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CAUCHY PROBLEM ON THE VLASOV-FOKKER-PLANCK EQUATION COUPLED WITH THE COMPRESSIBLE EULER EQUATIONS THROUGH THE FRICTION FORCE

Renjun Duan

Department of Mathematics, The Chinese University of Hong Kong
Shatin, Hong Kong

Shuangqian Liu

Department of Mathematics, Jinan Unviersity
Guangdong, P.R. China

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Abstract. We are concerned with a two-phase flow system consisting of the Vlasov-Fokker-Planck equation for particles coupled to the compressible Euler equations for the fluid through the friction force. Global well-posedness of the Cauchy problem is established in perturbation framework, and rates of convergence of solutions toward equilibrium, which are algebraic in the whole space and exponential on torus, are also obtained under some additional conditions on initial data. The proof is based on the classical energy estimates.

1. **Main result.** Consider the Vlasov-Fokker-Planck equation with friction force

$$\partial_t F + \xi \cdot \nabla_x F = \nu \nabla_\xi \cdot [(\xi - u)F + \beta \nabla_\xi F], \tag{1.1}$$

coupled with the compressible Euler equations

$$\partial_t n + \nabla_x \cdot (nu) = 0, \tag{1.2}$$

$$\partial_t (nu) + \nabla_x (nu \otimes u) + \nabla_x P = \mu \int_{\mathbb{R}^3} (\xi - u) F \, d\xi. \tag{1.3}$$

Here, the unknowns are $F = F(t, x, \xi) \geq 0$ for $t \geq 0, x \in \Omega, \xi \in \mathbb{R}^3$, denoting the density distribution function of particles in the phase space, and $n = n(t, x)$ and $u = u(t, x) \in \mathbb{R}^3$, for $t \geq 0, x \in \Omega$, denoting respectively the mass density and velocity field of the fluid. The spatial domain $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$. Initial data

$$F(0, x, \xi) = F_0(x, \xi), \quad n(0, x) = n_0(x), \quad u(0, x) = u_0(x),$$

are given. In (1.3), P is the pressure function depending only on n with $P'(\cdot) > 0$, for instance, $P(n) = c_\gamma n^\gamma$ with $\gamma > 1$ and $c_\gamma > 0$. In (1.1) and (1.3), $\nu > 0$, $\beta > 0$ and $\mu > 0$ are generic physical constants.

It is obvious to see that system (1.1), (1.2), (1.3) admits a trivial steady state

$$F = M_{[\rho_\infty, u_\infty, \beta]}, \quad n = n_\infty, \quad u = u_\infty, \tag{1.4}$$

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with $\rho_\infty > 0$, $n_\infty > 0$ and $u_\infty = (u_{\infty 1}, u_{\infty 2}, u_{\infty 3})$ being constants, where for (ρ, v, θ) the Maxwellian $M_{[\rho,v,\theta]}$ denotes

$$M_{[\rho,v,\theta]} = M_{[\rho,v,\theta]}(\xi) = \frac{\rho}{(2\pi\theta)^{3/2}} e^{-\frac{|\xi-v|^2}{2\theta}}.$$

Moreover, it is also straightforward to verify that the coupled system admits the following local dissipation identity

$$\partial_t \left\{ \frac{\nu}{\beta\mu} nE + \int_{\mathbb{R}^3} F \ln \frac{F}{M_{[\rho_\infty,u_\infty,\beta]}} \,d\xi \right\} + \nabla_x \cdot \left\{ \frac{\nu}{\beta\mu}(nE+P)u + \int_{\mathbb{R}^3} \xi F \ln \frac{F}{M_{[\rho_\infty,u_\infty,\beta]}} \,d\xi \right\} + \frac{\nu}{\beta} \int_{\mathbb{R}^3} \frac{1}{F} |\beta \nabla_\xi F + (\xi - u)F|^2 \,d\xi = 0,$$

where E denotes

$$E = \frac{1}{2}|u|^2 + \int^n \frac{P(\eta)}{\eta^2}\,d\eta.$$

Hence it is natural to ask if the constant steady state (1.4) is stable for small smooth perturbations. It will be seen later on that even though the compressible Euler equations in general develop singularity in finite time, the coupling to the Vlasov-Fokker-Planck equation through the friction force can assure that solutions with small initial data are time-asymptotically stable. Thus, the friction term plays a key role for the stability of the coupled system.

For the above purpose, let us reformulate the Cauchy problem in the framework of perturbations. Without loss of generality and for brevity of presentation, we suppose that all involving constants ν, β and μ are normalized to be one and we also choose $\rho_\infty = n_\infty = 1$ and $u_\infty = 0$. Denote $M = M_{[1,0,1]}$. Set

$$F = M + M^{1/2} f$$

so that the reformulated Cauchy problem reads

$$\partial_t f + \xi \cdot \nabla_x f + u \cdot \nabla_\xi f - \frac{1}{2} u \cdot \xi f - u \cdot \xi M^{1/2} = Lf, \tag{1.5}$$

$$\partial_t n + (n+1)\nabla_x \cdot u + u \cdot \nabla_x n = 0, \tag{1.6}$$

$$\partial_t u + u \cdot \nabla_x u + \frac{P'(n+1)}{n+1} \nabla_x n = \frac{b - u - au}{n+1}, \tag{1.7}$$

with initial data

$$f(0,x,\xi) = f_0(x,\xi) \equiv M^{-1/2}(F_0 - M), \quad n(0,x) = n_0(x), \quad u(0,x) = u_0(x). \tag{1.8}$$

Here, for simplicity of notations we have used $n+1$ to still replace n. L is the linearized Fokker-Planck operator defined by

$$Lf = \frac{1}{M^{1/2}} \nabla_\xi \cdot \left[M \nabla_\xi \left(\frac{f}{M^{1/2}} \right) \right],$$

and $a = a^f, b = b^f$ depending on f are the moments of f defined by

$$a^f(t,x) = \int_{\mathbb{R}^3} M^{1/2} f(t,x,\xi)\,d\xi, \quad b^f(t,x) = \int_{\mathbb{R}^3} \xi M^{1/2} f(t,x,\xi)\,d\xi.$$

The main results of the paper, concerning the global well-posedness and convergence rates for the above Cauchy problem on the coupled Vlassov-Fokker-Planck equations with the compressible Euler equations, are stated as follows.

