THE VLASOV-POISSON-BOLTZMANN SYSTEM WITHOUT ANGULAR CUTOFF

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Abstract. This paper is concerned with the Vlasov-Poisson-Boltzmann system for plasma particles of two species in three space dimensions. The Boltzmann collision kernel is assumed to be angular non-cutoff with $-3 < \gamma < -2s$ and $1/2 \le s < 1$, where γ, s are two parameters describing the kinetic and angular singularities, respectively. We establish the global existence and convergence rates of classical solutions to the Cauchy problem when initial data is near Maxwellians. This extends the results in [10, 11] for the cutoff kernel with $-2 \le \gamma \le 1$ to the case $-3 < \gamma < -2$ as long as the angular singularity exists instead and is strong enough, i.e., s is close to 1. The proof is based on the time-weighted energy method building also upon the recent studies of the non cutoff Boltzmann equation in [13] and the Vlasov-Poisson-Landau system in [21].

Dedicated to Professor Seiji Ukai (1940-2012)

Contents

1. Introduction

1.1. **Problem.** We consider the following Vlasov-Poisson-Boltzmann system describing the motion of plasma particles of two species (e.g. ions and electrons) in the whole space $\mathbf{R}^3$, cf. [23]:

$$\begin{aligned} &\partial_t F_+ + v\cdot\nabla_x F_+ + E\cdot\nabla_v F_+ = Q(F_+,F_+) + Q(F_-,F_+),\\ &\partial_t F_- + v\cdot\nabla_x F_- - E\cdot\nabla_v F_- = Q(F_-,F_-) + Q(F_+,F_-). \end{aligned} \tag{1.1}$$

The self-consistent electrostatic field takes the form of $E(t,x) = -\nabla_x\phi$, with the electric potential ϕ satisfying

$$-\Delta\phi = \int_{\mathbf{R}^3} (F_+ - F_-)\,dv, \quad \phi\to 0 \ \text{ as } \ |x|\to\infty. \tag{1.2}$$

The initial data of the system is given as

$$F_\pm(0,x,v) = F_{\pm,0}(x,v). \tag{1.3}$$

Here, the unknown $F_\pm(t,x,v)\ge 0$ stand for the velocity distribution functions for the particles with position $x=(x_1,x_2,x_3)\in\mathbf{R}^3$ and velocity $v=(v_1,v_2,v_3)\in\mathbf{R}^3$ at time $t\ge 0$. The bilinear collision operator $Q(F,G)$ on the right-hand side of (1.1) is defined by

$$Q(F,G)(v) = \int_{\mathbf{R}^3}\int_{\mathbf{S}^2} B(v-u,\sigma)[F(u')G(v') - F(u)G(v)]\,du d\sigma,$$

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where in terms of velocities v and u before the collision, velocities v' and u' after the collision are defined by

$$v' = \frac{v+u}{2} + \frac{|v-u|}{2}\sigma, \quad u' = \frac{v+u}{2} - \frac{|v-u|}{2}\sigma.$$

The Boltzmann collision kernel $B(v-u,\sigma) \geq 0$ depends only on the relative velocity $|v-u|$ and on the deviation angle θ given by $\cos\theta = \langle \sigma, (v-u)/|v-u| \rangle$, where $\langle\, , \,\rangle$ is the usual dot product in $\mathbf{R}^3$. As in [13], without loss of generality, we suppose that $B(v-u,\sigma)$ is supported on $\cos\theta \geq 0$. Notice also that all the physical parameters, such as the particle masses and the light speed, and all other involving constants, have been chosen to be unit for simplicity of presentation. Throughout the paper, the collision kernel is further supposed to satisfy the following assumptions:

- $B(v-u,\sigma)$ takes the product form in its argument as
$$B(v-u,\sigma) = \Phi(|v-u|)\mathbb{b}(\cos\theta)$$
with Φ and $\mathbb{b}$ being nonnegative functions.
- The angular function $\sigma \to \mathbb{b}(\langle \sigma, (v-u)/|v-u| \rangle)$ is not integrable on $\mathbf{S}^2$, i.e.
$$\int_{\mathbf{S}^2} \mathbb{b}(\cos\theta)\, d\sigma = 2\pi \int_0^{\frac{\pi}{2}} \sin\theta \mathbb{b}(\cos\theta)\, d\theta = \infty.$$
Moreover, there are $c_{\mathbb{b}} > 0$, $0 < s < 1$ such that
$$\frac{c_{\mathbb{b}}}{\theta^{1+2s}} \leq \sin\theta \mathbb{b}(\cos\theta) \leq \frac{1}{c_{\mathbb{b}}\theta^{1+2s}}, \ \forall\, 0 < \theta \leq \frac{\pi}{2}.$$
- The kinetic function $z \to \Phi(|z|)$ satisfies
$$\Phi(|z|) = C_\Phi |z|^\gamma$$
for a contant $C_\Phi > 0$, where the exponent $\gamma > -3$ is determined by the intermolecular interactive mechanism.

