0\|_{L^2} + \|\rho_0^\alpha - \rho_0\|_{L^2} + \alpha\|\nabla(u_0^\alpha - u_0)\|_{L^2}\right),\end{aligned}$$

where C is a positive constant depending only on $\|u\|_{C([0,T],H^k)}$, $\|u\|_{C^1([0,T];H^2)}$ *and* $\|\rho\|_{C([0,T];H^{k-1})}$.

Remark 1.2. *In particular, Theorem 1.4 (i) indicates that when* $(u_{0n}, \rho_{0n}) \to (u_0, 0)$ *in* $H^1 \times L^2$ *as* $n \to \infty$*, the solution* (u_n, ρ_n) *of* (1.1) *will converge to the solution* $(u, 0)$ *of* (1.3) *in* $L^\infty\big((0,T); H^1\big) \times L^\infty\big((0,T); L^2\big)$*. Theorem 1.4 (ii) shows that as* $\alpha \to 0$*, the solution* (u^α, ρ^α) *of* (1.1) *will converge to the solution* (u, ρ) *of* (1.2) *in* $L^\infty\big((0,T); H^1\big) \times L^\infty\big((0,T); L^2\big)$.

The rest of this paper is organized as follows. In Section 2, the local well-posedness of the initial-value problem associated with equations (1.1) and (1.2) is established. Sections 3 and 4 are devoted to establishing the blow-up criterion and to showing the existence of blow-up solutions and global solutions.Then in Section 5, we consider the approximation problem and prove Theorem 1.4. And in the last section, Section 6, we will prove a Liouville type theorem for the stationary weak solutions to equations (1.1) and (1.2).

Notations: Sometimes we will use $X \lesssim Y$ to denote $X \leq CY$ for some uniform $C > 0$, which may be different on different lines.

2. Local well-posedness

In this section, we shall establish the local existence and uniqueness of the classical solutions for the two-component Euler-Poincaré equations (1.1) and (1.2) by using the energy methods.

Proof of Theorem 1.1. (i) We first consider the local existence. Let $\eta \in C_0^\infty(\mathbb{R}^N)$ be the standard mollifier supported in the unit ball $|x| \leq 1$ with $\int_{\mathbb{R}^N} \eta(x)dx = 1$. Set $\eta_n(x) = \frac{1}{n^N}\eta(\frac{x}{n})$ and $(u^0, m^0, \rho^0) = (0,0,0)$. Then we can construct a sequence of smooth functions $\big\{(u^{n+1}, \rho^{n+1})\big\}_{n\in\mathbb{N}}$ by solving the linear equations

$$\left\{\begin{array}{l} m_t^{n+1} + u^n \cdot \nabla m^{n+1} + (\nabla u^n)^T m^{n+1} + m^{n+1}\nabla \cdot u^n = -\rho^n \nabla \rho^{n+1},\\ \rho_t^{n+1} + \rho^n \nabla \cdot u^{n+1} + u^n \cdot \nabla \rho^{n+1} = 0,\\ m^{n+1} = (1-\alpha^2\Delta)u^{n+1},\\ m^{n+1}(x,0) = m_0^{n+1}(x), \quad \rho^{n+1}(x,0) = \rho_0^{n+1}(x), \end{array}\right.$$

for $0 < t < T$ and $x \in \mathbb{R}^N$where $m_0^{n+1} = (1-\alpha^2\Delta)u_0^{n+1}$ with $(u_0^{n+1}, \rho_0^{n+1}) := \Big(\eta_{n+1} * u_0, \rho_{n+1} * \rho_0\Big)$ converging to (u_0, ρ_0) in $H^{k+1} \times H^k$ as $n \to \infty$. The basic idea is to prove that some subsequence of $\big\{(u^{n+1}, \rho^{n+1})\big\}$ will converge to a solution (u, ρ) of equations (1.1). For this purpose, we can first show that $\big\{(u^{n+1}, \rho^{n+1})\big\}$ is uniformly bounded in $X_k(0,T)$ and then prove that it is a Cauchy sequence in $C([0,T); H^k(\mathbb{R}^N)) \times C([0,T); H^{k-1}(\mathbb{R}^N))$, which will converge to some limit function $(u, \rho) \in C([0,T); H^k(\mathbb{R}^N)) \times C([0,T); H^{k-1}(\mathbb{R}^N))$. Thus the proof of local existence can be completed by checking that (u, ρ) belongs to $X_k(0,T)$ indeed and solves equations (1.1).

Since the above procedure is standard, here we only derive the key local in time *a priori* estimates for solutions (u,ρ) to equations (1.1). That is, for some $T>0$, there exists a positive constant C depending only on $\|u_0\|_{H^{k+1}}$ and $\|\rho_0\|_{H^k}$ such that

$$\|m(t)\|^2_{H^k}+\|\rho(t)\|^2_{H^{k+1}}\le C \qquad \text{for any} \quad 0<t<T.$$

For this purpose, applying D^β to both sides of the first equation of (1.1) and taking the $L^2(\mathbb{R}^N)$ inner product with $D^\beta m$ with $|\beta|\le k-1$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|D^\beta m\|^2_{L^2} &= -\int D^\beta m\cdot D^\beta(u\cdot\nabla m)\\ &\quad -\int D^\beta m\cdot\left(D^\beta\left((\nabla u)^T m\right)+D^\beta(m\nabla\cdot u)\right)-\int D^\beta m\cdot D^\beta(\rho\nabla\rho)\\ &:= I+II+III.\end{aligned}$$

We estimate I, II and III one by one. For the term I, we use the commutator estimates to deduce that

$$\begin{aligned}I &= -\int D^\beta m\cdot\left(D^\beta(u\cdot\nabla m)-u\cdot\nabla D^\beta m\right)-\int D^\beta m\cdot(u\cdot\nabla D^\beta m)\\ &\le \left\|D^\beta m\right\|_{L^2}\left\|D^\beta(u\cdot\nabla m)-u\cdot\nabla D^\beta m\right\|_{L^2}+\frac{1}{2}\int |D^\beta m|^2\ \nabla\cdot u\\ &\lesssim \left\|D^\beta m\right\|_{L^2}\left(\|D^\beta u\|_{L^2}\|\nabla m\|_{L^\infty}+\|D^\beta m\|_{L^2}\|\nabla u\|_{L^\infty}\right)+\|D^\beta m\|^2_{L^2}\|\nabla u\|_{L^\infty}.\end{aligned}$$

Notice that $u=(1-\alpha^2\Delta)^{-1}m$ implies that $\|u\|_{H^s}\lesssim\|m\|_{H^{s-2}}$ for any $s\in\mathbb{R}$. Thus, for any given $k>\frac{N}{2}+2$, we can take $0<\delta<k-\frac{N}{2}-2$ and use Sobolev embedding to obtain that I is bounded up to a constant by

$$\|m\|_{H^{k-1}}\left(\|u\|_{H^{k-1}}\|m\|_{H^{\frac{N}{2}+1+\delta}}+\|m\|_{H^{k-1}}\|u\|_{H^{\frac{N}{2}+1+\delta}}\right)+C\|m\|^2_{H^{k-1}}\|u\|_{H^{\frac{N}{2}+1+\delta}},$$

and hence, in terms of $\|u\|_{H^{k-1}}\lesssim\|m\|_{H^{k-3}}$, it holds

$$I\lesssim\|m\|^3_{H^{k-1}}.$$

Similarly, for the terms II and III, it follows from Sobolev embedding that

$$\begin{aligned}II &\le \left\|D^\beta m\right\|_{L^2}\left\|D^\beta\left((\nabla u)^T m\right)+D^\beta(m\nabla\cdot u)\right\|_{L^2}\\ &\lesssim \|m\|_{H^{k-1}}\|m\nabla u\|_{H^{k-1}}\\ &\lesssim \|m\|_{H^{k-1}}\left(\|m\|_{H^{k-1}}\|\nabla u\|_{L^\infty}+\|m\|_{L^\infty}\|\nabla u\|_{H^{k-1}}\right)\\ &\lesssim \|m\|_{H^{k-1}}\left(\|m\|_{H^{k-1}}\|u\|_{H^{\frac{N}{2}+1+\delta}}+\|m\|_{H^{\frac{N}{2}+\delta}}\|u\|_{H^k}\right)\\ &\lesssim \|m\|_{H^{k-1}}\left(\|m\|_{H^{k-1}}\|m\|_{H^{\frac{N}{2}-1+\delta}}+\|m\|_{H^{\frac{N}{2}+\delta}}\|m\|_{H^{k-2}}\right)\\ &\lesssim \|m\|^3_{H^{k-1}},\end{aligned}$$

and

$$\begin{aligned}III &\le \left\|D^\beta m\right\|_{L^2}\left\|D^\beta(\rho\nabla\rho)\right\|_{L^2}\le\|m\|_{H^{k-1}}\|\rho\nabla\rho\|_{H^{k-1}}\\ &\lesssim \|m\|_{H^{k-1}}\left(\|\rho\|_{H^{k-1}}\|\nabla\rho\|_{L^\infty}+\|\rho\|_{L^\infty}\|\nabla\rho\|_{H^{k-1}}\right)\\ &\lesssim \|m\|_{H^{k-1}}\left(\|\rho\|_{H^{k-1}}\|\rho\|_{H^{\frac{N}{2}+1+\delta}}+\|\rho\|_{H^{\frac{N}{2}+\delta}}\|\rho\|_{H^k}\right)\\ &\lesssim \|m\|_{H^{k-1}}\|\rho\|^2_{H^k}.\end{aligned}$$

Here we have used the algebra property of H^{k-1} by $k > \frac{N}{2} + 2$. Summarily, we have

$$\frac{1}{2}\frac{d}{dt}\|m\|_{H^{k-1}}^2 \lesssim \|m\|_{H^{k-1}} \left(\|m\|_{H^{k-1}}^2 + \|\rho\|_{H^k}^2\right). \tag{2.1}$$

We still need to estimate ρ. To this end, for any $|\gamma| \leq k$, we apply D^γ to both sides of equation $(1.1)_2$ and take the $L^2(\mathbb{R}^N)$ inner product with $D^\gamma \rho$ to have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|D^\gamma \rho\|_{L^2}^2 &= -\int D^\gamma \rho D^\gamma (u \cdot \nabla \rho) - \int D^\gamma \rho D^\gamma (\rho \nabla \cdot u) \\ &:= IV + V.\end{aligned}$$

Similar to the term I, we can estimate IV as follows

$$\begin{aligned}IV &= -\int D^\gamma \rho \cdot \Big(D^\gamma (u \cdot \nabla \rho) - u \cdot \nabla D^\gamma \rho\Big) - \int D^\gamma \rho \cdot (u \cdot \nabla D^\gamma \rho) \\ &\leq \|D^\gamma \rho\|_{L^2} \|D^\gamma (u \cdot \nabla \rho) - u \cdot \nabla D^\gamma \rho\|_{L^2} + \frac{1}{2}\int |D^\gamma \rho|^2 \ \nabla \cdot u \\ &\lesssim \|D^\gamma \rho\|_{L^2} \Big(\|D^\gamma u\|_{L^2}\|\nabla \rho\|_{L^\infty} + \|D^\gamma \rho\|_{L^2}\|\nabla u\|_{L^\infty}\Big) + \|D^\gamma \rho\|_{L^2}^2 \|\nabla u\|_{L^\infty} \\ &\lesssim \|\rho\|_{H^k} \Big(\|u\|_{H^k}\|\rho\|_{H^{\frac{N}{2}+1+\delta}} + \|\rho\|_{H^k}\|u\|_{H^{\frac{N}{2}+1+\delta}}\Big) + \|\rho\|_{H^k}^2 \|u\|_{H^{\frac{N}{2}+1+\delta}} \\ &\lesssim \|m\|_{H^{k-1}} \|\rho\|_{H^k}^2 .\end{aligned}$$

