STABILITY OF THE NONRELATIVISTIC VLASOV-MAXWELL-BOLTZMANN SYSTEM FOR ANGULAR NON-CUTOFF POTENTIALS

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Abstract. Although there recently have been extensive studies on the perturbation theory of the angular non-cutoff Boltzmann equation (cf. [4] and [17]), it remains mathematically unknown when there is a self-consistent Lorentz force coupled with the Maxwell equations in the nonrelativistic approximation. In the paper, for perturbative initial data with suitable regularity and integrability, we establish the large time stability of solutions to the Cauchy problem on the Vlasov-Maxwell-Boltzmann system with physical angular non-cutoff intermolecular collisions including the inverse power law potentials, and also obtain as a byproduct the convergence rates of solutions. The proof is based on a refined time-velocity weighted energy method with two key technical parts: one is to introduce the exponentially weighted estimates into the non-cutoff Boltzmann operator and the other to design a delicate temporal energy $X(t)$-norm to obtain its uniform bound. The result also extends the case of the hard sphere model considered by Guo (Invent. Math. 153(3): 593–630 (2003)) to the general collision potentials.

Contents

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1. Introduction

1.1. **The Cauchy problem.** The motion of ionized nonrelativistic plasmas consisting of two species particles (e.g., electrons and ions) with collisional effects is governed by the Boltzmann equations

$$\begin{aligned}&\partial_t F_+ + \xi\cdot\nabla_x F_+ + (E+\xi\times B)\cdot\nabla_\xi F_+ = Q(F_+,F_+) + Q(F_+,F_-),\\&\partial_t F_- + \xi\cdot\nabla_x F_- - (E+\xi\times B)\cdot\nabla_\xi F_- = Q(F_-,F_+) + Q(F_-,F_-).\end{aligned} \tag{1.1}$$ V

The self-consistent electromagnetic field satisfies the Maxwell equations

$$\begin{aligned}&\partial_t E - \nabla_x\times B = -\int_{\mathbb{R}^3}\xi(F_+-F_-)\,d\xi,\\&\partial_t B + \nabla_x\times E = 0,\\&\nabla_x\cdot E = \int_{\mathbb{R}^3}(F_+-F_-)\,d\xi,\quad \nabla_x\cdot B = 0.\end{aligned} \tag{1.2}$$ M

Here $F_\pm = F_\pm(t,x,\xi)\geq 0$ stands for the number densities of ions $(+)$ and electrons $(-)$ which have position $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ and velocity $\xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3$ at time $t\geq 0$, and $E(t,x)$, $B(t,x)$ denote the electro and magnetic fields, respectively. The initial data of the coupled system above is given by

$$F_\pm(0,x,\xi) = F_{0,\pm}(x,\xi),\quad E(0,x)=E_0(x),\quad B(0,x)=B_0(x) \tag{1.3}$$ VM.id

satisfying the compatibility conditions

$$\nabla_x\cdot E_0 = \int_{\mathbb{R}^3}(F_{0,+}-F_{0,-})\,d\xi,\quad \nabla_x\cdot B_0 = 0.$$

The Boltzmann collision operator $Q(\cdot,\cdot)$ in (1.1) takes the form of

$$Q(F,G) = \int_{\mathbb{R}^3\times\mathbb{S}^2} q(\xi-\xi_*,\sigma)\,[F(\xi')G(\xi'_*) - F(\xi)G(\xi_*)]\;d\xi_* d\sigma,$$

where in terms of velocity pair (ξ,ξ_*) before collisions and velocity pair (ξ',ξ'_*) after collisions is defined by

$$\xi' = \frac{\xi+\xi_*}{2} + \frac{|\xi-\xi_*|}{2}\sigma,\quad \xi'_* = \frac{\xi+\xi_*}{2} - \frac{|\xi-\xi_*|}{2}\sigma.$$

The Boltzmann collision kernel $q(\xi-\xi_*,\sigma)\geq 0$ depends only on the relative velocity $|\xi-\xi_*|$ and on the deviation angle θ given by $\cos\theta = \langle\sigma,(\xi-\xi_*)/|\xi-\xi_*|\rangle$. As in [4, 17], without loss of generality, one can suppose that $q(\xi-\xi_*,\sigma)$ is supported on $\cos\theta\geq 0$. Throughout the paper, the Boltzmann collision kernel is further assumed to satisfy the following assumption: $q(\xi-\xi_*,\sigma)$ takes the product form

$$q(\xi-\xi_*,\sigma) = C_q|\xi-\xi_*|^\gamma \mathbb{b}(\cos\theta)$$

for a constant $C_q>0$, where in the kinetic part $|\xi-\xi_*|^\gamma$, the exponent $\gamma>-3$ is determined by the intermolecular interactive mechanism, and in the angular part, there are $C_{\mathbb{b}}>0$, $0<s<1$ such that

$$\frac{1}{C_{\mathbb{b}}\theta^{1+2s}} \leq \sin\theta\,\mathbb{b}(\cos\theta) \leq \frac{C_{\mathbb{b}}}{\theta^{1+2s}}$$

holds for all θ in $(0,\frac{\pi}{2}]$. Here, it is convenient to call soft potentials when $-3<\gamma<-2s$, and hard potentials when $\gamma+2s\geq 0$. The current work is restricted to the case of

$$-3<\gamma<-2s,\quad 1/2\leq s<1. \tag{1.4}$$ ass.main

Notice that all the physical parameters, such as the particle masses, the light speed, and all other involving constants, have been chosen to be unit for simplicity of presentation and also without loss of generality. In addition, for the physical background of the system mentioned above, interested readers may refer to the textbook [23, Chapter 6].

Our goal of the paper is to study the large time asymptotic stability for the classical solutions to the Cauchy problem (1.1), (1.2), (1.3) of the Vlasov-Maxwell-Boltzmann system under the main assumption (1.4), provided that initial data is sufficiently close to equilibrium states with $F_\pm$ being the same global Maxwellian and (E, B) vanishing. See also a recent mathematical work [21].

Remark 1.1. *Recall that when the intermolecular interactive potential takes the inverse power law in the form of* $U(|x|) = |x|^{-(\ell-1)}$ *with* $2 < \ell < \infty$*, the Boltzmann collision kernel* $q(\xi - \xi_*, \sigma)$ *in three space dimensions satisfies the aforementioned assumptions with* $\gamma = \frac{\ell-5}{\ell-1}$ *and* $s = \frac{1}{\ell-1}$*, and our restriction of the paper corresponds to the condition* $2 < \ell < 3$ *in terms of the parameter* ℓ*. Note* $\gamma \to -3$ *and* $s \to 1$ *as* $\ell \to 2$ *in the limiting case, for which the grazing collisions between particles are dominated and the Boltzmann collision term has to be replaced by the classical Landau collision term for the Coulomb potential, cf.* [5, 24, 36]*; refer also to the recent works* [12, 20, 34, 37] *for discussions on the corresponding models.*

In what follows, we mention some works related to this paper. In the perturbative context, there have been extensive investigations on the Boltzmann and related equations, see the first result by Ukai [35] and also [2, 3, 4, 6, 17, 19, 26, 27, 28, 32] and reference therein. For the Vlasov-Maxwell-Boltzmann system, the global existence of solutions to the periodic initial boundary value problem near the global Maxwellian equilibrium states was firstly investigated by Guo [21] for the hard sphere model. Then, the rate of convergence to Maxwellians with any polynomial speed in large time was shown by Guo-Strain [31, 33] for the Vlasov-Maxwell-Boltzmann system on the periodic box in both the classical and relativistic situations. For the Cauchy problem in the whole space, the global in time classical solutions were constructed by Strain [29]. And recently, the large-time behavior of classical solutions to the Vlasov-Maxwell-Boltzmann system in the whole space was studied by Duan-Strain [14]. We would point out that all the works concerning the Vlasov-Maxwell-Boltzmann system mentioned above are focused on the cutoff collision kernels and the hard sphere model.

For the Vlasov-Maxwell-Boltzmann system without angular cutoff, it still remains a major open problem to construct global classical solutions near equilibrium. Very recently, Guo [20] made further progress in proving the global existence of classical solutions to the Vlasov-Poisson-Landau system in a periodic box for the most important Coulomb potential. One of the key points in the proof there is to design a new velocity weight depending on the order of space and velocity derivatives so as to capture the anisotropic dissipation property of the linearized Landau operator. Due to the recent study of the non cutoff Boltzmann equation independently by AMUXY [2, 3, 4] and Gressman-Strain [17, 18], it is now well known that the linearized Boltzmann operator without angular cutoff has the similar anisotropic dissipation phenomenon compared to the Landau, see [9, 19]. Therefore, as pointed in [20], it is also interesting to see whether or not the approach in [20] can be applied to the non cutoff Vlasov-Maxwell-Boltzmann system for the non hard-sphere model; see also [9, 34] and [25] for three recent applications.

On the other hand, basing on the time weighted energy method, another approach was developed recently in [10, 11, 12] and [8] to deal with the Vlasov-Poisson-Boltzmann, Vlasov-Poisson-Landau and Vlasov-Maxwell-Landau systems for general collision potentials. The main difference with [20] is to introduce another kind of time-velocity dependent weight function which can induce the extra dissipation mechanism to compensate the weaker dissipation of the linearized collision operator in the case of non hard-sphere models, particularly physically interesting soft potentials.

1.2. **Reformulation, weight and norm.** We now turn to the presentation of our main result. For that, write the normalized global Maxwellian, without loss of generality, as

$$\mu = \mu(\xi) = (2\pi)^{-3/2} e^{-|\xi|^2/2},$$

and set the perturbation in the standard way

$$F_\pm(t,x,\xi) = \mu + \mu^{1/2} f_\pm(t,x,\xi).$$

Use $[\cdot,\cdot]$ to denote the column vector in $\mathbb{R}^2$. Set $F=[F_+,F_-]$ and $f=[f_+,f_-]$. Then the Cauchy problem (1.1), (1.2), (1.3) can be reformulated as

$$\begin{cases} \partial_t f + \xi\cdot\nabla_x f + q_0(E+\xi\times B)\cdot\nabla_\xi f - E\cdot\xi\mu^{1/2}q_1 + Lf \\ \qquad\qquad = \dfrac{q_0}{2}E\cdot\xi u + \Gamma(f,f), \\ \partial_t E - \nabla_x\times B = -\displaystyle\int_{\mathbb{R}^3} \xi\mu^{1/2}(f_+-f_-)\,d\xi, \\ \partial_t B + \nabla_x\times E = 0, \\ \nabla_x\cdot E = \displaystyle\int_{\mathbb{R}^3}\mu^{1/2}(f_+-f_-)\,d\xi, \quad \nabla_x\cdot B = 0 \end{cases} \tag{1.5}$$

with initial data

$$f_\pm(0,x,\xi) = f_{0,\pm}(x,\xi), \quad E(0,x) = E_0(x), \quad B(0,x)=B_0(x) \tag{1.6}$$

satisfying the compatibility condition

$$\nabla_x\cdot E_0 = \int_{\mathbb{R}^3}\mu^{1/2}(f_{0,+}-f_{0,-})\,d\xi, \quad \nabla_x\cdot B_0 = 0. \tag{1.7}$$

Here, $q_0=\mathrm{diag}(1,-1)$, $q_1=[1,-1]$, and the linearized collision term Lf and the nonlinear collision term $\Gamma(f,f)$ are respectively defined by

$$Lf = [L_+f, L_-f], \quad \Gamma(f,g) = [\Gamma_+(f,g),\Gamma_-(f,g)],$$

with

$$L_\pm f = -2\mu^{-1/2}Q\left(\mu^{1/2}f_\pm,\mu\right) - \mu^{-1/2}Q\left(\mu,\mu^{1/2}\{f_\pm+f_\mp\}\right),$$

$$\Gamma_\pm(f,g) = \mu^{-1/2}Q\left(\mu^{1/2}f_\pm,\mu^{1/2}g_\pm\right) + \mu^{-1/2}Q\left(\mu^{1/2}f_\pm,\mu^{1/2}g_\mp\right).$$

As in [21], the null space of the linearized operator L is given by

$$\mathcal{N} = \mathrm{span}\left\{[1,0]\mu^{1/2},\ [0,1]\mu^{1/2},\ [\xi_i,\xi_i]\mu^{1/2}\ (1\le i\le 3),\ [|\xi|^2,|\xi|^2]\mu^{1/2}\right\}.$$

Let $\mathbf{P}$ be the orthogonal projection from $L^2_\xi\times L^2_\xi$ to $\mathcal{N}$. Given $f(t,x,\xi)$, one can write $\mathbf{P}$ as $\mathbf{P}f=[\mathbf{P}_+f,\mathbf{P}_-f]$ with

$$\mathbf{P}_\pm f = \left\{a_\pm(t,x) + b(t,x)\cdot\xi + c(t,x)(|\xi|^2-3)\right\}\mu^{1/2},$$

where the coefficient functions are determined by f in the way that

$$
\begin{aligned}
a_\pm &= \left\langle \mu^{1/2}, f_\pm \right\rangle = \left\langle \mu^{1/2}, \mathbf{P}_\pm f \right\rangle, \\
b_i &= \frac{1}{2}\left\langle \xi_i \mu^{1/2}, f_+ + f_- \right\rangle = \left\langle \xi_i \mu^{1/2}, \mathbf{P}_\pm f \right\rangle, \\
c &= \frac{1}{12}\left\langle (|\xi|^2-3)\mu^{1/2}, f_+ + f_- \right\rangle = \frac{1}{6}\left\langle (|\xi|^2-3)\mu^{1/2}, \mathbf{P}_\pm f \right\rangle.
\end{aligned}
$$

In what follows, we introduce the weight functions and norms used throughout the paper. First of all, define

$$
w_{\tau,\lambda} = w_{\tau,\lambda}(t,\xi) = \langle \xi \rangle^{\gamma\tau} \exp\left\{ \frac{\lambda}{(1+t)^\vartheta} \langle \xi \rangle \right\},\ 0 < \vartheta \le \frac{1}{4}, \tag{1.8}
$$ wgt

where constants $\tau \in \mathbb{R}$ and $\lambda \ge 0$ are two parameters which may vary in different places and $\langle \xi \rangle = (1+|\xi|^2)^{1/2}$. Note that the dependence of $w_{\tau,\lambda}$ on parameters γ and ϑ has been neglected without any confusion. For simplicity, we write $w_\tau = w_{\tau,0}$ when $\lambda = 0$. For any function $f(t,x,\xi)$, define

$$
|f(x)|^2_{\tau,\lambda} = \int_{\mathbb{R}^3} w^2_{\tau,\lambda}(t,\xi)|f|^2\, d\xi, \quad \|f\|^2_{\tau,\lambda} = \int_{\mathbb{R}^3} |f(x)|^2_{\tau,\lambda}\, dx.
$$

As in [2], introduce

$$
\begin{aligned}
|f|^2_{\mathbf{D}} &= \int_{\mathbb{R}^6\times\mathbb{S}^2} q(\xi-\xi_*,\sigma)\mu_*(f'-f)^2 d\xi_* d\xi d\sigma \\
&\quad + \int_{\mathbb{R}^6\times\mathbb{S}^2} q(\xi-\xi_*,\sigma) f_*^2 \left(\sqrt{\mu_*'} - \sqrt{\mu_*}\right)^2 d\xi_* d\xi d\sigma,
\end{aligned} \tag{1.9}
$$ def.DB

and define

$$
|f|^2_{\mathbf{D},\tau,\lambda} = |w_\tau f|^2_{\mathbf{D}}, \quad \|f\|^2_{\mathbf{D},\tau,\lambda} = \int_{\mathbb{R}^3} |f|^2_{\mathbf{D},\tau,\lambda}\, dx.
$$

For simplicity, we also use the notation

$$
|f(\xi)|^2_{L^2_\ell} = \left|\langle \xi \rangle^\ell f(\xi)\right|^2_{L^2} \quad \text{and} \quad \|f(\xi)\|^2_{L^2_\ell} = \int_{\mathbb{R}^3} |f(\xi)|^2_{L^2_\ell} dx,
$$

where $\ell \in \mathbb{R}$. And the weighted fractional Sobolev norm $|f(\xi)|^2_{H^s_\ell} = \left|\langle \xi \rangle^\ell f(\xi)\right|^2_{H^s}$ is given by

$$
|f|^2_{H^s_\ell} = |f|^2_{L^2_\ell} + \int_{\mathbb{R}^3} d\xi \int_{\mathbb{R}^3} d\xi' \, \frac{\left[\langle \xi \rangle^\ell f(\xi) - \langle \xi' \rangle^\ell f(\xi')\right]^2}{|\xi-\xi'|^{3+2s}} \chi_{|\xi-\xi'|\le 1},
$$

which turns out to be equivalent with

$$
|f|^2_{H^s_\ell} = \int_{\mathbb{R}^3} d\xi \left|(1-\Delta_\xi)^{\frac{s}{2}} \left(w^\ell(\xi) f(\xi)\right)\right|^2.
$$

Moreover $\|f\|_{H^s_\ell}$ is given by $\|f\|^2_{H^s_\ell} = \int_{\mathbb{R}^3} |f|^2_{H^s_\ell} dx$. We also use $\|\cdot\|_{H^N}$ to denote the standard Sobolev norm in $\mathbb{R}^3$ with respect to the variables x.

1.3. **Main result.** Let $\mathbf{I}=[\mathbf{I}_+,\mathbf{I}_-]$ with $\mathbf{I}_\pm f=f_\pm$. To study the global existence by means of the energy method, inspired by [8], the temporal energy functionals and the corresponding dissipation rate functionals are defined by

$$\begin{aligned}\mathcal{E}_{N,\ell,\lambda}(t)\sim&\sum_{|\alpha|\leq N}\|\partial^\alpha(a_\pm,b,c)\|+\sum_{|\alpha|+|\beta|\leq N}\left\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f(t)\right\|^2_{|\alpha|+|\beta|-\ell,\lambda}\\&+\|(E,B)\|^2_{H^N},\end{aligned}\tag{1.10}$$

def.e

$$\overline{\mathcal{E}}_N(t)\sim\sum_{|\alpha|\leq N}\|\partial^\alpha f(t)\|^2+\|(E,B)\|^2_{H^N},\tag{1.11}$$

def.e0

and

$$\begin{aligned}\mathcal{D}_{N,\ell,\lambda}(t)=&\sum_{|\alpha|+|\beta|\leq N}\left\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f(t)\right\|^2_{\mathbf{D},|\alpha|+|\beta|-\ell,\lambda}\\&+\sum_{|\alpha|\leq N-1}\|\nabla_x\partial^\alpha(a_\pm,b,c)\|^2\\&+\|a_+-a_-\|^2+\|E\|^2_{H^{N-1}}+\|\nabla_xB\|^2_{H^{N-2}}\\&+\frac{\lambda}{(1+t)^{1+\vartheta}}\sum_{|\alpha|+|\beta|\leq N}\left\|\langle\xi\rangle^{1/2}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f(t)\right\|^2_{|\alpha|+|\beta|-\ell,\lambda},\end{aligned}\tag{1.12}$$

def.dr

$$\begin{aligned}\overline{\mathcal{D}}_N(t)=&\sum_{|\alpha|\leq N}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f(t)\|^2_{\mathbf{D}}+\sum_{|\alpha|\leq N-1}\|\nabla_x\partial^\alpha(a_\pm,b,c)\|^2\\&+\|a_+-a_-\|^2+\|E\|^2_{H^{N-1}}+\|\nabla_xB\|^2_{H^{N-2}},\end{aligned}\tag{1.13}$$

def.dr0

where the integer $N\geq 0$ and $\ell\geq 0$ are parameters which may differ in different places and also satisfy $\ell-N\geq 0$. Note that $\overline{\mathcal{E}}_N(t)$ and $\overline{\mathcal{D}}_N(t)$ contain no velocity differentiation and no extra velocity weight, and also that the last term of $\mathcal{D}_{N,\ell,\lambda}(t)$ on the right-hand side of (1.12) disappears when $\lambda=0$.

Let constants $N_1\geq 14$, $\ell_1\geq 1+N_1$, $\lambda_0>0$, $0<\vartheta\leq\frac14$ and $\epsilon_0>0$ be given; the exact choice of N_1, ℓ_1 λ_0, ϑ and ϵ_0 can be seen in the later proof. In terms of those given constants, the temporal energy norm $X(t)$ is defined by

$$\begin{aligned}X(t)=&\sup_{0\leq s\leq t}\left\{\overline{\mathcal{E}}_{N_1}(s)+(1+s)^{\frac32}\overline{\mathcal{E}}_{N_1-2}(s)\right\}\\&+\sup_{0\leq s\leq t}\Big\{(1+s)^{-\frac{1+\epsilon_0}{2}}\mathcal{E}_{N_1,\ell_1,\lambda_0}(s)+\mathcal{E}_{N_1-1,\ell_1,\lambda_0}(s)\\&\qquad+(1+s)^{\frac32}\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(s)\Big\}\\&+\sup_{0\leq s\leq t}\left\{(1+s)^{2(1+\vartheta)}\|\nabla_x(E,B)(s)\|^2_{H^5}\right\}.\end{aligned}\tag{1.14}$$

def.X

The main result of the paper is stated as follows.

⟨thm.gl⟩ **Theorem 1.1.** *Assume* $\max\left\{-3,-\frac32-2s\right\}<\gamma<-2s$, $\frac12\leq s<1$ *and* $\vartheta=\frac14$. *Take* $N_1\geq 14$, $\ell_1\geq 1+N_1$, $\ell_2>\frac{15(\gamma+2s)}{4\gamma}$, $\lambda_0>0$ *and take also* $\epsilon_0>0$ *small enough. Let* $f_0=[f_{0,+},f_{0,-}]$ *satisfy* $F_\pm(0,x,\xi)=\mu(\xi)+\mu^{1/2}(\xi)f_{0,\pm}(x,\xi)\geq 0$. *If*

$$Y_0=\sum_{|\alpha|+|\beta|\leq N_1}\left\|\partial^\alpha_\beta f_0\right\|_{|\alpha|+|\beta|-\ell_1,\lambda_0}+\|(E_0,B_0)\|_{H^{N_1}\cap L^1}+\|w_{-\ell_2}f_0\|_{Z_1}\tag{1.15}$$

def.Y0

is sufficiently small, then there exist properly defined energy functionals $\mathcal{E}_{N,\ell,\lambda}(t)$ *and* $\overline{\mathcal{E}}_N(t)$ *in the definition* (1.14) *of* $X(t)$*-norm such that the Cauchy problem* (1.5), (1.6), (1.7) *of the Vlasov-Maxwell-Boltzmann system admits a unique global solution* $\big(f(t,x,\xi),E(t,x),B(t,x)\big)$ *satisfying* $F_\pm(t,x,\xi)=\mu(\xi)+\mu^{1/2}(\xi)f_\pm(t,x,\xi)\geq 0$ *and*

$$X(t)\lesssim Y_0^2 \tag{1.16}$$

thm.gl.1

for all time $t\geq 0$.

Now we give several remarks concerning this theorem.

Remark 1.2. *Similar results hold in the case of hard potentials, i.e.* $\gamma+2s\geq 0$, *for which the proof could be much simpler. In addition, in the case of soft potentials* $-3<\gamma<-2s$ *under consideration, the additional restriction* $\gamma>\frac{3}{2}-2s$ *is due to the technique of proofs used in the paper, see also Remark 2.3. But it is still possible to get rid of it by using the different approach as in* [9].

Remark 1.3. *The choice of parameters* $s\geq 1/2$, $l_1\geq N_1+1$ *and* $\vartheta=1/4$ *is critical in our proof of Theorem 1.1. We believe, however, that the regularity index* $N_1\geq 14$ *imposed in Theorem 1.1 is not optimal. Since our main concern in this paper is to show the global solvability of the Vlasov-Maxwell-Boltzmann system near Maxellians through the perturbation method, the problem on determining the critical value of* N_1 *is beyond the scope of this manuscript.*

Remark 1.4. *Note that when* $s\to 1$, *the basis* $\langle\xi\rangle^\gamma$ *in the algebraic part of the velocity weight* (1.8) *introduced in this paper does not coincide with that of* [20] *and* [8], *where the basis* $\langle\xi\rangle^{\gamma+2s}$ *was essentially used. The choice of* $\langle\xi\rangle^\gamma$ *is due to the fractional diffusive property of the Boltzmann operator. It is known that the linearized Landau operator has the anisotropic dissipation property which includes the integer* $\xi-$*derivatives, so that one can distribute different weight according to different derivatives. However, in the case of the non-cutoff Boltzmann operator, the fractional derivative can not be split in a direct way. Instead, our trick is to apply the Fourier transform to overcome this difficulty, and as a compensation, the basis function* $\langle\xi\rangle^\gamma$ *naturally comes out, see* (3.20) *and also cf.* [9].