It is convenient to call soft potentials when $-3 < \gamma < -2s$, and hard potentials when $\gamma + 2s \geq 0$. The current work will be restricted to the case of $-3 < \gamma < -2s$ and $1/2 \leq s < 1$. Recall that when the intermolecular interactive potential takes the inverse power law in the form of $U(|x|) = |x|^{-(\ell-1)}$ with $2 < \ell < \infty$, the collision kernel $B(v-u,\sigma)$ in three space dimensions satisfies the above assumptions with $\gamma = \frac{\ell-5}{\ell-1}$ and $s = \frac{1}{\ell-1}$, and our restriction corresponds to the condition $2 < \ell < 3$ in terms of ℓ. Note $\gamma \to -3$ and $s \to 1$ as $\ell \to 2$ in the limiting case, for which the grazing collisions between particles are dominated and the Boltzmann collision term has to be replaced by the classical Landau collision term for the Coulomb potential, cf. [35]. As far as the global classical solutions near Maxwellians to the pure Boltzmann equation with angular cutoff in the absence of any force are concerned, we only mention Ukai [33], Ukai-Asano [34], Caflisch [4], and Guo [16].

In the paper our goal is to establish the global existence of solutions to the Cauchy problem (1.1)-(1.3) of the Vlasov-Poisson-Boltzmann system near the global Maxwellian equilibrium states. This issue was firstly investigated by Guo [20] for the hard-sphere model of the Vlasov-Poisson-Boltzmann system in a periodic box. Since then, the robust energy method was also developed in [19] to deal with the hard-sphere Boltzmann equation even with the self-consistent electric and magnetic fields; see also [15, 17, 22, 24, 37]. However, the non hard-sphere case has remained open for general collision potentials either with the Grad's angular cutoff assumption or not. Until recently, Guo [21] made further progress in proving the global existence of classical solutions to the Vlasov-Poisson-Landau system in a periodic box for the most important Coulomb potential. One of the key points in the proof there is to design a new velocity weight depending on the order of space and velocity derivatives so as to capture the anisotropic dissipation property of the linearized Landau operator. Due to the recent study of the non cutoff Boltzmann equation independently by Gressman-Strain [13, 14] and AMUXY [1, 2, 3], it is now well known that the linearized Boltzmann operator without angular cutoff has the similar anisotropic dissipation phenomenon with the Landau, cf. [5, 18]. Therefore, as mentioned in [21], it is also interesting to see whether or not the approach in [21] can be applied to the non cutoff Vlasov-Poisson-Boltzmann system for the non hard-sphere model; see also [32] and [25] for two recent applications.

On the other hand, basing on the time weighted energy method, [10, 11, 12] recently developed another approach for the study of the Boltzmann or Landau equation with external forces for general collision potentials. The main difference with [21] is to introduce another kind of time-velocity dependent weight function which can induce the extra dissipation mechanism to compensate the weaker dissipation of the linearized collision operator in the case of non hard-sphere models, particularly physically interesting soft potentials. Unfortunately, for the Vlasov-Poisson-Boltzmann system with angular cutoff, the problem was solved only in the case of $-2 \leq \gamma \leq 1$ and is still left open for the very soft potential case $-3 < \gamma < -2$. In this paper, building on [13] and [29], we will extend the results in [10, 11] for the cutoff kernel with $-2 \leq \gamma \leq 1$ to the non cutoff case $-3 < \gamma < -2$ as long as the angular singularity exists instead and it is strong enough, i.e., s is close to 1.