Theorem 1.1. *Let $\Omega = \mathbb{R}^3$. Suppose that $\|f_0\|_{L^2_\xi(H^3_x)} + \|(n_0, u_0)\|_{H^3}$ is small enough and $F_0 = M + M^{1/2} f_0 \geq 0$. Then, the Cauchy problem (1.5), (1.6), (1.7), (1.8) admits a unique global solution $(f(t,x,\xi), n(t,x), u(t,x))$ satisfying*

$$f \in C([0,\infty); L^2_\xi(H^3_x)), \quad n, u \in C([0,\infty); H^3),$$

$$F = M + M^{1/2} f \geq 0,$$

$$\sup_{t\geq 0}(\|f(t)\|_{L^2_\xi(H^3_x)} + \|(n,u)(t)\|_{H^3}) \leq C(\|f_0\|_{L^2_\xi(H^3_x)} + \|(n_0,u_0)\|_{H^3}).$$

Moreover, if $\|f_0\|_{L^2_\xi(H^3_x \cap L^1_x)} + \|(n_0, u_0)\|_{H^3 \cap L^1}$ is sufficiently small then

$$\|f(t)\|_{L^2_\xi(H^3_x)} + \|(n,u)(t)\|_{H^3} \leq C(1+t)^{-\frac{3}{4}},$$

for all $t \geq 0$.

Theorem 1.2. *Let $\Omega = \mathbb{T}^3$. Suppose that $\|f_0\|_{L^2_\xi(H^3_x)} + \|(n_0, u_0)\|_{H^3}$ is small enough, $F_0 = M + M^{1/2} f_0 \geq 0$, and*

$$\int_{\mathbb{T}^3} a_0 \, dx = 0, \quad \int_{\mathbb{T}^3} n_0 \, dx = 0, \quad \int_{\mathbb{T}^3} [b_0 + (n_0+1)u_0] \, dx = 0. \tag{1.9}$$

Then, the Cauchy problem (1.5), (1.6), (1.7), (1.8) admits a unique global solution $(f(t,x,\xi), n(t,x), u(t,x))$ satisfying

$$f \in C([0,\infty); L^2_\xi(H^3_x)), \quad n, u \in C([0,\infty); H^3),$$

$$F = M + M^{1/2} f \geq 0,$$

$$\sup_{t\geq 0} e^{\lambda t}(\|f(t)\|_{L^2_\xi(H^3_x)} + \|(n,u)(t)\|_{H^3}) \leq C(\|f_0\|_{L^2_\xi(H^3_x)} + \|(n_0,u_0)\|_{H^3}),$$

where $\lambda > 0$ is a constant.

The background for the study of the coupled system under consideration is related to the modelling of fluid-particle interactions, for instance, to describe the behaviour of sprays, aerosols or more generically two phase flows where one phase (disperse) can be considered as a suspension of particles onto the other one (dense) thought as a fluid, cf. [1, 2, 3, 6, 17, 25].

Let us mention some previous mathematical work on the coupled kinetic-fluid systems related to the paper. [20] obtained global existence and large time behavior of solutions to the Vlasov-Stokes system, where the fluid is assumed to be viscous and incompressible and its velocity satisfies the Stokes equations with the same friction force as in (1.3). When the motion of the fluid is described by the incompressible Navier-Stokes equations, [15, 16] considered hydrodynamic limits of the Vlasov-Navier-Stokes system in different regimes (see also the recent work [18]), [10, 11, 13] dealt with similar singular perturbation problems, and [5] recently gave a proof of global existence of weak solutions on the periodic domain. [22, 23] provided a detailed study of the global existence and asymptotic analysis for the coupled system of the Vlasov-Fokker-Planck equation with the compressible Navier-Stokes equations in $\mathbb{R}^3$. [14] also proved global existence of classical solutions near equilibrium for the incompressible model, and [7] obtained global existence and convergence rates of solutions close to equilibrium when the fluid is modelled by the incompressible Euler equations. The extension of [14] to the compressible fluid was made by [9].

In the framework of the inviscid compressible flow under friction forces, existence of smooth solutions for short time was proved in [4] when there is no Brownian effect in the kinetic equation, see also the recent work [24]. When the velocity diffusion is considered, stability and asymptotic analysis were discussed in [8].

The goal of the paper is to extend the results in [7] for the incompressible Euler equations to the compressible case. The main additional difficulty comes from the appearance of the fluid density equation. We find that the coupled system in the case when the fluid is compressible has the similar dissipative structure, and the dissipation rate corresponding to the fluid density which is a hyperbolic component can be recovered in a way as for the damped compressible Euler equations, cf. [21]. The key point in the proof of global existence, also used in [7], is based on the fact that the local momentum component $b^f(t,x)$ of $f(t,x,\xi)$ behaves like an elliptic equation with the remaining terms which involve either the microscopic part of $f(t,x,\xi)$ or the relaxation term $b-u$.

Through the paper C denotes a positive (generally large) constant and λ a positive (generally small) constant, where both C and λ may take different values in different places. $A\sim B$ means $\lambda A\leq B\leq \frac{1}{\lambda}A$ for a generic constant $\lambda>0$.

2. **A priori estimate over the whole space.** The main tool, initiated by [19] and introduced in [12], is to decompose $f(t,x,\xi)$ as the sum of the fluid part $\mathbf{P}f$ and the particle part $\{\mathbf{I}-\mathbf{P}\}f$

$$f(t,x,\xi)=\mathbf{P}f+\{\mathbf{I}-\mathbf{P}\}f,$$

where the projection operator $\mathbf{P}$ is to be defined later on. Let $\sigma(\xi)=1+|\xi|^2$ and denote $|\cdot|_\sigma$ by

$$|f|_\sigma^2=\int_{\mathbb{R}^3}\left[|\nabla_\xi f(\xi)|^2+\sigma(\xi)|f|^2\right]\,d\xi,\quad f=f(\xi).$$

$\|\cdot\|_\sigma$ stands for the spatial integration of $|\cdot|_\sigma$. Notice that the operator L satisfies that there is a constant $\lambda_0>0$ such that

$$-\int_{\mathbb{R}^3} fLf\,d\xi\geq \lambda_0|\{\mathbf{I}-\mathbf{P}_0\}f|_\sigma^2,\tag{2.1}$$

for $f=f(\xi)$, where $\mathbf{P}_0 f=a^f M^{1/2}$. Moreover, define the velocity orthogonal projection $\mathbf{P}:\ L^2_\xi\to \mathrm{span}\{M^{1/2},\xi M^{1/2}\}$ by $\mathbf{P}=\mathbf{P}_0\oplus\mathbf{P}_1$ with $\mathbf{P}_1 f=b^f\cdot\xi M^{1/2}$. Therefore, Lf can be computed as

$$Lf=L\{\mathbf{I}-\mathbf{P}\}f+L\mathbf{P}f=L\{\mathbf{I}-\mathbf{P}\}f-\mathbf{P}_1 f,$$

which implies

$$-\int_{\mathbb{R}^3} fLf\,d\xi\geq \lambda_0|\{\mathbf{I}-\mathbf{P}\}f|_\sigma^2+|b^f|^2.$$