The proof of Theorem 1.1 is based on a refined energy method with the velocity weight $w_{\ell,\lambda}(t,\xi)$ containing the time-velocity-dependent exponential factor. The main difficulty in the proof is to control two types of nonlinearities: one is induced by the coupling term $(E+\xi\times B)\cdot\nabla_\xi F$ due to interactions between the self-consistent Lorentz force and gas particles in the nonrelativistic framework, and the other by the nonlinear Boltzmann collision operator for non-cutoff soft potentials. To overcome such difficulty, compared with the previous works [10, 11, 12] and [8], one of our main contributions of the paper is to introduce the exponential weight estimate into the angular non-cutoff Boltzmann operator, which as far as we know does not appear in any existing literature, and the other one is that it is much harder to apply the strategy of [8] through designing the $X(t)$-norm to obtain its closed global-in-time bound.

Let's expose the technical parts of the proof in more details. Unlike the case of the cutoff hard sphere model studied in [31, 33, 29, 14], the dissipation of the linearized Boltzmann collision operator for the physically interesting soft potentials is weaker in the sense that it is degenerate in the large-velocity domain. However, the introduction of the time-velocity-dependent weight $w_{\tau,\lambda}(t,\xi)$ to the energy norm

$\mathcal{E}_{N,\ell,\lambda}(t)$ can generate the extra dissipation corresponding to the last term in the energy dissipation rate functional $\mathcal{D}_{N,\ell,\lambda}(t)$ defined by (1.12), so that the weaker dissipation of the linearized collision operator is possibly compensated. For that, it is necessary to deduce the exponentially weighted estimates on the first equation of system (1.5). Therefore, the velocity-growth effect arising from the coupling term $(E+\xi\times B)\cdot\nabla_\xi F$ can be controlled through the extra dissipation balanced by the time-decay of the electromagnetic field.

Moreover, since system (1.5) contains the linearized collision operator L and the nonlinear collision operator Γ for non-cutoff soft potentials, we are also forced to make the exponentially weighted estimates on L and Γ. Then, new difficulties could occur due to the velocity differentiation of the exponential weight. In fact, the algebraic velocity weight enjoys a good property that its derivative decays in large velocity strictly faster than itself, which was essentially used in the weighted estimates for the study of the pure angular non-cutoff Boltzmann equation as in [4] and [17]. This property, however, fails for the exponential weight. On the other hand, notice that when comparing values of the weight $w_{\tau,\ell}(t,\xi)$ at two different points ξ' and ξ, one has to use the Taylor expansion like

$$\begin{aligned} w_{\ell,\lambda}(\xi')-w_{\ell,\lambda}(\xi)=(\xi'-\xi)\cdot\nabla_\xi w_{\ell,\lambda}(\xi)\\ +\frac{1}{2!}(\xi'-\xi)\otimes(\xi'-\xi):\nabla_\xi^2 w_{\ell,\lambda}(t,\xi)+\cdots. \end{aligned}$$

Then, it could be a problem to control the above derivatives of $w_{\tau,\ell}(t,\xi)$ because of the slower velocity-decay of the exponential factor compared to the usual algebraic factor. Fortunantely, depending on the order of velocity differentiation, those slower velocity-decay terms simultaneously contain the time-decay factor $(1+t)^{-\vartheta}$. Therefore, it could be still possible to control the high-order expansion terms by using the extra dissipation in the energy dissipation rate functional $\mathcal{D}_{N,\ell,\lambda}(t)$. Specifically, although some terms such as $|w_{\ell,\lambda_0}\partial^{\alpha_1}_{\beta_1}f|_{L^2_{\gamma/2+1}}$ and $|w_{\ell,\lambda_0}\partial^{\alpha}_{\beta}h|_{L^2_{\gamma/2+1}}$ on the right-hand side of (2.15) in Lemma 2.4 can not be directly controlled by the dissipation of L, i.e. the first five terms in $\mathcal{D}_{N,\ell,\lambda}(t)$, they can indeed be well controlled by the extra dissipation induced by the time-velocity-dependent exponential factor in the weight $w_{\ell,\lambda}(t,\xi)$ due to the main assumption (1.4).

Another trouble comes from the way to close the weighted energy estimates which is due to the regularity-loss property of the Vlasov-Maxwell-Boltzmann system. In fact, the weighted high-order energy functional $\mathcal{E}_{N_1,\ell_1,\lambda_0}(t)$ can not be bounded uniformly in time which can actually be seen from the proof of Lemma 3.5 for the linearized analysis, cf. [8]. On the other hand, for the weighted estimate on derivatives of the highest order N_1 for the linear term $E\cdot\xi\mu^{1/2}$, we can not hope again to deduce the time-decay of the derivatives of the electromagnetic field of the highest order. Moreover, although the velocity-growth in the nonlinear term containing the electromagnetic field could be treated through the time-dependent exponential factor in the weight function $w_{\ell,\lambda_0}(t,\xi)$, it is impossible when the electromagnetic field gains the differentiation of higher orders since they again could not decay in time. To overcome such a difficulty, a temporal energy norm $X(t)$, cf. (1.14), is carefully designed to refine the nonlinear estimates in order to use the time-decay property of the lower-order energy functional $\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)$ and time-growth property of the weighted high-order energy functional $\mathcal{E}_{N_1,\ell_1,\lambda_0}(t)$ so that the desired weighted energy estimates can indeed be closed. We would point out that the computations here are much harder than those in [8] for the Landau case.

The rest of the paper is arranged as follows. In Section 2, we carry out the weighted estimates on L and Γ, which not only is the key technical part of the paper but also has its own interest. In Section 3, we shall prove series of lemmas to obtain the closed estimate on $X(t)$-norm so as to conclude the proof of Theorem 1.1 based on the continuity argument.

Notations. Throughout this paper, C denotes some generic positive (generally large) constant and κ denotes some generic positive (generally small) constant, where both C and κ may take different values in different places. $A \lesssim B$ means that there is a generic constant $C > 0$ such that $A \leqslant CB$. $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. We use L^2 to denote the usual Hilbert spaces $L^2 = L^2_{x,\xi}$ or L^2_x with the norm $\|\cdot\|$, and use $\langle \cdot, \cdot \rangle$ to denote the inner product over $L^2_{x,\xi}$ or L^2_ξ. For $q \geq 1$, the mixed velocity-space Lebesgue space $Z_q = L^2_\xi(L^q_x) = L^2(\mathbb{R}^3_\xi; L^q(\mathbb{R}^3_x))$ is used. For multi-indices $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$, $\partial^\alpha_\beta = \partial^\alpha_x \partial^\beta_\xi = \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}\partial^{\beta_1}_{\xi_1}\partial^{\beta_2}_{\xi_2}\partial^{\beta_3}_{\xi_3}$. The length of α is $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and similar for $|\beta|$.

2. Weighted estimates on Γ and L

⟨sec2⟩ This section is devoted to deducing the key estimates on Γ and L with respect to the weight $w_{\ell,\lambda}(t,\xi)$, which is one of the main techniques used for the proof of the global stability of the Vlasov-Maxwell-Boltzmann system for the angular non-cutoff soft potentials. For that, we divide our discussions into two parts. The first part concerns the weighted estimates on Γ.

2.1. Weighted estimates on Γ.

For scalar functions g_1, g_2 and h, we use the following notations

$$\mathscr{T}(g_1, g_2) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} q(\xi - \xi_*, \sigma)\sqrt{\mu_*}\left[(g_1)'(g_2)'_* - g_1(g_2)_*\right] d\xi_* d\sigma,$$

and

$$\begin{aligned}\mathscr{L}h &= -\left\{\mathscr{T}(h, \sqrt{\mu}) + \mathscr{T}(\sqrt{\mu}, h)\right\}\\ &= -\int_{\mathbb{R}^3 \times \mathbb{S}^2} q(\xi - \xi_*, \sigma)\sqrt{\mu_*}\left[h'\sqrt{\mu'_*} - h\sqrt{\mu_*}\right] d\xi_* d\sigma\\ &\quad - \int_{\mathbb{R}^3 \times \mathbb{S}^2} q(\xi - \xi_*, \sigma)\sqrt{\mu_*}\left[\sqrt{\mu'}h'_* - \sqrt{\mu}h_*\right] d\xi_* d\sigma = \mathscr{L}_1 h + \mathscr{L}_2 h.\end{aligned}$$

With the above notations, it is straightforward to see

$$\Gamma_\pm(f, g) = \mathscr{T}(f_\pm, g_\pm) + \mathscr{T}(f_\pm, g_\mp), \quad L_\pm f = 2\mathscr{L}_1 f_\pm + \mathscr{L}_2(f_\pm + f_\mp).$$

Recalling (1.9), let us write

$$\begin{aligned}|g|^2_{\mathbf{D}} &= J_1 + J_2,\\ J_1 &= \int_{\mathbb{R}^6 \times \mathbb{S}^2} q(\xi - \xi_*, \sigma)\mu_*(g' - g)^2 d\xi_* d\xi d\sigma,\\ J_2 &= \int_{\mathbb{R}^6 \times \mathbb{S}^2} q(\xi - \xi_*, \sigma) g^2_*\left(\sqrt{\mu'_*} - \sqrt{\mu_*}\right)^2 d\xi_* d\xi d\sigma.\end{aligned}$$

With the help of Proposition A.5, we then analyze the triple inner product $\langle \mathscr{T}(f,g),h\rangle$ as follows. Let $\mu(\xi,\xi_*)$ be given in Proposition A.5, and write

$$\langle \mathscr{T}(f,g),h\rangle = \langle \mathscr{T}_\mu(f,g),h\rangle + \langle \mathscr{T}_{rest}(f,g),h\rangle\,, \tag{2.1}$$ ad.decom.Ga

where

$$\langle \mathscr{T}_\mu(f,g),h\rangle = \mathscr{T}(f,g) = \int_{\mathbb{R}^6\times\mathbb{S}^2} q(\xi-\xi_*,\sigma)\mu(\xi,\xi_*)\left[f'g'_*-fg_*\right]hd\xi_*d\xi d\sigma, \tag{2.2}$$ mu1op

and $\langle \mathscr{T}_{rest}(f,g),h\rangle$ is a finite combination of terms in the form of

$$\begin{aligned}\langle \mathscr{T}_{mod,i}(f,g),h\rangle =& \int_{\mathbb{R}^6\times\mathbb{S}^2} q(\xi-\xi_*,\sigma)\mu_*^{c_i}\mu^{c_i}\mu^{d_i}\left[f'g'_*-fg_*\right]hd\xi_*d\xi d\sigma\\ =&Q\left(\left(\mu^{c_i}f,\mu^{c_i}g\right),\mu^{d_i}h\right)\end{aligned} \tag{2.3}$$ mu2op

with $c_i>0$, $d_i>0$.

In order to make the weighted estimates on (1.5), particularly on L and Γ, in light of the basic estimates listed in Appendix (cf. Proposition A.4), it suffices to investigate the estimate on commutators with the weight w_{ℓ,λ_0}. Here, it is worth pointing out that, for the non-cutoff soft potentials, although there have been some weighted estimates with respect to the algebraic weight on those terms, the corresponding estimates with respect to the exponential weight so far does not appear in any existing literature.

The first result of this subsection is concerned with the commutator estimates on $\mathscr{T}$ in the case when $0>\gamma>\max\{-3,-3/2-2s\}$. Notice that the place of h in (2.4) in the following lemma will be weighted by $w_{\ell,\lambda_0}(t,\xi)$ for the later use of the weighted estimates on Γ.

⟨commu nonop⟩ **Lemma 2.1.** *Assume $0<s<1$, $0>\gamma>\max\{-3,-3/2-2s\}$, $\lambda_0>0$ and $\ell\le 0$. For some $\bar\lambda>0$, one has*

$$\begin{aligned}&\left|\langle w_{\ell,\lambda_0}\mathscr{T}(f,g)-\mathscr{T}(w_{\ell,\lambda_0}f,g),h\rangle\right|\\ &\lesssim \left|e^{\frac{\lambda_0\langle\xi\rangle}{(1+t)^{\vartheta}}}g\right|_{L^2}^2 |w_{\ell,\lambda_0}f|^2_{L^2_{\gamma/2}}|h|_{\mathbf{D}}\\ &\quad+\left|\mu^{\frac{\bar\lambda}{32}}g\right|_{L^2}|w_{\ell,\lambda_0}f|_{L^2_{\gamma/2+1/2}}|h|_{L^2_{\gamma/2+1/2}}\\ &\quad+(1+t)^{-\vartheta}\left|\mu^{\frac{\bar\lambda}{32}}g\right|_{L^2}|w_{\ell,\lambda_0}f|_{L^2_{\gamma/2+1}}|h|_{L^2_{\gamma/2+s}}\\ &\quad+(1+t)^{-2\vartheta}\left|\mu^{\frac{\bar\lambda}{32}}g\right|_{L^2}|w_{\ell,\lambda_0}f|_{L^2_{\gamma/2+1}}|h|_{L^2_{\gamma/2+1}}.\end{aligned} \tag{2.4}$$ ad.lem.nonop1

Proof. In view of the decomposition (2.1) given for $\mathscr{T}$, we first consider the commutator for $\mathscr{T}_\mu$ corresponding to (2.2) with

$$\mu(\xi,\xi_*)=\left(\mu^{\tilde\lambda}-\mu_*^{\tilde\lambda}\right)^k\mu_*^{\bar\lambda},$$

for $k\ge 4$ and some constants $\tilde\lambda,\bar\lambda>0$. Namely we shall estimate

$$\begin{aligned}&\left|\langle w_{\ell,\lambda_0}\mathscr{T}_\mu(f,g)-\mathscr{T}_\mu(w_{\ell,\lambda_0}f,g),h\rangle\right|\\ &=\int_{\mathbb{R}^6\times\mathbb{S}^2} q(\xi-\xi_*,\sigma)\mu(\xi,\xi_*)f'g'_*h\left[w_{\ell,\lambda_0}-w'_{\ell,\lambda_0}\right]d\xi_*d\xi d\sigma,\end{aligned}$$

which can be rewritten as

$$\begin{aligned}&\int_{\mathbb{R}^6\times\mathbb{S}^2} q(\xi-\xi_*,\sigma)\mu(\xi,\xi_*)f'g'_*h\left[w_{\ell,\lambda_0}-w'_{\ell,\lambda_0}\right]d\xi_*d\xi d\sigma\\ &\quad=\underbrace{\int_{\mathbb{R}^6\times\mathbb{S}^2} q(\xi-\xi_*,\sigma)\mu(\xi,\xi_*)f'g'_*\left[w_{\ell,\lambda_0}-w'_{\ell,\lambda_0}\right](h-h')d\xi_*d\xi d\sigma}_{I_1}\\ &\quad\quad+\underbrace{\int_{\mathbb{R}^6\times\mathbb{S}^2} q(\xi-\xi_*,\sigma)\left[\mu(\xi',\xi'_*)-\mu(\xi,\xi_*)\right]fg_*h\left[w'_{\ell,\lambda_0}-w_{\ell,\lambda_0}\right]d\xi_*d\xi d\sigma}_{I_2}\\ &\quad\quad+\underbrace{\int_{\mathbb{R}^6\times\mathbb{S}^2} q(\xi-\xi_*,\sigma)\mu(\xi,\xi_*)fg_*h\left[w'_{\ell,\lambda_0}-w_{\ell,\lambda_0}\right]d\xi_*d\xi d\sigma}_{I_3}\end{aligned}$$

by the usual change of variables. Those three terms I_j $(j=1,2,3)$ on the right can be treated as follows.

Estimate on I_1. For I_1, we use the Cauchy-Schwarz inequality to get

$$\begin{aligned}|I_1|&\leq\int_{\mathbb{R}^6\times\mathbb{S}^2} q(\xi-\xi_*,\sigma)\mu(\xi,\xi_*)\left|f'g'_*\left[w_{\ell,\lambda_0}-w'_{\ell,\lambda_0}\right](h-h')\right|d\xi_*d\xi d\sigma\\ &\lesssim\left(\int_{\mathbb{R}^6\times\mathbb{S}^2} q(\xi-\xi_*,\sigma)\left(\mu^{\tilde{\lambda}}-\mu_*^{\tilde{\lambda}}\right)^{2k}\mu_*^{\bar{\lambda}}|f'|^2|g'_*|^2\left[w_{\ell,\lambda_0}-w'_{\ell,\lambda_0}\right]^2d\xi_*d\xi d\sigma\right)^{\frac12}|h|_{\mathbf{D}}\\ &\lesssim I_{1,1}^{\frac12}|h|_{\mathbf{D}},\end{aligned}$$

where

$$I_{1,1}=\int_{\mathbb{R}^6\times\mathbb{S}^2}\langle\xi-\xi_*\rangle^{\gamma}\mathbb{b}(\cos\theta)(\mu'_*)^{\bar{\lambda}}|f|^2|g_*|^2\left[w_{\ell,\lambda_0}-w'_{\ell,\lambda_0}\right]^2d\xi_*d\xi d\sigma. \tag{2.5}$$ U

Furthermore, letting $\xi(\tau)=\xi+\tau(\xi'-\xi)$ for $0<\tau<1$, and using Taylor's formula, one has

$$\begin{aligned}w'_{\ell,\lambda_0}-w_{\ell,\lambda_0}&=\int_0^1(\xi'-\xi)\cdot\nabla_\xi w_{\ell,\lambda_0}(\xi)|_{\xi=\xi(\tau)}d\tau\\ &=\int_0^1(\xi'-\xi)\cdot\left[\gamma\ell\frac{\xi(\tau)}{\langle\xi(\tau)\rangle^2}\langle\xi(\tau)\rangle^{\gamma\ell}e^{\frac{\lambda_0\langle\xi(\tau)\rangle}{(1+t)^{\vartheta}}}\right.\\ &\quad\left.+\frac{\lambda_0}{(1+t)^{\vartheta}}\frac{\xi(\tau)}{\langle\xi(\tau)\rangle}\langle\xi(\tau)\rangle^{\gamma\ell}e^{\frac{\lambda_0\langle\xi(\tau)\rangle}{(1+t)^{\vartheta}}}\right]d\tau.\end{aligned} \tag{2.6}$$ diff weight

Since

$$(\xi'-\xi)\cdot(\xi-\xi'_*)=0, \tag{2.7}$$ vcross

it follows that

$$(\xi'-\xi)\cdot\xi(\tau)=(\xi'-\xi)\cdot(\xi+\tau(\xi'-\xi))=(\xi'-\xi)\cdot\xi'_*+\tau|\xi'-\xi|^2.$$

For $\langle\xi(\tau)\rangle$, from some elementary analysis, one can see that

$$\frac{1}{\langle\xi_*\rangle}\lesssim\frac{\langle\xi(\tau)\rangle}{\langle\xi\rangle}\lesssim\langle\xi_*\rangle,\quad \frac{1}{\langle\xi_*'\rangle}\lesssim\frac{\langle\xi(\tau)\rangle}{\langle\xi\rangle}\lesssim\langle\xi_*'\rangle, \tag{2.8}$$

meanxi

and

$$\langle\xi(\tau)\rangle\le(1-\tau)\langle\xi\rangle+\tau\langle\xi'\rangle\le(1-\tau)\langle\xi\rangle+\tau(\langle\xi\rangle+\langle\xi_*\rangle)=\langle\xi\rangle+\tau\langle\xi_*\rangle. \tag{2.9}$$

exivip

Therefore one has

$$\begin{aligned}
\left|w_{\ell,\lambda_0}'-w_{\ell,\lambda_0}\right|\lesssim&\int_0^1|\xi'-\xi|\langle\xi\rangle^{\gamma\ell-1}\langle\xi_*'\rangle^{\gamma\ell+1}e^{\frac{\lambda_0\langle\xi\rangle}{(1+t)^\vartheta}+\frac{\tau\lambda_0\langle\xi_*\rangle}{(1+t)^\vartheta}}d\tau\\
&+\int_0^1|\xi'-\xi||\xi_*'|\langle\xi\rangle^{\gamma\ell-1}\langle\xi_*'\rangle^{\gamma\ell+1}(1+t)^{-\vartheta}e^{\frac{\lambda_0\langle\xi\rangle}{(1+t)^\vartheta}+\frac{\tau\lambda_0\langle\xi_*\rangle}{(1+t)^\vartheta}}d\tau\\
&+\int_0^1|\xi'-\xi|\langle\xi_*'\rangle\langle\xi_*\rangle\langle\xi\rangle^{\gamma\ell-1}\langle\xi_*'\rangle^{\gamma\ell+1}(1+t)^{-\vartheta}e^{\frac{\lambda_0\langle\xi\rangle}{(1+t)^\vartheta}+\frac{\tau\lambda_0\langle\xi_*\rangle}{(1+t)^\vartheta}}d\tau\\
\lesssim&|\xi'-\xi|\langle\xi\rangle^{\gamma\ell-1}\langle\xi_*'\rangle^{\gamma\ell+1}e^{\frac{\lambda_0\langle\xi\rangle}{(1+t)^\vartheta}+\frac{\lambda_0\langle\xi_*\rangle}{(1+t)^\vartheta}}\\
&+|\xi'-\xi||\xi_*'|\langle\xi\rangle^{\gamma\ell-1}\langle\xi_*'\rangle^{\gamma\ell+1}(1+t)^{-\vartheta}e^{\frac{\lambda_0\langle\xi\rangle}{(1+t)^\vartheta}+\frac{\lambda_0\langle\xi_*\rangle}{(1+t)^\vartheta}}\\
&+|\xi'-\xi|\langle\xi_*'\rangle\langle\xi\rangle^{\gamma\ell-1}\langle\xi_*'\rangle^{\gamma\ell+1}(1+t)^{-\vartheta}e^{\frac{\lambda_0\langle\xi\rangle}{(1+t)^\vartheta}+\frac{\lambda_0\langle\xi_*\rangle}{(1+t)^\vartheta}},
\end{aligned} \tag{2.10}$$

diff weight1

where in the last inequality we have used the fact that

$$\int_0^1\lambda_0\langle\xi_*\rangle(1+t)^{-\vartheta}e^{\frac{-(1-\tau)\lambda_0\langle\xi_*\rangle}{(1+t)^\vartheta}}d\tau=\int_0^{\frac{\lambda_0\langle\xi_*\rangle}{(1+t)^\vartheta}}e^{-\eta}d\eta\le1.$$

Remark 2.1. *We emphasize that the geometric property that $\xi'-\xi$ is perpendicular to $\xi-\xi_*'$ (cf. (2.7)) plays a vital role in the estimates of (2.6), since the growth of ξ is transferred to the growth of ξ_*' which can be controlled by the exponential weight of ξ_*'. The idea that we used here is much similar to that of deducing the estimates of the Landau operator in [19, 32].*

Next plugging (2.10) into (2.5), we obtain

$$\begin{aligned}
I_{1,1}\lesssim&\int_{\mathbb{R}^6\times\mathbb{S}^2}\langle\xi-\xi_*'\rangle^\gamma\mathbb{b}(\cos\theta)(\mu_*')^{\frac{a}{2}}|f|^2e^{\frac{2\lambda_0\langle\xi_*\rangle}{(1+t)^\vartheta}}\\
&\times|g_*|^2|\xi-\xi_*'|^2\theta^2\langle\xi\rangle^{2\gamma\ell-2}e^{\frac{2\lambda_0\langle\xi\rangle}{(1+t)^\vartheta}}d\xi_*d\xi d\sigma\\
\lesssim&\left|e^{\frac{\lambda_0\langle\xi\rangle}{(1+t)^\vartheta}}g\right|_{L^2}^2|w_{\ell,\lambda_0}f|_{L^2_{\gamma/2}}^2,
\end{aligned}$$

where we have used the fact that $|\xi-\xi_*'|\le|\xi-\xi_*|$ and $|\xi'-\xi|=|\xi-\xi_*'|\tan\theta/2$. Thus

$$|I_1|\lesssim\left|e^{\frac{\lambda_0\langle\xi\rangle}{(1+t)^\vartheta}}g\right|_{L^2}^2|w_{\ell,\lambda_0}f|_{L^2_{\gamma/2}}^2|h|_{\mathbf{D}}.$$

Estimate on I_2. As to I_2, at first, we use the following estimate as given in [4]

$$
\begin{aligned}
&|\xi-\xi_*|^\gamma\langle\xi-\xi_*\rangle^{-\gamma/2}\,|\mu(\xi',\xi'_*)-\mu(\xi,\xi_*)|\\
&\qquad\lesssim\langle\xi-\xi_*\rangle^{\gamma/2}\left(\mu_*^{\bar\lambda/2}+(\mu'_*)^{\bar\lambda/2}\right)\min\left\{|\xi-\xi'_*|\theta,|\xi-\xi_*|\theta,1\right\}.
\end{aligned}
$$

By means of Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
|I_2|\lesssim&\int_{\mathbb{R}^6\times\mathbb{S}^2}\langle\xi-\xi_*\rangle^\gamma\mathbb{b}(\cos\theta)(\mu'_*)^{\frac{\bar\lambda}{2}}\min\left\{|\xi-\xi'_*|\theta,1\right\}\\
&\times|g_*||f|\left[w'_{\ell,\lambda_0}-w_{\ell,\lambda_0}\right]hd\xi_*d\xi d\sigma\\
&+\int_{\mathbb{R}^6\times\mathbb{S}^2}\langle\xi-\xi_*\rangle^\gamma\mathbb{b}(\cos\theta)\mu_*^{\frac{\bar\lambda}{2}}\min\left\{|\xi-\xi_*|\theta,1\right\}\\
&\times|g_*||f|\left[w'_{\ell,\lambda_0}-w_{\ell,\lambda_0}\right]hd\xi_*d\xi d\sigma\\
\lesssim&\underbrace{\left(\int_{\mathbb{R}^6\times\mathbb{S}^2}\langle\xi-\xi_*\rangle^\gamma\mathbb{b}(\cos\theta)(\mu'_*)^{\frac{\bar\lambda}{2}}|g_*|^2|f|^2\left[w'_{\ell,\lambda_0}-w_{\ell,\lambda_0}\right]^2d\xi_*d\xi d\sigma\right)^{1/2}}_{I_{2,1}^{1/2}}\\
&\times\underbrace{\left(\int_{\mathbb{R}^6\times\mathbb{S}^2}\langle\xi-\xi_*\rangle^\gamma\mathbb{b}(\cos\theta)(\mu'_*)^{\frac{\bar\lambda}{2}}\min\left\{|\xi-\xi'_*|^2\theta^2,1\right\}h^2d\xi_*d\xi d\sigma\right)^{1/2}}_{I_{2,2}^{1/2}}\\
&+\underbrace{\left(\int_{\mathbb{R}^6\times\mathbb{S}^2}\langle\xi-\xi_*\rangle^\gamma\mathbb{b}(\cos\theta)\mu_*^{\frac{\bar\lambda}{2}}|g_*|^2|f|^2\left[w'_{\ell,\lambda_0}-w_{\ell,\lambda_0}\right]^2d\xi_*d\xi d\sigma\right)^{1/2}}_{I_{2,3}^{1/2}}\\
&\times\underbrace{\left(\int_{\mathbb{R}^6\times\mathbb{S}^2}\langle\xi-\xi_*\rangle^\gamma\mathbb{b}(\cos\theta)\mu_*^{\frac{\bar\lambda}{2}}\min\left\{|\xi-\xi_*|^2\theta^2,1\right\}h^2d\xi_*d\xi d\sigma\right)^{1/2}}_{I_{2,4}^{1/2}}.
\end{aligned}
\tag{2.11}
$$

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Noticing that

$$
\int_{\mathbb{S}^2}\mathbb{b}(\cos\theta)\min\left\{|\xi-\xi_*|^2\theta^2,1\right\}d\sigma\lesssim\langle\xi-\xi_*\rangle^{2s},
$$

we obtain $I_{2,4}\lesssim|h|^2_{L^2_{\gamma/2+s}}$ and the same estimate for $I_{2,2}$ holds if one uses the regular change of variables $\xi_*\to\xi'_*$, for instance, cf. [1].