1.2. **Reformulation.** In what follows we will reformulate the problem as in [20, 19]. Denote a normalized global Maxwellian μ by

$$\mu(v) = \frac{1}{(2\pi)^{\frac{3}{2}}} \exp\left(-|v|^2/2\right).$$

Set $F_\pm(t,x,v) = \mu(v) + \sqrt{\mu(v)} f_\pm(t,x,v)$. Denote by $[\cdot,\cdot]$ the column vectors $F = [F_+, F_-]$, $f = [f_+, f_-]$ and $f_0 = [f_{0,+}, f_{0,-}]$. Then the Cauchy problem (1.1)-(1.3) can be reformulated as

$$\partial_t f + v\cdot\nabla_x f - q\nabla_x\phi\cdot\nabla_v f + \nabla_x\phi\cdot v\sqrt{\mu}q_1 + Lf = \Gamma(f,f) - \frac{q}{2}v\cdot\nabla_x\phi f, \tag{1.4}$$

$$-\Delta\phi = \int_{\mathbf{R}^3}(f_+ - f_-)\sqrt{\mu(v)}\,dv, \quad \phi\to 0 \ \text{ as } \ |x|\to\infty, \tag{1.5}$$

with given initial data

$$f(0,x,v) = f_0(x,v). \tag{1.6}$$

Here, $q = diag(1,-1)$, $q_1 = [1,-1]$. L are Γ are the linearized and nonlinear collision operators, respectively. For $f = [f_+, f_-]$ and $g = [g_+, g_-]$,

$$Lf = [L_+ f, L_- f],$$
$$L_\pm f = -\mu^{-1/2}\left\{2Q\left(\mu, \mu^{1/2} f_\pm\right) + Q\left(\mu^{1/2}\{f_\pm + f_\mp\}, \mu\right)\right\},$$

and

$$\Gamma(f,g) = [\Gamma_+(f,g), \Gamma_-(f,g)],$$
$$\Gamma_\pm(f,g) = \mu^{-1/2}\left\{Q\left(\mu^{1/2} f_\pm, \mu^{1/2} g_\pm\right) + Q\left(\mu^{1/2} f_\mp, \mu^{1/2} g_\pm\right)\right\}.$$

For later use, it is convenient to introduce the bilinear operator $\mathscr{T}$ by

$$\begin{aligned}\mathscr{T}(g_1,g_2) =& \mu^{-1/2}Q\left(\mu^{1/2}g_1, \mu^{1/2}g_2\right)\\ =& \int_{\mathbf{R}^3} du \int_{\mathbf{S}^2} d\sigma\, B(v-u,\sigma)\mu^{1/2}(u)\left[g_1(u')g_2(v') - g_1(u)g_2(v)\right]\end{aligned} \tag{1.7}$$

for two scalar functions g_1, g_2, and thus $L = [L_+, L_-]$ and $\Gamma = [\Gamma_+, \Gamma_-]$ can be rewritten as

$$\begin{aligned}L_\pm f &= -\left\{2\mathscr{T}\left(\mu^{1/2}, f_\pm\right) + \mathscr{T}\left(f_\pm + f_\mp, \mu^{1/2}\right)\right\},\\ \Gamma_\pm(f,g) &= \mathscr{T}(f_\pm, g_\pm) + \mathscr{T}(f_\mp, g_\pm).\end{aligned} \tag{1.8}$$

1.3. **Basic properties of L.** For scalar functions $f_\pm$, the first part of the linearized Boltzmann collision term $L_\pm f$ in (1.8) can be splitted as

$$-2\mathscr{T}\left(\mu^{1/2}, f_\pm\right) = -2\int_{\mathbf{R}^3} du \int_{\mathbf{S}^2} d\sigma\, B(v-u,\sigma)(f_\pm(v') - f_\pm(v))\mu^{1/2}(u)\mu^{1/2}(u') + 2\tilde{\nu}(v) f_\pm(v), \tag{1.9}$$

where

$$\tilde{\nu}(v) = \int_{\mathbf{R}^3} du \int_{\mathbf{S}^2} d\sigma\, B(v-u,\sigma)\left(\mu^{1/2}(u) - \mu^{1/2}(u')\right)\mu^{1/2}(u).$$

The first term on the right-hand side of (1.9) contains a crucial Hilbert space structure, while for the second term, Pao's splitting

$$\tilde{\nu}(v) = \nu_1(v) + \nu_2(v)$$

holds true, cf. [28], with the following known asymptotics

$$\nu_1(v) \sim (1+|v|^2)^{\frac{\gamma+2s}{2}}, \quad \nu_2(v) \lesssim (1+|v|^2)^{\frac{\gamma}{2}}.$$