Now, we begin to make the global a priori estimates in the case of the whole space $\Omega=\mathbb{R}^3$ under the assumption

$$\sup_{0\leq t<T}\{\|f(t)\|_{L^2_\xi(H^3_x)}+\|(n,u)\|_{H^3}\}\leq\delta,\tag{2.2}$$

where $0<\delta<1$ is a generic constant small enough and $(f(t,x,\xi),n(t,x),u(t,x))$ is the smooth solution to the Cauchy problem (1.5), (1.6), (1.7) and (1.8) on $0\leq t<T$ for $T>0$.

Lemma 2.1. *It holds that*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|f\|^2+\|n\|^2+\|u\|^2)+\lambda(\|\{\mathbf{I}-\mathbf{P}\}f\|_\sigma^2+\|b-u\|^2)\\&\leq C\|(n,u)\|_{H^1}(\|b-u\|^2+\|\nabla_x(a,b,n,u)\|^2)+C\|u\|_{H^2}\|\{\mathbf{I}-\mathbf{P}\}f\|_\sigma^2,\end{aligned}\tag{2.3}$$

for all $0 \le t < T$.

Proof. Multiplying (1.5), (1.6), (1.7) by f, n and u respectively and then taking integrations and the sum, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|f\|^2+\|n\|^2+\|u\|^2)+\int_{\mathbb{R}^3}\langle -L\{\mathbf{I}-\mathbf{P}\}f,f\rangle\,dx+\int_{\mathbb{R}^3}|b-u|^2\,dx\\
&=\int_{\mathbb{R}^3}u\cdot\langle\frac{1}{2}\xi f,f\rangle\,dx-\int_{\mathbb{R}^3}(u\cdot\nabla_x u)\cdot u\,dx-\int_{\mathbb{R}^3}\frac{n(b-u)\cdot u+au\cdot u}{n+1}\,dx\\
&\quad+\int_{\mathbb{R}^3}\left[\frac{P'(n+1)}{n+1}-P'(1)\right]\nabla_x n\cdot u\,dx. \qquad (2.4)
\end{aligned}$$

The first and third terms on the right can be estimated as follows. As in [7], we notice

$$\langle\frac{1}{2}\xi f,f\rangle=ab+\langle\xi\mathbf{P}f,\{\mathbf{I}-\mathbf{P}\}f\rangle+\langle\frac{1}{2}\xi,|\{\mathbf{I}-\mathbf{P}\}f|^2\rangle.$$

Furthermore, by Hölder and Sobolev inequalities as well as (2.2), it holds that

$$\begin{aligned}
&\int_{\mathbb{R}^3}u\cdot(ab)\,dx-\int_{\mathbb{R}^3}\frac{au\cdot u}{n+1}\,dx=\int_{\mathbb{R}^3}au\cdot(b-u)\,dx+\int_{\mathbb{R}^3}\frac{an|u|^2}{n+1}\,dx\\
&\le\|a\|_{L^6}\|u\|_{L^3}\|b-u\|_{L^2}+2\|a\|_{L^6}\|n\|_{L^3}\big\||u|^2\big\|_{L^3}\\
&\le C\|\nabla_x a\|_{L^2}\|u\|_{H^1}\|b-u\|_{L^2}+C\|\nabla_x a\|_{L^2}\|n\|_{H^1}\|\nabla_x u\|_{L^2}^2\\
&\le C\|(n,u)\|_{H^1}(\|b-u\|^2+\|\nabla_x(a,u)\|^2),
\end{aligned}$$

$$\begin{aligned}
\int_{\mathbb{R}^3}u\cdot\langle\xi\mathbf{P}f,\{\mathbf{I}-\mathbf{P}\}f\rangle\,dx&\le C\int_{\mathbb{R}^3}|u|\cdot|(a,b)|\cdot\|\{\mathbf{I}-\mathbf{P}\}f\|_{L^2_\xi}\,dx\\
&\le C\|u\|_{L^3}\|(a,b)\|_{L^6}\|\{\mathbf{I}-\mathbf{P}\}f\|_{L^2_{x,\xi}}\\
&\le C\|u\|_{H^1}(\|\nabla_x(a,b)\|^2+\|\{\mathbf{I}-\mathbf{P}\}f\|^2),
\end{aligned}$$

$$\begin{aligned}
\int_{\mathbb{R}^3}u\cdot\langle\frac{1}{2}\xi,|\{\mathbf{I}-\mathbf{P}\}f|^2\rangle\,dx&\le C\|u\|_{L^\infty}\||\xi|^{1/2}\{\mathbf{I}-\mathbf{P}\}f\|^2\\
&\le C\|\nabla_x u\|_{H^1}\|\{\mathbf{I}-\mathbf{P}\}f\|_\sigma^2,
\end{aligned}$$

and

$$-\int_{\mathbb{R}^3}\frac{n(b-u)\cdot u}{n+1}\,dx\le C\|n\|_{L^6}\|b-u\|_{L^2}\|u\|_{L^3}\le C\|u\|_{H^1}(\|\nabla_x n\|^2+\|b-u\|^2).$$

Therefore, the first and third terms on the right of (2.4) are bounded by

$$C\|(n,u)\|_{H^1}(\|b-u\|^2+\|\nabla_x(a,b,u)\|^2)+C\|u\|_{H^2}\|\{\mathbf{I}-\mathbf{P}\}f\|_\sigma^2.$$

For the remaining terms, one also has

$$\begin{aligned}
&-\int_{\mathbb{R}^3}(u\cdot\nabla_x u)\cdot u\,dx+\int_{\mathbb{R}^3}\left[\frac{P'(n+1)}{n+1}-P'(1)\right]\nabla_x n\cdot u\,dx\\
&\le C\|u\|_{L^3}\|\nabla_x u\|_{L^2}\|u\|_{L^6}+C\|n\|_{L^3}\|\nabla_x n\|_{L^2}\|u\|_{L^6}\\
&\le C\|(n,u)\|_{H^1}\|\nabla_x(n,u)\|^2.
\end{aligned}$$

Then, (2.3) follows by plugging all estimates above into (2.4) and applying (2.1), and hence Lemma 2.1 is proved. □

Lemma 2.2. *It holds that*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{1\le|\alpha|\le 3}(\|\partial_x^\alpha f\|^2+\|\frac{\sqrt{P'(n+1)}}{n+1}\partial_x^\alpha n\|^2+\|\partial_x^\alpha u\|^2)\\
&+\lambda\sum_{1\le|\alpha|\le 3}(\|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\sigma^2+\|\partial_x^\alpha(b-u)\|^2)\\
&\le C\|(a,b,n,u)\|_{H^3}(\|\nabla_x(a,b,n,u)\|_{H^2}^2+\sum_{1\le|\alpha|\le 3}\|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\sigma^2),
\end{aligned}\tag{2.5}$$

for all $0\le t<T$.