It is easy to see that $I_{2,1}$ shares the same upper bound as I_1. For $I_{2,3}$, we get from (2.6), (2.8) and (2.9) that

$$
\begin{aligned}
I_{2,3} \lesssim & \int_{\mathbb{R}^6\times\mathbb{S}^2} \langle \xi-\xi_*\rangle^\gamma \mathbb{b}(\cos\theta)\mu_*^{\frac{\bar\lambda}{4}}|g_*|^2|f|^2|\xi-\xi_*|^2\theta^2\langle\xi\rangle^{2\gamma\ell-2}e^{\frac{2\lambda_0\langle\xi\rangle}{(1+t)^\vartheta}}d\xi_*d\xi d\sigma\\
&+(1+t)^{-2\vartheta}\int_{\mathbb{R}^6\times\mathbb{S}^2} \langle \xi-\xi_*\rangle^\gamma \mathbb{b}(\cos\theta)\mu_*^{\frac{\bar\lambda}{4}}|g_*|^2\\
&\times|f|^2|\xi-\xi_*|^2\theta^2\langle\xi\rangle^{2\gamma\ell}e^{\frac{2\lambda_0\langle\xi\rangle}{(1+t)^\vartheta}}d\xi_*d\xi d\sigma\\
\lesssim & \left|\mu^{\frac{\bar\lambda}{32}}g\right|_{L^2}^2|w_{\ell,\lambda_0}f|^2_{L^2_{\gamma/2}}+(1+t)^{-2\vartheta}\left|\mu^{\frac{\bar\lambda}{32}}g\right|_{L^2}^2|w_{\ell,\lambda_0}f|^2_{L^2_{\gamma/2+1}}.
\end{aligned}
$$

Plugging the estimates above into (2.11), we have

$$
\begin{aligned}
|I_2|\lesssim & \left|e^{\frac{\lambda_0\langle\xi\rangle}{(1+t)^\vartheta}}g\right|_{L^2}^2|w_{\ell,\lambda_0}f|^2_{L^2_{\gamma/2}}|h|_{L^2_{\gamma/2+s}}\\
&+\left|\mu^{\frac{\bar\lambda}{32}}g\right|_{L^2}|w_{\ell,\lambda_0}f|_{L^2_{\gamma/2}}|h|_{L^2_{\gamma/2+s}}\\
&+(1+t)^{-\vartheta}\left|\mu^{\frac{\bar\lambda}{32}}g\right|_{L^2}|w_{\ell,\lambda_0}f|_{L^2_{\gamma/2+1}}|h|_{L^2_{\gamma/2+s}}.
\end{aligned}
$$

Estimate on I_3. For the term I_3, we use the Taylor expansion for $w_\ell'-w_\ell$ up to the second order, in the form of

$$
w_{\ell,\lambda_0}(\xi')-w_{\ell,\lambda_0}(\xi)=(\xi'-\xi)\cdot\nabla_\xi w_{\ell,\lambda_0}(\xi)+\frac{1}{2}(\xi'-\xi)\otimes(\xi'-\xi):\nabla_\xi^2 w_{\ell,\lambda_0}|_{\xi=\xi(\tau)},
$$

where : is defined as $\mathbb{M}:\mathbb{N}=\sum\limits_{i,j=1}^n m_{ij}n_{ij}$ for two $n\times n$ matrices $\mathbb{M}$ and $\mathbb{N}$.

As for the first order term in the expansion, notice that

$$
\xi-\xi'=\frac{|\xi-\xi_*|}{2}(\sigma-(\mathbf{k}\cdot\sigma)\mathbf{k})+((\mathbf{k}\cdot\sigma)-1)\frac{\xi-\xi_*}{2},\quad \mathbf{k}=\frac{\xi-\xi_*}{|\xi-\xi_*|} \tag{2.12}
$$

id.diff

and the spherical integral corresponding to the first term on the right-hand side of (2.12) vanishes because of the symmetry on $\mathbb{S}^2$. For the term involving the second term on the right-hand side of (2.12), one has

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^6\times\mathbb{S}^2} q(\xi-\xi_*,\sigma)\mu(\xi,\xi_*)g_*fh\left[\frac{1}{2}((\mathbf{k}\cdot\sigma)-1)(\xi-\xi_*)\cdot\nabla_\xi w_{\ell,\lambda_0}(\xi)\right]d\xi_*d\xi d\sigma\right|\\
&\lesssim\int_{\mathbb{R}^6\times\mathbb{S}^2}\langle\xi-\xi_*\rangle^\gamma\mathbb{b}(\cos\theta)\mu_*^{\bar\lambda/4}|\xi-\xi_*|\theta^2\langle\xi\rangle^{\gamma\ell}e^{\frac{\lambda_0\langle\xi\rangle}{(1+t)^\vartheta}}|g_*||f||h|d\xi_*d\xi d\sigma\\
&\lesssim\left(\int_{\mathbb{R}^6\times\mathbb{S}^2}\langle\xi-\xi_*\rangle^\gamma\mathbb{b}(\cos\theta)\mu_*^{\bar\lambda/4}|\xi-\xi_*|\theta^2\langle\xi\rangle^{2\gamma\ell}e^{\frac{2\lambda_0\langle\xi\rangle}{(1+t)^\vartheta}}|g_*|^2|f|^2d\xi_*d\xi d\sigma\right)^{1/2}\\
&\quad\times\left(\int_{\mathbb{R}^6\times\mathbb{S}^2}\langle\xi-\xi_*\rangle^\gamma\mathbb{b}(\cos\theta)\mu_*^{\bar\lambda/4}|\xi-\xi_*|\theta^2h^2d\xi_*d\xi d\sigma\right)^{1/2}\\
&\lesssim\left|\mu^{\frac{\bar\lambda}{32}}g\right|_{L^2}|w_{\ell,\lambda_0}f|_{L^2_{\gamma/2+1/2}}|h|_{L^2_{\gamma/2+1/2}}.
\end{aligned}
$$

For the second order term in the expansion above, we compute

$$\begin{aligned}\nabla_\xi^2 w_\ell|_{\xi=\xi(\tau)} =&(\gamma\ell)^2\frac{\xi(\tau)\otimes\xi(\tau)}{\langle\xi(\tau)\rangle^4}\langle\xi(\tau)\rangle^{\gamma\ell}e^{\frac{\lambda_0\langle\xi(\tau)\rangle}{(1+t)^\vartheta}}\\&+\gamma\ell\frac{\langle\xi(\tau)\rangle^2\mathbf{E}-2\xi(\tau)\otimes\xi(\tau)}{\langle\xi(\tau)\rangle^4}\langle\xi(\tau)\rangle^{\gamma\ell}e^{\frac{\lambda_0\langle\xi(\tau)\rangle}{(1+t)^\vartheta}}\\&+\frac{\lambda_0}{(1+t)^\vartheta}\frac{\langle\xi(\tau)\rangle^2\mathbf{E}+(2\gamma\ell-1)\xi(\tau)\otimes\xi(\tau)}{\langle\xi(\tau)\rangle^3}\langle\xi(\tau)\rangle^{\gamma\ell}e^{\frac{\lambda_0\langle\xi(\tau)\rangle}{(1+t)^\vartheta}}\\&+\left[\frac{\lambda_0}{(1+t)^\vartheta}\right]^2\frac{\xi(\tau)\otimes\xi(\tau)}{\langle\xi(\tau)\rangle^2}\langle\xi(\tau)\rangle^{\gamma\ell}e^{\frac{\lambda_0\langle\xi(\tau)\rangle}{(1+t)^\vartheta}},\end{aligned}$$

where $\mathbf{E}$ denotes the 3×3 unit matrix. Thus we get from (2.8) and (2.9) that

$$\begin{aligned}&\left|(\xi'-\xi)\otimes(\xi'-\xi):\nabla_\xi^2 w_{\ell,\lambda_0}|_{\xi=\xi(\tau)}\right|\\&\lesssim|\xi'-\xi|^2\langle\xi\rangle^{\gamma\ell-2}\langle\xi_*\rangle^{\gamma\ell+2}e^{\frac{\lambda_0\langle\xi\rangle}{(1+t)^\vartheta}+\frac{\tau q\langle\xi_*\rangle}{(1+t)^\vartheta}}\\&\quad+|\xi'-\xi|^2\langle\xi\rangle^{\gamma\ell-1}\langle\xi_*\rangle^{\gamma\ell+1}(1+t)^{-\vartheta}e^{\frac{\lambda_0\langle\xi\rangle}{(1+t)^\vartheta}+\frac{\tau\lambda_0\langle\xi_*\rangle}{(1+t)^\vartheta}}\\&\quad+|\xi'-\xi|^2\langle\xi\rangle^{\gamma\ell}\langle\xi_*\rangle^{\gamma\ell}(1+t)^{-2\vartheta}e^{\frac{\lambda_0\langle\xi\rangle}{(1+t)^\vartheta}+\frac{\tau\lambda_0\langle\xi_*\rangle}{(1+t)^\vartheta}}.\end{aligned}$$

Therefore one has

$$\begin{aligned}&\int_{\mathbb{R}^6\times\mathbb{S}^2}\left|q(\xi-\xi_*,\sigma)\mu(\xi,\xi_*)g_*fh\left[(\xi'-\xi)\otimes(\xi'-\xi):\nabla_\xi^2 w_{\ell,\lambda_0}|_{\xi=\xi(\tau)}\right]\right|d\xi_*d\xi d\sigma\\&\lesssim\underbrace{\int_{\mathbb{R}^6\times\mathbb{S}^2}\langle\xi-\xi_*\rangle^\gamma\mathbb{b}(\cos\theta)\mu_*^{\frac{\bar\lambda}{4}}|\xi-\xi_*|^2\theta^2\langle\xi\rangle^{\gamma\ell-2}e^{\frac{\lambda_0\langle\xi\rangle}{(1+t)^\vartheta}}|g_*||f||h|d\xi_*d\xi d\sigma}_{I_{3,1}}\\&\quad+\underbrace{\frac{1}{(1+t)^\vartheta}\int_{\mathbb{R}^6\times\mathbb{S}^2}\langle\xi-\xi_*\rangle^\gamma\mathbb{b}(\cos\theta)\mu_*^{\frac{\bar\lambda}{4}}|\xi-\xi_*|^2\theta^2\langle\xi\rangle^{\gamma\ell-1}e^{\frac{\lambda_0\langle\xi\rangle}{(1+t)^\vartheta}}|g_*||f||h|d\xi_*d\xi d\sigma}_{I_{3,2}}\\&\quad+\underbrace{\frac{1}{(1+t)^{2\vartheta}}\int_{\mathbb{R}^6\times\mathbb{S}^2}\langle\xi-\xi_*\rangle^\gamma\mathbb{b}(\cos\theta)\mu_*^{\frac{\bar\lambda}{4}}|\xi-\xi_*|^2\theta^2\langle\xi\rangle^{\gamma\ell}e^{\frac{\lambda_0\langle\xi\rangle}{(1+t)^\vartheta}}|g_*||f||h|d\xi_*d\xi d\sigma}_{I_{3,3}}.\end{aligned}$$

By virtue of Cauchy-Schwarz's inequality, we further obtain

$$
\begin{aligned}
I_{3,1} \lesssim & \Big(\int_{\mathbb{R}^6\times\mathbb{S}^2} \langle \xi-\xi_*\rangle^\gamma \mathbb{b}(\cos\theta)\mu_*^{\bar{\lambda}/4}|\xi-\xi_*|^2\theta^2\langle\xi\rangle^{2\gamma\ell-2}e^{\frac{2\lambda_0\langle\xi\rangle}{(1+t)^\vartheta}} \\
& \times |g_*|^2|f|^2 d\xi_* d\xi d\sigma\Big)^{1/2} \\
& \times \left(\int_{\mathbb{R}^6\times\mathbb{S}^2} \langle \xi-\xi_*\rangle^\gamma \mathbb{b}(\cos\theta)\mu_*^{\bar{\lambda}/4}|\xi-\xi_*|^2\theta^2\langle\xi\rangle^{-2}h^2 d\xi_* d\xi d\sigma\right)^{1/2} \\
\lesssim & \left|\mu^{\frac{\bar{\lambda}}{32}}g\right|_{L^2} |w_{\ell,\lambda_0}f|_{L^2_{\gamma/2}} |h|_{L^2_{\gamma/2}},\\
I_{3,2} \lesssim & (1+t)^{-\vartheta}\left(\int_{\mathbb{R}^6\times\mathbb{S}^2} \langle \xi-\xi_*\rangle^\gamma \mathbb{b}(\cos\theta)\mu_*^{\bar{\lambda}/4}|\xi-\xi_*|^2\theta^2\langle\xi\rangle^{-1}h^2 d\xi_* d\xi d\sigma\right)^{1/2} \\
& \times \Big(\int_{\mathbb{R}^6\times\mathbb{S}^2} \langle \xi-\xi_*\rangle^\gamma \mathbb{b}(\cos\theta)\mu_*^{\bar{\lambda}/4}|\xi-\xi_*|^2\theta^2\langle\xi\rangle^{2\gamma\ell-1}e^{\frac{2\lambda_0\langle\xi\rangle}{(1+t)^\vartheta}} \\
& \times |g_*|^2|f|^2 d\xi_* d\xi d\sigma\Big)^{1/2} \\
\lesssim & (1+t)^{-\vartheta}\left|\mu^{\frac{\bar{\lambda}}{32}}g\right|_{L^2} |w_{\ell,\lambda_0}f|_{L^2_{\gamma/2+1/2}} |h|_{L^2_{\gamma/2+1/2}},\\
I_{3,3} \lesssim & (1+t)^{-2\vartheta}\Big(\int_{\mathbb{R}^6\times\mathbb{S}^2} \langle \xi-\xi_*\rangle^\gamma \mathbb{b}(\cos\theta)\mu_*^{\bar{\lambda}/4}|\xi-\xi_*|^2\theta^2\langle\xi\rangle^{2\gamma\ell}e^{\frac{2\lambda_0\langle\xi\rangle}{(1+t)^\vartheta}} \\
& \times |g_*|^2|f|^2 d\xi_* d\xi d\sigma\Big)^{1/2} \\
& \times \left(\int_{\mathbb{R}^6\times\mathbb{S}^2} \langle \xi-\xi_*\rangle^\gamma \mathbb{b}(\cos\theta)\mu_*^{\bar{\lambda}/4}|\xi-\xi_*|^2\theta^2 h^2 d\xi_* d\xi d\sigma\right)^{1/2} \\
\lesssim & (1+t)^{-2\vartheta}\left|\mu^{\frac{\bar{\lambda}}{32}}g\right|_{L^2} |w_{\ell,\lambda_0}f|_{L^2_{\gamma/2+1}} |h|_{L^2_{\gamma/2+1}}.
\end{aligned}
$$

Note that in all the above estimates on $\mathscr{T}_\mu$, the assumption $\gamma > -3/2-2s$ was not used.

It remains to consider the commutator for $\mathscr{T}_{mod,i}$ corresponding to (2.3). Namely, we need to estimate

$$
\langle w_{\ell,\lambda_0}\mathscr{T}_{mod,i}(f,g) - \mathscr{T}_{mod,i}(w_{\ell,\lambda_0}f,g),h\rangle = \langle w_{\ell,\lambda_0}Q(f_\mu,g_\mu) - Q(w_{\ell,\lambda_0}f_\mu,g_\mu),h_\mu\rangle
$$

with $f_\mu=\mu^{c_i}f$, $g_\mu=\mu^{c_i}g$, $h_\mu=\mu^{d_i}h$ for some $c_i,\ d_i>0$. Since f_μ,g_μ,h_μ contain Gaussians, if $\gamma>-\frac{3}{2}-2s$, one can show by performing similar calculations as in the proof of the estimate (3.7) in [4] that

$$
|\langle w_{\ell,\lambda_0}Q\left(f_\mu,g_\mu\right) - Q\left(w_{\ell,\lambda_0}f_\mu,g_\mu\right),h_\mu\rangle| \lesssim |w_{\ell,\lambda_0}f_\mu|_{L^2}|w_{\ell,\lambda_0}g_\mu|_{L^2}|h_\mu|_{H^s}.
$$

This completes the proof of Lemma 2.1. □

In the proof of Lemma 2.1, it is in dealing with the term $\langle w_{\ell,\lambda_0}Q\left(f_\mu,g_\mu\right) - Q\left(w_{\ell,\lambda_0}f_\mu,g_\mu\right),h_\mu\rangle$ that the assumption $\gamma>-3/2-2s$ was used. We note, however, that even for the case $\gamma>-3$, we can deduce the following estimate on such a term by repeating the calculations used in the proof of Corollary 3.15 of [4]

$$
\begin{aligned}
&|\langle w_{\ell,\lambda_0} Q(f_\mu, g_\mu) - Q(w_{\ell,\lambda_0} f_\mu, g_\mu), h_\mu\rangle| \\
&\quad \lesssim \min\left\{ \left|\mu^{\frac{1}{40}} g\right|_{L^\infty} \left|\mu^{\frac{1}{40}} f\right|_{H^s}, \left|\mu^{\frac{1}{40}} g\right|_{L^2} \left|\mu^{\frac{1}{60}} f\right|_{H^1} \right\} |h|_{\mathbf{D}} \\
&\qquad + |w_{\ell,\lambda_0} f_\mu|_{L^2} |w_{\ell,\lambda_0} g_\mu|_{L^2} |h_\mu|_{L^2}.
\end{aligned}
$$

The above estimate together with the proof of Lemma 2.1 yield the following result on the commutator estimate for the case $-3<\gamma<-2s,\ 0<s<1$.

⟨commu nonopimp⟩ **Lemma 2.2.** *Assume that $0<s<1$, $-2s>\gamma>-3$ and $\ell\leq 0$. Then one has*

$$
\begin{aligned}
&|\langle w_{\ell,\lambda_0}\mathscr{T}(f,g) - \mathscr{T}(w_{\ell,\lambda_0} f, g), h\rangle| \\
&\lesssim \left| e^{\frac{\lambda_0\langle\xi\rangle}{(1+t)^\vartheta}} g\right|_{L^2}^2 |w_{\ell,\lambda_0} f|_{L^2_{\gamma/2}}^2 |h|_{\mathbf{D}} \\
&\quad + \left|\mu^{\frac{\bar\lambda}{32}} g\right|_{L^2} |w_{\ell,\lambda_0} f|_{L^2_{\gamma/2+1/2}} |h|_{L^2_{\gamma/2+1/2}} \\
&\quad + (1+t)^{-\vartheta} \left|\mu^{\frac{\bar\lambda}{32}} g\right|_{L^2} |w_{\ell,\lambda_0} f|_{L^2_{\gamma/2+1}} |h|_{L^2_{\gamma/2+s}} \\
&\quad + (1+t)^{-2\vartheta} \left|\mu^{\frac{\bar\lambda}{32}} g\right|_{L^2} |w_{\ell,\lambda_0} f|_{L^2_{\gamma/2+1}} |h|_{L^2_{\gamma/2+1}} \\
&\quad + \min\left\{ \left|\mu^{\frac{1}{40}} g\right|_{L^\infty} \left|\mu^{\frac{1}{40}} f\right|_{H^s}, \left|\mu^{\frac{1}{40}} g\right|_{L^2} \left|\mu^{\frac{1}{60}} f\right|_{H^1} \right\} |h|_{\mathbf{D}}.
\end{aligned}
\tag{2.13}
$$
ad.lem1

⟨lem.ad⟩ **Lemma 2.3.** *Lemma 2.2 will be used in the only proof of Lemma 2.6 concerning the weighted estimate on the linearized operator L. Notice that (2.13) does not apply to the estimate on the nonlinear term Γ since the last term on the right-hand side of (2.13) must contain the increased velocity differentiation on either g or f.*

For some suitable functions f, g, v, define $\mathscr{T}_0(f,g,v)$ as

$$
\mathscr{T}_0(f,g,v) = \int_{\mathbb{R}^3\times\mathbb{S}^2} q(\xi-\xi_*,\sigma) v_* \left[g_*' f' - g_* f\right] d\xi_* d\sigma. \tag{2.14}
$$
nonop0

Then $\mathscr{T}(f,g) = \mathscr{T}_0(f,g,\sqrt{\mu})$ and

$$
\begin{aligned}
&\left\langle \partial^\alpha_\beta\ \mathscr{T}(f,g), w_{\ell,\lambda_0} h\right\rangle \\
&= \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2+\beta_3=\beta}} C_\alpha^{\alpha_1,\alpha_2} C_\beta^{\beta_1,\beta_2,\beta_3} \left\langle \mathscr{T}_0(\partial^{\alpha_1}_{\beta_1} f, \partial^{\alpha_2}_{\beta_2} g, \partial_{\beta_3}\sqrt{\mu}), w_{\ell,\lambda_0} h\right\rangle \\
&= \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2+\beta_3=\beta}} C_\alpha^{\alpha_1,\alpha_2} C_\beta^{\beta_1,\beta_2,\beta_3} \left\langle \mathscr{T}_0(w_{\ell,\lambda_0}\partial^{\alpha_1}_{\beta_1} f, \partial^{\alpha_2}_{\beta_2} g, \partial_{\beta_3}\sqrt{\mu}), h\right\rangle \\
&\quad + \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2+\beta_3=\beta}} C_\alpha^{\alpha_1,\alpha_2} C_\beta^{\beta_1,\beta_2,\beta_3} \Big\{ \left\langle \mathscr{T}_0(\partial^{\alpha_1}_{\beta_1} f, \partial^{\alpha_2}_{\beta_2} g, \partial_{\beta_3}\sqrt{\mu}), w_{\ell,\lambda_0} h\right\rangle \\
&\quad - \left\langle \mathscr{T}_0(w_{\ell,\lambda_0}\partial^{\alpha_1}_{\beta_1} f, \partial^{\alpha_2}_{\beta_2} g, \partial_{\beta_3}\sqrt{\mu}), h\right\rangle \Big\}.
\end{aligned}
$$

Note that all the upper estimates on $\mathscr{T}$ established in the paper apply also to $\mathscr{T}_0$ as long as v is a product of a polynomial in ξ and a positive power of the Maxwellian $\mu(\xi)$.