For the term V, we use the algebra property of H^k to obtain

$$\begin{aligned}V &\leq \|D^\gamma \rho\|_{L^2} \|D^\gamma (\rho \nabla \cdot u)\|_{L^2} \leq \|\rho\|_{H^k} \|\rho \nabla u\|_{H^k} \\ &\lesssim \|\rho\|_{H^k} \left(\|\rho\|_{H^k} \|\nabla u\|_{L^\infty} + \|\rho\|_{L^\infty} \|\nabla u\|_{H^k}\right) \\ &\lesssim \|\rho\|_{H^k} \Big(\|\rho\|_{H^k} \|u\|_{H^{\frac{N}{2}+1+\delta}} + \|\rho\|_{H^{\frac{N}{2}+\delta}} \|u\|_{H^{k+1}}\Big) \\ &\lesssim \|m\|_{H^{k-1}} \|\rho\|_{H^k}^2 .\end{aligned}$$

Combining the estimates for IV and V, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\rho\|_{H^k}^2 \lesssim \|m\|_{H^{k-1}} \|\rho\|_{H^k}^2 ,$$

which together with (2.1) yield that

$$\frac{d}{dt}\left(\|m\|_{H^{k-1}}^2 + \|\rho\|_{H^k}^2\right) \lesssim \|m\|_{H^{k-1}} \Big(\|m\|_{H^{k-1}}^2 + \|\rho\|_{H^k}^2\Big) \lesssim \Big(\|m\|_{H^{k-1}}^2 + \|\rho\|_{H^k}^2\Big)^{\frac{3}{2}} .$$

Then the further calculation gives that

$$\|m\|_{H^{k-1}}^2 + \|\rho\|_{H^k}^2 \leq \left(\Big(\|m_0\|_{H^{k-1}}^2 + \|\rho_0\|_{H^k}^2\Big)^{-\frac{1}{2}} - Ct\right)^2 .$$

Thus, by taking $T := C^{-1}\Big(\|m_0\|_{H^{k-1}}^2 + \|\rho_0\|_{H^k}^2\Big)^{-\frac{1}{2}}$, we complete the proof of the existence.

We now turn to consider the uniqueness. Let (u_1, ρ_1) and (u_2, ρ_2) be two solution pairs of equations (1.1) with the same initial data (u_0, ρ_0). We set $u = u_1 - u_2$, $m = m_1 - m_2 := (1-\alpha^2\Delta)u_1 - (1-\alpha^2\Delta)u_2$ and $\rho = \rho_1 - \rho_2$. Then we can deduce that

$$\begin{cases} m_t + u_1 \cdot \nabla m + u \cdot \nabla m_2 + (\nabla u_1)^T m + (\nabla u)^T m_2 + m \nabla \cdot u_1 + m_2 \nabla \cdot u \\ \qquad = -\rho \nabla \rho_1 - \rho_2 \nabla \rho, \\ \rho_t + u_1 \cdot \nabla \rho + u \cdot \nabla \rho_1 + \rho \nabla \cdot u_1 + \rho_2 \nabla \cdot u = 0. \end{cases} \tag{2.2}$$

For any $p > N$, by taking the $L^2(\mathbb{R}^N)$ inner product of equation $(2.2)_1$ with $|m|^{p-2}m$ and using Sobolev embedding, we have

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\|m\|_{L^p}^p = & -\frac{1}{p}\int u_1 \cdot \nabla |m|^p - \int |m|^p \nabla \cdot u_1 - \int |m|^{p-2} m u \cdot \nabla m_2 \\ & - \frac{1}{p}\int |m|^{p-2} m \cdot m_2 \nabla \cdot u - \int |m|^{p-2} m (\nabla u_1)^T m \\ & - \int |m|^{p-2} m (\nabla u)^T m_2 - \int |m|^{p-2} m \cdot \rho \nabla \rho_1 \\ & - \int |m|^{p-2} m \cdot \rho_2 \nabla \rho, \end{aligned}$$

which implies

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\|m\|_{L^p}^p \lesssim & \|Du_1\|_{L^\infty}\|m\|_{L^p}^p + \|m\|_{L^p}^{p-1}\|u\|_{L^p}\|Dm_2\|_{L^\infty} + \|m\|_{L^p}^{p-1}\|m_2\|_{L^\infty}\|Du\|_{L^p} \\ & + \|m\|_{L^p}^{p-1}\|\rho\|_{L^p}\|D\rho_1\|_{L^\infty} + \|m\|_{L^p}^{p-1}\|\rho_2\|_{L^\infty}\|D\rho\|_{L^p} \\ \lesssim & \left(\|u_1\|_{H^{k+1}} + \|u_2\|_{H^{k+1}} + \|\rho_1\|_{H^k} + \|\rho_2\|_{H^k}\right)\left(\|m\|_{L^p}^p + \|u\|_{W^{1,p}}^p + \|\rho\|_{W^{1,p}}^p\right). \end{aligned} \tag{2.3}$$

Similarly, we take the $L^2(\mathbb{R}^N)$ inner product of equation $(2.2)_2$ with $|\rho|^{p-2}\rho$ to obtain

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\|\rho\|_{L^p}^p = & -\frac{1}{p}\int u_1 \cdot \nabla |\rho|^p - \int |\rho|^{p-2}\rho u \cdot \nabla \rho_2 - \int |\rho|^p \nabla \cdot u_1 \\ & - \frac{1}{p}\int |\rho|^{p-2}\rho \cdot \rho_2 \nabla \cdot u \\ \lesssim & \|Du_1\|_{L^\infty}\|\rho\|_{L^p}^p + \|\rho\|_{L^p}^{p-1}\|u\|_{L^p}\|D\rho_2\|_{L^\infty} + \|\rho\|_{L^p}^{p-1}\|\rho_2\|_{L^\infty}\|D\rho\|_{L^p} \\ \lesssim & \left(\|u_1\|_{H^{k+1}} + \|\rho_2\|_{H^k}\right)\left(\|u\|_{L^p}^p + \|\rho\|_{W^{1,p}}^p\right). \end{aligned} \tag{2.4}$$

On the other hand, applying D to both sides of equation $(2.2)_2$ and taking the $L^2(\mathbb{R}^N)$ inner product of $(2.2)_2$ with $|D\rho|^{p-2}D\rho$, we can use the integration by parts and Sobolev embedding to obtain

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\|D\rho\|_{L^p}^p \lesssim & \|Du_1\|_{L^\infty}\|D\rho\|_{L^p}^p + \|D\rho\|_{L^p}^{p-1}\|Du\|_{L^p}\|D\rho_1\|_{L^\infty} \\ & + \|D\rho\|_{L^p}^{p-1}\|u\|_{L^p}\|D^2\rho_1\|_{L^\infty} + \|D\rho\|_{L^p}^{p-1}\|\rho\|_{L^p}\|D^2 u_1\|_{L^\infty} \\ & + \|D\rho\|_{L^p}^{p-1}\|D\rho_2\|_{L^\infty}\|Du\|_{L^p} + \|D\rho\|_{L^p}^{p-1}\|\rho_2\|_{L^\infty}\|D^2 u\|_{L^p} \\ \lesssim & \left(\|u_1\|_{H^{k+1}} + \|\rho_1\|_{H^k} + \|\rho_2\|_{H^k}\right)\left(\|u\|_{W^{2,p}}^p + \|\rho\|_{W^{1,p}}^p\right). \end{aligned}$$

This together with (2.3) and (2.4) yield that

$$\frac{d}{dt}\left(\|m\|_{L^p}^p+\|\rho\|_{W^{1,p}}^p\right)\le C\left(\|u_1\|_{H^{k+1}}+\|u_2\|_{H^{k+1}}+\|\rho_1\|_{H^k}+\|\rho_2\|_{H^k}\right)$$
$$\times\left(\|m\|_{L^p}^p+\|u\|_{W^{2,p}}^p+\|\rho\|_{W^{1,p}}^p\right).$$

Notice that $\|u\|_{W^{2,p}}\lesssim\|m\|_{L^p}$ (see Proposition 5, Page 251, [31]). It then follows from Gronwall's inequality that

$$\left(\|m\|_{L^p}^p+\|\rho\|_{W^{1,p}}^p\right)\le\left(\|m(0)\|_{L^p}^p+\|\rho(0)\|_{W^{1,p}}^p\right)e^{C\int_0^t\left(\|u_1\|_{H^{k+1}}+\|u_2\|_{H^{k+1}}+\|\rho_1\|_{H^k}+\|\rho_2\|_{H^k}\right)d\tau}.$$

Since $m(0)=0$ and $\rho(0)=0$, the uniqueness of solutions to equations (1.1) with $\alpha>0$ holds in the class $L^1\left(0,T;H^{k+1}(\mathbb{R}^N)\times H^k(\mathbb{R}^N)\right)$ with $k>\frac{N}{2}+2$.

(ii) In this case, we can rewrite equation (1.2) as a symmetric hyperbolic system. For instance, we take $N=3$. By setting $S=(m_1,m_2,m_3,\rho)^T$, and

$$A=\begin{pmatrix}3m_1&m_2&m_3&\rho\\ m_2&m_1&0&0\\ m_3&0&m_1&0\\ \rho&0&0&m_1\end{pmatrix},\quad B=\begin{pmatrix}m_2&m_1&0&0\\ m_1&3m_2&m_3&\rho\\ 0&m_3&m_2&0\\ 0&\rho&0&m_2\end{pmatrix},$$

and

$$C=\begin{pmatrix}m_3&0&m_1&0\\ 0&m_3&m_2&0\\ m_1&m_2&3m_3&\rho\\ 0&0&\rho&m_3\end{pmatrix},$$

we see that equation (1.2) is equivalent to the following symmetric quasilinear hyperbolic system

$$S_t+AS_{x_1}+BS_{x_2}+CS_{x_3}=0. \tag{2.5}$$

Then the local existence and uniqueness of classical solutions to this system follows directly from Majda [28]. □

3. Blow-up criteria

In this section, we turns to establish the blow-up criteria for equations (1.1) and prove Theorem 1.2. The basic idea is still to use the energy method.

Proof of Theorem 1.2. Recalling the estimates for $I,II,\cdots,V$ in the proof of Theorem 1.1, we have

$$\frac{d}{dt}\left(\|m\|_{H^{k-1}}^2+\|\rho\|_{H^k}^2\right)\le C\left(\|m\|_{L^\infty}+\|\nabla m\|_{L^\infty}+\|\nabla u\|_{L^\infty}\right.$$
$$\left.+\|\rho\|_{L^\infty}+\|\nabla\rho\|_{L^\infty}\right)\left(\|m\|_{H^{k-1}}^2+\|\rho\|_{H^k}^2\right),$$

which implies that

$$\|m\|_{H^{k-1}}^2+\|\rho\|_{H^k}^2\le\left(\|m_0\|_{H^{k-1}}^2+\|\rho_0\|_{H^k}^2\right)$$
$$\times e^{C\int_0^t\left(\|m\|_{L^\infty}+\|\nabla m\|_{L^\infty}+\|\nabla u\|_{L^\infty}+\|\rho\|_{L^\infty}+\|\nabla\rho\|_{L^\infty}\right)d\tau}, \tag{3.1}$$

by Gronwall's inequality. Thus it is sufficient to prove that each time integral in the exponential function on the right hand side is bounded under the assumption of Theorem 1.2.