Proof. Applying ∂_x^α with $1\le|\alpha|\le 3$ to (1.5), (1.6) and (1.7), one has

$$\begin{aligned}
&\partial_t(\partial_x^\alpha f)+\xi\cdot\nabla_x(\partial_x^\alpha f)+u\cdot\nabla_\xi(\partial_x^\alpha f)-\partial_x^\alpha u\cdot\xi M^{1/2}-L\partial_x^\alpha f\\
&\quad=[-\partial_x^\alpha,u\cdot\nabla_\xi]f+\partial_x^\alpha(\frac{1}{2}u\cdot\xi f),
\end{aligned}\tag{2.6}$$

$$\begin{aligned}
&\partial_t(\partial_x^\alpha n)+(n+1)\nabla_x\cdot\partial_x^\alpha u+u\cdot\nabla_x\partial_x^\alpha n\\
&\quad=[-\partial_x^\alpha,n\nabla_x\cdot]u+[-\partial_x^\alpha,u\cdot\nabla_x]n,
\end{aligned}\tag{2.7}$$

$$\begin{aligned}
&\partial_t(\partial_x^\alpha u)+u\cdot\nabla_x(\partial_x^\alpha u)+\frac{P'(n+1)}{n+1}\nabla\partial_x^\alpha n-\partial_x^\alpha(b-u)\\
&\quad=\partial_x^\alpha\left[-\frac{n(b-u)+au}{n+1}\right]+[-\partial_x^\alpha,(u\cdot\nabla_x)]u+[-\partial_x^\alpha,\frac{P'(n+1)}{n+1}\nabla_x]n,
\end{aligned}\tag{2.8}$$

where $[\cdot,\cdot]$ stands for $[A,B]=AB-BA$ for two operators A and B. Further multiplying (2.6), (2.7), (2.8) by $\partial_x^\alpha f$, $\frac{P'(n+1)}{(n+1)^2}\partial_x^\alpha n$ and $\partial_x^\alpha u$, respectively and then taking integration and the sum, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\partial_x^\alpha f\|^2+\|\frac{\sqrt{P'(n+1)}}{n+1}\partial_x^\alpha n\|^2+\|\partial_x^\alpha u\|^2)\\
&+\int_{\mathbb{R}^3}\langle -L\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}f,\partial_x^\alpha f\rangle\,dx+\int_{\mathbb{R}^3}|\partial_x^\alpha(b-u)|^2\,dx=I_1+I_2+I_3,
\end{aligned}\tag{2.9}$$

where I_1, I_2, I_3 denote

$$\begin{aligned}
I_1&=\int_{\mathbb{R}^3}\langle[-\partial_x^\alpha,u\cdot\nabla_\xi]f,\partial_x^\alpha f\rangle\,dx+\int_{\mathbb{R}^3}\langle\partial_x^\alpha(\frac{1}{2}u\cdot\xi f),\partial_x^\alpha f\rangle\,dx,\\
I_2&=\int_{\mathbb{R}^3}\frac{1}{2}\partial_t\left[\frac{P'(n+1)}{(n+1)^2}\right]|\partial_x^\alpha n|^2\,dx\\
&\quad+\int_{\mathbb{R}^3}\frac{P'(n+1)}{(n+1)^2}\{[-\partial_x^\alpha,n\nabla_x\cdot]u+[-\partial_x^\alpha,u\cdot\nabla_x]n\}\partial_x^\alpha n\,dx,\\
I_3&=\int_{\mathbb{R}^3}\{[-\partial_x^\alpha,(u\cdot\nabla_x)]u+[-\partial_x^\alpha,\frac{P'(n+1)}{n+1}\nabla_x]n\}\cdot\partial_x^\alpha u\,dx\\
&\quad+\int_{\mathbb{R}^3}\partial_x^\alpha\left[-\frac{n(b-u)+au}{n+1}\right]\cdot\partial_x^\alpha u\,dx.
\end{aligned}$$

Each term can be estimated as follows. Notice $1\le|\alpha|\le 3$. For I_1, as in [7, Lemma 2.3], it holds that

$$I_1\le C\|u\|_{H^3}(\|\nabla_x(a,b)\|_{H^2}^2+\sum_{1\le|\alpha|\le 3}\|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\sigma^2).$$

For I_2, recalling (1.5) as well as (2.2), one has

$$\sup_{0\le t<T, x\in\mathbb{R}^3} |\partial_t n(t,x)| \le (\|n\|_{L^\infty}+1)\|\nabla_x\cdot u\|_{L^\infty} + \|u\|_{L^\infty}\|\nabla_x n\|_{L^\infty} \le C\|u\|_{H^3},$$

and hence it follows from Hölder and Sobolev inequalities that

$$I_2 \le C\|(n,u)\|_{H^3}\|\nabla_x(n,u)\|_{H^2}^2.$$

For I_3, in a similar way, it holds that

$$I_3 \le C\|(a,b,n,u)\|_{H^3}\|\nabla_x(n,u)\|_{H^2}^2.$$

Plugging these estimates into (2.9), using (2.1) and then taking the sum over $1\le |\alpha|\le 3$, (2.5) follows and thus Lemma 2.2 is proved. $\square$