We now collect some basic properties of the linearized collision operator L as follows:

(i) As in [13], L can be decomposed as $L = \mathcal{N} + \mathcal{K}$. Here for $f = [f_+, f_-]$, $\mathcal{N} f = [\mathcal{N}_+ f, \mathcal{N}_- f]$ is the "norm part", given by

$$
\begin{aligned}
\mathcal{N}_\pm f =& -2\mathscr{T}\left(\mu^{1/2}, f_\pm\right) - 2\nu_2(v) f_\pm \\
=& -2\int_{\mathbf{R}^3} du \int_{\mathbf{S}^2} d\sigma\, B(v-u,\sigma)(f_\pm(v') - f_\pm(v))\mu^{1/2}(u)\mu^{1/2}(u') + 2\nu_1(v) f_\pm(v).
\end{aligned}
$$

Thus by using the pre-post collisional change of variables, $\mathcal{N}_\pm g$ satisfies the identity

$$
\begin{aligned}
\langle \mathcal{N}_\pm f, f_\pm\rangle =& \int_{\mathbf{R}^3} du \int_{\mathbf{R}^3} dv \int_{\mathbf{S}^2} d\sigma\, B(v-u,\sigma)(f_\pm(v') - f_\pm(v))^2 \mu^{1/2}(u)\mu^{1/2}(u') \\
&+ 2\int_{\mathbf{R}^3} dv\, \nu_1(v)|f_\pm(v)|^2.
\end{aligned}
$$

Moreover, $\mathcal{K} f = [\mathcal{K}_+ f, \mathcal{K}_- f]$ is the "compact part", given by

$$
\begin{aligned}
\mathcal{K}_\pm f =& -\mathscr{T}\left(f_+ + f_-, \mu^{1/2}\right) + 2\nu_2(v) f_\pm \\
=& -\int_{\mathbf{R}^3} du \int_{\mathbf{S}^2} d\sigma\, B(v-u,\sigma)\mu^{1/2}(u)\left[(f_+ + f_-)(u')\mu^{1/2}(v') - (f_+ + f_-)(u)\mu^{1/2}(v)\right] \\
&+ 2\nu_2(v) f_\pm(v).
\end{aligned}
$$

(ii) As in [19], the null space of L is given by

$$
\mathscr{N} = \ker L = \operatorname{span}\left\{[1,0]\mu^{1/2}, [0,1]\mu^{1/2}, [v_i, v_i]\mu^{1/2} (1 \le i \le 3), \left[|v|^2, |v|^2\right]\mu^{1/2}\right\}.
$$

For given $f(t,x,v)$, one can decompose $f(t,x,v)$ uniquely as

$$
f = \mathbf{P} f + (\mathbf{I} - \mathbf{P}) f.
$$

Here, $\mathbf{P}$ denotes the orthogonal projection from $L^2_v \times L^2_v$ to $\mathscr{N}$, defined by

$$
\mathbf{P} f = \left\{a_+(t,x)[1,0] + a_-(t,x)[0,1] + v\cdot b(t,x)[1,1] + \left(|v|^2 - 3\right) c(t,x)[1,1]\right\}\sqrt{\mu}, \tag{1.10}
$$

or equivalently $\mathbf{P} = [\mathbf{P}_+, \mathbf{P}_-]$ with

$$
\mathbf{P}_\pm f = \left\{a_\pm(t,x) + v\cdot b(t,x) + \left(|v|^2 - 3\right) c(t,x)\right\}\sqrt{\mu}.
$$

Notice that

$$
\int_{\mathbf{R}^3} \psi(v)\cdot(\mathbf{I} - \mathbf{P}) f\, dv = 0, \quad \forall\, \psi = [\psi_+, \psi_-] \in \mathscr{N}.
$$

(iii) For any fixed (t,x), L is nonnegative and further L is known to be locally coercive in the sense that there is a constant $\lambda > 0$ such that, cf. [26, 27]

$$
\langle f, Lf\rangle = \langle (\mathbf{I} - \mathbf{P}) f, L(\mathbf{I} - \mathbf{P}) f\rangle \ge \lambda \left\| (1+|v|^2)^{\frac{\gamma}{4}} (\mathbf{I} - \mathbf{P}) f \right\|^2_{L^2_v}.
$$