We first assume that $\int_0^T \|\nabla u\|_{L^\infty} d\tau < \infty$ so as to control other time integrals on the right hand side of (3.1). Firstly, we show that ρ can be bounded by ∇u indeed. To this end, we take the $L^2(\mathbb{R}^N)$ inner product of equation $(1.1)_2$ with $|\rho|^{p-2}\rho$, $(p > 2)$, and use the integration by parts to have

$$\frac{1}{p}\frac{d}{dt}\|\rho\|_{L^p}^p = -\int |\rho|^p \nabla\cdot u - \frac{1}{p}\int u\cdot\nabla|\rho|^p = \left(\frac{1}{p}-1\right)\int |\rho|^p\nabla\cdot u \le 2\|\rho\|_{L^p}^p\|\nabla\cdot u\|_{L^\infty},$$

and thus

$$\frac{d}{dt}\|\rho\|_{L^p} \le 2\|\rho\|_{L^p}\|\nabla\cdot u\|_{L^\infty}. \tag{3.2}$$

Then Gronwall's inequality yields that

$$\|\rho\|_{L^p} \le \|\rho_0\|_{L^p} e^{2\int_0^t \|\nabla\cdot u\|_{L^\infty} d\tau}.$$

Letting $p \to \infty$, we obtain

$$\|\rho\|_{L^\infty} \le \|\rho_0\|_{L^\infty} e^{2\int_0^t \|\nabla\cdot u\|_{L^\infty} d\tau}, \tag{3.3}$$

which is bounded by the assumption. Then we turn to $\nabla\rho$ and m. Applying D to both sides of the second equation of (1.1) and taking the $L^2(\mathbb{R}^N)$ inner product with $|D\rho|^{p-2}D\rho$, we deduce that

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|D\rho\|_{L^p}^p &= -\int |D\rho|^p\nabla\cdot u - \int |D\rho|^{p-2}D\rho\rho\nabla\cdot Du - \int |D\rho|^{p-2}D\rho Du\cdot\nabla\rho\\ &\quad + \frac{1}{p}\int |D\rho|^p\nabla\cdot u\\ &\le 3\|D\rho\|_{L^p}^p\|Du\|_{L^\infty} + \|D\rho\|_{L^p}^{p-1}\|\rho\|_{L^\infty}\|D^2u\|_{L^p}\\ &\le 3\|D\rho\|_{L^p}^p\|Du\|_{L^\infty} + \|D\rho\|_{L^p}^{p-1}\|\rho\|_{L^\infty}\|m\|_{L^p},\end{aligned}$$

which implies that

$$\frac{d}{dt}\|D\rho\|_{L^p} \le 3\left(\|D\rho\|_{L^p}\|Du\|_{L^\infty} + \|\rho\|_{L^\infty}\|m\|_{L^p}\right). \tag{3.4}$$

Similarly, we take the $L^2(\mathbb{R}^N)$ inner product of the first equation of (1.1) with $|m|^{p-2}m$ to get

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|m\|_{L^p}^p &= -\int |m|^{p-2}m\cdot(u\cdot\nabla)m - \int |m|^{p-2}m\cdot(\nabla u)^T m - \int |m|^{p-2}m\nabla\cdot u\\ &\quad - \int |m|^{p-2}m\cdot\rho\nabla\rho\\ &\le 3\|m\|_{L^p}^p\|Du\|_{L^\infty} + \|m\|_{L^p}^{p-1}\|\rho\|_{L^\infty}\|D\rho\|_{L^p},\end{aligned}$$

and thus

$$\frac{d}{dt}\|m\|_{L^p} \le 3\left(\|m\|_{L^p}\|Du\|_{L^\infty} + \|\rho\|_{L^\infty}\|D\rho\|_{L^p}\right),$$

which together with (3.4) yields that

$$\frac{d}{dt}\left(\|m\|_{L^p} + \|D\rho\|_{L^p}\right) \le 3\left(\|Du\|_{L^\infty} + \|\rho\|_{L^\infty}\right)\left(\|m\|_{L^p} + \|D\rho\|_{L^p}\right). \tag{3.5}$$

It then follows from Gronwall's inequality that

$$\|m\|_{L^p} + \|D\rho\|_{L^p} \le \left(\|m_0\|_{L^p} + \|D\rho_0\|_{L^p}\right) e^{3\int_0^t (\|Du\|_{L^\infty} + \|\rho\|_{L^\infty})d\tau}, \tag{3.6}$$

which is bounded by the assumption and (3.3). By letting $p \to \infty$, we also have

$$\|m\|_{L^\infty} + \|D\rho\|_{L^\infty} \le \left(\|Dm_0\|_{L^\infty} + \|D\rho_0\|_{L^\infty}\right) e^{3\int_0^t (\|Du\|_{L^\infty} + \|\rho\|_{L^\infty})d\tau} < \infty. \tag{3.7}$$

Next, we turn to bound $\|Dm\|_{L^\infty}$. For this purpose, we apply D to both sides of the first equation of (1.1), take the $L^2(\mathbb{R}^N)$ inner product of $|Dm|^{p-2}Dm$ and then deduce that

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|Dm\|_{L^p}^p \le{}& 4\int |Dm|^p|Du| + 2\int |Dm|^{p-1}|m||D^2u| + \int |Dm|^{p-1}|D\rho|^2\\ &+\int |Dm|^{p-1}|\rho||D^2\rho|\\ \le{}& 4\|Dm\|_{L^p}^p\|Du\|_{L^\infty} + 2\|Dm\|_{L^p}^{p-1}\|m\|_{L^\infty}\|D^2u\|_{L^p}\\ &+\|Dm\|_{L^p}^{p-1}\|D\rho\|_{L^\infty}\|D\rho\|_{L^p} + \|Dm\|_{L^p}^{p-1}\|\rho\|_{L^\infty}\|D^2\rho\|_{L^p}.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\frac{d}{dt}\|Dm\|_{L^p} \le{}& 4\left(\|Dm\|_{L^p}\|Du\|_{L^\infty} + \|\rho\|_{L^\infty}\|D^2\rho\|_{L^p}\right)\\ &+2\left(\|m\|_{L^\infty}\|m\|_{L^p} + \|D\rho\|_{L^\infty}\|D\rho\|_{L^p}\right). \qquad (3.8)\end{aligned}$$

To close this inequality, we apply D^2 to both sides of the second equation of (1.1), take the $L^2(\mathbb{R}^N)$ inner product with $|D^2\rho|^{p-2}D^2\rho$ and then have

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|D^2\rho\|_{L^p}^p \le{}& 3\int |D^2\rho|^p|Du| + \int |D^2\rho|^{p-1}|D\rho||D^2u| + \int |D^2\rho|^{p-1}|\rho||D^3u|\\ \le{}& 3\|D^2\rho\|_{L^p}^p\|Du\|_{L^\infty} + \|D^2\rho\|_{L^p}^{p-1}\|D\rho\|_{L^\infty}\|D^2u\|_{L^p} + \|D^2\rho\|_{L^p}^{p-1}\|\rho\|_{L^\infty}\|D^3u\|_{L^p},\end{aligned}$$

which implies that

$$\frac{d}{dt}\|D^2\rho\|_{L^p} \le 3\left(\|D^2\rho\|_{L^p}\|Du\|_{L^\infty} + \|\rho\|_{L^\infty}\|Dm\|_{L^p}\right) + \|D\rho\|_{L^\infty}\|m\|_{L^p}. \qquad (3.9)$$

Combining (3.8) with (3.9) yields that

$$\begin{aligned}\frac{d}{dt}\left(\|Dm\|_{L^p} + \|D^2\rho\|_{L^p}\right) \le{}& 4\left(\|Du\|_{L^\infty} + \|\rho\|_{L^\infty}\right)\left(\|Dm\|_{L^p} + \|D^2\rho\|_{L^p}\right)\\ &+2\left(\|m\|_{L^\infty} + \|D\rho\|_{L^\infty}\right)\left(\|m\|_{L^p} + \|D\rho\|_{L^p}\right). \qquad (3.10)\end{aligned}$$

Thus it follows from Gronwall's inequality that

$$\begin{aligned}&\|Dm\|_{L^p} + \|D^2\rho\|_{L^p}\\ &\le \left(\|m_0\|_{L^p} + \|D\rho_0\|_{L^p} + \int_0^t 2\left(\|m\|_{L^\infty} + \|D\rho\|_{L^\infty}\right)\left(\|m\|_{L^p} + \|D\rho\|_{L^p}\right)d\tau\right)\\ &\qquad\times e^{4\int_0^t(\|Du\|_{L^\infty} + \|\rho\|_{L^\infty})d\tau},\end{aligned}$$

which is bounded by the assumption, (3.3) and (3.6). In particular, by letting $p \to \infty$, we get

$$\begin{aligned}\|Dm\|_{L^\infty} \le{}& \left(\|m_0\|_{L^\infty} + \|D\rho_0\|_{L^\infty} + 2\int_0^t \left(\|m\|_{L^\infty} + \|D\rho\|_{L^\infty}\right)^2 d\tau\right)\\ &\times e^{4\int_0^t(\|Du\|_{L^\infty} + \|\rho\|_{L^\infty})d\tau} < \infty. \qquad (3.11)\end{aligned}$$

We substitute (3.3), (3.7) and (3.11) into (3.1) and then complete the proof of the conclusion

$$\lim_{t\to T}\left(\|u(t)\|_{H^{k+1}} + \|\rho(t)\|_{H^k}\right) < \infty \quad \text{if and only if} \quad \int_0^T \|\nabla u(\tau)\|_{L^\infty}d\tau < \infty.$$

Now we consider the $\dot{B}^0_{\infty,\infty}$-norm case and assume that

$$\int_0^T \left(\|Du(\tau)\|_{\dot{B}^0_{\infty,\infty}} + \|\rho(\tau)\|_{\dot{B}^0_{\infty,\infty}} \right) d\tau < \infty. \tag{3.12}$$

It follows from (3.1) and Sobolev embedding that

$$\|m\|^2_{H^{k-1}} + \|\rho\|^2_{H^k} \le \left(\|m_0\|^2_{H^{k-1}} + \|\rho_0\|^2_{H^k} \right) e^{C\int_0^t \left(\|m\|_{W^{2,p}} + \|\rho\|_{W^{2,p}} \right) d\tau},$$

for any $p > N$. Thus it is sufficient to prove that the two integrals on the right hand side are bounded under the assumption (3.12). For this purpose, we first recall the following logarithmic Sobolev inequality

$$\|f\|_{L^\infty(\mathbb{R}^N)} \le C\left(1 + \|f\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^N)} \log\left(1 + \|f\|_{W^{1,p}(\mathbb{R}^N)}\right)\right)$$

for any $N < p < \infty$ (see e.g. [32]). Applying this inequality to (3.2) and (3.5) yields

$$\begin{aligned}\frac{d}{dt}\left(\|m\|_{L^p} + \|\rho\|_{W^{1,p}}\right) \le& 3\left(\|Du\|_{L^\infty} + \|\rho\|_{L^\infty}\right)\left(\|m\|_{L^p} + \|\rho\|_{W^{1,p}}\right) \\ \le& C\left(1 + \|Du\|_{\dot{B}^0_{\infty,\infty}} + \|\rho\|_{\dot{B}^0_{\infty,\infty}}\right)\log\left(1 + \|m\|_{L^p} + \|\rho\|_{W^{1,p}}\right) \\ &\times\left(\|m\|_{L^p} + \|\rho\|_{W^{1,p}}\right).\end{aligned} \tag{3.13}$$

It then follows from Gronwall's inequality and the assumption (3.12) that

$$\|m(t)\|_{L^p} + \|\rho(t)\|_{W^{1,p}} < +\infty \qquad \text{for any} \quad 0 < t < T. \tag{3.14}$$