Combing Lemma 2.1 with Proposition A.4, we can deduce the following weighted estimates on the nonlinear collision term with respect to the time-velocity weight $w_{\ell,\lambda_0}(t,\xi)$ which will play an important role in performing the nonlinear energy estimates in the next section.

⟨nonopxvder⟩ **Lemma 2.4.** *For all $0<s<1$, $-2s>\gamma>\max\{-3,-3/2-2s\}$, $\lambda_0>0$ and $\ell\leq 0$. Then one has*

$$
\begin{aligned}
&\left|\left\langle \partial^\alpha_\beta \mathscr{T}(f,g), w^2_{\ell,\lambda_0}\partial^\alpha_\beta h\right\rangle\right|\\
&\lesssim \sum\left\{\left|w_{\ell,\lambda_0}\partial^{\alpha_1}_{\beta_1}f\right|_{L^2_{s+\gamma/2}}\left|\partial^{\alpha_2}_{\beta_2}g\right|_{\mathbf{D}}+\left|\partial^{\alpha_2}_{\beta_2}g\right|_{L^2_{s+\gamma/2}}\left|w_{\ell,\lambda_0}\partial^{\alpha_1}_{\beta_1}f\right|_{\mathbf{D}}\right\}\left|w_{\ell,\lambda_0}\partial^\alpha_\beta h\right|_{\mathbf{D}}\\
&\quad+\min\left\{\left|w_{\ell,\lambda_0}\partial^{\alpha_1}_{\beta_1}f\right|_{L^2}\left|\partial^{\alpha_2}_{\beta_2}g\right|_{L^2_{s+\gamma/2}},\left|\partial^{\alpha_2}_{\beta_2}g\right|_{L^2}\left|w_{\ell,\lambda_0}\partial^{\alpha_1}_{\beta_1}f\right|_{L^2_{s+\gamma/2}}\right\}\left|w_{\ell,\lambda_0}\partial^\alpha_\beta h\right|_{\mathbf{D}}\\
&\quad+\sum\left|e^{\frac{\lambda_0\langle\xi\rangle}{(1+t)^\vartheta}}\partial^{\alpha_2}_{\beta_2}g\right|_{L^2}\left|w_{\ell,\lambda_0}\partial^{\alpha_1}_{\beta_1}f\right|_{L^2_{\gamma/2}}\left|w_{\ell,\lambda_0}\partial^\alpha_\beta h\right|_{\mathbf{D}}\\
&\quad+\sum\left|\mu^{\frac{\bar\lambda}{128}}\partial^{\alpha_2}_{\beta_2}g\right|_{L^2}\left|w_{\ell,\lambda_0}\partial^{\alpha_1}_{\beta_1}f\right|_{L^2_{\gamma/2+1/2}}\left|w_{\ell,\lambda_0}\partial^\alpha_\beta h\right|_{L^2_{\gamma/2+1/2}}\\
&\quad+\sum(1+t)^{-\vartheta}\left|\mu^{\frac{\bar\lambda}{128}}\partial^{\alpha_2}_{\beta_2}g\right|_{L^2}\left|w_{\ell,\lambda_0}\partial^{\alpha_1}_{\beta_1}f\right|_{L^2_{\gamma/2+1}}\left|w_{\ell,\lambda_0}\partial^\alpha_\beta h\right|_{L^2_{\gamma/2+s}}\\
&\quad+\sum(1+t)^{-2\vartheta}\left|\mu^{\frac{\bar\lambda}{128}}\partial^{\alpha_2}_{\beta_2}g\right|_{L^2}\left|w_{\ell,\lambda_0}\partial^{\alpha_1}_{\beta_1}f\right|_{L^2_{\gamma/2+1}}\left|w_{\ell,\lambda_0}\partial^\alpha_\beta h\right|_{L^2_{\gamma/2+1}}=\sum_{i=1}^6 H_n,
\end{aligned}
\tag{2.15}
$$
nonopxvder1

where the summation is taken over $\alpha_1+\beta_1+\alpha_2+\beta_2\leq\alpha+\beta$ and H_n $(1\leq n\leq 6)$ denote those six terms on the right-hand side of (2.15) respectively.

Remark 2.2. *As mentioned in the introduction, although some terms such as $|w_{\ell,\lambda_0}\partial^{\alpha_1}_{\beta_1}f|_{L^2_{\gamma/2+1}}$ and $|w_{\ell,\lambda_0}\partial^\alpha_\beta h|_{L^2_{\gamma/2+1}}$ on the right-hand side of (2.15) can not be directly controlled by the dissipation of L, i.e. the first five terms in $\mathcal{D}_{N,\ell,\lambda}(t)$, they can be controlled by the extra dissipation corresponding to the last term of $\mathcal{D}_{N,\ell,\lambda}(t)$ given by (1.12). See the proof of Lemma 2.5 for details, especially the estimate on the term $I_{9,3}$.*

Next, we turn to control the weighted estimates on the nonlinear collision term Γ with respect to the time-velocity exponential weight $w_{\ell,\lambda_0}(t,\xi)$ in terms of the temporal energy functionals $\mathcal{E}_{N,\ell,\lambda}(t)$, $\overline{\mathcal{E}}_N(t)$ defined by (1.10) and (1.11) and the corresponding entropy dissipation rate $\mathcal{D}_{N,\ell,\lambda}(t)$, $\overline{\mathcal{D}}_N(t)$ defined by (1.12) and (1.13). For this issue, we prove the following

⟨nonopxvint⟩ **Lemma 2.5.** *For all $1/2\leq s<1$, $\max\{-3,-3/2-2s\}<\gamma<-2s$, and assume $l-1\geq N\geq 10$, $\lambda_0>0$, $0<\vartheta\leq 1/4$, $|\alpha|+|\beta|\leq N$. Then one has*

$$
|\langle\partial^\alpha\Gamma_\pm(f,f),\partial^\alpha\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\rangle|\lesssim\mathcal{E}^{1/2}_{N-3,l-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\overline{\mathcal{D}}_N(t),\tag{2.16}
$$
nonopwep

$$\begin{aligned}&\left|\left\langle w^2_{|\alpha|+|\beta|-l,\lambda_0}\ \partial^\alpha_\beta\Gamma_\pm(f,f),\partial^\alpha_\beta\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\right\rangle\right|\\ &\qquad\lesssim(1+t)^{1-\vartheta}\mathcal{E}^{1/2}_{N-3,l-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\mathcal{D}_{N,l,\lambda_0}(t).\end{aligned}\tag{2.17}$$ nonopall

Moreover, if $|\alpha|>0$*, it follows that*

$$\begin{aligned}&\left|\left\langle w^2_{|\alpha|-l,\lambda_0}\ \partial^\alpha\Gamma_\pm(f,f),\partial^\alpha f_\pm\right\rangle\right|\\ &\qquad\lesssim(1+t)^{1-\vartheta}\mathcal{E}^{1/2}_{N-3,l-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\mathcal{D}_{N,l,\lambda_0}(t).\end{aligned}\tag{2.18}$$ nonopxep

Here and in the sequel, in the case when an undetermined energy functional $\mathcal{E}_{N,\ell,\lambda}(t)$ *appears on the right-hand side of inequalities, it is always understood to take exactly the right-hand expression of* (1.10).

Proof. Noticing $l-1\ge N\ge 10$, by virtue of Proposition A.4 and Sobolev's inequality, one can prove (2.16) without any difficulty. Thus, to complete the proof of Lemma 2.5, we only prove (2.17) detailedly in the following part, since the proof of (2.18) is similar and easier. Recalling Lemma 2.4, it suffices to estimate

$$I_{n+3}=\int_{\mathbb{R}^3}H_n dx,\quad 1\le n\le 6,$$

where H_n $(1\le n\le 6)$ are defined in (2.15). In what follows, for brevity of presentation, we compute only I_9 corresponding to the spatial integral of H_6, since the estimates for other terms are quite similar. By splitting $f_\pm=\mathbf{P}_\pm f+\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f$, we have

$$I_9\lesssim I_{9,1}+I_{9,2}+I_{9,3}+I_{9,4},$$

with

$$\begin{aligned}I_{9,1}=&(1+t)^{-2\vartheta}\int_{\mathbb{R}^3}|\partial^{\alpha_1}(a_\pm,b,c)||\partial^{\alpha_2}(a_\pm,b,c)|\\ &\times\left|w_{|\alpha|+|\beta|-l,\lambda_0}\partial^\alpha_\beta\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\right|_{L^2_{\gamma/2+1}}dx,\\ I_{9,2}=&(1+t)^{-2\vartheta}\int_{\mathbb{R}^3}|\partial^{\alpha_1}(a_\pm,b,c)|\left|\mu^{\frac{\bar\lambda}{128}}\partial^{\alpha_2}_{\beta_2}\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\right|_{L^2}\\ &\times|w_{|\alpha|+|\beta|-l,\lambda_0}\partial^\alpha_\beta\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f|_{L^2_{\gamma/2+1}}dx,\\ I_{9,3}=&(1+t)^{-2\vartheta}\int_{\mathbb{R}^3}\left|w_{|\alpha|+|\beta|-l,\lambda_0}\partial^{\alpha_1}_{\beta_1}\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\right|_{L^2_{\gamma/2+1}(\mathbf{R}^3)}|\partial^{\alpha_2}(a_\pm,b,c)|\\ &\times\left|w_{|\alpha|+|\beta|-l,\lambda_0}\partial^\alpha_\beta\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\right|_{L^2_{\gamma/2+1}}dx,\\ I_{9,4}=&(1+t)^{-2\vartheta}\int_{\mathbb{R}^3}\left|\mu^{\frac{\bar\lambda}{128}}\partial^{\alpha_2}_{\beta_2}\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\right|_{L^2}\left|w_{|\alpha|+|\beta|-l,\lambda_0}\partial^{\alpha_1}_{\beta_1}\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\right|_{L^2_{\gamma/2+1}}\\ &\times\left|w_{|\alpha|+|\beta|-l,\lambda_0}\partial^\alpha_\beta\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\right|_{L^2_{\gamma/2+1}}dx.\end{aligned}$$

Next, we only present the estimates for $I_{9,1}$ and $I_{9,3}$, the others being similar. Our purpose is to prove

$$I_{9,1},\quad I_{9,3}\lesssim(1+t)^{1-\vartheta}\mathcal{E}^{1/2}_{N-3,l-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\mathcal{D}_{N,l,\lambda_0}(t).\tag{2.19}$$ I91.I93

For this, the following fact will be repeatedly used. Since

$$\gamma < -2s, \quad 1/2 \le s < 1, \quad \gamma + 2 \le 1,$$

one can see that

$$(1+t)^{-\frac{1+\vartheta}{2}} \left\| \langle \xi \rangle^{\gamma/2+1} w_{|\alpha|+|\beta|-l,\lambda_0} \partial^\alpha_\beta \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f \right\| \lesssim \mathcal{D}^{1/2}_{N,l,\lambda_0}(t). \tag{2.20}$$ expodis

Next we divide our computations into the following three cases.

Case 1. $|\alpha|+|\beta| \le N/2$. Noticing that $\alpha_1+\beta_1+\alpha_2+\beta_2 \le \alpha+\beta$, in this case, L^∞_x-norm can be used to control both functions involving differentiations $\partial^{\alpha_1}_{\beta_1}$ and $\partial^{\alpha_2}_{\beta_2}$. Thus, by applying Sobolev's inequality and (2.20), we obtain

$$\begin{aligned} I_{9,1} \lesssim & \sum_{|\alpha_1| \le N/2} \|\nabla_x \partial^{\alpha_1}(a_\pm, b, c)\|_{H^1} \|\partial^{\alpha_2}(a_\pm, b, c)\| (1+t)^{\frac{1-3\vartheta}{2}} (1+t)^{-\frac{1+\vartheta}{2}} \\ & \times \left\| \langle \xi \rangle^{\gamma/2+1} w_{|\alpha|+|\beta|-l,\lambda_0} \partial^\alpha_\beta \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f \right\| \\ \lesssim & (1+t)^{\frac{1-3\vartheta}{2}} \mathcal{E}^{1/2}_{N-3,l-\frac{\gamma+2s}{\gamma},\lambda_0}(t) \mathcal{D}_{N,l,\lambda_0}(t), \end{aligned}$$

where the fact that $N \ge 10$ and $N/2+2 \le N-3$ has been used.

Similarly, one has

$$\begin{aligned} I_{9,3} \lesssim & \sum_{|\alpha_2| \le N/2} \left\| \langle \xi \rangle^{\gamma/2+1} w_{|\alpha|+|\beta|-l,\lambda_0} \partial^{\alpha_1}_{\beta_1} \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f \right\| \|\partial^{\alpha_2} \nabla_x (a_\pm, b, c)\|_{H^1} \\ & \times (1+t)^{1-\vartheta} (1+t)^{-1-\vartheta} \left\| \langle \xi \rangle^{\gamma/2+1} w_{|\alpha|+|\beta|-l,\lambda_0} \partial^\alpha_\beta \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f \right\| \\ \lesssim & (1+t)^{1-\vartheta} \mathcal{E}^{1/2}_{N-3,l-\frac{\gamma+2s}{\gamma},\lambda_0}(t) \mathcal{D}_{N,l,\lambda_0}(t). \end{aligned}$$

Case 2. $|\alpha|+|\beta| \ge N/2$ and $|\alpha_1|+|\beta_1| \ge |\alpha_2|+|\beta_2|$. In this case, $|\alpha_2|+|\beta_2| \le N/2$ and $|\alpha_1|+|\beta_1| \ge N/4$.

For $I_{9,1}$, if $|\alpha_1| > 0$, since $N/2+2 \leqslant N-3$, we see that

$$\begin{aligned} & \|\partial^{\alpha_1}(a_\pm, b, c)\| \lesssim \mathcal{D}^{1/2}_{N,l,\lambda_0}(t), \\ & \|\partial^{\alpha_2}(a_\pm, b, c)\|_{L^\infty} \lesssim \|\partial^{\alpha_2}(a_\pm, b, c)\|_{H^2} \lesssim \mathcal{E}^{1/2}_{N-3,l-\frac{\gamma+2s}{\gamma},\lambda_0}(t), \end{aligned}$$

which imply that

$$\begin{aligned} I_{9,1} \lesssim & \sum_{0<|\alpha_1| \le N} \|\partial^{\alpha_1}(a_\pm, b, c)\| \sum_{|\alpha_2| \le N/2} \|\partial^{\alpha_2}(a_\pm, b, c)\|_{L^\infty} \\ & \times (1+t)^{\frac{1-3\vartheta}{2}} (1+t)^{-\frac{1+\vartheta}{2}} \left\| \langle \xi \rangle^{\gamma/2+1} w_{|\alpha|+|\beta|-l,\lambda_0} \partial^\alpha_\beta \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f \right\| \\ \lesssim & (1+t)^{\frac{1-3\vartheta}{2}} \mathcal{E}^{1/2}_{N-3,l-\frac{\gamma+2s}{\gamma},\lambda_0}(t) \mathcal{D}_{N,l,\lambda_0}(t). \end{aligned}$$

If $|\alpha_1| = 0$, then $\alpha_2 = \alpha$. We find that

$$\|(a_\pm, b, c)\|_{L^\infty} \lesssim \|\nabla_x (a_\pm, b, c)\|_{H^1} \lesssim \mathcal{D}^{1/2}_{N,l,\lambda_0}(t), \ \|\partial^{\alpha_2}(a_\pm, b, c)\| \lesssim \mathcal{E}^{1/2}_{N-3,l-\frac{\gamma+2s}{\gamma},\lambda_0}(t).$$

Therefore

$$
\begin{aligned}
I_{9,1} \lesssim & \|(a_\pm,b,c)\|_{L^\infty} \sum_{|\alpha|=|\alpha_2|\le N/2} \|\partial^{\alpha_2}(a_\pm,b,c)\| \\
& \times (1+t)^{\frac{1-3\vartheta}{2}}(1+t)^{-\frac{1+\vartheta}{2}} \left\|\langle\xi\rangle^{\gamma/2+1} w_{|\alpha|+|\beta|-l,\lambda_0}\partial^\alpha_\beta\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\right\| \\
\lesssim & (1+t)^{\frac{1-3\vartheta}{2}}\mathcal{E}^{1/2}_{N-3,l-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\mathcal{D}_{N,l,\lambda_0}(t).
\end{aligned}
$$

As to $I_{9,3}$, it follows form Sobolev's inequality and (2.20) that

$$
\begin{aligned}
I_{9,3} \lesssim & \sum_{|\alpha_2|\le N/2} \left\|\langle\xi\rangle^{\gamma/2+1} w_{|\alpha|+|\beta|-l,\lambda_0}\partial^{\alpha_1}_{\beta_1}\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\right\| \|\partial^{\alpha_2}(a_\pm,b,c)\|_{H^2} \\
& \times (1+t)^{1-\vartheta}(1+t)^{-1-\vartheta} \left\|\langle\xi\rangle^{\gamma/2+1} w_{|\alpha|+|\beta|-l,\lambda_0}\partial^\alpha_\beta\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\right\| \\
\lesssim & (1+t)^{1-\vartheta}\mathcal{E}^{1/2}_{N-3,l-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\mathcal{D}_{N,l,\lambda_0}(t).
\end{aligned}
$$

Case 3. $|\alpha|+|\beta|\ge N/2$ and $|\alpha_1|+|\beta_1|\le|\alpha_2|+|\beta_2|$. In this case, $|\alpha_1|+|\beta_1|\le N/2$ and $|\alpha_2|+|\beta_2|\ge N/4$. It is easy to see that the estimates on $I_{9,1}$ in this case are the same as that of Case 2, and we thus omit the details of its proof for brevity.

Now we shall estimate $I_{9,3}$ carefully. Firstly, since $|\alpha|+|\beta|\ge N/2\ge 5$ and $|\alpha_1|+|\beta_1|\le|\alpha_2|+|\beta_2|$, we see that $|\alpha|+|\beta|-|\alpha_1|-|\beta_1|-2\ge 1$, and hence

$$
\langle\xi\rangle^{\gamma/2+1} w_{|\alpha|+|\beta|-l,\lambda_0}(\xi) \lesssim w_{|\alpha_1|+|\beta_1|+2-l+\frac{\gamma+2s}{\gamma},\lambda_0}(\xi).
$$

Then it follows that

$$
\begin{aligned}
& \sum_{\substack{|\alpha_1|+|\beta_1|\le N/2\\ 2|\alpha_1|+2|\beta_1|\le|\alpha|+|\beta|}} \left\|\left|\langle\xi\rangle^{\gamma/2+1} w_{|\alpha|+|\beta|-l,\lambda_0}\partial^{\alpha_1}_{\beta_1}\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\right|_{L^2_\xi}\right\|_{L^\infty} \\
& \qquad \lesssim \sum_{\substack{|\alpha_1|+|\beta_1|\le N/2\\ 2|\alpha_1|+2|\beta_1|\le|\alpha|+|\beta|}} \left\|\left|\langle\xi\rangle^{\gamma/2+1} w_{|\alpha|+|\beta|-l,\lambda_0}\partial^{\alpha_1}_{\beta_1}\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\right|_{L^2_\xi}\right\|_{H^2} \\
& \qquad \lesssim \min\left\{(1+t)^{\frac{1+\vartheta}{2}}\mathcal{D}^{1/2}_{N,l,\lambda_0}(t), \mathcal{E}^{1/2}_{N-3,l-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\right\},
\end{aligned}
$$

which yields that for $\alpha_2>0$

$$
\begin{aligned}
I_{9,3} \lesssim & \sum_{\substack{|\alpha_1|+|\beta_1|\le N/2\\ 2|\alpha_1|+2|\beta_1|\le|\alpha|+|\beta|}} \left\|\left|\langle\xi\rangle^{\gamma/2+1} w_{|\alpha|+|\beta|-l,\lambda_0}\partial^{\alpha_1}_{\beta_1}\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\right|_{L^2_\xi}\right\|_{H^2} \|\partial^{\alpha_2}(a_\pm,b,c)\| \\
& \times (1+t)^{\frac{1-3\vartheta}{2}}(1+t)^{-\frac{1+\vartheta}{2}} \left\|\langle\xi\rangle^{\gamma/2+1} w_{|\alpha|+|\beta|-l,\lambda_0}\partial^\alpha_\beta\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\right\| \\
\lesssim & (1+t)^{\frac{1-3\vartheta}{2}}\mathcal{E}^{1/2}_{N-3,l-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\mathcal{D}_{N,l,\lambda_0}(t),
\end{aligned}
$$

and for $\alpha_2 = 0$

$$I_{9,3} \lesssim \sum_{|\alpha_1|+|\beta_1|\le N/2} \left\| \langle\xi\rangle^{\gamma/2+1} w_{|\alpha|+|\beta|-l,\lambda_0}\partial^{\alpha_1}_{\beta_1}\{\mathbf{I}_\pm - \mathbf{P}_\pm\}f \right\| \|(a_\pm, b, c)\|_{H^2}$$

$$\times (1+t)^{1-\vartheta}(1+t)^{-1-\vartheta} \left\| \langle\xi\rangle^{\gamma/2+1} w_{|\alpha|+|\beta|-l,\lambda_0}\partial^{\alpha}_{\beta}\{\mathbf{I}_\pm - \mathbf{P}_\pm\}f \right\|$$

$$\lesssim (1+t)^{1-\vartheta}\mathcal{E}^{1/2}_{N-3,l-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\mathcal{D}_{N,l,\lambda_0}(t),$$

Thus (2.19) holds true. This completes the proof of Lemma 2.5. □

We conclude this subsection with the following

Remark 2.3. *For the proof of Lemma 2.5, to control those terms* $|w_{\ell,\lambda_0}\partial^{\alpha_1}_{\beta_1}f|_{L^2_{\gamma/2+1}}$ *and* $|w_{\ell,\lambda_0}\partial^{\alpha}_{\beta}h|_{L^2_{\gamma/2+1}}$*, the extra dissipation corresponding to the last term in* $\mathcal{D}_{N,\ell,\lambda}(t)$ *defined by (1.12) has to be used, This in turn leads to the appearance of the time-growth factor* $(1+t)^{1-\vartheta}$*. However, the low-order energy functional* $\mathcal{E}^{1/2}_{N-3,l-\frac{\gamma+2s}{\gamma},\lambda_0}(t)$ *can be employed to kill this time-growth factor since one can obtain the proper time-decay of* $\mathcal{E}^{1/2}_{N-3,l-\frac{\gamma+2s}{\gamma},\lambda_0}(t)$*. In fact, as long as one chooses* $\vartheta = 1/4$*, i.e.* $1-\vartheta = 3/4$*, then*

$$(1+t)^{1-\vartheta}\mathcal{E}^{1/2}_{N-3,l-\frac{\gamma+2s}{\gamma},\lambda_0}(t)$$

can be expected to be bounded uniformly in time.

2.2. **Weighted estimates on L.** In this subsection, we deduce some weighted estimates on the linearized collision operator L with respect to the time-velocity exponential weight $w_{\ell,\lambda}(t,\xi)$.