1.4. **Notations.** Through the paper, C denotes some positive constant (generally large) and λ denotes some positive constant (generally small), where both C and λ may take different values in different places. $A \lesssim B$ means that there is a generic constant $C > 0$ such that $A \leqslant CB$. $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. For multi-indices $\alpha = [\alpha_1, \alpha_2, \alpha_3]$ and $\beta = [\beta_1, \beta_2, \beta_3]$, $\partial^\alpha_\beta = \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}\partial^{\beta_1}_{v_1}\partial^{\beta_2}_{v_2}\partial^{\beta_3}_{v_3}$, and the length of α is denoted by $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. $\alpha' \le \alpha$ means that no component of α' is greater than the component of α, and $\alpha' < \alpha$ means that $\alpha' \le \alpha$ and $|\alpha'| < |\alpha|$. $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product in $\mathbf{R}^3_v$, with the L^2 norm $|\cdot|_2$ or $|\cdot|$. For notational simplicity, $(\cdot,\cdot)$ denotes the L^2 inner product either in $\mathbf{R}^3_x \times \mathbf{R}^3_v$ or in $\mathbf{R}^3_x$ with the L^2 norm $\|\cdot\|$. For $\ell \in \mathbf{R}$, H^ℓ denotes the usual Sobolev space. For $q \ge 1$, Z_q denotes the space $Z_q = L^2(\mathbf{R}^3_v; L^q(\mathbf{R}^3_x))$ with the norm $\|f\|_{Z_q} = \left\| \|f\|_{L^q_x} \right\|_{L^2_v}$.

We now list series of notations introduced in [13]. Let $\mathcal{S}'(\mathbf{R}^3)$ be the space of the tempered distribution functions. N^s_γ denotes the weighted geometric fractional Sobolev space

$$
N^s_\gamma = \{f \in \mathcal{S}'(\mathbf{R}^3) : |f|_{N^s_\gamma} < \infty\},
$$

with the anisotropic norm

$$|f|_{N^s_\gamma}^2=|f|^2_{L^2_{\gamma+2s}}+\int_{\mathbf{R}^3}dv\int_{\mathbf{R}^3}dv'\,(\langle v\rangle\langle v'\rangle)^{\frac{\gamma+2s+1}{2}}\,\frac{(f(v)-f(v'))^2}{d(v,v')^{3+2s}}\chi_{d(v,v')\le 1},$$

where $\langle v\rangle=\sqrt{1+|v|^2}$, L^2_ℓ for $\ell\in\mathbf{R}$ denotes the weighted space with the norm

$$|f|^2_{L^2_\ell}=\int_{\mathbf{R}^3}dv\,\langle v\rangle^\ell|f(v)|^2,$$

the anisotropic metric $d(v,v')$ measuring the fractional differentiation effects is given by

$$d(v,v')=\sqrt{|v-v'|^2+\frac{1}{4}(|v|^2-|v'|^2)^2},$$

and χ_A is the indicator function of a set A. In $\mathbf{R}^3\times\mathbf{R}^3$, we use $\|\cdot\|_{N^s_\gamma}=\Big\||\cdot|_{N^s_\gamma}\Big\|_{L^2_x}$.

To the end, the velocity weight function $w=w(v)$ always denotes

$$w(v)=\langle v\rangle^{-\gamma}. \tag{1.11}$$

Let $\ell\in\mathbb{R}$. The weighted fractional Sobolev norm is given by

$$\left|w^\ell f\right|^2_{H^s_\gamma}=\left|w^\ell h\right|^2_{L^2_\gamma}+\int_{\mathbf{R}^3}dv\int_{\mathbf{R}^3}dv'\,\frac{\left[\langle v\rangle^{\frac{\gamma}{2}}w^\ell(v)f(v)-\langle v'\rangle^{\frac{\gamma}{2}}w^\ell(v')f(v')\right]^2}{|v-v'|^{3+2s}}\chi_{|v-v'|\le 1},$$

which turns out to be equivalent with

$$\left|w^\ell f\right|^2_{H^s_\gamma}=\int_{\mathbf{R}^3}dv\,\langle v\rangle^\gamma\left|(1-\Delta_v)^{\frac{s}{2}}\left(w^\ell(v)f(v)\right)\right|^2.$$