Similarly, by (3.10), we have

$$\begin{aligned}&\frac{d}{dt}\left(\|Dm\|_{L^p} + \|D^2\rho\|_{L^p}\right) \\ \le& 4\left(\|Du\|_{L^\infty} + \|\rho\|_{L^\infty}\right)\left(\|Dm\|_{L^p} + \|D^2\rho\|_{L^p}\right) \\ &+ 2\left(\|m\|_{L^\infty} + \|D\rho\|_{L^\infty}\right)\left(\|m\|_{L^p} + \|D\rho\|_{L^p}\right) \\ \le& C\left(1 + \|Du\|_{\dot{B}^0_{\infty,\infty}} + \|\rho\|_{\dot{B}^0_{\infty,\infty}}\right)\log\left(1 + \|m\|_{L^p} + \|\rho\|_{W^{1,p}}\right) \\ &\times\left(\|Dm\|_{L^p} + \|D^2\rho\|_{L^p}\right) + C\left(\|m\|_{L^p} + \|D\rho\|_{L^p}\right)\left(\|m\|_{W^{1,p}} + \|\rho\|_{W^{2,p}}\right),\end{aligned}$$

which together with (3.13) yields that

$$\begin{aligned}&\frac{d}{dt}\left(\|m\|_{W^{1,p}} + \|\rho\|_{W^{2,p}}\right) \\ &\le C\left(1 + \|Du\|_{\dot{B}^0_{\infty,\infty}} + \|\rho\|_{\dot{B}^0_{\infty,\infty}} + \|m\|_{L^p} + \|D\rho\|_{L^p}\right) \\ &\quad\times\left(\log\left(1 + \|m\|_{W^{1,p}} + \|\rho\|_{W^{2,p}}\right)\right)\left(\|m\|_{W^{1,p}} + \|\rho\|_{W^{2,p}}\right).\end{aligned} \tag{3.15}$$

Then by Gronwall's inequality, (3.14) and the assumption (3.12), we see that

$$\|m(t)\|_{W^{1,p}} + \|\rho(t)\|_{W^{2,p}} < +\infty \qquad \text{for any} \quad 0 < t < T. \tag{3.16}$$

It remains to bound $\|D^2m\|_{L^p}$. This can be done with the aid of bounding $\|D^3\rho\|_{L^p}$. Indeed, we can apply D^3 and D^2 to both sides of equation $(1.1)_2$ and $(1.1)_1$, and take the $L^2(\mathbb{R}^N)$ inner product of $|D^3\rho|^{p-2}D^3\rho$ and of $|D^2m|^{p-2}D^2m$, respectively, to deduce that

$$\begin{aligned}\frac{d}{dt}&\left(\|D^2m\|_{L^p}+\|D^3\rho\|_{L^p}\right)\\&\le C\left(\|Du\|_{L^\infty}+\|D^2u\|_{L^\infty}+\|m\|_{L^\infty}+\|\rho\|_{L^\infty}+\|D\rho\|_{L^\infty}\right)\\&\qquad\times\left(\|Dm\|_{L^p}+\|D^2m\|_{L^p}+\|D^2\rho\|_{L^p}+\|D^3\rho\|_{L^p}\right).\end{aligned}$$

Combining this inequality with (3.15), we obtain

$$\begin{aligned}\frac{d}{dt}&\left(\|m\|_{W^{2,p}}+\|\rho\|_{W^{3,p}}\right)\\&\le C\left(1+\|Du\|_{\dot B^0_{\infty,\infty}}+\|\rho\|_{\dot B^0_{\infty,\infty}}+\|m\|_{W^{1,p}}+\|\rho\|_{W^{2,p}}\right)\\&\qquad\times\left(\log\left(1+\|m\|_{W^{2,p}}+\|\rho\|_{W^{3,p}}\right)\right)\left(\|m\|_{W^{2,p}}+\|\rho\|_{W^{3,p}}\right).\end{aligned}$$

Hence, by Gronwall's inequality, (3.16) and the assumption (3.12), we have

$$\|m(t)\|_{W^{2,p}}+\|\rho(t)\|_{W^{3,p}}<+\infty \qquad \text{for any} \quad 0<t<T.$$

Then we see

$$\lim_{t\to T}\left(\|u(t)\|_{H^{k+1}}+\|\rho(t)\|_{H^k}\right)<\infty$$

if and only if

$$\int_0^T\left(\|\nabla u(\tau)\|_{\dot B^0_{\infty,\infty}}+\|\rho(\tau)\|_{\dot B^0_{\infty,\infty}}\right)d\tau<\infty.$$

This completes the proof of Theorem 1.2. $\square$

4. Blow-up solutions and global solutions

In this section, we will show that for a large class of smooth initial data with some concentration property, the solutions to equations (1.1) will blow up in finite time. For the decoupled system (1.3), we also obtain the precise blow-up estimates and a global existence result. The class of functions that we consider here was first introduced by [27] for the one-component Euler-Poincaré equations (1.3), but their argument cannot directly apply to our case. This is because the appearance of the component ρ makes the discussion on $\psi(0,t)$ inconvenient, which is the key in [27]. To overcome this difficulty, our basic strategy is to transfer the higher dimensional problem to a one-dimensional problem, from which a nonlocal integral arises. For the resulting one-dimensional problem, we will first use the following classical lemma to construct an ordinary differential inequality, whose solution will yield the desired result.

Lemma 4.1 (Constantin-Escher [8]). *Let $T>0$ and $\omega\in C^1([0,T);H^2(\mathbb{R}))$. Then for every $t\in[0,T)$, there exists at least one point $\xi(t)\in\mathbb{R}$ with*

$$\mathfrak{m}(t):=\inf_{x\in\mathbb{R}}\omega_x(x,t)=\omega_x(\xi(t),t).$$

The function $\mathfrak{m}(t)$ is absolutely continuous on $(0,T)$ with

$$\frac{d\mathfrak{m}}{dt}=\omega_{tx}(\xi(t),t), \quad \text{a.e. on} \quad (0,T).$$

Then to deal with the nonlocal integral, we need to use the Littlewood-Paley decomposition theory. Finally, we also need to recall a basic fact as follows. For smooth solutions of equations (1.1) with enough spatial decay, the following conservation law holds

$$\int \left(|u|^2+\alpha^2|\nabla u|^2+2\rho^2\right)(t)=\int \left(|u_0|^2+\alpha^2|\nabla u_0|^2+2\rho_0^2\right), \tag{4.1}$$

for all $t\geq 0$, which can be deduced by integrating by parts in equations (1.1).

Proof of Theorem 1.3. Without loss of generality, we may set $\alpha=1$. We will use ∂_r to denote the radial derivative whenever there is no confusion. The notation $(1-\Delta)^{-1}f(r)$ means $(1-\Delta)^{-1}f(x)$ is a radial function and the notation $\Delta(1-\Delta)^{-1}f(r)$ can be similarly understood.

Let $(\partial_r\psi(r,t),\rho(r,t))$ be the unique solution of the first order partial differential equations

$$\begin{cases} \partial_t\partial_r\psi(r,t)-\psi(r,t)\partial_r\psi(r,t)+\left((1-\Delta)^{-1}\psi\right)(r,t)\partial_r\psi(r,t) \\ \qquad\qquad +\partial_r\left(\partial_r\left((1-\Delta)^{-1}\psi\right)\partial_r\psi\right)(r,t)=-\dfrac{1}{2}\partial_r\rho^2(r,t), \\ \partial_t\rho(r,t)=-\partial_r\left((1-\Delta)^{-1}\psi\right)(r,t)\partial_r\rho(r,t) \\ \qquad\qquad +\rho(r,t)\left((-\Delta)(1-\Delta)^{-1}\psi\right)(r,t) \end{cases} \tag{4.2}$$

in $[0,+\infty)\times[0,T)$ with initial data $(\partial_r\psi_0,\rho_0)$. Set $u:=\nabla(1-\Delta)^{-1}\psi$, or equivalently, $\psi(x,t)=\Delta^{-1}\mathrm{div}\, m=\Delta^{-1}(1-\Delta)\mathrm{div}\, u$. Then (u,ρ) is radial and will solve equation (1.1). By uniqueness, (u,ρ) is the unique solution of equation (1.1) with initial data (u_0,ρ_0).

We claim that

$$\int_0^{T^*}\|\nabla\cdot u\|_{L^\infty}d\tau=+\infty \quad \text{for some} \quad T^*\geq T,$$

which implies that (u,ρ) will blow up at T^* by the blow-up criterion (see Theorem 1.2). We will prove our claim by contradiction argument. Indeed, if the calim is false, we may assume that

$$\int_0^{T_0}\|\nabla\cdot u\|_{L^\infty}d\tau<+\infty \quad \text{for any} \quad T_0>0. \tag{4.3}$$

To deduce a contradiction, we integrate the first equation of (4.2) on $[r,+\infty)$ and obtain

$$\begin{aligned}\partial_t\psi(r,t)=\frac{1}{2}\psi^2(r,t)+\int_r^\infty &\left((1-\Delta)^{-1}\psi\right)(s,t)\partial_s\psi(s,t)ds \\ &-\partial_r\left((1-\Delta)^{-1}\psi\right)(r,t)\partial_r\psi(r,t)-\frac{1}{2}\rho^2(r,t).\end{aligned}$$

Define $\omega(r,t):=\int_0^r\psi(s,t)ds$ for $r\geq 0$ and extend $\omega(r,t)$ to all of $r\in\mathbb{R}$ by odd reflection, that is,

$$\omega(r,t)=\begin{cases} \displaystyle\int_0^r\psi(s,t)ds, & \text{for} \quad r\geq 0; \\ \displaystyle-\int_0^{-r}\psi(s,t)ds, & \text{for} \quad r<0. \end{cases}$$

One has from Theorem 1.1 and direct computations that $\omega \in C^1([0,T_0);H^2(\mathbb{R}))$. Thus by Lemma 4.1, we see that there exists $\xi(t) \geq 0$ such that

$$\mathcal{M}(t) := \psi(\xi(t),t) = \sup_{r\geq 0} \psi(r,t), \quad \text{for any } t \in [0,T_0) \tag{4.4}$$

and

$$\frac{d\mathcal{M}}{dt} = \partial_t \psi(\xi(t),t), \quad \text{a. e. on} \quad (0,T_0). \tag{4.5}$$

We now prove that $\mathcal{M}(t)$ blows up at some finite time T_1. Notice

$$\partial_r \psi(\xi(t),t) = 0, \quad \text{for a. e. } \ t \in (0,T_0),$$

which together with (4.2) yield that at $r = \xi(t)$,

$$\partial_t \psi(\xi(t),t) = \frac{1}{2}\psi^2(\xi(t),t) + \int_{\xi(t)}^{\infty} \left((1-\Delta)^{-1}\psi\right)(s,t)\partial_s\psi(s,t)ds - \frac{1}{2}\rho^2(\xi(t),t).$$

Then by (4.5) we have

$$\frac{d\mathcal{M}}{dt} = \frac{1}{2}\mathcal{M}^2(t) + \int_{\xi(t)}^{\infty} \left((1-\Delta)^{-1}\psi\right)(s,t)\partial_s\psi(s,t)ds - \frac{1}{2}\rho^2(\xi(t),t). \tag{4.6}$$