⟨L1key⟩ **Lemma 2.6.** *Assume* $0 < s < 1$, $-3 < \gamma < -2s$, $|\beta| \ge 1$, $\ell \in \mathbb{R}$ *and* $\lambda \ge 0$*. One has*

$$\left\langle w^2_{\ell,\lambda}\partial_\beta \mathscr{L}_1 g, \partial_\beta g\right\rangle \gtrsim |w_{\ell,\lambda}\partial_\beta g|^2_{\mathbf{D}} - C\sum_{\beta_1<\beta}|w_{\ell,\lambda}\partial_{\beta_1}g|^2_{\mathbf{D}} - C|w_{\ell,\lambda}\partial_\beta g|^2_{L^2_{\gamma/2}}. \quad (2.21)$$ L1wlqlcoer2

For $\beta = 0$*, one also has*

$$\left\langle w^2_{\ell,\lambda}\mathscr{L}_1 g, g\right\rangle \gtrsim |w_{\ell,\lambda}g|^2_{\mathbf{D}} - C|w_{\ell,\lambda}g|^2_{L^2_{\gamma/2}}. \quad (2.22)$$ L1wlqlcoer1

Proof. To prove (2.22), we write

$$\left\langle w^2_{\ell,\lambda}\mathscr{L}_1 g, g\right\rangle = \langle \mathscr{L}_1 w_{\ell,\lambda}g, w_{\ell,\lambda}g\rangle + \left\{ \left\langle w^2_{\ell,\lambda}\mathscr{L}_1 g, g\right\rangle - \langle \mathscr{L}_1 w_{\ell,\lambda}g, w_{\ell,\lambda}g\rangle \right\}. \quad (2.23)$$ L1wlq

By Proposition A.2, the first part on the right-hand side of (2.23) has the lower-bound as

$$\langle \mathscr{L}_1 w_{\ell,\lambda}g, w_{\ell,\lambda}g\rangle \ge \delta|w_{\ell,\lambda}g|^2_{\mathbf{D}} - C|w_{\ell,\lambda}g|^2_{L^2_{\gamma/2}}, \quad (2.24)$$ L1wlqlower

for some constants $\delta > 0$, $C > 0$. Moreover, from simple calculations, by denoting the right-hand second part of (2.23) as I_{10}, we have

$$
\begin{aligned}
I_{10} =& \left\langle w_{\ell,\lambda}^2 \mathscr{L}_1 g, g\right\rangle - \langle \mathscr{L}_1 w_{\ell,\lambda} g, w_{\ell,\lambda} g\rangle \\
=& -\int_{\mathbb{R}^6\times\mathbb{S}^2} q(\xi-\xi_*,\sigma)\sqrt{\mu_*}\sqrt{\mu_*'}gg' w_{\ell,\lambda}\left[w_{\ell,\lambda}-w_{\ell,\lambda}'\right] d\xi_* d\xi d\sigma \\
=& -\frac{1}{2}\int_{\mathbb{R}^6\times\mathbb{S}^2} q(\xi-\xi_*,\sigma)\sqrt{\mu_*}\sqrt{\mu_*'}gg'\left[w_{\ell,\lambda}-w_{\ell,\lambda}'\right]^2 d\xi_* d\xi d\sigma.
\end{aligned}
$$

Therefore, the commutator part I_{10} can be bounded by

$$
|I_{10}| \lesssim \int_{\mathbb{R}^6\times\mathbb{S}^2} q(\xi-\xi_*,\sigma)\sqrt{\mu_*}\sqrt{\mu_*'}g^2\left[w_{\ell,\lambda}-w_{\ell,\lambda}'\right]^2 d\xi_* d\xi d\sigma. \tag{2.25}
$$

X

Next plugging (2.10) into (2.25), we obtain

$$
\begin{aligned}
|I_{10}| \lesssim& \int_{\mathbb{R}^6\times\mathbb{S}^2} q(\xi-\xi_*,\sigma)\mu_*^{1/4} g^2 |\xi-\xi_*|^2\theta^2\langle\xi\rangle^{2\gamma\ell-2} e^{\frac{2\lambda\langle\xi\rangle}{(1+t)^\vartheta}} d\xi_* d\xi d\sigma \\
\lesssim& \int_{\mathbb{R}^6} |\xi-\xi_*|^{\gamma+2}\mu_*^{1/4} g^2\langle\xi\rangle^{2\gamma\ell-2} e^{\frac{2\lambda\langle\xi\rangle}{(1+t)^\vartheta}} d\xi_* d\xi \lesssim |w_{\ell,\lambda}g|_{L^2_{\gamma/2}}^2.
\end{aligned}
\tag{2.26}
$$

Idone

Substituting (2.26) and (2.24) into (2.23), one has

$$
\left\langle w_{\ell,\lambda}^2 \mathscr{L}_1 g, g\right\rangle \gtrsim |w_{\ell,\lambda} g|_{\mathbf{D}}^2 - C|w_{\ell,\lambda} g|_{L^2_{\gamma/2}}^2.
$$

This completes the proof of (2.22).

Now we turn to prove (2.21). For this, we write

$$
\begin{aligned}
\left\langle w_{\ell,\lambda}^2 \partial_\beta \mathscr{L}_1 g, \partial_\beta g\right\rangle =& \left\langle w_{\ell,\lambda}^2 \mathscr{L}_1 \partial_\beta g, \partial_\beta g\right\rangle \\
&- \sum_{\beta_1<\beta} C_\beta^{\beta_1,\beta_2,\beta_3}\left\langle w_{\ell,\lambda}^2 \mathscr{T}_0(\partial_{\beta_1} g, \partial_{\beta_2}\sqrt{\mu}, \partial_{\beta_3}\sqrt{\mu}), \partial_\beta g\right\rangle,
\end{aligned}
\tag{2.27}
$$

L1dvsplit

where $\mathscr{T}_0$ is defined by (2.14). Recalling (2.22), we have

$$
\left\langle w_{\ell,\lambda}^2 \mathscr{L}_1 \partial_\beta g, \partial_\beta g\right\rangle \gtrsim |w_{\ell,\lambda}\partial_\beta g|_{\mathbf{D}}^2 - C|w_{\ell,\lambda}\partial_\beta g|_{L^2_{\gamma/2}}^2. \tag{2.28}
$$

L1 dvcoer3

We use I_{11} to denote the second term on the right-hand side of (2.27), and write

$$
\begin{aligned}
I_{11} =& \sum_{\beta_1<\beta} C_\beta^{\beta_1,\beta_2,\beta_3}\langle \mathscr{T}_0(w_{\ell,\lambda}\partial_{\beta_1} g, \partial_{\beta_2}\sqrt{\mu}, \partial_{\beta_3}\sqrt{\mu}), w_{\ell,\lambda}\partial_\beta g\rangle \\
&+ \sum_{\beta_1<\beta} C_\beta^{\beta_1,\beta_2,\beta_3}\Big\{ \left\langle w_{\ell,\lambda}^2 \mathscr{T}_0(\partial_{\beta_1} g, \partial_{\beta_2}\sqrt{\mu}, \partial_{\beta_3}\sqrt{\mu}), \partial_\beta g\right\rangle \\
&- \langle \mathscr{T}_0(w_{\ell,\lambda}\partial_{\beta_1} g, \partial_{\beta_2}\sqrt{\mu}, \partial_{\beta_3}\sqrt{\mu}), w_{\ell,\lambda}\partial_\beta g\rangle \Big\} := I_{11,1} + I_{11,2}.
\end{aligned}
$$

For $I_{11,1}$, we get from Proposition 3.1 in [4] that

$$
|I_{11,1}| \lesssim \sum_{\beta_1<\beta} |w_{\ell,\lambda}\partial_{\beta_1} g|_{\mathbf{D}} \cdot |w_{\ell,\lambda}\partial_\beta g|_{\mathbf{D}}.
$$

As to $I_{11,2}$, we have

$$\begin{aligned} I_{11,2} = -\int_{\mathbb{R}^6\times\mathbb{S}^2} & B(\xi-\xi_*,\sigma)\,[\partial_{\beta_3}\sqrt{\mu_*}]\left[\partial_{\beta_2}\sqrt{\mu_*'}\right][\partial_{\beta_1}g']w_{\ell,\lambda}\partial_\beta g \\ & \times\left[w_{\ell,\lambda}-w_{\ell,\lambda}'\right]d\xi_*d\xi d\sigma. \end{aligned} \tag{2.29}$$

XIsplit2

The estimates for (2.29) shares the same commutator properties as Γ. Thus following the same trick as in Lemma 2.1 and 2.2, one gets

$$|I_{11,2}| \lesssim \sum_{\beta_1<\beta} |w_{\ell,\lambda}\partial_{\beta_1}g|_{\mathbf{D}}\cdot|w_{\ell,\lambda}\partial_\beta g|_{\mathbf{D}}. \tag{2.30}$$

XI2

Plugging (2.28), (2.29) and (2.30) into (2.27), one can see that (2.21) holds true. Therefore, Lemma 2.6 is proved. □

As a direct application of Propositions A.2, A.3 and Lemma 2.6, we have the following weighted coercivity estimates on the linearized collision operator L with respect to the time-velocity exponential weight $w_{\ell,\lambda}(t,\xi)$.

Lemma 2.7. *Assume $0<s<1$, $-3<\gamma<-2s$, $\ell\in\mathbb{R}$ and $\lambda\geq 0$. One has*

$$\left\langle w_{\ell,\lambda}^2\mathscr{L}g,g\right\rangle \gtrsim |w_{\ell,\lambda}g|_{\mathbf{D}}^2 - C|w_{\ell,\lambda}g|^2_{L^2_{\gamma/2}}.$$

For $|\beta|\geq 1$, one also has

$$\left\langle w_{\ell,\lambda}^2\partial_\beta\mathscr{L}g,\partial_\beta g\right\rangle \gtrsim |w_{\ell,\lambda}\partial_\beta g|_{\mathbf{D}}^2 - C\sum_{\beta_1<\beta}|w_{\ell,\lambda}\partial_{\beta_1}g|^2_{\mathbf{D}} - C|w_{\ell,\lambda}\partial_\beta g|^2_{L^2_{\gamma/2}}.$$

By applying Sobolev's embedding theorem, and recalling the definitions of L and $\mathscr{L}$, one can further obtain

Lemma 2.8. *For $0<s<1$, $-3<\gamma<-2s$, $\ell\in\mathbb{R}$ and $\lambda\geq 0$. It holds that*

$$\sum_{|\beta|\leq N}\left\langle w_{\ell,\lambda}^2\partial_\beta Lg,\partial_\beta g\right\rangle \gtrsim \sum_{|\beta|\leq N}|w_{\ell,\lambda}\partial_\beta g|_{\mathbf{D}}^2 - C|g|^2_{L^2_{B_C}},$$

where g is a vector function in $\mathbb{R}^2$, and B_C denotes the closed ball in $\mathbb{R}^3_\xi$ with center zero and radius a constant C.

3. Global a priori estimates

⟨sec3⟩ In this section we are going to prove the main result of the paper Theorem 1.1. The key point is to deduce the uniform-in-time *a priori* estimates of solutions to the Vlasov-Maxwell-Boltzmann system

$$\begin{cases} \partial_t f + \xi\cdot\nabla_x f - E\cdot\xi\mu^{1/2}q_1 + Lf = S, \\ \partial_t E - \nabla_x\times B = -\left\langle \xi\mu^{1/2}, f_+ - f_-\right\rangle, \\ \partial_t B + \nabla_x\times E = 0, \\ \nabla_x\cdot E = \left\langle \mu^{1/2}, f_+-f_-\right\rangle, \quad \nabla_x\cdot B = 0, \end{cases} \tag{3.1}$$

ns

where the nonlinear term $S=[S_+,S_-]$ is given by

$$S = \Gamma(f,f) + \frac{1}{2}q_0E\cdot\xi f - q_0(E+\xi\times B)\cdot\nabla_\xi f. \tag{3.2}$$

def.S

For that, let (f, E, B) be a smooth solution to (3.1) over the time interval $0 \le t \le T$ with initial data (f_0, E_0, B_0) for $0 < T \le \infty$, and further suppose that (f, E, B) satisfies

$$X(t) \le \delta^2, \tag{3.3}$$ apX

where $X(t)$ is given in (1.14) and the constant $\delta > 0$ is sufficiently small. Here recall that $X(t)$ also depends on parameters N_1, ℓ_1, λ_0, ϑ and ϵ_0 which will be fixed in the proof.

3.1. Macro structure and macro dissipation.

In the first subsection, we consider the macroscopic structure of system (3.1) in terms of the Grad's moment method [15, 16] in order to find out the macroscopic dissipation. As in [21], by taking velocity integrations of the first equation of (3.1) with respect to the velocity moments

$$\mu^{1/2}, \quad \xi_i \mu^{1/2} \ (i = 1, 2, 3), \quad \frac{1}{6}(|\xi|^2 - 3)\mu^{1/2},$$

one has

$$\begin{aligned}
&\partial_t a_\pm + \nabla_x \cdot b + \nabla_x \cdot \left\langle \xi \mu^{1/2}, \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f \right\rangle = \left\langle \mu^{1/2}, S_\pm \right\rangle, \qquad (3.4) \\
&\partial_t \left[b_i + \left\langle \xi_i \mu^{1/2}, \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f \right\rangle \right] + \partial_i (a_\pm + 2c) \mp E_i \\
&\qquad + \nabla_x \cdot \left\langle \xi \xi_i \mu^{1/2}, \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f \right\rangle = \left\langle \xi_i \mu^{1/2}, S_\pm - L_\pm f \right\rangle, \\
&\partial_t \left[c + \frac{1}{6} \left\langle (|\xi|^2 - 3)\mu^{1/2}, \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f \right\rangle \right] + \frac{1}{3} \nabla_x \cdot b \\
&\qquad + \frac{1}{6} \nabla_x \cdot \left\langle \left(|\xi|^2 - 3\right) \xi \mu^{1/2}, \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f \right\rangle \\
&= \frac{1}{6} \left\langle (|\xi|^2 - 3)\mu^{1/2}, S_\pm - L_\pm f \right\rangle.
\end{aligned}$$ m0

As in [13], define the high-order moment functions $\Theta(f_\pm) = (\Theta_{ij}(f_\pm))_{3\times 3}$ and $\Lambda(f_\pm) = (\Lambda_1(f_\pm), \Lambda_2(f_\pm), \Lambda_3(f_\pm))$ by

$$\Theta_{ij}(f_\pm) = \left\langle (\xi_i \xi_j - 1)\mu^{1/2}, f_\pm \right\rangle, \quad \Lambda_i(f_\pm) = \frac{1}{10} \left\langle (|\xi|^2 - 5)\xi_i \mu^{1/2}, f_\pm \right\rangle.$$

Further taking velocity integrations of the first equation of (3.1) with respect to the above high-order moments one has

$$\begin{aligned}
&\partial_t \left[\Theta_{ii}(\{\mathbf{I}_\pm - \mathbf{P}_\pm\} f) + 2c \right] + 2\partial_i b_i = \Theta_{ii}(r_\pm + S_\pm), \\
&\partial_t \Theta_{ij}(\{\mathbf{I}_\pm - \mathbf{P}_\pm\} f) + \partial_j b_i + \partial_i b_j + \nabla_x \cdot \left\langle \xi \mu^{1/2}, \{\mathbf{I}_\pm - \mathbf{P}_\pm\} f \right\rangle \\
&\qquad = \Theta_{ij}(r_\pm + S_\pm) + \left\langle \mu^{1/2}, S_\pm \right\rangle, \quad i \ne j, \qquad (3.5) \\
&\partial_t \Lambda_i(\{\mathbf{I}_\pm - \mathbf{P}_\pm\} f) + \partial_i c = \Lambda_i(r_\pm + S_\pm),
\end{aligned}$$ m2ij

where $r_\pm = -\xi\cdot\nabla_x\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f - L_\pm f$. Notice that we have used (3.4) to derive (3.5). Moreover, it is straightforward to compute from integration by parts that

$$
\begin{aligned}
\left\langle \mu^{1/2}, S_\pm \right\rangle &= 0,\\
\left\langle \xi\mu^{1/2}, S_\pm \right\rangle &= \pm E a_\pm \pm b\times B \pm \left\langle \xi\mu^{1/2}, \{\mathbf{I}_\pm-\mathbf{P}_\pm\}f \right\rangle \times B\\
&\quad + \left\langle \xi\mu^{1/2}, \Gamma_\pm(f,f) \right\rangle,\\
\frac{1}{6}\left\langle \left(|\xi|^2-3\right)\mu^{1/2}, S_\pm \right\rangle &= \pm\frac{1}{3} b\cdot E \pm \frac{1}{3}\left\langle \xi\mu^{1/2}, \{\mathbf{I}_\pm-\mathbf{P}_\pm\}f \right\rangle \cdot E\\
&\quad + \frac{1}{6}\left\langle \left(|\xi|^2-3\right)\mu^{1/2}, \Gamma_\pm(f,f) \right\rangle.
\end{aligned}
$$

Now we define the macro dissipation $\mathcal{D}_{N,mac}(t)$ by

$$
\begin{aligned}
\mathcal{D}_{N,mac}(t) = \sum_{|\alpha|\le N-1} &\|\nabla_x\partial^\alpha(a_\pm,b,c)\|^2 + \|a_+-a_-\|^2\\
&+ \|E\|^2_{H^{N-1}} + \|\nabla_x B\|^2_{H^{N-2}}.
\end{aligned}
$$

With the above macro structure of the system (3.1) in hand, we have

⟨lem.mad⟩ **Lemma 3.1.** *For any integer N with $8\le N\le N_1$, there is an interactive energy functional $\mathcal{E}^{int}_N(t)$ such that*

$$
|\mathcal{E}^{int}_N(t)| \lesssim \sum_{|\alpha|\le N}\left(\|\partial^\alpha f\|^2 + \|\partial^\alpha(E,B)\|^2\right) \tag{3.6}
$$
lem.mad.ad1

and

$$
\frac{d}{dt}\mathcal{E}^{int}_N(t) + \kappa\mathcal{D}_{N,mac}(t) \lesssim \sum_{|\alpha|\le N}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2_{\mathbf{D}} + \overline{\mathcal{E}}_N(t)\overline{\mathcal{D}}_N(t) \tag{3.7}
$$
ma-mi

for $0\le t\le T$.

Proof. Basing on the previous work [13] and [14] and combing Lemma A.1, it is a quite standard process to obtain (3.7) with (3.6) being satisfied. We hence omit the details for brevity. □

3.2. **Uniform spatial energy estimate.** In this section we derive the basic energy estimates on $\overline{\mathcal{E}}_{N_1}(t)$ which contains only the spatial derivatives.

⟨lem.n1⟩ **Lemma 3.2.** *Let $l_1-1\ge N_1\ge 14$. There is an energy functional $\overline{\mathcal{E}}_{N_1}(t)$ such that*

$$
\frac{d}{dt}\overline{\mathcal{E}}_{N_1}(t) + \kappa\overline{\mathcal{D}}_{N_1}(t) \lesssim \frac{\delta}{(1+t)^{1+\vartheta}}\mathcal{D}_{N_1,\ell_1,\lambda_0}(t) + \overline{\mathcal{E}}_{N_1}(t)\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t) \tag{3.8}
$$
lem.n1.1

for $0\le t\le T$.

Proof. It is straightforward to establish the energy identities

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|\le N_1}\left(\|\partial^\alpha f\|^2 + \|\partial^\alpha(E,B)\|^2\right) + \sum_{|\alpha|\le N_1}\langle L\partial^\alpha f, \partial^\alpha f\rangle&\\
= \sum_{|\alpha|\le N_1}\langle \partial^\alpha S, \partial^\alpha f\rangle.&
\end{aligned}
\tag{3.9}
$$
lem.n1.p1

Moreover, from Lemma 3.1 as well as (3.3),

$$\frac{d}{dt}\mathcal{E}^{int}_{N_1}(t)+\kappa\mathcal{D}_{N_1,mac}(t)\lesssim\sum_{|\alpha|\le N_1}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2_{\mathbf{D}}+\delta^2\overline{\mathcal{D}}_{N_1}(t). \tag{3.10}$$ lem.n1.p2

Then, since $\delta>0$ can be small enough, the proper linear combination of (3.9) and (3.10) implies that there is an energy functional $\overline{\mathcal{E}}_{N_1}(t)$ satisfying (1.10) such that

$$\frac{d}{dt}\overline{\mathcal{E}}_{N_1}(t)+\kappa\overline{\mathcal{D}}_{N_1}(t)\lesssim\mathcal{I}^{(1)}_{N_1}(t), \tag{3.11}$$ lem.n1.p3

where

$$\mathcal{I}^{(1)}_{N_1}(t)=\langle S,f\rangle+\sum_{1\le|\alpha|\le N_1}\langle\partial^\alpha S,\partial^\alpha f\rangle.$$

Finally, we claim that

$$\begin{aligned}\mathcal{I}^{(1)}_{N_1}(t)\lesssim\mathcal{E}^{1/2}_{N_1-3,l_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\overline{\mathcal{D}}_{N_1}(t)+\frac{\delta}{(1+t)^{1+\vartheta}}\mathcal{D}_{N_1,\ell_1,\lambda_0}(t)\\+\overline{\mathcal{E}}^{1/2}_{N_1}(t)\mathcal{E}^{1/2}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\overline{\mathcal{D}}^{1/2}_{N_1}(t).\end{aligned} \tag{3.12}$$ lem.n1.p4

Therefore, the desired estimate (3.8) follows from plugging (3.12) into (3.11) and applying (3.3) and the Cauchy-Schwarz inequality to the first and third terms on the right-hand side of (3.12), respectively. This then completes the proof of Lemma 3.2. □

Proof of (3.12). We first consider the estimate of $\mathcal{I}^{(1)}_{N_1}(t)$ corresponding to $\Gamma(f,f)$ in the nonlinear term S. Using Lemma 2.5, it directly follows that it is bounded up to a generic constant by $\mathcal{E}^{1/2}_{N_1-3,l_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\overline{\mathcal{D}}_{N_1}(t)$. Recall from the definition of $X(t)$, and hence,

$$\|\nabla_x(E,B)\|_{H^5}\le\frac{X^{1/2}(t)}{(1+t)^{1+\vartheta}}\le\frac{\delta}{(1+t)^{1+\vartheta}}.$$

For the zero-order term related to the electromagnetic field, it holds that

$$\begin{aligned}&\frac12\langle q_0E\cdot\xi f-q_0(E+\xi\times B)\cdot\nabla_\xi f,f\rangle=\frac12\langle q_0E\cdot\xi f,f\rangle\\&\lesssim\iint_{\mathbb{R}^3\times\mathbb{R}^3}|E|\cdot|\xi|(|\mathbf{P}f|^2+|\{\mathbf{I}-\mathbf{P}\}f|^2)\,dxd\xi\\&\lesssim\|E\|\cdot\|(a_\pm,b,c)\|_{L^\infty}\|(a_\pm,b,c)\|+\|E\|_{L^\infty}\iint_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|\cdot|\{\mathbf{I}-\mathbf{P}\}f|^2\,dxd\xi\\&\lesssim\overline{\mathcal{E}}^{1/2}_{N_1}(t)\overline{\mathcal{D}}_{N_1}(t)+\frac{\delta}{(1+t)^{1+\vartheta}}\mathcal{D}_{N_1,\ell_1,\lambda_0}(t).\end{aligned}$$

For the ∂^α derivative term related to (E,B) with $1\le|\alpha|\le N_1$, we write

$$\begin{aligned}&\langle\partial^\alpha(E\cdot\xi f),\partial^\alpha f\rangle=\sum_{\alpha_1\le\alpha}C^\alpha_{\alpha_1}\langle\partial^{\alpha_1}E\cdot\xi\partial^{\alpha-\alpha_1}f,\partial^\alpha f\rangle\\&=\sum_{\alpha_1\le\alpha}C^\alpha_{\alpha_1}\langle\partial^{\alpha_1}E\cdot\xi\partial^{\alpha-\alpha_1}\mathbf{P}f,\partial^\alpha f\rangle+\sum_{\alpha_1\le\alpha}C^\alpha_{\alpha_1}\langle\partial^{\alpha_1}E\cdot\xi\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f,\partial^\alpha f\rangle.\end{aligned} \tag{3.13}$$ Epurex

By Sobolev's inequality, one can easily prove that the first term on the right-hand side of (3.13) is bounded by

$$C\,\overline{\mathcal{E}}_{N_1}^{1/2}(t)\overline{\mathcal{D}}_{N_1}(t).$$

Now we turn to estimate the second term on the right hand side of (3.13). Notice that

$$l_1-1\geq N_1,\quad 1>s\geq 1/2,$$

which implies

$$\langle\xi\rangle^{1-(\gamma+2s)}\leq w_{N_1-l_1}(\xi).$$

Therefore if $|\alpha_1|\leq 4$, it follows that

$$\begin{aligned}
&\sum_{|\alpha_1|\leq 4,\,\alpha_1\leq\alpha}\langle\partial^{\alpha_1}E\cdot\xi\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f,\partial^\alpha f\rangle\\
&\lesssim\sum_{|\alpha_1|\leq 4}\|\partial^{\alpha_1}E\|_{L^\infty}\left\||\xi|\langle\xi\rangle^{-\frac{\gamma+2s}{2}}\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f\right\|\cdot\left\|\langle\xi\rangle^{\frac{\gamma+2s}{2}}\partial^\alpha f\right\|\\
&\lesssim\frac{\delta}{(1+t)^{1+\vartheta}}\mathcal{D}_{N_1,\ell_1,\lambda_0}(t).
\end{aligned}$$

On the other hand, for $|\alpha_1|\geq 5$ and $N_1\geq 14$, we also see that

$$|\alpha-\alpha_1|+2\leq N_1-3,\quad |\xi|\langle\xi\rangle^{-(\gamma+2s)}\leq w_{|\alpha-\alpha_1|+2-l_1+\frac{\gamma+2s}{\gamma}}(\xi).$$

With the above observation, we obtain

$$\begin{aligned}
&\sum_{|\alpha_1|\geq 5,\,\alpha_1\leq\alpha}\langle\partial^{\alpha_1}E\cdot\xi\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f,\partial^\alpha f\rangle\\
&\lesssim\sum_{|\alpha_1|\geq 5,\,\alpha_1\leq\alpha}\|\partial^{\alpha_1}E\|\cdot\sup_x\left||\xi|\langle\xi\rangle^{-\frac{\gamma+2s}{2}}\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f\right|_{L^2_\xi}\left\|\langle\xi\rangle^{\frac{\gamma+2s}{2}}\partial^\alpha f\right\|\\
&\lesssim\overline{\mathcal{E}}_{N_1}^{1/2}(t)\mathcal{E}^{1/2}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\overline{\mathcal{D}}_{N_1}^{1/2}(t),
\end{aligned}$$

where the Sobolev inequality $\|g\|_{L^\infty}\leq C\|\nabla_x g\|_{H^1}$ for any function $g=g(x)\in H^2$ has been used. Combing both cases, one can see that

$$\begin{aligned}
\sum_{|\alpha|\leq N_1}\langle\partial^\alpha(E\cdot\xi f),\partial^\alpha f\rangle\lesssim&\frac{\delta}{(1+t)^{1+\vartheta}}\mathcal{D}_{N_1,\ell_1,\lambda_0}(t)\\
&+\overline{\mathcal{E}}_{N_1}^{1/2}(t)\mathcal{E}^{1/2}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\overline{\mathcal{D}}_{N_1}^{1/2}(t).
\end{aligned}$$

Remark 3.1. *We would here point out that $l_1-1\geq N_1$ is needed in order to control the worst term*

$$\sum_{|\alpha|=N_1}\langle E\cdot\xi\partial^\alpha f,\partial^\alpha f\rangle.$$

And $|\alpha_1|\geq 5$ can not be improved if one intends to control the term

$$\sup_x\left||\xi|\langle\xi\rangle^{-\frac{\gamma+2s}{2}}\partial^{\alpha-\alpha_1}f\right|_{L^2_\xi}$$

by $\mathcal{E}^{1/2}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)$. This implies that derivatives of the electromagnetic field (E,B) which enjoys the explicit time-decay rate can be up to order six.