To include $\|\nabla_x(a,b)\|_{H^2}^2$ into the energy dissipation rate, as in [7], we need to study the following equations of a and b

$$\partial_t a + \nabla_x\cdot b = 0, \tag{2.10}$$

$$\partial_t b_i + \partial_{x_i} a + \sum_j \partial_{x_j}\Gamma_{ij}(\{\mathbf{I}-\mathbf{P}\}f) = -b_i + u_i(1+a), \tag{2.11}$$

$$\partial_t\Gamma_{ij}(\{\mathbf{I}-\mathbf{P}\}f) + \partial_{x_i} b_j + \partial_{x_j} b_i - (u_i b_j + u_j b_i) = \Gamma_{ij}(\ell + r), \tag{2.12}$$

for $1\le i,j\le 3$, where Γ_{ij} is the moment functional defined by $\Gamma_{ij}(g) = \langle(\xi_i\xi_j - 1)M^{1/2}, g\rangle$, for any $g=g(\xi)$, and ℓ, r denote

$$\ell = -\xi\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}f + L\{\mathbf{I}-\mathbf{P}\}f,$$
$$r = -u\cdot\nabla_\xi\{\mathbf{I}-\mathbf{P}\}f + \frac{1}{2}u\cdot\xi\{\mathbf{I}-\mathbf{P}\}f.$$

See [7] for the derivation of (2.10), (2.11) and (2.12). Define a temporal functional $\mathcal{E}_0(t)$ by

$$\begin{aligned}\mathcal{E}_0(t) = &\sum_{|\alpha|\le 2}\sum_{ij}\int_{\mathbb{R}^3}\partial_x^\alpha(\partial_{x_i}b_j + \partial_{x_j}b_i)\partial_x^\alpha\Gamma_{ij}(\{\mathbf{I}-\mathbf{P}\}f)\,dx\\ &-\sum_{|\alpha|\le 2}\int_{\mathbb{R}^3}\partial_x^\alpha a\partial_x^\alpha\nabla_x\cdot b\,dx,\end{aligned} \tag{2.13}$$

The following lemma was proved in [7].

Lemma 2.3. *It holds that*

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_0(t) + \lambda\|\nabla_x(a,b)\|_{H^2}^2 \le\, &C(\|\{\mathbf{I}-\mathbf{P}\}f\|_{L^2_\xi(H^3_x)}^2 + \|u-b\|_{H^2}^2)\\ &+C\|u\|_{H^2}^2\Big\{\|\nabla_x(a,b)\|_{H^2}^2 + \|\nabla_x\{\mathbf{I}-\mathbf{P}\}f\|_{L^2_\xi(H^2_x)}^2\Big\},\end{aligned} \tag{2.14}$$

for all $0\le t<T$.

Moreover, since $b-u$ is dissipative, it is straightforward to derive from (1.6) and (1.7) the dissipation rate of $n(t,x)$.

Lemma 2.4. *It holds that*

$$\begin{aligned}&\frac{d}{dt}\sum_{|\alpha|\le 2}\int_{\mathbb{R}^3}\partial_x^\alpha\nabla_x n\cdot\partial_x^\alpha u\,dx + \lambda\sum_{|\alpha|\le 2}\|\nabla_x\partial_x^\alpha n\|^2\\ &\le C(\|\nabla_x u\|_{H^2}^2 + \|b-u\|_{H^2}^2) + C\|(a,n,u)\|_{H^3}^3\|\nabla_x(n,u)\|_{H^2}^2,\end{aligned} \tag{2.15}$$

for all $0\le t<T$.

Proof. Let $|\alpha|\le 2$. By (1.7), one can compute

$$\begin{aligned}
&P'(1)\|\nabla_x\partial_x^\alpha n\|^2\\
&=\int_{\mathbb{R}^3}\nabla_x\partial_x^\alpha n\cdot\partial_x^\alpha(-\partial_t u)\,dx+\int_{\mathbb{R}^3}\nabla_x\partial_x^\alpha n\cdot\partial_x^\alpha\left\{\frac{b-u}{n+1}\right\}dx\\
&\quad+\int_{\mathbb{R}^3}\nabla_x\partial_x^\alpha n\cdot\partial_x^\alpha\left\{-u\cdot\nabla_x u-\frac{au}{n+1}-\left[\frac{P'(n+1)}{n+1}-P'(1)\right]\nabla_x n\right\}dx\\
&=I_4+I_5+I_6.
\end{aligned}\tag{2.16}$$

For I_4, by (1.5), one has

$$\begin{aligned}
I_4&=-\frac{d}{dt}\int_{\mathbb{R}^3}\partial_x^\alpha\nabla_x n\cdot\partial_x^\alpha u\,dx+\int_{\mathbb{R}^3}\partial_x^\alpha\nabla_x\partial_t n\cdot\partial_x^\alpha u\,dx\\
&=-\frac{d}{dt}\int_{\mathbb{R}^3}\partial_x^\alpha\nabla_x n\cdot\partial_x^\alpha u\,dx+\int_{\mathbb{R}^3}\partial_x^\alpha[(n+1)\nabla_x\cdot u+u\cdot\nabla_x n]\partial_x^\alpha\nabla_x\cdot u\,dx\\
&\le-\frac{d}{dt}\int_{\mathbb{R}^3}\partial_x^\alpha\nabla_x n\cdot\partial_x^\alpha u\,dx+\|\partial_x^\alpha\nabla_x\cdot u\|^2+C\|(n,u)\|_{H^3}\|\nabla_x u\|^2.
\end{aligned}$$

For I_5 and I_6, due to (2.2), it can be directly estimated by Cauchy-Schwarz inequality that

$$\begin{aligned}
I_5&\le\frac{P'(1)}{4}\|\nabla_x\partial_x^\alpha n\|^2+C\|b-u\|_{H^2}^2,\\
I_6&\le\frac{P'(1)}{4}\|\nabla_x\partial_x^\alpha n\|^2+C\|(a,n,u)\|_{H^3}^2\|\nabla_x(n,u)\|_{H^2}^2.
\end{aligned}$$

Putting these estimates into (2.16) and then taking the sum over $|\alpha|\le 2$ gives (2.15). Lemma 2.4 is proved. $\square$