Notice that (3.3) and (4.3) imply that $\|\rho\|_{L^\infty} \leq \|\rho_0\|_{L^\infty} e^{2\int_0^{T_0}\|\nabla\cdot u\|_{L^\infty}d\tau} < (2C_0)^{\frac{1}{2}}$ for some C_0 and any $t \in [0,T_0]$, which together with (4.6) yield that

$$\frac{d\mathcal{M}}{dt} \geq \frac{1}{2}\mathcal{M}^2(t) + \int_{\xi(t)}^{\infty} \left((1-\Delta)^{-1}\psi\right)(s,t)\partial_s\psi(s,t)ds - C_0. \tag{4.7}$$

We need to estimate the nonlocal integration in (4.7). For this purpose, we use the integration by parts to obtain

$$\begin{aligned}
&\int_{\xi(t)}^{\infty} \left((1-\Delta)^{-1}\psi\right)(s,t)\partial_s\psi(s,t)ds \\
=&\psi(s,t)\left((1-\Delta)^{-1}\psi\right)(s,t)\big|_{s=\xi(t)}^{\infty} - \int_{\xi(t)}^{\infty} \partial_s\left((1-\Delta)^{-1}\psi\right)(s,t)\psi(s,t)ds \\
=& -\mathcal{M}(t)\left((1-\Delta)^{-1}\psi\right)(\xi(t),t) - \int_{\xi(t)}^{\infty} \partial_s\left((1-\Delta)^{-1}\psi\right)(s,t)\psi(s,t)ds.
\end{aligned}$$

It is straightforward to see by $u = (1-\Delta)^{-1}\nabla\psi$ that

$$\begin{aligned}
&\left|\int_{\xi(t)}^{\infty} \partial_s\left((1-\Delta)^{-1}\psi\right)(s,t)\psi(s,t)ds\right| \\
\lesssim& \int_{\mathbb{R}^N} \frac{\left|\left((1-\Delta)^{-1}\nabla\psi\right)(x,t)\right||\psi(x,t)|}{|x|^{N-1}}dx = \int_{\mathbb{R}^N} \frac{|u(x,t)||\psi(x,t)|}{|x|^{N-1}}dx \\
\leq& \int_{|x|\leq 1} \frac{|u(x,t)||\psi(x,t)|}{|x|^{N-1}}dx + \|u(t)\|_{L^2}\|\psi(t)\|_{L^2} := K_1 + K_2.
\end{aligned} \tag{4.8}$$

We first estimate K_2. By the conservation law (4.1), one has

$$|K_2| \lesssim \left(\|u_0\|_{H^1} + \|\rho_0\|_{L^2}\right)\|\psi(t)\|_{L^2} \lesssim \left(\|\psi_0\|_{L^2} + \|\rho_0\|_{L^2}\right)\|\psi(t)\|_{L^2}.$$

To control the L^2 norm of ψ, we will use the Littlewood-Paley decomposition to decompose ψ into low frequency parts and high frequency ones. Indeed, for any $t \in [0,T_0)$, we have

$$\|\psi(t)\|_{L^2} \leq \|\Delta_0\psi(t)\|_{L^2} + \|(1-\Delta_0)\psi(t)\|_{L^2}. \tag{4.9}$$

Denoting

$$T_{ij}=u_iu_j+\frac{1}{2}\delta_{ij}|u|^2+\nabla u_i\cdot\nabla u_j-\partial_i u\cdot\partial_j u+\frac{1}{2}\delta_{ij}|\nabla u|^2,$$

and using $(1.1)_1$, we have

$$\partial_i\psi(t)=\partial_i\psi(0)-\sum_{j=1}^{N}\int_0^t\partial_jT_{ij}(\tau)d\tau-\frac{1}{2}\int_0^t\partial_i\rho^2(\tau)d\tau.$$

It then follows from Young's inequality and the conservation law (4.1) that

$$\begin{aligned}\|\Delta_0\psi\|_{L^2}\lesssim&\sum_{i,j=1}^{N}\|\Delta_0\Delta^{-1}\partial_i\partial_j\psi_0\|_{L^2}\\&\quad+\sum_{i,j=1}^{N}\int_0^t\left(\|\Delta_0\Delta^{-1}\partial_i\partial_jT_{ij}(\tau)\|_{L^2}+\|\Delta_0\Delta^{-1}\partial_i\partial_j\rho^2(\tau)\|_{L^2}\right)d\tau\\&\lesssim\|\psi_0\|_{L^2}+\int_0^t\left(\|T(\tau)\|_{L^1}+\|\rho^2(\tau)\|_{L^1}\right)d\tau\\&\lesssim\|\psi_0\|_{L^2}+\int_0^t\left(\|u(\tau)\|_{L^2}^2+\|\nabla u(\tau)\|_{L^2}^2+\|\rho(\tau)\|_{L^2}^2\right)d\tau\\&\lesssim\|\psi_0\|_{L^2}+\left(\|u_0\|_{L^2}^2+\|\nabla u_0\|_{L^2}^2+\|\rho_0\|_{L^2}^2\right)t\\&\lesssim\|\psi_0\|_{L^2}+\left(\|\psi_0\|_{L^2}^2+\|\rho_0\|_{L^2}^2\right)t.\end{aligned}\tag{4.10}$$

By using the conservation law (4.1) again, we have

$$\sum_{i,j=1}^{N}\|(1-\Delta)^{-1}\partial_i\partial_j\psi(t)\|_{L^2}\lesssim\|\nabla u(t)\|_{L^2}\lesssim\|u_0\|_{L^2}+\|\nabla u_0\|_{L^2}+\|\rho_0\|_{L^2}\lesssim\|\psi_0\|_{L^2}+\|\rho_0\|_{L^2},$$

which implies that

$$\|(1-\Delta_0)\psi(t)\|_{L^2}\lesssim\|\psi_0\|_{L^2}+\|\rho_0\|_{L^2}.\tag{4.11}$$

Plugging the estimates (4.10) and (4.11) into (4.9), we obtain

$$\|\psi(t)\|_{L^2}\lesssim\left(\|\psi_0\|_{L^2}+\|\rho_0\|_{L^2}\right)\left(1+t\left(\|\psi_0\|_{L^2}+\|\rho_0\|_{L^2}\right)\right),\tag{4.12}$$

which implies that

$$|K_2|\lesssim\left(\|\psi_0\|_{L^2}^2+\|\rho_0\|_{L^2}^2\right)\left(1+t\left(\|\psi_0\|_{L^2}+\|\rho_0\|_{L^2}\right)\right).$$

To estimate K_1, we first take p, q and s such that

$$2<p<6,\quad q>2,\quad s(N-1)<N,\quad \frac{1}{p}+\frac{1}{q}+\frac{1}{s}=1.$$

Then Hölder's inequality yields that

$$|K_1|\le\|u\|_{L^p}\|\psi\|_{L^q}\left\||x|^{-(N-1)}\right\|_{L^s(B(0,1))}\lesssim\|u\|_{L^p}\|\psi\|_{L^q}.$$

Since

$$\|u\|_{L^p}\le C\|u\|_{L^2}^{1+\frac{N}{p}-\frac{N}{2}}\|\nabla u\|_{L^2}^{\frac{N}{2}-\frac{N}{p}}\le C$$

and

$$\|\psi\|_{L^q}\le\|\psi\|_{L^2}^{\frac{2}{q}}\|\psi\|_{L^\infty}^{\frac{q-2}{q}}\le C(\mathcal{M}(t)+1)\left(1+\|\psi_0\|_{L^2}+\|\rho_0\|_{L^2}\right)\left(1+t\left(\|\psi_0\|_{L^2}+\|\rho_0\|_{L^2}\right)\right)$$

by the interpolation, the conservation law (4.1) and (4.12), we have

$$|K_1| \le \|u\|_{L^p}\|\psi\|_{L^q}\left\||x|^{-(N-1)}\right\|_{L^s(B(0,1))} \lesssim \|u\|_{L^p}\|\psi\|_{L^q},$$

which together with the estimates (4.7)-(4.8) yields that

$$\begin{aligned}\frac{d\mathcal{M}}{dt} \ge \frac{1}{3}\mathcal{M}^2(t) - \mathcal{M}(t)\left((1-\Delta)^{-1}\psi\right)(\xi(t),t)& \\ - C\left(\|\psi_0\|_{L^2}^2 + \|\rho_0\|_{L^2}^2\right)\left(1 + t\left(\|\psi_0\|_{L^2} + \|\rho_0\|_{L^2}\right)\right).& \quad (4.13)\end{aligned}$$

We will still need the estimates for $\left((1-\Delta)^{-1}\psi\right)(\xi(t),t)$. This can be done as follows. By the Littlewood-Paley decomposition and Bernstein's inequality, we have

$$\begin{aligned}\|(1-\Delta)^{-1}\psi(t)\|_{L^\infty} &\le \|\Delta_K(1-\Delta)^{-1}\psi(t)\|_{L^\infty} + \|(1-\Delta_K)(1-\Delta)^{-1}\psi(t)\|_{L^\infty} \\ &\le C\|\psi(t)\|_{L^2} + \frac{1}{12}\|\psi(t)\|_{L^\infty}\end{aligned} \quad (4.14)$$

for some K large enough. Then plugging (4.4), (4.12) and (4.14) into (4.13), we obtain

$$\begin{aligned}\frac{d\mathcal{M}}{dt} \ge \frac{1}{4}\mathcal{M}^2(t) - C\mathcal{M}(t)\left(\|\psi_0\|_{L^2} + \|\rho_0\|_{L^2}\right)\left(1 + t\left(\|\psi_0\|_{L^2} + \|\rho_0\|_{L^2}\right)\right)& \\ - C\left(\|\psi_0\|_{L^2}^2 + \|\rho_0\|_{L^2}^2\right)\left(1 + t\left(\|\psi_0\|_{L^2} + \|\rho_0\|_{L^2}\right)\right).& \quad (4.15)\end{aligned}$$

If $\mathcal{M}(0) = \psi_0(0) \ge C\left(\|\psi_0\|_{L^2} + \|\rho_0\|_{L^2}\right)$ for some C large enough, the inequality (4.15) implies that $\mathcal{M}(t)$ will blow up at some finite time T_1. Moreover, we have

$$\mathcal{M}(t) \ge c(T_1 - t)^{-1} \quad (4.16)$$

for some $c > 0$.

Since $\nabla\cdot u = -\psi + (1-\Delta)^{-1}\psi$, we can use (4.12), (4.14) and (4.16) to obtain

$$\begin{aligned}\|\nabla\cdot u\|_\infty \ge \|\psi\|_\infty - \|(1-\Delta)^{-1}\psi\|_\infty \ge \frac{3}{4}\|\psi\|_\infty - C\|\psi\|_{L^2} \ge \frac{3}{4}\mathcal{M}(t) - C(T_1+1)& \\ \ge c(T_1-t)^{-1},&\end{aligned}$$

which implies that

$$\int_0^{T_1}\|\nabla\cdot u\|_{L^\infty}d\tau = +\infty.$$

This contradicts to the assumption (4.3). Thus we complete the proof of the claim and then that of Theorem 1.3. □

Proof of Proposition 1.1. Similar to the proof of Theorem 1.3, let $\psi(r,t)$, be the unique solution of the partial differential equation

$$\begin{aligned}\partial_t\partial_r\psi(r,t) - \psi(r,t)\partial_r\psi(r,t) + \left((1-\Delta)^{-1}\psi\right)(r,t)\partial_r\psi(r,t)& \\ + \partial_r\left(\partial_r\left((1-\Delta)^{-1}\psi\right)\partial_r\psi\right)(r,t) = 0,& \quad (4.17)\end{aligned}$$

with initial data ψ_0. Set $u := \nabla(1-\Delta)^{-1}\psi$, or equivalently, $\psi(x,t) = \Delta^{-1}\mathrm{div}\, m = \Delta^{-1}(1-\Delta)\mathrm{div}\, u$. Then u is radial and will solve equation (1.3). By uniqueness, u is the unique solution of equation (1.3) with initial data u_0. Thus, we pay our attention to ψ.