Finally, in a similar way as before, it holds that

$$
\begin{aligned}
\langle \partial^\alpha\{(\xi\times B)\cdot\nabla_\xi f\},\partial^\alpha f\rangle &= \sum_{0<\alpha_1\le\alpha} C^\alpha_{\alpha_1}\langle(\xi\times\partial^{\alpha_1}B)\cdot\nabla_\xi\partial^{\alpha-\alpha_1}f,\partial^\alpha f\rangle\\
&\lesssim \frac{\delta}{(1+t)^{1+\vartheta}}\mathcal{D}_{N_1,\ell_1,\lambda_0}(t)+\overline{\mathcal{E}}^{1/2}_{N_1}(t)\mathcal{E}^{1/2}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\overline{\mathcal{D}}^{1/2}_{N_1}(t).
\end{aligned}
$$

Therefore, the claimed inequality (3.12) follows by collecting all the estimates. □

3.3. **The highest-order energy estimate with weight.** In this section we turn to the weighted energy estimates on $\mathcal{E}_{N_1,\ell_1,\lambda_0}(t)$. As pointed out in [8], due to the regularity-loss property of the whole system, two difficulties naturally come out, that is, the weighted highest-order energy functional $\mathcal{E}_{N_1,\ell_1,\lambda_0}(t)$ can only be expected to increase in time and it is also a problem to obtain the weighted estimate on derivatives of the highest order N_1 for the linear term $E\cdot\xi\mu^{1/2}$. To overcome the first difficulty, we shall refine in the following lemma the nonlinear estimates in order to make use of the time-decay property of the lower-order energy functional $\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)$, and postpone the estimate on $E\cdot\xi\mu^{1/2}$ to Lemma 3.4 in terms of the trick firstly introduced in [22].

⟨lem.n1w⟩ **Lemma 3.3.** *There is an energy functional $\mathcal{E}_{N_1,\ell_1,\lambda_0}(t)$ with $\lambda_0>0$, $\vartheta=\frac14$ and $l_1-1\ge N_1\ge 14$ such that*

$$
\frac{d}{dt}\mathcal{E}_{N_1,\ell_1,\lambda_0}(t)\ +\ \kappa\mathcal{D}_{N_1,\ell_1,\lambda_0}(t)\ \ \lesssim\ \sum_{|\alpha|=N_1}\left\langle\partial^\alpha E\cdot\xi\mu^{1/2},w^2_{|\alpha|-\ell_1,\lambda_0}\partial^\alpha f\right\rangle \tag{3.14}
$$
lem.n1w.1

for $0\le t\le T$.

Proof. Starting from the first equation of (3.1), the energy estimate on $\partial^\alpha f$ with $1\le|\alpha|\le N_1$ weighted by the time-velocity dependent function $w_{|\alpha|-\ell_1,\lambda_0}=w_{|\alpha|-\ell_1,\lambda_0}(t,\xi)$ gives

$$
\begin{aligned}
&\frac12\frac{d}{dt}\sum_{1\le|\alpha|\le N_1}\|\partial^\alpha f\|^2_{|\alpha|-\ell_1,\lambda_0}+\sum_{1\le|\alpha|\le N_1}\left\langle L\partial^\alpha f,w^2_{|\alpha|-\ell_1,\lambda_0}\partial^\alpha f\right\rangle\\
&\quad+\frac{\vartheta\lambda_0}{(1+t)^{1+\vartheta}}\left\|\langle\xi\rangle^{1/2}\partial^\alpha f\right\|^2_{|\alpha|-\ell_1,\lambda_0}=\sum_{1\le|\alpha|\le N_1}\left\langle\partial^\alpha S,w^2_{|\alpha|-\ell_1,\lambda_0}\partial^\alpha f\right\rangle\\
&\qquad\qquad+\sum_{1\le|\alpha|\le N_1}\left\langle\partial^\alpha E\cdot\xi\mu^{1/2},w^2_{|\alpha|-\ell_1,\lambda_0}\partial^\alpha f\right\rangle.
\end{aligned}\tag{3.15}
$$
lem.n1w.p1

Similarly, from the first equation of (3.1), one has the weighted energy estimate on $\{\mathbf{I}-\mathbf{P}\}f$

$$
\begin{aligned}
&\frac12\frac{d}{dt}\|\{\mathbf{I}-\mathbf{P}\}f\|^2_{-\ell_1,\lambda_0}+\kappa\|\mathbf{I}-\mathbf{P}f\|^2_{\mathbf{D},-\ell_1,\lambda_0}\\
&\quad+\frac{\vartheta\lambda_0}{(1+t)^{1+\vartheta}}\left\|\langle\xi\rangle^{1/2}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{-\ell_1,\lambda_0}\\
&\lesssim\left\langle S,w^2_{-\ell_1,\lambda_0}\{\mathbf{I}-\mathbf{P}\}f\right\rangle+\overline{\mathcal{D}}_{N_1}(t)\overline{\mathcal{E}}_{N_1}(t)+\overline{\mathcal{D}}_{N_1}(t).
\end{aligned}\tag{3.16}
$$
lem.n1w.p2

and the weighted energy estimate on $\{\mathbf{I}-\mathbf{P}\}\partial^\alpha_\beta f$ with $|\alpha|+|\beta|\le N_1$ and $|\beta|\ge 1$

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{\substack{|\beta|\ge 1\\|\alpha|+|\beta|\le N_1}}\left\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{|\alpha|+|\beta|-\ell_1,\lambda_0}\\
&+\kappa\sum_{\substack{|\beta|\ge 1\\|\alpha|+|\beta|\le N_1}}\left(\left\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{\mathbf{D},|\alpha|+|\beta|-\ell_1,\lambda_0}+\frac{\lambda_0}{(1+t)^{1+\vartheta}}\left\|\langle\xi\rangle^{1/2}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{|\alpha|+|\beta|-\ell_1,\lambda_0}\right)\\
&\lesssim\sum_{|\alpha|\le N_1}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2_{\mathbf{D},|\alpha|-\ell_1,\lambda_0}+\sum_{|\alpha|\le N_1-1}\left(\|\nabla_x\partial^\alpha(a_\pm,b,c)\|^2+\|\partial^\alpha E\|^2\right)\\
&\quad+\overline{\mathcal{E}}_{N_1}(t)\overline{\mathcal{D}}_{N_1}(t)+\sum_{\substack{|\beta|\ge 1\\|\alpha|+|\beta|\le N_1}}\left\langle\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f,w^2_{|\alpha|+|\beta|-\ell_1,\lambda_0}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\rangle\\
&\quad+\sum_{\substack{|\beta|\ge 1\\|\alpha|+|\beta|\le N_1}}\left\langle\partial^\alpha_\beta S,w^2_{|\alpha|+|\beta|-\ell_1,\lambda_0}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\rangle,
\end{aligned}
\tag{3.17}
$$
lem.n1w.p3

where e_i denotes the multi-index with the ith element unit and the rest ones zeros.

To be continued, we need to deal with the term

$$
\sum_{\substack{|\beta|\ge 1\\|\alpha|+|\beta|\le N_1}}\left\langle\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f,w^2_{|\alpha|+|\beta|-\ell_1,\lambda_0}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\rangle
$$

carefully. For doing this, we write

$$
\begin{aligned}
&\sum_{\substack{|\beta|\ge 1\\|\alpha|+|\beta|\le N_1}}\left\langle\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f,w^2_{|\alpha|+|\beta|-\ell_1,\lambda_0}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\rangle\\
&=\sum_{\substack{|\beta|\ge 1\\|\alpha|+|\beta|\le N_1}}\Big\langle w_{|\alpha|+|\beta|+\frac12-\ell_1,\lambda_0}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f,\\
&\qquad\qquad\partial_{e_i}\left[w_{|\alpha|+|\beta-e_i|+\frac12-\ell_1,\lambda_0}\partial^\alpha_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\right]\Big\rangle\\
&\quad-\sum_{\substack{|\beta|\ge 1\\|\alpha|+|\beta|\le N_1}}\Big\langle w_{|\alpha|+|\beta|+\frac12-\ell_1,\lambda_0}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f,\\
&\qquad\qquad\partial_{e_i}\left[w_{|\alpha|+|\beta-e_i|+\frac12-\ell_1,\lambda_0}\right]\partial^\alpha_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\Big\rangle.
\end{aligned}
\tag{3.18}
$$
transterm1

It is easy to see that the second inner product on the right-hand side of (3.18) is bounded by

$$
\begin{aligned}
&C\left\|w_{|\alpha|+|\beta|-\ell_1,\lambda_0}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_{\gamma/2}}\left\|w_{|\alpha|+|\beta-e_i|-\ell_1,\lambda_0}\partial^\alpha_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L^2_{\gamma/2}}\\
&\qquad\lesssim\sum_{\substack{|\bar\beta|=|\beta|-1,\ |\beta|\ge 1\\|\alpha|+|\bar\beta|\le N_1}}\|\partial^\alpha_{\bar\beta}\{\mathbf{I}-\mathbf{P}\}f\|^2_{\mathbf{D},|\alpha|+|\bar\beta|-\ell_1,\lambda_0}.
\end{aligned}
\tag{3.19}
$$
transterm11

Next noticing $1/2 \leq s < 1$, applying the Parseval identity, one can find that the first inner product on the right-hand side of (3.18) can be dominated by

$$
\begin{aligned}
&\int_{\mathbf{R}^3}\Big|\int_{\mathbf{R}^3} \mathbb{i}k_i\mathcal{F}_\xi\left[w_{|\alpha|+|\beta|+\frac{1}{2}-\ell_1,\lambda_0}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\right] \\
&\qquad\times\overline{\mathcal{F}_\xi\left[w_{|\alpha|+|\beta-e_i|+\frac{1}{2}-\ell_1,\lambda_0}\partial^{\alpha}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\right]}dk\Big|dx \\
&\lesssim \int_{\mathbf{R}^3}\left|\langle k\rangle^{\frac{1}{2}}\mathcal{F}_\xi\left[w_{|\alpha|+|\beta|+\frac{1}{2}-\ell_1,\lambda_0}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\right]\right|_{L^2_k} \\
&\qquad\times\left|\langle k\rangle^{\frac{1}{2}}\mathcal{F}_\xi\left[w_{|\alpha|+|\beta-e_i|+\frac{1}{2}-\ell_1,\lambda_0}\partial^{\alpha}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\right]\right|_{L^2_k}dx \\
&\lesssim \int_{\mathbf{R}^3}\left|w_{|\alpha|+|\beta|+\frac{1}{2}-\ell_1,\lambda_0}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\right|_{H^s_\xi}\left|[w_{|\alpha|+|\beta-e_i|+\frac{1}{2}-\ell_1,\lambda_0}\partial^{\alpha}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\right|_{H^s_\xi}dx \\
&\lesssim \left\|w_{|\alpha|+|\beta|-\ell_1,\lambda_0}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\right\|_{H^s_\gamma}\left\|[w_{|\alpha|+|\beta-e_i|-\ell_1,\lambda_0}\partial^{\alpha}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\right\|_{H^s_\gamma} \\
&\lesssim \sum_{\substack{|\bar{\beta}|=|\beta|-1,\ |\beta|\geq 1\\ |\alpha|+|\bar{\beta}|\leq N_1}}\|\partial^{\alpha}_{\bar{\beta}}\{\mathbf{I}-\mathbf{P}\}f\|^2_{\mathbf{D},|\alpha|+|\bar{\beta}|-\ell_1,\lambda_0},
\end{aligned}
\tag{3.20}
$$
transterm12

where $\mathcal{F}_\xi$ means the Fourier transform with respect to ξ-variable with k the corresponding frequency variable, $\bar{\cdot}$ denotes the complex conjugate, and $\mathbb{i}=\sqrt{-1}\in\mathbb{C}$ is the pure imaginary unit.

Then, the proper linear combination of (3.9), (3.10), (3.15), (3.16), (3.17), (3.18), (3.19) and (3.20) implies that there is an energy functional $\mathcal{E}_{N_1,\ell_1,\lambda_0}(t)$ satisfying (1.10) such that

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E}_{N_1,\ell_1,\lambda_0}(t)+\kappa\mathcal{D}_{N_1,\ell_1,\lambda_0}(t) &\lesssim \mathcal{I}^{(2)}_{N_1,\ell_1,\lambda_0}(t) \\
&\quad+\sum_{|\alpha|=N_1}\left\langle\partial^\alpha E\cdot\xi\mu^{1/2},w^2_{|\alpha|-\ell_1,\lambda_0}\partial^\alpha f\right\rangle,
\end{aligned}
\tag{3.21}
$$
lem.n1w.p4

where

$$
\begin{aligned}
\mathcal{I}^{(2)}_{N_1,\ell_1,\lambda_0}(t) &= \langle S,f\rangle+\sum_{1\leq|\alpha|\leq N_1}\langle\partial^\alpha S,\partial^\alpha f\rangle \\
&\quad+\left\langle S,w^2_{-\ell_1,\lambda_0}\{\mathbf{I}-\mathbf{P}\}f\right\rangle+\underbrace{\sum_{1\leq|\alpha|\leq N_1}\left\langle\partial^\alpha S,w^2_{|\alpha|-\ell_1,\lambda_0}\partial^\alpha f\right\rangle}_{\mathscr{A}} \\
&\quad\underbrace{+\sum_{m=1}^{N_1}C_m\sum_{\substack{|\beta|=m\\|\alpha|+|\beta|\leq N_1}}\left\langle\partial^\alpha_\beta S,w^2_{|\alpha|+|\beta|-\ell_1,\lambda_0}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\rangle}_{\mathscr{B}}.
\end{aligned}
\tag{3.22}
$$
def.i2

We now claim that

$$\mathcal{I}^{(2)}_{N_1,\ell_1,\lambda_0}(t) \lesssim (1+t)^{1-\vartheta}\mathcal{E}^{1/2}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\mathcal{D}_{N_1,\ell_1,\lambda_0}(t) + \overline{\mathcal{E}}^{1/2}_{N_1}(t)\mathcal{D}_{N_1,\ell_1,\lambda_0}(t) + \frac{1}{\lambda_0}(1+t)^{1+\vartheta}\|\nabla_x(E,B)\|_{H^4}\mathcal{D}_{N_1,\ell_1,\lambda_0}(t). \tag{3.23}$$

lem.n1w.p5

By the assumption $X(t)\le\delta^2$, one gets that

$$\mathcal{I}^{(2)}_{N_1,\ell_1,\lambda_0}(t) \lesssim \delta\mathcal{D}_{N_1,\ell_1,\lambda_0}(t). \tag{3.24}$$

lem.n1w.p6

Then putting (3.24) into (3.21), we see that the desired estimate (3.14) follows. This completes the proof of Lemma 3.3. □

Proof of (3.23). For brevity, we only present the estimate of $\mathscr{A}$ and $\mathscr{B}$ on the right-hand side of (3.22) since the estimate on other terms is simpler or follows in the completely same way. Take α with $1\le|\alpha|\le N_1$. For the inner product term related to $\partial^\alpha\Gamma(f,f)$, by using (2.18) in Lemma 2.5, it follows that

$$\left\langle \partial^\alpha\Gamma(f,f), w^2_{|\alpha|-\ell_1,\lambda_0}\partial^\alpha f\right\rangle \lesssim (1+t)^{1-\vartheta}\mathcal{E}^{1/2}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\mathcal{D}_{N_1,\ell_1,\lambda_0}(t).$$

Next, for the term $E\cdot\xi f$ in S, one has

$$\begin{aligned}
&\left\langle \partial^\alpha(E\cdot\xi f), w^2_{|\alpha|-\ell_1,\lambda_0}\partial^\alpha f\right\rangle = \sum_{\alpha_1\le\alpha} C^\alpha_{\alpha_1}\left\langle \partial^{\alpha_1}E\cdot\xi\partial^{\alpha-\alpha_1}f, w^2_{|\alpha|-\ell_1,\lambda_0}\partial^\alpha f\right\rangle\\
&\lesssim \sum_{|\alpha_1|\le 2}\|\partial^{\alpha_1}E\|_{L^\infty}\iint_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|w^2_{|\alpha|-\ell_1,\lambda_0}\left(|\partial^{\alpha-\alpha_1}f|^2+|\partial^\alpha f|^2\right)dxd\xi\\
&\quad+\sum_{\substack{|\alpha_1|\ge 3\\ \text{or } \alpha_1=\alpha}}\|\partial^{\alpha_1}E\|\cdot\sup_x\left||\xi|\langle\xi\rangle^{-\frac{\gamma+2s}{2}}w_{|\alpha|-\ell_1,\lambda_0}\partial^{\alpha-\alpha_1}f\right|_{L^2_\xi}\left\|\langle\xi\rangle^{\frac{\gamma+2s}{2}}w_{|\alpha|-\ell_1,\lambda_0}\partial^\alpha f\right\|\\
&\lesssim \frac{1}{\lambda_0}(1+t)^{1+\vartheta}\|\nabla_x(E,B)\|_{H^3}\mathcal{D}_{N_1,\ell_1,\lambda_0}(t)+\overline{\mathcal{E}}^{1/2}_{N_1}(t)\mathcal{D}_{N_1,\ell_1,\lambda_0}(t),
\end{aligned}$$

where we have used the Sobolev inequality $\|g\|_{L^\infty}\lesssim\|\nabla g\|_{H^1}$ and the fact that $1-2s\le 0$.

For the term $(E+\xi\times B)\cdot\nabla_\xi f$ in S, the difference point is that it contains the velocity derivative of order one. Our goal is to prove

$$\left\langle \partial^\alpha[(E+\xi\times B)\cdot\nabla_\xi f], w^2_{|\alpha|-\ell_1,\lambda_0}\partial^\alpha f\right\rangle \lesssim \overline{\mathcal{E}}^{1/2}_{N_1}(t)\mathcal{D}_{N_1,\ell_1,\lambda_0}(t)+\frac{1}{\lambda_0}(1+t)^{1+\vartheta}\|\nabla_x(E,B)\|_{H^4}\mathcal{D}_{N_1,\ell_1,\lambda_0}(t). \tag{3.25}$$

EBxw

In fact, one can deduce that

$$\begin{aligned}
&\left\langle \partial^\alpha[(E+\xi\times B)\cdot\nabla_\xi f], w^2_{|\alpha|-\ell_1,\lambda_0}\partial^\alpha f\right\rangle\\
&=\left\langle (E+\xi\times B)\cdot\nabla_\xi w^2_{|\alpha|-\ell_1,\lambda_0}, -\frac{1}{2}|\partial^\alpha f|^2\right\rangle\\
&\quad+\sum_{0<\alpha_1\le\alpha}C^\alpha_{\alpha_1}\left\langle (\partial^{\alpha_1}E+\xi\times\partial^{\alpha_1}B)\cdot\nabla_\xi\partial^{\alpha-\alpha_1}f, w^2_{|\alpha|-\ell_1,\lambda_0}\partial^\alpha f\right\rangle.
\end{aligned} \tag{3.26}$$

i2.p1

Here, it is straightforward to see that the first term on the right is bounded in a rough way by

$$
\begin{aligned}
C\|E\|_{L^\infty}&\iint_{\mathbb{R}^3\times\mathbb{R}^3}\left(\langle\xi\rangle^{-1}+(1+t)^{-\vartheta}\right)w^2_{|\alpha|-\ell_1,\lambda_0}|\partial^\alpha f|^2\,dxd\xi\\
&\lesssim\frac{1}{\lambda_0}(1+t)^{1+\vartheta}\|\nabla_x E\|_{H^1}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\lambda_0}{(1+t)^{1+\vartheta}}w^2_{|\alpha|-\ell_1,\lambda_0}|\partial^\alpha f|^2\,dxd\xi\\
&\qquad\qquad\lesssim\frac{1}{\lambda_0}(1+t)^{1+\vartheta}\|\nabla_x E\|_{H^1}\mathcal{D}_{N_1,\ell_1,\lambda_0}(t).
\end{aligned}
$$

We split the second term on the right-hand side of (3.26) into

$$
\begin{aligned}
&\sum_{0<\alpha_1\le\alpha}C^\alpha_{\alpha_1}\left\langle(\partial^{\alpha_1}E+\xi\times\partial^{\alpha_1}B)\cdot\nabla_\xi\partial^{\alpha-\alpha_1}f,w^2_{|\alpha|-\ell_1,\lambda_0}\partial^\alpha f\right\rangle\\
&\qquad=\sum_{0<\alpha_1\le\alpha}C^\alpha_{\alpha_1}\left\langle(\partial^{\alpha_1}E+\xi\times\partial^{\alpha_1}B)\cdot\nabla_\xi\partial^{\alpha-\alpha_1}\mathbf{P}f,w^2_{|\alpha|-\ell_1,\lambda_0}\partial^\alpha f\right\rangle\\
&+\sum_{0<\alpha_1\le\alpha}C^\alpha_{\alpha_1}\left\langle(\partial^{\alpha_1}E+\xi\times\partial^{\alpha_1}B)\cdot\nabla_\xi\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f,w^2_{|\alpha|-\ell_1,\lambda_0}\partial^\alpha f\right\rangle.
\end{aligned}\tag{3.27}
$$

Epurexw

In a similar same way to estimate (3.13), we only compute the second term on the right-hand side of (3.27). For doing this, when $|\alpha_1|\le 3$, it is bounded by

$$
\begin{aligned}
&C\,\|\partial^{\alpha_1}(E,B)\|_{L^\infty}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\langle\xi\rangle\left(w^2_{|\alpha|-\ell_1,\lambda_0}|\nabla_\xi\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f|^2+w^2_{|\alpha|-\ell_1,\lambda_0}|\partial^\alpha f|^2\right)dxd\xi\\
&\lesssim\|\nabla_x\partial^{\alpha_1}(E,B)\|_{H^1}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\langle\xi\rangle\left(|w_{1+|\alpha-\alpha_1|-\ell_1,\lambda_0}\nabla_\xi\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f|^2+|w_{|\alpha|-\ell_1,\lambda_0}\partial^\alpha f|^2\right)dxd\xi\\
&\lesssim\frac{1}{\lambda_0}(1+t)^{1+\vartheta}\|\nabla_x(E,B)\|_{H^4}\mathcal{D}_{N_1,\ell_1,\lambda_0}(t),
\end{aligned}
$$

where we have used the fact that $|\alpha-\alpha_1|+1\le|\alpha|$.