Proof of global existence: It is now immediate to obtain the global a priori estimates. In fact, define the temporal energy functional

$$\begin{aligned}
\mathcal{E}(t)=&\|f\|^2+\|n\|^2+\|u\|^2\\
&+\sum_{1\le|\alpha|\le 3}\left(\|\partial_x^\alpha f\|^2+\|\frac{\sqrt{P'(n+1)}}{n+1}\partial_x^\alpha n\|^2+\|\partial_x^\alpha u\|^2\right)\\
&+\kappa_1\mathcal{E}_0(t)+\kappa_2\sum_{|\alpha|\le 2}\int_{\mathbb{R}^3}\partial_x^\alpha u\cdot\partial_x^\alpha\nabla_x n\,dx,
\end{aligned}\tag{2.17}$$

and the corresponding dissipation rate functional

$$\mathcal{D}(t)=\|\nabla_x(a,b,n,u)\|_{H^2}^2+\|b-u\|_{H^3}^2+\sum_{|\alpha|\le 3}\|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\sigma^2,\tag{2.18}$$

where $0<\kappa_1,\kappa_2\ll 1$ are constants. Notice that since $\kappa_1>0$ and $\kappa_2>0$ are sufficiently small, under the assumption (2.2), recalling (2.13), it holds that

$$\mathcal{E}(t)\sim\|f(t)\|_{L_\xi^2(H_x^3)}^2+\|(n,u)(t)\|_{H^3}^2,$$

uniformly for all $0\le t<T$. Moreover, by suitably choosing constants κ_1 and κ_2 with $0<\kappa_2\ll\kappa_1\ll 1$, the sum of equations (2.3), (2.5), $\kappa_1\times$(2.14) and $\kappa_2\times$(2.15) gives

$$\frac{d}{dt}\mathcal{E}(t)+\lambda\mathcal{D}(t)\le C[\mathcal{E}^{1/2}(t)+\mathcal{E}(t)]\mathcal{D}(t),\tag{2.19}$$

for all $0 \le t < T$. By (2.2), one has $\mathcal{E}^{1/2}(t) + \mathcal{E}(t) \le C(\delta^2 + \delta)$. Thus, as long as $0 < \delta < 1$ is small enough, the time integration of (2.19) yields

$$\mathcal{E}(t) + \lambda \int_0^t \mathcal{D}(s)\, ds \le \mathcal{E}(0), \tag{2.20}$$

for all $0 \le t < T$. Besides, (2.2) can be justified by choosing

$$\mathcal{E}(0) \sim \|f_0\|^2_{L^2_\xi(H^3_x)} + \|(n_0, u_0)\|^2_{H^3}$$

sufficiently small. For brevity, the proof for local existence of smooth solutions is omitted. Then, the global existence of solutions follows by the obtained global a priori estimates as well the continuity argument, and also (2.20) holds true for all $t \ge 0$. □

3. **Time-decay of solutions.** We deal with the time-decay of the obtained global solutions under the additional assumption that L^1 norm of initial data is bounded. First of all, we consider the following linearized Cauchy problem

$$\partial_t f + \xi \cdot \nabla_x f - u \cdot \xi M^{1/2} - Lf = S, \tag{3.1}$$

$$\partial_t n + \nabla_x \cdot u = 0, \tag{3.2}$$

$$\partial_t u + P'(1)\nabla_x n + u - b = 0, \tag{3.3}$$

with initial data

$$f(0, x, \xi) = f_0(x, \xi), \quad n(0, x) = n_0(x), \quad u(0, x) = u_0(x). \tag{3.4}$$

Here, for the later use in order to handle the velocity differentiation in the nonlinear term, the source term S in the linearized Fokker-Planck equation takes the form of

$$S = \nabla_\xi G - \frac{1}{2}\xi \cdot G + h,$$

where $G = G(t, x, \xi) \in \mathbb{R}^3$ and $h = h(t, x, \xi) \in \mathbb{R}$ satisfy

$$\mathbf{P}_0 G = 0, \quad \mathbf{P} h = 0.$$

Let us introduce some notations for simplicity of presentation. Denote $U(t) = (f(t), n(t), u(t))$ to be the solution to the Cauchy problem (3.1), (3.2), (3.3) and (3.4), and denote $U_0 = (f_0, n_0, u_0)$. Define $\mathbb{A}(t)$ to be the solution operator in the case of $S = 0$. Then, by Duhamel's principle,

$$U(t) = \mathbb{A}(t)U_0 + \int_0^t \mathbb{A}(t-s)(S(s), 0, 0)\, ds,$$

for all $t \ge 0$. Set $Z_q = L^2_\xi(L^q_x)$ for $q \ge 1$. Define norms $\|\cdot\|_{\mathcal{H}^m}$, $\|\cdot\|_{\mathcal{Z}_q}$ by

$$\|U\|^2_{\mathcal{H}^m} = \|f\|^2_{L^2_\xi(H^m_x)} + \|n\|^2_{H^m} + \|u\|^2_{H^m}, \quad \|U\|_{\mathcal{Z}_q} = \|f\|_{Z_q} + \|n\|_{L^q} + \|u\|_{L^q},$$

and set $\mathcal{L}^2 = \mathcal{H}^0$ when $m = 0$.

Lemma 3.1. *Let $1 \le q \le 2$. For any α, α' with $\alpha' \le \alpha$ and $m = |\alpha - \alpha'|$,*

$$\|\partial_x^\alpha \mathbb{A}(t)U_0\|_{\mathcal{L}^2} \le C(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})-\frac{m}{2}}(\|\partial_x^{\alpha'} U_0\|_{\mathcal{Z}_q} + \|\partial_x^\alpha U_0\|_{\mathcal{L}^2}), \tag{3.5}$$

and

$$\begin{aligned}\left\|\partial_x^\alpha \int_0^t \mathbb{A}(t-s)(S(s), 0, 0)\, ds\right\|^2_{\mathcal{L}^2} \le C\int_0^t (1+t-s)^{-3(\frac{1}{q}-\frac{1}{2})-m} \\ \times \{\|\partial_x^{\alpha'}(G(s), \sigma^{-1/2}h(s))\|^2_{Z_q} + \|\partial_x^\alpha (G(s), \sigma^{-1/2}h(s))\|^2_{L^2}\}\, ds,\end{aligned} \tag{3.6}$$

hold for all $t \geq 0$.

Proof. The proof is quite similar to the case of the incompressible Euler equations given by [7, Theorem 3.1]. In fact, denote $\widehat{\cdot}$ to be the Fourier transform with respect to space variable. Then, as for proving [7, Equations (49) and (50) in Theorem 3.1], one has

$$\frac{1}{2}\partial_t(\|\widehat{f}\|^2_{L^2_\xi} + P'(1)|\widehat{n}|^2 + |\widehat{u}|^2) + \lambda(|\{\mathbf{I}-\mathbf{P}\}\widehat{f}|^2_\sigma + |\widehat{b-u}|^2) \leq C(\|\widehat{G}\|^2_{L^2_\xi} + \|\sigma^{-1/2}h\|^2_{L^2_\xi}), \quad (3.7)$$

and

$$\partial_t \Re\widetilde{\mathcal{E}}_1(t) + \frac{\lambda|k|^2}{1+|k|^2}|\widehat{(a,b)}|^2 \leq C(|\{\mathbf{I}-\mathbf{P}\}\widehat{f}|^2_{L^2_\xi} + |\widehat{b-u}|^2) + C(\|\widehat{G}\|^2_{L^2_\xi} + \|\sigma^{-1/2}h\|^2_{L^2_\xi}), \quad (3.8)$$

where $\widetilde{\mathcal{E}}_1(t)$ is defined by

$$\widetilde{\mathcal{E}}_1(t) = \frac{1}{1+|k|^2}\sum_{ij}(ik_i\widehat{b}_j + ik_j\widehat{b}_i|\Gamma_{ij}(\{\mathbf{I}-\mathbf{P}\}\widehat{f})) - \frac{1}{1+|k|^2}(\widehat{a}|ik\widehat{b}).$$