(i) To obtain the blow-up estimates, we integrate (4.17) on $[r,+\infty)$ and then obtain

$$\partial_t\psi(r,t)=\frac{1}{2}\psi^2(r,t)+\int_r^\infty\left((1-\Delta)^{-1}\psi\right)(s,t)\partial_s\psi(s,t)ds \\ -\partial_r\left((1-\Delta)^{-1}\psi\right)(r,t)\partial_r\psi(r,t). \quad (4.18)$$

Define $\omega(r,t):=\int_0^r\psi(s,t)ds$ for $r\geq 0$ and extend $\omega(r,t)$ to all of $r\in\mathbb{R}$ by odd reflection. It then follows from Theorem 1.1 and a direct computation that $\omega\in C^1([0,T);H^2(\mathbb{R}))$. Thus by Lemma 4.1, we see that there exists $\xi(t)\geq 0$ such that

$$\mathcal{M}(t):=\psi(\xi(t),t)=\sup_{r\geq 0}\psi(r,t),\quad \text{for any } t\in[0,T)$$

and

$$\frac{d\mathcal{M}}{dt}=\partial_t\psi(\xi(t),t),\quad \text{a. e. on}\quad (0,T).$$

To show that u blows up at some time T^* and obtain its blow-up estimate, we notice that

$$\|\nabla\cdot u\|_\infty\geq\|\psi\|_\infty-\|(1-\Delta)^{-1}\psi\|_\infty\geq\frac{3}{4}\|\psi\|_\infty-C\|\psi\|_{L^2}\geq\frac{3}{4}\mathcal{M}(t)-C(T^*+1),$$

and

$$\|\nabla\cdot u\|_\infty\leq\|\psi\|_\infty+\|(1-\Delta)^{-1}\psi\|_\infty\leq\frac{5}{4}\|\psi\|_\infty+C\|\psi\|_{L^2}\leq\frac{5}{4}\mathcal{M}(t)+C(T^*+1),$$

which can be deduced by using a similar estimate as the proof of Theorem 1.3. Thus we just need to establish the blowup of $\mathcal{M}(t)$ and its blow-up estimate. For this purpose, we take $r=\xi(t)$ in (4.18) and then obtain

$$\frac{d\mathcal{M}}{dt}=\frac{1}{2}\mathcal{M}^2(t)+\int_{\xi(t)}^\infty\left((1-\Delta)^{-1}\psi\right)(s,t)\partial_s\psi(s,t)ds.$$

Thus we can use a similar procedure as the proof of Theorem 1.3 to estimate the integral term and then obtain

$$\frac{d\mathcal{M}}{dt}\geq\frac{1}{4}\mathcal{M}^2(t)-C\mathcal{M}(t)\|\psi_0\|_{L^2}\left(1+t\|\psi_0\|_{L^2}\right)-C\|\psi_0\|_{L^2}^2\left(1+t\|\psi_0\|_{L^2}\right)$$

and

$$\frac{d\mathcal{M}}{dt}\leq\frac{3}{4}\mathcal{M}^2(t)+C\mathcal{M}(t)\|\psi_0\|_{L^2}\left(1+\|\psi_0\|_{L^2}\right)+C\|\psi_0\|_{L^2}^2\left(1+t\|\psi_0\|_{L^2}\right),$$

which implies that $\mathcal{M}(t)$ will blow up at some finite time T^* provided that $\psi_0(0)\geq C\|\psi_0\|_{L^2}$ for some $C>0$ large enough. Moreover, the blow-up rate estimate is given by

$$c(T^*-t)^{-1}\leq\mathcal{M}(t)\leq C(T^*-t)^{-1}$$

for some $C>c>0$. This completes the proof of Proposition 1.1 (i).

(ii) Now we show the existence of global solutions. We will repeat some derivations similar to (i) by setting $\zeta(r,t):=-\psi(r,t)$. Instead of (4.17), we can obtain

$$\partial_t\partial_r\zeta(r,t)+\zeta(r,t)\partial_r\zeta(r,t)-\left((1-\Delta)^{-1}\zeta\right)(r,t)\partial_r\zeta(r,t) \\ -\partial_r\left(\partial_r\left((1-\Delta)^{-1}\zeta\right)\partial_r\zeta\right)(r,t)=0.$$

Integrating this equation on $[r,+\infty)$ yields that

$$\partial_t\zeta(r,t) = -\frac{1}{2}\zeta^2(r,t) - \int_r^\infty \left((1-\Delta)^{-1}\zeta\right)(s,t)\partial_s\zeta(s,t)ds \\ + \partial_r\left((1-\Delta)^{-1}\zeta\right)(r,t)\partial_r\zeta(r,t). \quad (4.19)$$

It follows from Lemma 4.1 that there exists $\eta(t)\geq 0$ such that

$$\mathsf{M}(t) := \zeta(\eta(t),t) = \sup_{r\geq 0}\zeta(r,t), \quad \text{for any } t\in[0,T)$$

and then

$$\frac{d\mathsf{M}}{dt} = -\frac{1}{2}\mathsf{M}^2(t) - \int_{\eta(t)}^\infty \left((1-\Delta)^{-1}\zeta\right)(s,t)\partial_s\zeta(s,t)ds.$$

Similar to the proof of (i), we can deduce that

$$\begin{aligned}\frac{d\mathsf{M}}{dt} &\leq -\frac{1}{4}\mathsf{M}^2(t) + C\mathsf{M}(t)\|\psi_0\|_{L^2}\left(1+t\|\psi_0\|_{L^2}\right) + C\|\psi_0\|_{L^2}^2\left(1+t\|\psi_0\|_{L^2}\right)\\ &\leq -\frac{1}{8}\mathsf{M}^2(t) + C\|\psi_0\|_{L^2}^2\left(1+t^2\|\psi_0\|_{L^2}^2\right).\end{aligned}$$

Notice that $\mathsf{M}(0) = -\psi(0) > 0$. A simple bootstrap argument implies that on any $[0,T]$, $\mathsf{M}(t)$ can be bounded above. On the other hand, if we rewrite equation (4.19) as

$$\partial_t\zeta(r,t) + \left(\frac{1}{2}\zeta(r,t) + \zeta^{-1}(r,t)\int_r^\infty \left((1-\Delta)^{-1}\zeta\right)(s,t)\partial_s\zeta(s,t)ds\right)\zeta(r,t) \\ - \partial_r\left((1-\Delta)^{-1}\zeta\right)(r,t)\partial_r\zeta(r,t) = 0,$$

then the method of characteristics argument yields $\zeta(r,t) > 0$, since $\zeta(r,0) = -\psi(r) > 0$ for any $r\geq 0$. Thus $\mathsf{M}(t)$ can also be bounded below. Indeed, if we check the proof of Lemma 4.1 in [8], we have $\mathsf{M}(t)\geq 0$.

We now use the blow-up criterion to conclude the proof of the global existence. By Bernstein's inequality and Sobolev embedding, we have

$$\|Du(t)\|_{\dot{B}^0_{\infty,\infty}} = \left\|D^2(1-\Delta)^{-1}\psi\right\|_{\dot{B}^0_{\infty,\infty}} \leq C\|\psi(t)\|_{\dot{B}^0_{\infty,\infty}} \leq C\|\psi(t)\|_{L^\infty} \leq C|\mathsf{M}(t)| \leq C,$$

for any $0\leq t\leq T$. Notice that

$$\begin{aligned}\|\nabla\cdot u(t)\|_{L^\infty} &= \left\|\Delta(1-\Delta)^{-1}\psi\right\|_{L^\infty} = C\left\|\int_{\mathbb{R}^N} K(y)\left(\psi(x-y)-\psi(x)\right)dy\right\|_{L^\infty}\\ &\leq C\int_{\mathbb{R}^N} K(y)dx\,\|\psi(t)\|_{L^\infty} = C|\mathsf{M}(t)|\\ &\leq C \quad \text{for any} \quad 0\leq t\leq T,\end{aligned}$$

where K is the Bessel potential and is defined by the Fourier transform $\mathcal{F}(K)(\xi) = (1+|\xi|^2)^{-1}$. Summarily, we have

$$\|Du(t)\|_{\dot{B}^0_{\infty,\infty}} \leq C \quad \text{for any} \quad 0\leq t\leq T.$$

By the blow-up criterion, we conclude that the corresponding solution u exists for all time $t>0$. This completes the proof of Proposition 1.1 (ii). □

5. Limit problem

In this section, we show that the two-component Euler-Poincaré equations (1.1) can be regarded as an approximation of the one-component Euler-Poincaré equations (1.3) or a dispersion regularization of the limited equations (1.2) in some sense. To do this, our basic strategy is to establish the energy estimates for the difference of the approximation solution and the limit solution.

Proof of Theorem 1.4. (i) By setting $\mathrm{u}_n := u_n - u$ and $\mathrm{m}_n := m_n - m$, we have $\mathrm{m}_n = \mathrm{u}_n - \alpha^2 \Delta \mathrm{u}_n$. After a simple calculation, we see that (u_n, ρ_n) satisfies

$$\begin{cases} \mathrm{m}_{nt} + \mathrm{u}_n \cdot \nabla m_n + u \cdot \nabla \mathrm{m}_n + (\nabla \mathrm{u}_n)^T m_n + (\nabla u)^T \mathrm{m}_n \\ \qquad\qquad + m_n \mathrm{div} \mathrm{u}_n + \mathrm{m}_n \mathrm{div} u + \rho_n \nabla \rho_n = 0, \\ \rho_{nt} + \nabla \rho_n \cdot \mathrm{u}_n + \nabla \rho_n \cdot u + \rho_n \mathrm{div} \mathrm{u}_n + \rho_n \mathrm{div} u = 0. \end{cases} \tag{5.1}$$

Taking the $L^2(\mathbb{R}^N)$ inner product of the first equation of (5.1) with u_n and integrating by parts, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\mathrm{u}_n\|_{L^2}^2 + \alpha^2 \|\nabla \mathrm{u}_n\|_{L^2}^2\right) \\ &= -\int \mathrm{u}_n \cdot (\mathrm{u}_n \cdot \nabla) m_n - \int \mathrm{u}_n \cdot (u \cdot \nabla) \mathrm{m}_n - \int \mathrm{u}_n \cdot (\nabla \mathrm{u}_n)^T m_n - \int \mathrm{u}_n \cdot (\nabla u)^T \mathrm{m}_n \\ &\quad - \int \mathrm{u}_n \cdot \mathrm{m}_n \mathrm{div} \mathrm{u}_n - \int \mathrm{u}_n \cdot \mathrm{m}_n \mathrm{div} u - \int \mathrm{u}_n \cdot \rho_n \nabla \rho_n \\ &= -\int \mathrm{u}_n \cdot (u \cdot \nabla) \mathrm{m}_n - \int \mathrm{u}_n \cdot (\nabla u)^T \mathrm{m}_n - \int \mathrm{u}_n \cdot \mathrm{m}_n \mathrm{div} u - \int \mathrm{u}_n \cdot \rho_n \nabla \rho_n, \end{aligned} \tag{5.2}$$

where we used the identity

$$\int \mathrm{u}_n \cdot (\mathrm{u}_n \cdot \nabla) m_n + \int \mathrm{u}_n \cdot (\nabla \mathrm{u}_n)^T m_n + \int \mathrm{u}_n \cdot \mathrm{m}_n \mathrm{div} \mathrm{u}_n = 0,$$

which can be obtained by the integration by parts. Similarly, we take the $L^2(\mathbb{R}^N)$ inner product of the second equation of (5.1) with ρ_n and then have

$$\frac{1}{2}\frac{d}{dt}\|\rho_n\|_{L^2}^2 = -\int \rho_n \nabla \rho_n \cdot \mathrm{u}_n - \int \rho_n \nabla \rho_n \cdot u - \int \rho_n^2 \mathrm{div} \mathrm{u}_n - \int \rho_n^2 \mathrm{div} u. \tag{5.3}$$