When $|\alpha_1|\ge 4$, one can see that

$$
\langle\xi\rangle^{1-\frac{\gamma+2s}{2}}w_{|\alpha|-\ell_1,\lambda_0}(t,\xi)\lesssim\langle\xi\rangle^{\frac{\gamma+2s}{2}}w_{3+|\alpha-\alpha_1|-\ell_1,\lambda_0}(t,\xi),
$$

which implies that the second term on the right-hand side of (3.27) can be dominated by

$$
\begin{aligned}
&C\|\partial^{\alpha_1}(E,B)\|\cdot\sup_x\left|\langle\xi\rangle^{1-\frac{\gamma+2s}{2}}w_{|\alpha|-\ell_1,\lambda_0}\nabla_\xi\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f\right|_{L^2_\xi}\|\langle\xi\rangle^{\frac{\gamma+2s}{2}}w_{|\alpha|-\ell_1,\lambda_0}\partial^\alpha f\|\\
&\lesssim\overline{\mathcal{E}}^{1/2}_{N_1}(t)\sum_{|\alpha'|\le2}\left\|\langle\xi\rangle^{\frac{\gamma+2s}{2}}w_{1+|\alpha-\alpha_1+\alpha'|-\ell_1,\lambda_0}\nabla_\xi\partial^{\alpha-\alpha_1+\alpha'}\{\mathbf{I}-\mathbf{P}\}f\right\|\mathcal{D}^{1/2}_{N_1,\ell_1,\lambda_0}(t)\\
&\lesssim\overline{\mathcal{E}}^{1/2}_{N_1}(t)\mathcal{D}_{N_1,\ell_1,\lambda_0}(t).
\end{aligned}
$$

Therefore (3.25) is true. Collecting all the above estimates, we see that

$$
\begin{aligned}
\mathscr{A}\lesssim&(1+t)^{1-\vartheta}\mathcal{E}^{1/2}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\mathcal{D}_{N_1,\ell_1,\lambda_0}(t)\\
&+\overline{\mathcal{E}}^{1/2}_{N_1}(t)\mathcal{D}_{N_1,\ell_1,\lambda_0}(t)+\frac{1}{\lambda_0}(1+t)^{1+\vartheta}\|\nabla_x(E,B)\|_{H^4}\mathcal{D}_{N_1,\ell_1,\lambda_0}(t).
\end{aligned}
$$

As to $\mathscr{B}$, letting $|\alpha|+|\beta|\le N_1$ with $|\beta|\ge 1$, applying (2.17) in Lemma 2.5, we obtain

$$\left\langle \partial^\alpha_\beta \Gamma(f,f), w^2_{|\alpha|+|\beta|-\ell_1,\lambda_0}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\rangle \lesssim (1+t)^{1-\vartheta}\mathcal{E}^{1/2}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\mathcal{D}_{N_1,\ell_1,\lambda_0}(t).$$

Next, for the term $E\cdot\xi f$ in S, we only consider the estimates by

$$\begin{aligned}
&\left\langle \partial^\alpha_\beta (E\cdot\xi\{\mathbf{I}-\mathbf{P}\}f), w^2_{|\alpha|+|\beta|-\ell_1,\lambda_0}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\rangle\\
&=\sum_{\alpha_1\le\alpha,|\beta_1|\le 1} C^{\alpha,\beta}_{\alpha_1,\beta_1}\left\langle \partial^{\alpha_1}E\cdot\partial_{\beta_1}\xi\partial^{\alpha-\alpha_1}_{\beta-\beta_1}\{\mathbf{I}-\mathbf{P}\}f, w^2_{|\alpha|+|\beta|-\ell_1,\lambda_0}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\rangle\\
&\lesssim \sum_{|\alpha_1|\le 2}\|\partial^{\alpha_1}E\|_{L^\infty}\iint_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|w^2_{|\alpha|+|\beta|-\ell_1,\lambda_0}\left(|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}\{\mathbf{I}-\mathbf{P}\}f|^2+\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f|^2\right)dxd\xi\\
&\quad+\sum_{\substack{|\alpha_1|\ge 3\\ \text{or } \alpha_1=\alpha}}\|\partial^{\alpha_1}E\|\cdot\sup_x\left||\xi|\langle\xi\rangle^{-\frac{\gamma+2s}{2}}w_{|\alpha|+|\beta|-\ell_1,\lambda_0}\partial^{\alpha-\alpha_1}_{\beta-\beta_1}\{\mathbf{I}-\mathbf{P}\}f\right|_{L^2_\xi}\\
&\times\left\|\langle\xi\rangle^{\frac{\gamma+2s}{2}}w^2_{|\alpha|+|\beta|-\ell_1,\lambda_0}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}\right\|\\
&\lesssim \frac{1}{\lambda_0}(1+t)^{1+\vartheta}\|\nabla_x(E,B)\|_{H^3}\mathcal{D}_{N_1,\ell_1,\lambda_0}(t)+\overline{\mathcal{E}}^{1/2}_{N_1}(t)\mathcal{D}_{N_1,\ell_1,\lambda_0}(t).
\end{aligned}$$

For the term $(E+\xi\times B)\cdot\nabla_\xi f$ in S, the difference point is that it contains the velocity derivative of order one and the growth of ξ. For brevity, we only estimate the following term

$$\begin{aligned}
&\left\langle \partial^\alpha_\beta[\xi\times B\cdot\nabla_\xi\{\mathbf{I}-\mathbf{P}\}f], w^2_{|\alpha|+|\beta|-\ell_1,\lambda_0}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\rangle\\
&=\sum_{0<\alpha_1+\beta_1} C^{\alpha,\beta}_{\alpha_1,\beta_1}\Big\langle(\partial_{\beta_1}\xi\times\partial^{\alpha_1}B)\cdot\nabla_\xi\partial^{\alpha-\alpha_1}_{\beta-\beta_1}\{\mathbf{I}-\mathbf{P}\}f,\\
&\qquad w^2_{|\alpha|+|\beta|-\ell_1,\lambda_0}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\Big\rangle. \qquad (3.28)\ \text{i2.p2}
\end{aligned}$$

When $|\alpha_1|\le 3$, (3.28) is bounded by

$$\begin{aligned}
&C\|\partial^{\alpha_1}B\|_{L^\infty}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\langle\xi\rangle\Big(w^2_{|\alpha|+|\beta|-\ell_1,\lambda_0}|\nabla_\xi\partial^{\alpha-\alpha_1}_{\beta-\beta_1}\{\mathbf{I}-\mathbf{P}\}f|^2\\
&\quad+w^2_{|\alpha|+|\beta|-\ell_1,\lambda_0}|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f|^2\Big)dxd\xi\\
&\lesssim\|\nabla_x\partial^{\alpha_1}B\|_{H^1}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\langle\xi\rangle\Big(|w_{|\alpha-\alpha_1|+|\beta-\beta_1|+1-\ell_1,\lambda_0}\nabla_\xi\partial^{\alpha-\alpha_1}_{\beta-\beta_1}\{\mathbf{I}-\mathbf{P}\}f|^2\\
&\quad+|w_{|\alpha|+|\beta|-\ell_1,\lambda_0}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f|^2\Big)dxd\xi\\
&\lesssim\frac{1}{\lambda_0}(1+t)^{1+\vartheta}\|\nabla_x(E,B)\|_{H^4}\mathcal{D}_{N_1,\ell_1,\lambda_0}(t).
\end{aligned}$$

When $|\alpha_1|\geq 4$, (3.28) is dominated by

$$
\begin{aligned}
&C\,\|\partial^{\alpha_1}B\|\cdot\sup_x\left|\langle\xi\rangle^{1-\frac{\gamma+2s}{2}}w_{|\alpha|+|\beta|-\ell_1,\lambda_0}|\nabla_\xi\partial^{\alpha-\alpha_1}_{\beta-\beta_1}\{\mathbf{I}-\mathbf{P}\}f|\right|_{L^2_\xi}\\
&\quad\times\left\|\langle\xi\rangle^{\frac{\gamma+2s}{2}}w_{|\alpha|+|\beta|-\ell_1,\lambda_0}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|\\
&\lesssim\overline{\mathcal{E}}^{1/2}_{N_1}(t)\sum_{|\alpha'|\leq 2}\left\|\langle\xi\rangle^{\frac{\gamma+2s}{2}}w_{1+|\alpha-\alpha_1+\alpha'|+|\beta-\beta_1|-\ell_1,\lambda_0}\nabla_\xi\partial^{\alpha-\alpha_1+\alpha'}_{\beta-\beta_1}\{\mathbf{I}-\mathbf{P}\}f\right\|\\
&\quad\times\mathcal{D}^{1/2}_{N_1,\ell_1,\lambda_0}(t)\\
&\lesssim\overline{\mathcal{E}}^{1/2}_{N_1}(t)\mathcal{D}_{N_1,\ell_1,\lambda_0}(t).
\end{aligned}
$$

Thus (3.23) holds true for $\mathscr{B}$. This proves the desired inequality (3.23). □

Now we give a remark to explain the choice of our new weight $w_{|\alpha|+|\beta|-\ell_1,\lambda_0}$.

Remark 3.2. *In fact, the weight $w_{|\alpha|+|\beta|-\ell_1,\lambda_0}$ is designed to treat the delicate term*

$$\sum_{|\alpha_1|=1}(\xi\times\partial^{\alpha_1}B)\cdot\nabla_\xi\partial^{\alpha-\alpha_1}_\beta\{\mathbf{I}-\mathbf{P}\}f.$$

More precisely, the exponential part of the weight $w_{|\alpha|+|\beta|-\ell_1,\lambda_0}$ is needed to absorb the growth of ξ, and the algebraic weight $\langle\xi\rangle^{\gamma(|\alpha|+|\beta|-\ell_1)}$ is required to deal with the growth of $\xi-$derivatives.

At this point, we are ready to obtain the closed estimate on the first portion of the time-weighted energy norm $X(t)$ in the following

⟨lem.ce1⟩ **Lemma 3.4.** *Assume $l_1-1\geq N_1\geq 14$. It holds that*

$$
\begin{aligned}
&\sup_{0\leq s\leq t}\left\{\overline{\mathcal{E}}_{N_1}(s)+(1+s)^{-\frac{1+\epsilon_0}{2}}\mathcal{E}_{N_1,\ell_1,\lambda_0}(s)\right\}\\
&\qquad+\int_0^t\overline{\mathcal{D}}_{N_1}(s)\,ds+\int_0^t(1+s)^{-\frac{1+\epsilon_0}{2}}\mathcal{D}_{N_1,\ell_1,\lambda_0}(s)\,ds\lesssim Y_0^2+X^2(t)
\end{aligned}
\tag{3.29}
$$
ce1

for $0\leq t\leq T$.

Proof. In fact, the time integration of (3.8) gives

$$
\begin{aligned}
\overline{\mathcal{E}}_{N_1}(t)+\int_0^t\overline{\mathcal{D}}_{N_1}(s)\,ds&\lesssim Y_0^2+\delta\int_0^t(1+s)^{-1-\vartheta}\overline{\mathcal{D}}_{N_1,\ell_1,\lambda_0}(s)\,ds\\
&\quad+\int_0^t\overline{\mathcal{E}}_{N_1}(s)\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(s)\,ds.
\end{aligned}
\tag{3.30}
$$
ce1.p1

Furthermore, from multiplying (3.8) by $(1+t)^{-\epsilon_0}$ and then taking the time integration, it follows that

$$
\begin{aligned}
&(1+t)^{-\epsilon_0}\overline{\mathcal{E}}_{N_1}(t)+\int_0^t(1+s)^{-\epsilon_0}\overline{\mathcal{D}}_{N_1}(s)\,ds+\int_0^t(1+s)^{-1-\epsilon_0}\overline{\mathcal{E}}_{N_1}(s)\,ds\\
&\lesssim Y_0^2+\delta\int_0^t(1+s)^{-1-\vartheta-\epsilon_0}\mathcal{D}_{N_1,\ell_1,\lambda_0}(s)\,ds\\
&\quad+\int_0^t(1+s)^{-\epsilon_0}\overline{\mathcal{E}}_{N_1}(s)\mathcal{E}_{N_1,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(s)\,ds.
\end{aligned}
\tag{3.31}
$$
ce1.p2

Combining (3.30) and (3.31) gives

$$\begin{aligned}&\overline{\mathcal{E}}_{N_1}(t)+\int_0^t\overline{\mathcal{D}}_{N_1}(s)\,ds+\int_0^t(1+s)^{-1-\epsilon_0}\overline{\mathcal{E}}_{N_1}(s)\,ds\\ \lesssim\;& Y_0^2+\delta\int_0^t(1+s)^{-1-\vartheta}\mathcal{D}_{N_1,\ell_1,\lambda_0}(s)\,ds+\int_0^t\overline{\mathcal{E}}_{N_1}(s)\mathcal{E}_{N_1,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(s)\,ds\\ \lesssim\;& Y_0^2+X^2(t)+\delta\int_0^t(1+s)^{-1-\vartheta}\mathcal{D}_{N_1,\ell_1,\lambda_0}(s)\,ds,\end{aligned}\tag{3.32}$$ ce1.p3

where to obtain the second inequality, we have used

$$\sup_{0\le s\le t}\left\{\overline{\mathcal{E}}_{N_1}(s)+(1+s)^{\frac{3}{2}}\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(s)\right\}\le X(t).$$

From (3.14), multiplying it by $(1+t)^{-(1+\epsilon_0)/2}$ and taking the time integration yields

$$\begin{aligned}&(1+t)^{-\frac{1+\epsilon_0}{2}}\mathcal{E}_{N_1,\ell_1,\lambda_0}(t)+\int_0^t(1+s)^{-\frac{1+\epsilon_0}{2}}\mathcal{D}_{N_1,\ell_1,\lambda_0}(s)\,ds\\ &\qquad+\int_0^t(1+s)^{-\frac{3+\epsilon_0}{2}}\mathcal{E}_{N_1,\ell_1,\lambda_0}(s)\,ds\\ &\lesssim Y_0^2+\sum_{|\alpha|=N_1}\int_0^t(1+s)^{-\frac{1+\epsilon_0}{2}}\left\langle\partial^\alpha E\cdot\xi\mu^{1/2},w^2_{|\alpha|-\ell_1,\lambda_0}\partial^\alpha f\right\rangle.\end{aligned}\tag{3.33}$$ ce1.p4

By the Cauchy-Schwarz inequality, the right-hand second term of (3.33) is bounded up to a generic constant by

$$\begin{aligned}&\sum_{|\alpha|=N_1}\int_0^t\left((1+s)^{-1-\epsilon_0}\|\partial^\alpha E\|^2+\left\|\langle\xi\rangle^{\frac{\gamma+2s}{2}}\partial^\alpha f\right\|^2\right)ds\\ &\qquad\lesssim\int_0^t(1+s)^{-1-\epsilon_0}\overline{\mathcal{E}}_{N_1}(s)\,ds+\int_0^t\overline{\mathcal{D}}_{N_1}(s)\,ds.\end{aligned}$$

Then, in terms of the above estimates, taking the sum of (3.32) and (3.33) and using the fact that $\delta>0$ is small enough, we arrive at

$$\begin{aligned}&\overline{\mathcal{E}}_{N_1}(t)+(1+t)^{-\frac{1+\epsilon_0}{2}}\mathcal{E}_{N_1,\ell_1,\lambda_0}(t)+\int_0^t\overline{\mathcal{D}}_{N_1}(s)\,ds\\ &\quad+\int_0^t(1+s)^{-\frac{1+\epsilon_0}{2}}\mathcal{D}_{N_1,\ell_1,\lambda_0}(s)\,ds+\int_0^t(1+s)^{-1-\epsilon_0}\overline{\mathcal{E}}_{N_1}(s)\,ds\\ &\quad+\int_0^t(1+s)^{-\frac{3+\epsilon_0}{2}}\mathcal{E}_{N_1,\ell_1,\lambda_0}(s)\,ds\lesssim Y_0^2+X^2(t).\end{aligned}\tag{3.34}$$ ce1.p5

Therefore, (3.29) follows, and then this completes the proof of Lemma 3.4. □

3.4. **Decay of electromagnetic fields and macro components.** In this step, we will use directly the Duhamel's principle to obtain the time-decay of the electromagnetic field (E,B) and the macro components $(a_\pm,b,c)$ up to order six in terms of the time-decay of the weighted high-order energy function $\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)$ which follows from the boundedness of $X(t)$.

To consider the solution to the Cauchy problem

$$\begin{cases} \partial_t f + \xi\cdot\nabla_x f - E\cdot\xi\mu^{1/2}q_1 + Lf = S, \\ \partial_t E - \nabla_x\times B = -\left\langle \xi\mu^{1/2}, f_+ - f_-\right\rangle, \\ \partial_t B + \nabla_x\times E = 0, \\ \nabla_x\cdot E = \left\langle \mu^{1/2}, f_+ - f_-\right\rangle, \quad \nabla_x\cdot B = 0, \\ [f,E,B]|_{t=0} = [f_0,E_0,B_0], \end{cases} \tag{3.35}$$ ls

where initial data $[f_0,E_0,B_0]$ satisfies the compatibility condition

$$\nabla_x\cdot E_0 = \int_{\mathbb{R}^3}\mu^{1/2}(f_{0,+}-f_{0,-})\,d\xi, \quad \nabla_x\cdot B_0 = 0, \tag{3.36}$$ comp.con

we denote for simplicity $U=[f,E,B]$, $U_0=[f_0,E_0,B_0]$ so that one can formally write

$$U(t) = \mathbb{A}(t)U_0 + \int_0^t \mathbb{A}(t-s)[S(s),0,0]\,ds,$$

where $\mathbb{A}(t)$ is the linear solution operator for the Cauchy problem on the linearized homogeneous system corresponding to (3.35) in the case when $S=0$.

Following the completely same proof as in [8], one can prove

⟨thm.lb⟩ **Lemma 3.5.** *Let $S=0$, and let $[f,E,B]$ be the solution to the Cauchy problem (3.35), (3.36) of the linearized homogeneous system. Define the velocity weight function $W=W(\xi)$ by*

$$W(\xi) = \langle\xi\rangle^{-\frac{\gamma+2s}{2}} \text{ with } -3<\gamma<-2s,\ 0<s<1. \tag{3.37}$$ def.wl

Then, for $\ell\geq 0$ and $\alpha\geq 0$ with $m=|\alpha|$,

$$\begin{aligned} &\left\|W^\ell\partial^\alpha f\right\| + \|\partial^\alpha(E,B)\| \\ &\lesssim (1+t)^{-\sigma_m}\left(\left\|W^{\ell+\ell_*^{low}}f_0\right\|_{Z_1} + \|(E_0,B_0)\|_{L^1_x}\right) \\ &\qquad +(1+t)^{-j}\left(\left\|W^{\ell+\ell_*^{high}}\nabla_x^{j+1}\partial^\alpha f_0\right\| + \left\|\nabla_x^{j+1}\partial^\alpha(E_0,B_0)\right\|\right), \end{aligned}$$

where

$$\sigma_m = \frac{3}{4}+\frac{m}{2},\ \ell_*^{low} > 2\sigma_m,\ \ell_*^{high}>0,\ 0\leq j<\ell_*^{high}.$$

The following remarks are concerned with Lemma 3.5.

Remark 3.3. *The weight $W(\xi)$ is chosen as (3.37) so that $W^{-1}(\xi)$ is consistent with the weak weight in the dissipation norm (1.12) induced by the linear operator L. In fact, the similar result can be obtained even if the weight is replaced by the other algebraic weights such as $\langle\xi\rangle^{-\gamma}$.*

Remark 3.4. *The extra $(j+1)-th$ order derivative of the initial data is required in order to deduce the time decay rate of $[f,E,B]$. This results essentially from the coupling of the hyperbolic Maxwell equations but not due to the technique of the approach, see [7] for the analysis of the Green's function of the damping Euler-Maxwell system.*

⟨lem.emd⟩ **Lemma 3.6.** *Suppose $l_1 - 1 \geq N_1 \geq 14$. It holds that*

$$\sup_{0\leq s\leq t}\left\{(1+s)^{\frac{5}{2}}\|\nabla_x(E,B)\|_{H^5}^2+(1+s)^{\frac{3}{2}}\|(a_\pm,b,c,E,B)\|^2\right\}\lesssim Y_0^2+X^2(t) \quad (3.38)$$ lem.emd.1

for $0 \leq t \leq T$.

Proof. Recall the mild form

$$U(t)=\mathbb{A}(t)U_0+\int_0^t\mathbb{A}(t-s)[S(s),0,0]\,ds, \quad (3.39)$$ lem.emd.p1

which denotes the solutions to the Cauchy problem on system (3.1) with initial data $U_0=(f_0,E_0,B_0)$, where the nonlinear term S is given by (3.2). The linearized analysis for the homogeneous system in Lemma 3.5 implies

$$\begin{aligned}\|\nabla_x P_{E,B}\{\mathbb{A}(t)U_0\}\|_{H^5}\lesssim{}&(1+t)^{-\frac{5}{4}}\left(\left\|W^{\ell_3^{low}}f_0\right\|_{Z_1}+\|(E_0,B_0)\|_{L^1_x}\right)\\&+(1+t)^{-\frac{5}{4}}\sum_{1\leq|\alpha|\leq 6}\left(\left\|W^{\ell_3^{high}}\nabla_x^{\frac{9}{4}}\partial^\alpha f_0\right\|+\left\|\nabla_x^{\frac{9}{4}}\partial^\alpha(E_0,B_0)\right\|\right),\end{aligned}$$

where $P_{E,B}$ means the projection along the electro and magnetic components in the solution (f,E,B), $W=W(\xi)$ is defined by (3.37), and constants ℓ_3^{low}, ℓ_3^{high} are chosen to satisfy

$$\ell_3^{low}>\frac{15}{2},\quad \ell_3^{high}>\frac{5}{4},$$

and also ℓ_3^{low}, ℓ_3^{high} are sufficiently close to 15/2 and 5/4, respectively. By interpolation of derivatives,

$$\begin{aligned}\|\nabla_x P_{E,B}\{\mathbb{A}(t)U_0\}\|_{H^5}\lesssim{}&(1+t)^{-\frac{5}{4}}\left(\left\|W^{\ell_3^{low}}f_0\right\|_{Z_1}+\|(E_0,B_0)\|_{L^1_x}\right)\\&+(1+t)^{-\frac{5}{4}}\sum_{3\leq|\alpha|\leq 9}\left(\left\|W^{\ell_3^{high}}\partial^\alpha f_0\right\|+\|\partial^\alpha(E_0,B_0)\|\right).\end{aligned}$$

Applying this time-decay property to the mild form (3.39) gives

$$\begin{aligned}\|\nabla_x(E,B)\|_{H^5}\lesssim{}&(1+t)^{-\frac{5}{4}}Y_0+\int_0^t(1+t-s)^{-\frac{5}{4}}\left\|W^{\ell_3^{low}}S(s)\right\|_{Z_1}ds\\&+\int_0^t(1+t-s)^{-\frac{5}{4}}\sum_{3\leq|\alpha|\leq 9}\left\|W^{\ell_3^{high}}\partial^\alpha S(s)\right\|ds, \quad (3.40)\end{aligned}$$ lem.emd.p2

where we have used $W(\xi)=w_{-\frac{\gamma+2s}{2\gamma}}(\xi)\leq w_{-1/2}(\xi)$ and the definition (1.15) for Y_0. By applying Lemma A.2, it is straightforward to obtain

$$\left\|W^{\ell_3^{low}}S(t)\right\|_{Z_1}+\sum_{3\leq|\alpha|\leq 9}\left\|W^{\ell_3^{high}}\partial^\alpha S(t)\right\|\lesssim\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t). \quad (3.41)$$ Z1.L2.

Here, we have used the choice of $N_1-3\geq 11$, $\ell_1-1\geq N_1$.

Recall $X(t)$ norm, and hence

$$\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(s)\leq(1+s)^{-\frac{3}{2}}X(t),\quad 0\leq s\leq t.$$

Plugging these estimates into (3.40), the further computations yield

$$\sup_{0\le s\le t}\left\{(1+s)^{\frac{5}{2}}\|\nabla_x(E,B)\|_{H^5}^2\right\}\lesssim Y_0^2+X^2(t). \tag{3.42}$$ lem.emd.p3

Moreover, to obtain the time-decay of $\|(a_\pm,b,c,E,B)\|$, we use the linearized time-decay property

$$\begin{aligned}\|P_f\{\mathbb{A}(t)U_0\}\|+\|P_{E,B}\{\mathbb{A}(t)U_0\}\|\lesssim (1+t)^{-\frac{3}{4}}\left(\left\|W^{\ell_4^{low}}f_0\right\|_{Z_1}+\|(E_0,B_0)\|_{L^1_x}\right)\\ +(1+t)^{-\frac{3}{4}}\left(\left\|W^{\ell_4^{high}}\nabla_x^{\frac{7}{4}}f_0\right\|+\left\|\nabla_x^{\frac{7}{4}}(E_0,B_0)\right\|\right),\end{aligned}$$

where P_f means the projection along the f-component in the solution (f,E,B), and constants ℓ_4^{low}, ℓ_4^{high} are chosen to satisfy $\ell_4^{low}>3/2$, $\ell_4^{high}>3/4$ and also ℓ_4^{low}, ℓ_4^{high} are sufficiently close to 3/2 and 3/4, respectively. Therefore, in the completely same way for estimating $\|\nabla_x(E,B)\|_{H^5}$ in (3.42), one has

$$\sup_{0\le s\le t}\left\{(1+s)^{\frac{3}{2}}\|(a_\pm,b,c,E,B)\|^2\right\}\lesssim Y_0^2+X^2(t). \tag{3.43}$$ lem.emd.p4

Thus, combining (3.42) and (3.43) gives the desired estimate (3.38). This then completes the proof of Lemma 3.6. □

Remark 3.5. *Notice that in the proof of* (3.41)*, the inequality* $N_1-3\ge 11$ *was used, which then yields to require* $N_1\ge 14$ *in Theorem 1.1.*

3.5. **The compensating energy estimate with weight.** In this section we obtain the uniform-in-time boundedness of the energy functional $\mathcal{E}_{N_1-1,\ell_1,\lambda_0}(t)$. Notice that this is consistent with the estimation on the linearized system. The main observation in the nonlinear analysis is that those remaining terms in the energy inequalities are time-space integrable.