Here $(\cdot|\cdot)$ means the complex inner product. The additional consideration only occurs to gaining the dissipation of $\widehat{n}$. For that, one can compute from (3.2) and (3.3) that

$$\begin{aligned} P'(1)|k|^2|\widehat{n}|^2 &= (ik\widehat{n}| - \partial_t\widehat{u} + \widehat{b-u}) \\ &= -\partial_t(ik\widehat{n}|\widehat{u}) + (ik\partial_t\widehat{n}|\widehat{u}) + (ik\widehat{n}|\widehat{b-u}), \end{aligned}$$

which further implies

$$\partial_t\frac{\Re\,(ik\widehat{n}|\widehat{u})}{1+|k|^2} + \frac{1}{2}P'(1)\frac{|k|^2}{1+|k|^2}|\widehat{n}|^2 \leq \frac{|k\cdot\widehat{u}|^2}{1+|k|^2} + C|\widehat{b-u}|^2. \quad (3.9)$$

Now, as in (2.17), we define

$$\widetilde{\mathcal{E}}(t) = \|\widehat{f}\|^2_{L^2_\xi} + P'(1)|\widehat{n}|^2 + |\widehat{u}|^2 + \kappa_3\widetilde{\mathcal{E}}_1(t) + \kappa_4\frac{\Re\,(ik\widehat{n}|\widehat{u})}{1+|k|^2},$$

where $0<\kappa_3,\kappa_4 \ll 1$ are constants. It is also immediate to verify from (3.7), (3.8) and (3.9) that

$$\widetilde{\mathcal{E}}(t) \sim \|\widehat{f}(t)\|^2_{L^2_\xi} + |\widehat{n}(t)|^2 + |\widehat{u}(t)|^2,$$

and for $0<\kappa_4 \ll \kappa_3 \ll 1$,

$$\partial_t\widetilde{\mathcal{E}}(t) + \frac{\lambda|k|^2}{1+|k|^2}\widetilde{\mathcal{E}}(t) \leq C(\|\widehat{G}\|^2_{L^2_\xi} + \|\sigma^{-1/2}h\|^2_{L^2_\xi}).$$

The conclusions of Lemma 3.1 directly follows from the above estimate, and the detailed proof is omitted for brevity. □

Proof of rate of convergence: We can rewrite the nonlinear Cauchy problem (1.5), (1.6), (1.7) and (1.8) as

$$U(t) = \mathbb{A}(t)U_0 + \int_0^t \mathbb{A}(t-s)(S(s),G_1(s),G_2(s))\,ds, \quad (3.10)$$

with

$$
\begin{aligned}
S &= -u\cdot\nabla_\xi f - \frac{1}{2}u\cdot\xi f\\
&= \nabla_\xi G - \frac{1}{2}\xi\cdot G + u\cdot a\xi M^{1/2}, \quad G = -u\{\mathbf{I}-\mathbf{P}_0\}f,\\
G_1 &= -n\nabla_x\cdot u - u\cdot\nabla_x u,\\
G_2 &= -u\cdot\nabla_x n - \left[\frac{P'(n+1)}{n+1} - P'(1)\right]\nabla_x n - \frac{n(b-u)+au}{n+1}.
\end{aligned}
$$

To estimate $\|U(t)\|_{\mathcal{L}^2}$, we further rewrite (3.10) as

$$
U(t) = J_1(t) + J_2(t) + J_3(t) + J_4(t),
$$

with

$$
\begin{aligned}
J_1(t) &= \mathbb{A}(t)U_0,\\
J_2(t) &= \int_0^t \mathbb{A}(t-s)(S(s),0,0)\,ds,\\
J_3(t) &= \int_0^t \mathbb{A}(t-s)(u\cdot a\xi M^{1/2},0,0)\,ds,\\
J_4(t) &= \int_0^t \mathbb{A}(t-s)(0,G_1(s),G_2(s))\,ds.
\end{aligned}
$$

Define

$$
\mathcal{E}_\infty(t) = \sup_{0\le s\le t}(1+s)^{\frac{3}{2}}\mathcal{E}(s).
$$

One has that from (3.5),

$$
\|J_1(t)\|_{\mathcal{L}^2} \le C(1+t)^{-\frac{3}{4}}\|U_0\|_{\mathcal{L}^2\cap\mathcal{Z}_1},
$$

$$
\begin{aligned}
&\|J_3(t)\|_{\mathcal{L}^2} + \|J_4(t)\|_{\mathcal{L}^2}\\
&\le C\int_0^t (1+t-s)^{-\frac{3}{4}}(\|u\cdot a\xi M^{1/2}\|_{L^2\cap Z_1} + \|(G_1,G_2)(s)\|_{L^2\cap L^1}\,ds\\
&\le C\int_0^t (1+t-s)^{-\frac{3}{4}}\mathcal{E}(s)\,ds \le C\int_0^t (1+t-s)^{-\frac{3}{4}}(1+s)^{-\frac{3}{2}}\,ds\,\mathcal{E}_\infty(t)\\
&\le C(1+t)^{-\frac{3}{4}}\mathcal{E}_\infty(t),
\end{aligned}
$$

and from (3.6),

$$
\begin{aligned}
\|J_2(t)\|_{\mathcal{L}^2}^2 &\le C\int_0^t (1+t-s)^{-\frac{3}{2}}\|u\{\mathbf{I}-\mathbf{P}_0\}f\|_{\mathcal{L}^2\cap Z_1}^2\,ds\\
&\le C\int_0^t (1+t-s)^{-\frac{3}{2}}[\mathcal{E}(s)]^2\,ds \le C(1+t)^{-\frac{3}{2}}[\mathcal{E}_\infty(t)]^2.
\end{aligned}
$$