Adding (5.2) to (5.3) and integrating by parts, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\mathrm{u}_n\|_{L^2}^2 + \alpha^2 \|\nabla \mathrm{u}_n\|_{L^2}^2 + \|\rho_n\|_{L^2}^2\right) \\ &= -\int \mathrm{u}_n \cdot (u \cdot \nabla) \mathrm{m}_n - \left(\int \mathrm{u}_n \cdot (\nabla u)^T \mathrm{m}_n + \int \mathrm{u}_n \cdot \mathrm{m}_n \mathrm{div} u\right) - \frac{1}{2}\int \rho_n^2 \mathrm{div} u \\ &:= I_1 + I_2 + I_3. \end{aligned} \tag{5.4}$$

We now estimate I_1, I_2 and I_3 as follows. For I_1, we use the fact $\mathrm{m}_n = \mathrm{u}_n - \alpha^2\Delta\mathrm{u}_n$ and integration by parts to obtain

$$\begin{aligned}
I_1 &= -\int \mathrm{u}_n \cdot (u\cdot\nabla)\mathrm{u}_n + \alpha^2 \int \mathrm{u}_n\cdot(u\cdot\nabla)\Delta\mathrm{u}_n \\
&= \int |\mathrm{u}_n|^2 \mathrm{div} u + \alpha^2\left(\frac{1}{2}\int |\nabla\mathrm{u}_n|^2\mathrm{div}u + \sum_{j=1}^N \int \mathrm{u}_{nj}\nabla\mathrm{u}_{nj}\cdot\nabla\mathrm{div}u + \sum_{i,j=1}^N \int \partial_i\mathrm{u}_{nj}\nabla\mathrm{u}_{nj}\cdot\nabla u_i\right) \\
&\lesssim \|\nabla u\|_{L^\infty}\|\mathrm{u}_n\|_{L^2}^2 + \alpha^2\left(\|\nabla u\|_{L^\infty}\|\nabla\mathrm{u}_n\|_{L^2}^2 + \|\nabla^2 u\|_{L^\infty}\|\mathrm{u}_n\|_{L^2}\|\nabla\mathrm{u}_n\|_{L^2}\right) \\
&\lesssim \left(\|\nabla u\|_{L^\infty} + \|\nabla^2 u\|_{L^\infty}\right)\left(\|\mathrm{u}_n\|_{L^2}^2 + \alpha^2\|\nabla\mathrm{u}_n\|_{L^2}^2\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_2 &= -\left(\int \mathrm{u}_n\cdot(\nabla u)^T\mathrm{u}_n + \int |\mathrm{u}_n|^2\mathrm{div}u\right) + \alpha^2\int \mathrm{u}_n\cdot(\nabla u)^T\Delta\mathrm{u}_n + \alpha^2\int \mathrm{u}_n\cdot\Delta\mathrm{u}_n\mathrm{div}u \\
&= -\left(\int \mathrm{u}_n\cdot(\nabla u)^T\mathrm{u}_n + \int |\mathrm{u}_n|^2\mathrm{div}u\right) - \alpha^2\sum_{i,j=1}^N\left(\int \nabla\mathrm{u}_{ni}\cdot\nabla\mathrm{u}_{nj}\partial_i u_j + \int \mathrm{u}_{ni}\nabla\partial_i u_j\cdot\nabla\mathrm{u}_{nj}\right) \\
&\qquad - \alpha^2\sum_{i=1}^N\left(\int |\nabla\mathrm{u}_{ni}|^2\mathrm{div}u + \int \mathrm{u}_{ni}\nabla\mathrm{u}_{ni}\cdot\nabla\mathrm{div}u\right) \\
&\lesssim \|\nabla u\|_{L^\infty}\|\mathrm{u}_n\|_{L^2}^2 + \alpha^2\left(\|\nabla u\|_{L^\infty}\|\nabla\mathrm{u}_n\|_{L^2}^2 + \|\nabla^2 u\|_{L^\infty}\|\mathrm{u}_n\|_{L^2}\|\nabla\mathrm{u}_n\|_{L^2}\right) \\
&\lesssim \left(\|\nabla u\|_{L^\infty} + \|\nabla^2 u\|_{L^\infty}\right)\left(\|\mathrm{u}_n\|_{L^2}^2 + \alpha^2\|\nabla\mathrm{u}_n\|_{L^2}^2\right).
\end{aligned}$$

For I_3, it is direct to see that

$$I_3 \le \frac{1}{2}\|\nabla u\|_{L^\infty}\|\rho_n\|_{L^2}^2.$$

Plugging the estimates for I_1, I_2 and I_3 into (5.4), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\|\mathrm{u}_n\|_{L^2}^2 + \alpha^2\|\nabla\mathrm{u}_n\|_{L^2}^2 + \|\rho_n\|_{L^2}^2\right) \\
&\qquad\qquad \le C\left(1 + \|\nabla u\|_{L^\infty} + \|\nabla^2 u\|_{L^\infty}\right)\left(\|\mathrm{u}_n\|_{L^2}^2 + \alpha^2\|\nabla\mathrm{u}_n\|_{L^2}^2 + \|\rho_n\|_{L^2}^2\right),
\end{aligned}$$

which together with the Gronwall's inequality implies that

$$\begin{aligned}
&\|\mathrm{u}_n(t)\|_{L^2}^2 + \alpha^2\|\nabla\mathrm{u}_n(t)\|_{L^2}^2 + \|\rho_n(t)\|_{L^2}^2 \\
&\le \left(\|\mathrm{u}_{n0}\|_{L^2}^2 + \alpha^2\|\nabla\mathrm{u}_{n0}\|_{L^2}^2 + \|\rho_{n0}\|_{L^2}^2\right)e^{C\int_0^t\left(1+\|\nabla u(\tau)\|_{L^\infty}+\|\nabla^2 u(\tau)\|_{L^\infty}\right)d\tau} \\
&\le \left(\|\mathrm{u}_{n0}\|_{L^2}^2 + \alpha^2\|\nabla\mathrm{u}_{n0}\|_{L^2}^2 + \|\rho_{n0}\|_{L^2}^2\right)e^{C\int_0^t\left(1+\|u(\tau)\|_{H^k}\right)d\tau}.
\end{aligned}$$

If we drop all the terms involving α in the previous proof, the conclusion is still true. This completes the proof of Theorem 1.4 (i).

(ii) Set $m := u - \alpha^2\Delta u$. Then we can deduce that (u,ρ) satisfies

$$m_t + u\cdot\nabla m + (\nabla u)^T m + m\,\mathrm{div}u = -\alpha^2\left(\Delta u_t + u\cdot\nabla\Delta u + (\nabla u)^T\Delta u + \Delta u\,\mathrm{div}u\right) - \rho\nabla\rho.$$

To consider the desired limit, we denote $\bar{u} := u^\alpha - u$, $\bar{m} := m^\alpha - m = \bar{u} - \alpha^2\Delta\bar{u}$ and $\bar{\rho} := \rho^\alpha - \rho$. It then follows from the equations for (u, ρ) and (u^α, ρ^α) that

$$\begin{cases} \bar{m}_t + \bar{u}\cdot\nabla\bar{m} + \bar{u}\cdot\nabla m + u\cdot\nabla\bar{m} + (\nabla\bar{u})^T\bar{m} + (\nabla\bar{u})^T m + (\nabla u)^T\bar{m} + \bar{m}\,\mathrm{div}\bar{u} \\ \qquad\qquad\qquad\qquad\qquad +m\mathrm{div}\,\bar{u} + \bar{m}\,\mathrm{div}u \\ \qquad\qquad = \alpha^2\left(\Delta u_t + u\cdot\nabla\Delta u + (\nabla u)^T\Delta u + \Delta u\,\mathrm{div}u\right) - (\bar{\rho}\nabla\bar{\rho} + \nabla(\rho\bar{\rho})), \\ \bar{\rho}_t + \bar{u}\cdot\nabla\bar{\rho} + \bar{u}\cdot\nabla\rho + u\cdot\nabla\bar{\rho} + \bar{\rho}\nabla\cdot\bar{u} + \rho\nabla\cdot\bar{u} + \bar{\rho}\nabla\cdot u = 0. \end{cases} \tag{5.5}$$

Taking the $L^2(\mathbb{R}^N)$ inner product of $(5.5)_1$ and $(5.5)_2$ with $\bar{u}$ and $\bar{\rho}$, respectively, and then integrating by parts, we can find that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\bar{u}\|_{L^2}^2 + \alpha^2\|\nabla\bar{u}\|_{L^2}^2\right) \\ &\quad = -\int \bar{u}\cdot(u\cdot\nabla)\bar{m} - \int \bar{u}\cdot\bar{m}\,\mathrm{div}u - \int \bar{u}\cdot(\nabla u)^T\bar{m} \\ &\qquad + \alpha^2\left(\int \bar{u}\cdot\Delta u_t + \int \bar{u}\cdot(u\cdot\nabla)\Delta u + \int \bar{u}\cdot(\nabla u)^T\Delta u + \int \bar{u}\cdot\Delta u\,\mathrm{div}u\right) \\ &\qquad - \left(\int \bar{u}\cdot\bar{\rho}\nabla\bar{\rho} + \int \bar{u}\cdot\nabla(\rho\bar{\rho})\right) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\bar{\rho}\|_{L^2}^2 = &-\int \bar{\rho}\bar{u}\cdot\nabla\bar{\rho} - \int \bar{\rho}\bar{u}\cdot\nabla\rho - \int \bar{\rho}u\cdot\nabla\bar{\rho} \\ &\qquad - \int \bar{\rho}^2\nabla\cdot\bar{u} - \int \bar{\rho}\rho\nabla\cdot\bar{u} - \int \bar{\rho}^2\nabla\cdot u. \end{aligned}$$

Here we used the identities

$$\int \bar{u}\cdot(\bar{u}\cdot\nabla)\bar{m} + \int \bar{u}\cdot\bar{m}\,\mathrm{div}\bar{u} + \int \bar{u}\cdot(\nabla\bar{u})^T\bar{m} = 0,$$

and

$$\int \bar{u}\cdot(\bar{u}\cdot\nabla)m + \int \bar{u}\cdot m\,\mathrm{div}\bar{u} + \int \bar{u}\cdot(\nabla\bar{u})^T m = 0.$$

Combining the above two equalities and integrating by parts, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\bar{u}\|_{L^2}^2 + \|\bar{\rho}\|_{L^2}^2 + \alpha^2\|\nabla\bar{u}\|_{L^2}^2\right) \\ &= -\int \bar{u}\cdot(u\cdot\nabla)\bar{m} - \left(\int \bar{u}\cdot\bar{m}\,\mathrm{div}u + \int \bar{u}\cdot(\nabla u)^T\bar{m}\right) - \left(\int \bar{\rho}\bar{u}\cdot\nabla\rho + \frac{1}{2}\int \bar{\rho}^2\nabla\cdot u\right) \\ &\qquad + \alpha^2\left(\int \bar{u}\cdot\Delta u_t + \int \bar{u}\cdot(u\cdot)\nabla\Delta u + \int \bar{u}\cdot(\nabla u)^T\Delta u + \int \bar{u}\cdot\Delta u\,\mathrm{div}u\right) \\ &:= J_1 + J_2 + J_3 + J_4. \end{aligned}$$