⟨lem.n1bd⟩ **Lemma 3.7.** *Assume* $l_1-1\ge N_1\ge 14$*. It holds that*

$$\sup_{0\le s\le t}\mathcal{E}_{N_1-1,\ell_1,\lambda_0}(s)+\int_0^t\mathcal{D}_{N_1-1,\ell_1,\lambda_0}(s)\,ds\lesssim Y_0^2+X^2(t), \tag{3.44}$$ lem.n1bd.1

for $0\le t\le T$.

Proof. Similarly for obtaining (3.21), one has

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{N_1-1,\ell_1,\lambda_0}(t)+\kappa\mathcal{D}_{N_1-1,\ell_1,\lambda_0}(t)\lesssim \mathcal{I}^{(2)}_{N_1-1,\ell_1,\lambda_0}(t)\\ +\sum_{|\alpha|=N_1-1}\left\langle\partial^\alpha E\cdot\xi\mu^{1/2},w^2_{|\alpha|-\ell_1,\lambda_0}\partial^\alpha f\right\rangle,\end{aligned} \tag{3.45}$$ lem.n1bd.p1

where $\mathcal{I}^{(2)}_{N_1-1,\ell_1,\lambda_0}(t)$ is defined by (3.22) with N_1 replaced by N_1-1. Following the same way as in the proof of (3.23) one can obtain that

$$\begin{aligned}\mathcal{I}^{(2)}_{N_1-1,\ell_1,\lambda_0}(t)\lesssim (1+t)^{1-\vartheta}\mathcal{E}^{1/2}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\mathcal{D}_{N_1-1,\ell_1,\lambda_0}(t)\\ +\frac{1}{\lambda_0}(1+t)^{1+\vartheta}\|\nabla_x(E,B)\|_{H^4}\mathcal{D}_{N_1-1,\ell_1,\lambda_0}(t)\\ +\overline{\mathcal{E}}^{1/2}_{N_1-1}(t)\mathcal{D}_{N_1-1,\ell_1,\lambda_0}(t).\end{aligned} \tag{3.46}$$ lem.n1bd.p1-0

Further using

$$\sup_{0\leq s\leq t}\Big\{\overline{\mathcal{E}}_{N_1-1}(s)+\mathcal{E}_{N_1-1,\ell_1,\lambda_0}(s)+(1+s)^{2(1+\vartheta)}\|\nabla_x(E,B)\|_{H^4}^2 +(1+s)^{\frac{3}{2}}\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(s)\Big\}\leq X(t)\leq \delta^2,$$

it follows that

$$\mathcal{I}^{(2)}_{N_1-1,\ell_1,\lambda_0}(t)\lesssim \delta\mathcal{D}_{N_1-1,\ell_1,\lambda_0}(t)+(1+t)^{-\frac{3}{4}}X(t)\mathcal{D}^{1/2}_{N_1-1,\ell_1,\lambda_0}(t). \tag{3.47}$$

lem.n1bd.p2

From the Cauchy-Schwarz inequality, the right-hand second term of (3.45) is estimated by

$$\begin{aligned}\sum_{|\alpha|=N_1-1}\Big\langle\partial^\alpha E\cdot\xi\mu^{1/2},w^2_{|\alpha|-\ell_1,\lambda_0}\partial^\alpha f\Big\rangle &\lesssim \sum_{|\alpha|=N_1-1}\left(\eta\left\|\langle\xi\rangle^{\frac{\gamma+2s}{2}}\partial^\alpha f\right\|^2+\frac{1}{\eta}\|\partial^\alpha E\|^2\right)\\ &\lesssim \eta\mathcal{D}_{N_1-1,\ell_1,\lambda_0}(t)+\frac{1}{\eta}\overline{\mathcal{D}}_{N_1}(t),\end{aligned} \tag{3.48}$$

lem.n1bd.p3

for any $\eta>0$. Then, by applying again the Cauchy-Schwarz inequality with η to the right-hand second term of (3.47), plugging the resultant estimate together with (3.48) into (3.45), and choosing $\eta>0$ small enough, one has

$$\frac{d}{dt}\mathcal{E}_{N_1-1,\ell_1,\lambda_0}(t)+\kappa\mathcal{D}_{N_1-1,\ell_1,\lambda_0}(t)\lesssim\overline{\mathcal{D}}_{N_1}(t)+(1+t)^{-\frac{3}{2}}X^2(t). \tag{3.49}$$

lem.n1bd.p4

Recall that from (3.34),

$$\int_0^t\overline{\mathcal{D}}_{N_1}(s)\,ds\lesssim Y_0^2+X^2(t).$$

Therefore, (3.44) follows by the time integration of (3.49). This completes the proof of Lemma 3.7. □

3.6. **Decay of the lower order energy.** To obtain the closed estimate on the energy norm $X(t)$, it remains to obtain the time-decay of the lower-order energy functional $\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)$ and $\overline{\mathcal{E}}_{N_1-2}(t)$ through the time-weighted estimate as well as the iterative trick as in [8]. Notice that smoothness-loss and velocity-weight-loss in $\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)$ result from the regularity-loss of the electromagnetic field and the degeneration of collisional kernels for soft potentials. Here we emphasize that although the proof of the following lemma looks similar to that in [8], the full details will be provided since most of subscripts in the energy functional $\mathcal{E}_{N,\ell,\lambda}(t)$ take the completely different form, and one has to carefully check the validity of all the estimates.

⟨lem.n1wd⟩ **Lemma 3.8.** *It holds that*

$$\sup_{0\leq s\leq t}\Big\{(1+s)^{\frac{3}{2}}[\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(s)+\overline{\mathcal{E}}_{N_1-2}(s)]\Big\}\lesssim Y_0^2+X^2(t) \tag{3.50}$$

lem.n1wd.1

for $0\leq t\leq T$.

Proof. First recall from Lemma 3.4 and Lemma 3.7

$$\overline{\mathcal{E}}_{N_1}(t)+\mathcal{E}_{N_1-1,\ell_1,\lambda_0}(t)+\int_0^t\overline{\mathcal{D}}_{N_1}(s)+\mathcal{D}_{N_1-1,\ell_1,\lambda_0}(s)\,ds\lesssim Y_0^2+X^2(t). \tag{3.51}$$

lem.n1wd.p1

To obtain the time-decay of $\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)$ and $\overline{\mathcal{E}}_{N_1-2}(t)$, we will make the time-weighted estimate. For brevity of presentation we write

$$\mathcal{J}^{(2)}_{N,\ell,\lambda_0}(t)=\sum_{|\alpha|=N}\left\langle \partial^\alpha E\cdot\xi\mu^{1/2},w^2_{|\alpha|-\ell,\lambda_0}\partial^\alpha f\right\rangle.$$

From the proof of Lemma 3.2 and Lemma 3.3, cf. (3.8) and (3.21), one has the Lyapunov inequalities

$$\begin{cases}\dfrac{d}{dt}\overline{\mathcal{E}}_{N_1-1}(t)+\kappa\overline{\mathcal{D}}_{N_1-1}(t)\lesssim \mathcal{I}^{(1)}_{N_1-1}(t),\\ \dfrac{d}{dt}\mathcal{E}_{N_1-2,\ell_1-\frac{\gamma+2s}{2\gamma},\lambda_0}(t)+\kappa\mathcal{D}_{N_1-2,\ell_1-\frac{\gamma+2s}{2\gamma},\lambda_0}(t)\\ \qquad\qquad\lesssim \mathcal{I}^{(2)}_{N_1-2,\ell_1-\frac{\gamma+2s}{2\gamma},\lambda_0}(t)+\mathcal{J}^{(2)}_{N_1-2,\ell_1-\frac{\gamma+2s}{2\gamma},\lambda_0}(t).\end{cases}\tag{3.52}$$

lem.n1wd.p2

Those terms on the right can be estimated as follows. Similar to (3.12), it holds that

$$\mathcal{I}^{(1)}_{N_1-1}(t)\lesssim \mathcal{E}^{1/2}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\overline{\mathcal{D}}_{N_1-1}(t)+\frac{\delta}{(1+t)^{1+\vartheta}}\mathcal{D}_{N_1-1,\ell_1,\lambda_0}(t)\\ +\overline{\mathcal{E}}^{1/2}_{N_1-1}(t)\mathcal{E}^{1/2}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\overline{\mathcal{D}}^{1/2}_{N_1-1}(t).$$

Here, noticing that $\mathcal{E}^{1/2}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\leq X^{1/2}(t)\leq\delta$ is small enough for the first term on the right and applying the Cauchy-Schwarz inequality to the third term on the right, it then follows from the first equation of (3.52) that

$$\frac{d}{dt}\overline{\mathcal{E}}_{N_1-1}(t)+\kappa\overline{\mathcal{D}}_{N_1-1}(t)\\ \lesssim\frac{\delta}{(1+t)^{1+\vartheta}}\mathcal{D}_{N_1-1,\ell_1,\lambda_0}(t)+\overline{\mathcal{E}}_{N_1-1}(t)\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t).\tag{3.53}$$

lem.n1wd.p3

Moreover, similar to (3.46), it holds that

$$\mathcal{I}^{(2)}_{N_1-2,\ell_1-\frac{\gamma+2s}{2\gamma},\lambda_0}(t)\lesssim(1+t)^{1-\vartheta}\mathcal{E}^{1/2}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\mathcal{D}_{N_1-2,\ell_1-\frac{\gamma+2s}{2\gamma},\lambda_0}(t)\\ +\frac{1}{\lambda_0}(1+t)^{1+\vartheta}\|\nabla_x(E,B)\|_{H^4}\mathcal{D}_{N_1-2,\ell_1-\frac{\gamma+2s}{2\gamma},\lambda_0}(t)\\ +\overline{\mathcal{E}}^{1/2}_{N_1-2}(t)\mathcal{D}_{N_1-2,\ell_1-\frac{\gamma+2s}{2\gamma},\lambda_0}(t),$$

which by using $X(t)\leq\delta^2$ for the first two terms on the right and the Cauchy-Schwarz inequality for the last term, further implies

$$\mathcal{I}^{(2)}_{N_1-2,\ell_1-\frac{\gamma+2s}{2\gamma},\lambda_0}(t)\lesssim\delta\mathcal{D}_{N_1-2,\ell_1-\frac{\gamma+2s}{2\gamma},\lambda_0}(t).\tag{3.54}$$

lem.n1wd.p4

Again from the Cauchy-Schwarz inequality with $\eta>0$,

$$\mathcal{J}^{(2)}_{N_1-2,\ell_1-\frac{\gamma+2s}{2\gamma},\lambda_0}(t)\lesssim\sum_{|\alpha|=N_1-2}\left(\eta\left\|\langle\xi\rangle^{\frac{\gamma+2s}{2}}\partial^\alpha f\right\|^2+\frac{1}{\eta}\|\partial^\alpha E\|^2\right).\tag{3.55}$$

lem.n1wd.p5

Then, by plugging (3.54) and (3.55) into the second equation of (3.52), taking the sum of the resultant inequality multiplied by a proper small constant $\kappa_1>0$ and another inequality (3.53), and using smallness of $\delta>0$ and $\eta>0$, one has

$$\frac{d}{dt}\left\{\overline{\mathcal{E}}_{N_1-1}(t)+\kappa_1\mathcal{E}_{N_1-2,\ell_1-\frac{\gamma+2s}{2\gamma},\lambda_0}(t)\right\}+\kappa\left\{\overline{\mathcal{D}}_{N_1-1}(t)+\kappa_1\mathcal{D}_{N_1-2,\ell_1-\frac{\gamma+2s}{2\gamma},\lambda_0}(t)\right\}$$
$$\lesssim\frac{\delta}{(1+t)^{1+\vartheta}}\mathcal{D}_{N_1-1,\ell_1,\lambda_0}(t)+\overline{\mathcal{E}}_{N_1-1}(t)\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t). \quad (3.56) \text{ lem.n1wd.p6}$$

Further from multiplying it by $(1+t)^{\frac12+\epsilon}$ with $\epsilon>0$ fixed small enough and taking the time integration, it follows

$$(1+t)^{\frac12+\epsilon}\left\{\overline{\mathcal{E}}_{N_1-1}(t)+\mathcal{E}_{N_1-2,\ell_1-\frac{\gamma+2s}{2\gamma},\lambda_0}(t)\right\}$$
$$+\int_0^t(1+s)^{\frac12+\epsilon}\left\{\overline{\mathcal{D}}_{N_1-1}(s)+\mathcal{D}_{N_1-2,\ell_1-\frac{\gamma+2s}{2\gamma},\lambda_0}(s)\right\}ds$$
$$\lesssim Y_0^2+\int_0^t\delta(1+s)^{-\frac12-\vartheta+\epsilon}\mathcal{D}_{N_1-1,\ell_1,\lambda_0}(s)\,ds$$
$$+\int_0^t(1+s)^{\frac12+\epsilon}\overline{\mathcal{E}}_{N_1-1}(s)\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(s)\,ds$$
$$+\int_0^t(1+s)^{-\frac12+\epsilon}\left\{\overline{\mathcal{E}}_{N_1-1}(s)+\mathcal{E}_{N_1-2,\ell_1-\frac{\gamma+2s}{2\gamma},\lambda_0}(s)\right\}ds. \quad (3.57) \text{ lem.n1wd.p7}$$

Here, since $\epsilon>0$ is small enough, the second term on the right is bounded by $Y_0^2+X^2(t)$ directly by (3.51), the third term on the right is bounded by

$$\delta^2\sup_{0\le s\le t}\left\{(1+s)^{\frac12+\epsilon}\overline{\mathcal{E}}_{N_1-1}(s)\right\},$$

due to the fact that

$$\sup_{0\le s\le t}\left\{(1+s)^{\frac32}\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(s)\right\}\le X(t)\le\delta^2,$$

and the fourth term on the right is bounded by $Y_0^2+X^2(t)$ by noticing

$$\overline{\mathcal{E}}_{N_1-1}(t)+\mathcal{E}_{N_1-2,\ell_1-\frac{\gamma+2s}{2\gamma},\lambda_0}(t)\lesssim\overline{\mathcal{D}}_{N_1}(t)+\mathcal{D}_{N_1-1,\ell_1,\lambda_0}(t)+\|(a_\pm,b,c,B)\|^2,$$

and further using (3.51) as well as Lemma 3.6. Hence, we arrive from (3.57) at

$$\sup_{0\le s\le t}\left\{(1+s)^{\frac12+\epsilon}\left(\overline{\mathcal{E}}_{N_1-1}(s)+\mathcal{E}_{N_1-2,\ell_1-\frac{\gamma+2s}{2\gamma},\lambda_0}(s)\right)\right\}$$
$$+\int_0^t(1+s)^{\frac12+\epsilon}\left\{\overline{\mathcal{D}}_{N_1-1}(s)+\mathcal{D}_{N_1-2,\ell_1-\frac{\gamma+2s}{2\gamma},\lambda_0}(s)\right\}ds\lesssim Y_0^2+X^2(t). \quad (3.58) \text{ lem.n1wd.p8}$$

In a similar way to obtain (3.56), starting with the Lyapunov inequalities

$$\begin{cases}\dfrac{d}{dt}\overline{\mathcal{E}}_{N_1-2}(t)+\kappa\overline{\mathcal{D}}_{N_1-2}(t)\lesssim\mathcal{I}^{(1)}_{N_1-2}(t),\\ \dfrac{d}{dt}\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)+\kappa\mathcal{D}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\\ \qquad\lesssim\mathcal{I}^{(2)}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)+\mathcal{J}^{(2)}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t),\end{cases}$$

one can prove

$$\frac{d}{dt}\left\{\overline{\mathcal{E}}_{N_1-2}(t)+\kappa_2\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\right\}+\kappa\left\{\overline{\mathcal{D}}_{N_1-2}(t)+\kappa_2\mathcal{D}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\right\}$$
$$\lesssim \frac{\delta}{(1+t)^{1+\vartheta}}\mathcal{D}_{N_1-2,\ell_1-\frac{\gamma+2s}{2\gamma},\lambda_0}(t)+\overline{\mathcal{E}}_{N_1-2}(t)\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)$$

for a properly chosen constant $\kappa_2>0$.

Further multiplying it by $(1+t)^{\frac{3}{2}+\epsilon}$ and taking the time integration gives

$$\begin{aligned}
&(1+t)^{\frac{3}{2}+\epsilon}\left\{\overline{\mathcal{E}}_{N_1-2}(t)+\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)\right\}\\
&\qquad+\int_0^t(1+s)^{\frac{3}{2}+\epsilon}\left\{\overline{\mathcal{D}}_{N_1-2}(s)+\mathcal{D}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(s)\right\}ds\\
&\qquad\lesssim Y_0^2+\int_0^t\delta(1+s)^{\frac{1}{2}+\epsilon-\vartheta}\mathcal{D}_{N_1-2,\ell_1-\frac{\gamma+2s}{2\gamma},\lambda_0}(s)\,ds\\
&\qquad\quad+\int_0^t(1+s)^{\frac{3}{2}+\epsilon}\overline{\mathcal{E}}_{N_1-2}(s)\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(s)\,ds\\
&\qquad\quad+\int_0^t(1+s)^{\frac{1}{2}+\epsilon}\left\{\overline{\mathcal{E}}_{N_1-2}(s)+\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(s)\right\}ds. \qquad (3.59)
\end{aligned}$$

lem.n1wd.p9

Here, notice again that $\epsilon>0$ is a fixed constant small enough. Then, the second term on the right is bounded by $Y_0^2+X^2(t)$ by (3.58), the third term on the right is bounded by $X^2(t)$ due to

$$\sup_{0\le s\le t}\left\{(1+s)^{\frac{3}{2}}\left(\overline{\mathcal{E}}_{N_1-2}(s)+\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(s)\right)\right\}\le X(t),$$

and as before, the fourth term on the right is bounded by

$$C(1+t)^{\epsilon}\left(Y_0^2+X^2(t)\right)$$

by noticing

$$\overline{\mathcal{E}}_{N_1-2}(t)+\mathcal{E}_{N_1-3,\ell_1-\frac{\gamma+2s}{\gamma},\lambda_0}(t)$$
$$\lesssim \overline{\mathcal{D}}_{N_1-1}(t)+\mathcal{D}_{N_1-2,\ell_1-\frac{\gamma+2s}{2\gamma},\lambda_0}(t)+\|(a_\pm,b,c,B)\|^2,$$

and further using (3.58) as well as Lemma 3.6. Therefore, the desired inequality (3.50) follows by putting these estimates into (3.59). This then completes the proof of Lemma 3.8. □

3.7. **Global existence.** We are now in a position to complete the

Proof of Theorem 1.1. Recall $X(t)$-norm (1.14). From Lemma 3.4, Lemma 3.6, Lemma 3.7 and Lemma 3.8, it follows that

$$X(t)\lesssim Y_0^2+X^2(t).$$

Since Y_0 is sufficiently small, (1.16) holds true. The global existence follows further from the local existence (cf. [38]) and the continuity argument in the usual way. □

Appendix A

In this appendix, we collect some known basic estimates needed in the proof of the main result of the paper. We first list the following propositions which have been proved in [4].

Proposition A.1. *Suppose $0<s<1$ and $\gamma>-3$. Then, there exist two generic constants $C_1,\ C_2>0$ such that for any suitable function g,*

$$C_1|\{\mathbf{I}-\mathbf{P}\}g|^2_{\mathbf{D}}\le\langle\mathscr{L}g,g\rangle\le 2\langle\mathscr{L}_1 g,g\rangle\le C_2|g|^2_{\mathbf{D}},$$
$$C_1\left\{|g|^2_{H^s_{\gamma/2}}+|g|^2_{L^2_{s+\gamma/2}}\right\}\le|g|^2_{\mathbf{D}}\le C_2|g|^2_{H^s_{s+\gamma/2}},$$
$$C_1\left\{|\{\mathbf{I}-\mathbf{P}\}g|^2_{H^s_{\gamma/2}}+|\{\mathbf{I}-\mathbf{P}\}g|^2_{L^2_{s+\gamma/2}}\right\}\le\langle\mathscr{L}g,g\rangle\le C_2|g|^2_{H^s_{s+\gamma/2}}.$$

⟨L1L21⟩ **Proposition A.2.** *Suppose $0<s<1$ and $\gamma>-3$. Then, one has*

$$J_2\sim|g|^2_{L^2_{s+\gamma/2}},\quad C_1|g|^2_{H^s_{\gamma/2}}-C_2|g|^2_{L^2_{s+\gamma/2}}\le J_1\lesssim|g|^2_{H^s_{s+\gamma/2}},$$
$$|\langle\mathscr{L}_2 g,h\rangle|\lesssim\left|\mu^{1/10^3}g\right|_{L^2}\left|\mu^{1/10^3}h\right|_{L^2},\quad |g|^2_{\mathbf{D}}\ge\langle\mathscr{L}_1 g,g\rangle\ge\frac{1}{10}|g|^2_{\mathbf{D}}-C|g|^2_{L^2_{\gamma/2}}.$$

⟨L1L22⟩ **Proposition A.3.** *Assume $0<s<1$, $\gamma>-3$, $|\beta|\ge1$. Then, one has*

$$|\langle\partial_\beta\mathscr{L}_2 g,h\rangle|\lesssim\left|\mu^{1/10^4}g\right|_{H^{|\beta|}_\xi}\left|\mu^{1/10^4}h\right|_{L^2}.$$

⟨basic nonop⟩ **Proposition A.4.** *Assume $0<s<1$, $\gamma>\max\{-3,-3/2-2s\}$. Then, one has*

$$|\langle\mathscr{T}(f,g),h\rangle|\lesssim\Big\{|f|_{L^2_{s+\gamma/2}}|g|_{\mathbf{D}}+|g|_{L^2_{s+\gamma/2}}|f|_{\mathbf{D}}$$
$$+\min\left\{|f|_{L^2}|g|_{L^2_{s+\gamma/2}},|g|_{L^2}|f|_{L^2_{s+\gamma/2}}\right\}\Big\}|h|_{\mathbf{D}}.$$

⟨expo split⟩ **Proposition A.5.** *For any integer $k\ge2$ one can write*

$$\mu_*^{\frac12}=(\mu^{a_1}-\mu_*^{a_1})^k\sum_{i=1}^{k+2}\alpha_{i,2}\mu_*^{a_{i,2}}\mu^{b_{i,2}}+\sum_{i=1}^{k}\alpha_{i,3}\mu_*^{a_{i,3}}\mu^{b_{i,3}}$$
$$=\mu(\xi,\xi_*)+\sum_{i=1}^{k}\alpha_{i,3}\mu_*^{a_{i,3}}\mu^{b_{i,3}}.$$

Above, $\alpha_{i,j}$ are real numbers for all i and j, and the other exponents are strictly positive, at the exception of $b_{1,2}=0$, and with $b_{i,3}>a_{i,3}$.

With Proposition A.4 in hand, as in [17, pp.819, Porposition 6.1] one can prove

⟨estimates on nonop2⟩ **Lemma A.1.** *Let $\zeta(\xi)$ be a smooth function that decays in ξ exponentially, and let $|\alpha|\le N$, $N\ge8$. Writing*

$$\partial^\alpha\Gamma(f,f)=\sum_{\alpha_1+\alpha_2=\alpha}\Gamma(\partial^{\alpha_1}f,\partial^{\alpha_2}f),$$

one has

$$\left\|\int\Gamma(\partial^{\alpha_1}f,\partial^{\alpha_2}f)\zeta(\xi)\,d\xi\right\|\lesssim\overline{\mathcal{E}}_N^{1/2}(t)\overline{\mathcal{D}}_N^{1/2}(t).$$

We also borrow the following result from [30, Proposition 3.1].

⟨lem.non.z1⟩ **Lemma A.2.** *Let $\ell \geq 0$. It holds that*

$$|w_\ell \mathscr{T}(f,f)|_{L^2} \lesssim |w_\ell f|^2_{H^2_{\gamma/2+s}},$$

and

$$\|w_\ell \mathscr{T}(f,f)\|_{H^9_x} + \|w_\ell \mathscr{T}(f,f)\|_{Z_1} \lesssim \sum_{|\alpha|+|\beta|\leq 11} \left\| w_{\ell - \frac{\gamma+2s}{2\gamma}} \partial^\alpha_\beta f \right\|^2.$$

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