Therefore, it follows that

$$
\|U(t)\|_{\mathcal{L}^2}^2 \le C(1+t)^{-\frac{3}{2}}\{\|U_0\|_{\mathcal{L}^2\cap\mathcal{Z}_1}^2 + [\mathcal{E}_\infty(t)]^2\}. \tag{3.11}
$$

On the other hand, recall that (2.19) implies

$$
\frac{d}{dt}\mathcal{E}(t) + \lambda\mathcal{E}(t) \le C\|U(t)\|_{\mathcal{L}^2}^2. \tag{3.12}
$$

Then, by Gronwall's inequality, (3.12) together with (3.11) give

$$
\mathcal{E}(t) \le \mathcal{E}(0)e^{-\lambda t} + C(1+t)^{-\frac{3}{2}}\{\|U_0\|_{\mathcal{L}^2\cap\mathcal{Z}_1}^2 + [\mathcal{E}_\infty(t)]^2\},
$$

and hence

$$\mathcal{E}_\infty(t) \le C\{\|U_0\|^2_{\mathcal{L}^2\cap\mathcal{Z}_1} + [\mathcal{E}_\infty(t)]^2\}.$$

Thus, since $\|U_0\|_{\mathcal{L}^2\cap\mathcal{Z}_1}$ can be small enough, one has $\mathcal{E}_\infty(t) \le C\|U_0\|^2_{\mathcal{L}^2\cap\mathcal{Z}_1}$ for all $t \ge 0$, that is,

$$\mathcal{E}(t) \le C(1+t)^{-\frac{3}{2}}\|U_0\|^2_{\mathcal{L}^2\cap\mathcal{Z}_1}.$$

This completes the proof of Theorem 1.1. □

4. **Torus case.** In this section we consider the spatial domain $\Omega = \mathbb{T}^3$. In such case, we have the conservations of mass for both fluid and particles

$$\frac{d}{dt}\iint_{\mathbb{T}^3\times\mathbb{R}^3} F(t,x,\xi)\,dxd\xi = 0,$$
$$\frac{d}{dt}\int_{\mathbb{T}^3} n(t,x)\,dx = 0,$$

and also the conservation of the total momentum

$$\frac{d}{dt}\left[\iint_{\mathbb{T}^3\times\mathbb{R}^3} \xi F(t,x,\xi)\,dxd\xi + \int_{\mathbb{T}^3}(nu)(t,x)\,dx\right] = 0.$$

This implies that under the assumption (1.9), it holds that

$$\int_{\mathbb{T}^3} a(t,x)\,dx = 0,\ \int_{\mathbb{T}^3} n(t,x)\,dx = 0,\ \int_{\mathbb{T}^3}[b(t,x)+(n(t,x)+1)u(t,x)]\,dx = 0, \tag{4.1}$$

for all $t \ge 0$.

Proof of Theorem 1.2: We only give the proof of the global a priori estimates. Let the temporal energy functional $\mathcal{E}(t)$ and the corresponding dissipation rate functional $\mathcal{D}(t)$ be defined in the same way as in (2.17) and (2.18) for the case of the whole space $\Omega = \mathbb{R}^3$. The similar process by making the energy estimates leads to

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) + \lambda\mathcal{D}(t) \le{}& C(\|(a,b,n,u)\| + \|(a,b,n,u)\|^2)\\ &\times\left[\sum_{|\alpha|\le 3}\|\{\mathbf{I}-\mathbf{P}\}\partial_x^\alpha f\|_\sigma^2 + \|(a,b,n,u)\|^2_{H^3}\right].\end{aligned} \tag{4.2}$$

It remains to find out the zero-order dissipation of (a,b,n,u) by using the conservation laws (4.1) with the help of the Poincaré inequality. In fact, it is straightforward to obtain

$$\|a\|_{L^2} \le C\|\nabla_x a\|_{L^2},\quad \|n\|_{L^2} \le C\|\nabla_x n\|_{L^2}, \tag{4.3}$$

and

$$\begin{aligned}\|b+u\|_{L^2} &\le \|b+u+nu\|_{L^2} + \|nu\|_{L^2}\\ &\le C\|\nabla_x(b+u+nu)\|_{L^2} + \|u\|_{L^\infty}\|n\|_{L^2}\\ &\le C\|\nabla_x(b,u)\|_{L^2} + C\|\nabla_x(nu)\|_{L^2} + C\|u\|_{H^2}\|\nabla_x n\|_{L^2}\\ &\le C\|\nabla_x(b,u)\|_{L^2} + C\|u\|_{H^2}\|\nabla_x n\|_{L^2} + C\|n\|_{H^2}\|\nabla_x u\|_{L^2}.\end{aligned} \tag{4.4}$$

Define

$$\mathcal{D}_{\mathbb{T}}(t) = \mathcal{D}(t) + \kappa_5(\|a(t)\|^2 + \|n(t)\|^2) + \kappa_6\|(b+u)(t)\|^2,$$

where $0 < \kappa_5, \kappa_6 \ll 1$ are constants. Notice

$$\mathcal{D}_{\mathbb{T}}(t) \sim \sum_{|\alpha|\le 3}\|\{\mathbf{I}-\mathbf{P}\}\partial_x^\alpha f\|_\sigma^2 + \|(a,b,n,u)\|^2_{H^3}, \tag{4.5}$$

uniformly for all $t \geq 0$. Moreover, by choosing $0 < \kappa_6 \ll \kappa_5 \ll 1$ suitably small, it follows from (4.2) together with (4.3) and (4.4) that

$$\frac{d}{dt}\mathcal{E}(t) + \lambda \mathcal{D}_{\mathbb{T}}(t) \leq C[\mathcal{E}^{1/2}(t) + \mathcal{E}(t)]\mathcal{D}_{\mathbb{T}}(t),$$

which due to the fact that $\mathcal{E}(t)$ is small enough uniformly in time, implies

$$\frac{d}{dt}\mathcal{E}(t) + \lambda \mathcal{D}_{\mathbb{T}}(t) \leq 0.$$

Since $\mathcal{E}(t) \leq C\mathcal{D}_{\mathbb{T}}(t)$ by (4.5), one has

$$\frac{d}{dt}\mathcal{E}(t) + \lambda \mathcal{E}(t) \leq 0,$$

for all $t \geq 0$. This gives the exponential decay of $\mathcal{E}(t) \sim \|(f, n, u)(t)\|^2_{\mathcal{H}^3}$. The proof of Theorem 1.2 is complete. $\square$

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E-mail address: rjduan@math.cuhk.edu.hk
E-mail address: shqliusx@163.com