The estimates for J_1 and J_2 are similar to that for I_1 and I_2, respectively. Indeed, by using $\bar{m} = \bar{u} - \alpha^2 \Delta \bar{u}$ and integrating by parts, we can deduce that

$$\begin{aligned}
J_1 &= \int \bar{m} \cdot (u \cdot \nabla)\bar{u} + \int (\bar{m} \cdot \bar{u})(\nabla \cdot u) \\
&= \frac{1}{2} \int (\nabla \cdot u)|\bar{u}|^2 + \alpha^2 \int (\nabla \cdot u)|\nabla \bar{u}|^2 + \alpha^2 \sum_{i,k=1}^{N} \int \partial_k \bar{u} \cdot (\partial_k u_i \partial_i \bar{u} + \bar{u} \partial_i \partial_k u_i) \\
&\leq \|\bar{u}\|_{L^2}^2 \|Du\|_{L^\infty} + \alpha^2 \|\nabla \bar{u}\|_{L^2}^2 \|Du\|_{L^\infty} + \alpha^2 \|\nabla \bar{u}\|_{L^2} \|\bar{u}\|_{L^2} \|D^2 u\|_{L^\infty} \\
&\leq \left(\|Du\|_{L^\infty} + \|D^2 u\|_{L^\infty}\right) \left(\|\bar{u}\|_{L^2}^2 + \alpha^2 \|\nabla \bar{u}\|_{L^2}^2\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
J_2 = &- \int \left(|\bar{u}|^2 \mathrm{div} u + \bar{u} \cdot (\nabla u)^T \bar{u}\right) \\
&+ \alpha^2 \sum_{k=1}^{N} \int \left(\partial_k \bar{u} \cdot (\partial_k \bar{u}\, \mathrm{div} u + \bar{u} \partial_k \mathrm{div} u) + \left(\partial_k \bar{u} \cdot (\nabla u)^T + \bar{u} \cdot (\nabla \partial_k u)^T\right) \partial_k \bar{u}\right) \\
\leq{}& C \left(\|Du\|_{L^\infty} + \|D^2 u\|_{L^\infty}\right) \left(\|\bar{u}\|_{L^2}^2 + \alpha^2 \|\nabla \bar{u}\|_{L^2}^2\right).
\end{aligned}$$

For J_3 and J_4, we have

$$J_3 \leq \left(\|Du\|_{L^\infty} + \|D\rho\|_{L^\infty}\right) \left(\|\bar{u}\|_{L^2}^2 + \|\bar{\rho}\|_{L^2}^2\right),$$

and

$$\begin{aligned}
J_4 &\leq \alpha^2 \|\bar{u}\|_{L^2} \left(\|\Delta u_t\|_{L^2} + \|u\|_{L^\infty} \|D^3 u\|_{L^2} + \|Du\|_{L^\infty} \|D^2 u\|_{L^2}\right) \\
&\leq \|\bar{u}\|_{L^2}^2 + \alpha^4 \left(\|\Delta u_t\|_{L^2} + \|u\|_{L^\infty} \|D^3 u\|_{L^2} + \|Du\|_{L^\infty} \|D^2 u\|_{L^2}\right)^2.
\end{aligned}$$

Summarizing the above estimates, we have

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left(\|\bar{u}\|_{L^2}^2 + \|\bar{\rho}\|_{L^2}^2 + \alpha^2 \|\nabla \bar{u}\|_{L^2}^2\right) \\
&\quad \leq C \left(1 + \|Du\|_{L^\infty} + \|D^2 u\|_{L^\infty} + \|D\rho\|_{L^\infty}\right) \left(\|\bar{u}\|_{L^2}^2 + \|\bar{\rho}\|_{L^2}^2 + \alpha^2 \|\nabla \bar{u}\|_{L^2}^2\right) \\
&\qquad + \alpha^4 C \left(\|\Delta u_t\|_{L^2} + \|u\|_{L^\infty} \|D^3 u\|_{L^2} + \|Du\|_{L^\infty} \|D^2 u\|_{L^2}\right)^2 \\
&\quad \leq C \left(1 + \|u\|_{H^k} + \|\rho\|_{H^{k-1}}\right) \left(\|\bar{u}\|_{L^2}^2 + \|\bar{\rho}\|_{L^2}^2 + \alpha^2 \|\nabla \bar{u}\|_{L^2}^2\right) \\
&\qquad + \alpha^4 C \left(1 + \|u\|_{C^1([0,T];H^2)} + \|u\|_{H^k}^3\right)^2.
\end{aligned}$$

Then it follows from Gronwall's inequality that

$$\|\bar{u}\|_{L^2}^2 + \|\bar{\rho}\|_{L^2}^2 + \alpha^2 \|\nabla \bar{u}\|_{L^2}^2 \leq C \left(\alpha^4 + \|\bar{u}_0\|_{L^2}^2 + \|\bar{\rho}_0\|_{L^2}^2 + \alpha^2 \|\nabla \bar{u}_0\|_{L^2}^2\right),$$

where C is a positive constant depending only on $\|u\|_{C([0,T],H^k)}$, $\|u\|_{C^1([0,T];H^2)}$ and $\|\rho\|_{C([0,T];H^{k-1})}$. This completes the proof of Theorem 1.4 (ii). □

6. Liouville type result for the stationary solutions

In this section, we prove a Liouville type result for the weak stationary solutions to equations (1.1) and (1.2). We first introduce the definition of the weak stationary solutions as follows.

Definition 6.1. *A stationary weak solution to equation (1.1) is a pair $(u, \rho) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ such that for $i = 1, 2, \cdots, N$, the identities*

$$\sum_{j=1}^{N} \int T_{ij}\partial_j\phi_i + \sum_{j,k=1}^{N} \int S_{ijk}\partial_j\partial_k\phi_i + \frac{1}{2}\int \rho^2\partial_i\phi_i = 0 \quad \text{and} \quad \int \rho u \cdot \nabla\phi = 0 \tag{6.1}$$

hold for any test functions $(\phi_1, \phi_2, \cdots, \phi_N)$ and $\phi \in C_0^\infty(\mathbb{R}^N)$, where

$$T_{ij} = u_iu_j + \frac{1}{2}\delta_{ij}|u|^2 + \alpha\nabla u_i \cdot \nabla u_j - \alpha\partial_i u \cdot \partial_j u + \frac{1}{2}\alpha\delta_{ij}|\nabla u|^2, \quad S_{ijk} = \alpha u_j\partial_k u_i.$$

Similarly, a stationary weak solution to equation (1.2) is a pair $(u, \rho) \in L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ such that for $i = 1, 2, \cdots, N$, the indentities

$$\sum_{j=1}^{N} \int \left(u_iu_j + \frac{1}{2}\delta_{ij}|u|^2\right)\partial_j\phi_i + \frac{1}{2}\int \rho^2\partial_i\phi_i = 0 \quad \text{and} \quad \int \rho u \cdot \nabla\phi = 0$$

hold for any test functions $(\phi_1, \phi_2, \cdots, \phi_N)$ and $\phi \in C_0^\infty(\mathbb{R}^N)$.

Theorem 6.1. *Suppose $(u,\ \rho)$ is a stationary weak solution to equation (1.1) or (1.2). Then $u = 0$ and $\rho = 0$.*

Remark 6.1. *Theorem 6.1 shows that a stationary weak solution is vacuum, $\rho = 0$, provided that the spatial infinity is vacuum, i.e., $\rho \to 0$ as $|x| \to \infty$.*

Proof of Theorem 6.1. The method is similar to [4, 5], where the case $\rho = 0$ is investigated. Indeed, to prove the conclusion, we do not use the equation of continuity $(6.1)_2$ and the term related to ρ in $(6.1)_1$ is harmful. Here, we give a sketch of the proof for completeness.

We first consider the case $\alpha > 0$. Take $\phi_i(x) := x_i\varphi_R(x) := x_i\varphi\left(\frac{x}{R}\right)$, where $\varphi(x) \in C_0^\infty(\mathbb{R}^N)$ is a radial bump function supported in the ball $|x| \leq 2$ and equal to one on the ball $|x| \leq 1$, that is,

$$\varphi(x) = 1 \text{ for } |x| \leq 1, \quad \varphi(x) = 0 \text{ for } |x| \geq 2, \quad \text{and} \quad 0 \leq \varphi(x) \leq 1 \text{ for } 1 < |x| < 2.$$

Then a direct computation yields

$$\begin{aligned}
&\sum_{i=1}^{N} \int T_{ii}\varphi_R(x) \\
&= -\sum_{i,j=1}^{N} \int T_{ij}x_j\partial_i\varphi_R(x) - \sum_{i,k=1}^{N} \int S_{iik}\partial_k\varphi_R(x) - \sum_{i,j=1}^{N} \int S_{iji}\partial_j\varphi_R(x) \\
&\quad - \sum_{i,j,k=1}^{N} \int S_{iik}x_i\partial_j\partial_k\varphi_R(x) - \frac{N}{2}\int \rho^2\varphi_R(x) - \frac{1}{2}\sum_{i=1}^{N} \int \rho^2 x_i\partial_i\varphi_R(x).
\end{aligned}$$

It follows from the integration by parts that

$$\begin{aligned}
&\frac{1}{2}\int \left((N+2)|u|^2+N\alpha|\nabla u|^2+N\rho^2\right)\varphi_R(x)\\
&=-\sum_{i,j=1}^{N}\int T_{ij}x_j\partial_i\varphi_R(x)-\sum_{i,k=1}^{N}\int \left(S_{iik}+S_{iki}\right)\partial_k\varphi_R(x)\\
&\quad-\sum_{i,j,k=1}^{N}\int S_{iik}x_i\partial_j\partial_k\varphi_R(x)-\frac{1}{2}\sum_{i=1}^{N}\int \rho^2x_i\partial_i\varphi_R(x)\\
&:=J_1+J_2+J_3+J_4.
\end{aligned}$$

We estimate J_1, J_2, J_3 and J_4 term by term. For J_1, we have

$$|J_1|\le \frac{1}{R}\int\int_{R\le|x|\le 2R}|T||x||\nabla\varphi|\le 2\|\nabla\varphi\|_{L^\infty}\int_{R\le|x|\le 2R}|T|\to 0,$$

as $R\to\infty$, where $T=(T_{ij})$. Similarly, we denote by $S=(S_{ijk})$ and deduce that

$$|J_2|\le \frac{2}{R}\int |S||\nabla\varphi|\le \frac{1}{R}\|\nabla\varphi\|_{L^\infty}\int |S|\to 0,$$

and

$$|J_3|\le \frac{1}{R^2}\int |S||x||\nabla^2\varphi|\le \frac{1}{R}\|\nabla^2\varphi\|_{L^\infty}\int |S|\to 0,$$

as $R\to\infty$. For J_4, we have

$$|J_4|\le \frac{1}{R}\int\int_{R\le|x|\le 2R}\rho^2|x||\nabla\varphi|\le 2\|\nabla\varphi\|_{L^\infty}\int_{R\le|x|\le 2R}\rho^2\to 0$$

as $R\to\infty$. Summarily, we obtain

$$\lim_{R\to\infty}\frac{1}{2}\int \left((N+2)|u|^2+N\alpha|\nabla u|^2+N\rho^2\right)\varphi_R(x)=0.$$

That is,

$$\int \left((N+2)|u|^2+N\alpha|\nabla u|^2+N\rho^2\right)=0,$$

which implies that $u=0$ and $\rho=0$.

If we drop all the terms involving α in the previous proof, the conclusion is still true. This completes the proof of Theorem 6.1. □

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