The velocity-weighted $|\cdot|_{N^s_\gamma}$-norm is given by

$$\left|w^\ell f\right|^2_{N^s_\gamma}=\left|w^\ell f\right|^2_{L^2_{\gamma+2s}}+\int_{\mathbf{R}^3}dv\int_{\mathbf{R}^3}dv'\,(\langle v\rangle\langle v'\rangle)^{\frac{\gamma+2s+1}{2}}\,w^{2\ell}(v)\frac{(f(v)-f(v'))^2}{d(v,v')^{3+2s}}\chi_{d(v,v')\le 1}.$$

Notice that, cf. [13, 14],

$$\left|w^\ell f\right|^2_{L^2_{\gamma+2s}}+\left|w^\ell f\right|^2_{H^s_\gamma}\lesssim\left|w^\ell f\right|^2_{N^s_\gamma}\lesssim\left|w^\ell f\right|^2_{H^s_{\gamma+2s}}.$$

In $\mathbf{R}^3\times\mathbf{R}^3$, $\left\|w^\ell f\right\|_{H^s_\gamma}=\Big\|\left|w^\ell f\right|_{H^s_\gamma}\Big\|_{L^2_x}$ and $\left\|w^\ell f\right\|_{N^s_\gamma}=\Big\|\left|w^\ell f\right|_{N^s_\gamma}\Big\|_{L^2_v}$ are used. For the integer $K\ge 0$, we also use the anisotropic space $N^s_{\gamma,K}(\mathbf{R}^3\times\mathbf{R}^3)$ containing the space-velocity derivatives, given by

$$\left\|w^\ell f\right\|^2_{N^s_{\gamma,K}}=\left\|w^\ell f\right\|^2_{N^s_{\gamma,K}(\mathbf{R}^3\times\mathbf{R}^3)}=\sum_{|\alpha|+|\beta|\le K}\left\|w^\ell\partial^\alpha_\beta f\right\|^2_{N^s_\gamma(\mathbf{R}^3\times\mathbf{R}^3)},$$

where we write

$$\left|w^\ell f\right|^2_{N^s_{\gamma,K}}=\left|w^\ell f\right|^2_{N^s_{\gamma,K}(\mathbf{R}^3)}=\sum_{|\beta|\le K}\left|w^\ell\partial_\beta f\right|^2_{N^s_\gamma(\mathbf{R}^3)}$$

whenever only the velocity derivatives are involved. For integer $K\ge 0$, we define the Sobolev space

$$|f|_{H^K}=\sum_{|\beta|\le K}\left|\partial_\beta f\right|_{L^2(\mathbf{R}^3)},\quad \|f\|_{H^K}=\sum_{|\alpha|+|\beta|\le K}\left\|\partial^\alpha_\beta f\right\|_{L^2(\mathbf{R}^3\times\mathbf{R}^3)}.$$

For integer $K\ge 0$ and $\ell\in\mathbf{R}$, we define the weighted Sobolev space

$$\left|w^\ell f\right|_{H^K}=\sum_{|\beta|\le K}\left|w^\ell\partial_\beta f\right|_{L^2(\mathbf{R}^3)},\quad \left\|w^\ell f\right\|_{H^K}=\sum_{|\alpha|+|\beta|\le K}\left\|w^\ell\partial^\alpha_\beta f\right\|_{L^2(\mathbf{R}^3\times\mathbf{R}^3)},$$

and

$$\left|w^\ell f\right|_{H^K_\gamma}=\sum_{|\beta|\le K}\left|w^\ell\langle v\rangle^{\frac{\gamma}{2}}\partial_\beta f\right|_{L^2(\mathbf{R}^3)},\quad \left\|w^\ell f\right\|_{H^K_\gamma}=\sum_{|\alpha|+|\beta|\le K}\left\|w^\ell\langle v\rangle^{\frac{\gamma}{2}}\partial^\alpha_\beta f\right\|_{L^2(\mathbf{R}^3\times\mathbf{R}^3)}.$$

Finally, we define $B_C\subset\mathbf{R}^3$ to be the ball with center origin and radius C, and use $L^2(B_C)$ to denote the space L^2 over B_C and likewise for other spaces.

1.5. **Main results.** To state the result of the paper, we introduce more notations. As in [21], for $l \geq 0$, α and β, we define the velocity weight function $w_l(\alpha,\beta)$ by

$$w_l(\alpha,\beta)(v) = w^{l+K-|\alpha|-|\beta|}(v) = \langle v\rangle^{(-\gamma)(l+K-|\alpha|-|\beta|)}, \tag{1.12}$$

where K is an integer. Corresponding to given $f = f(t,x,v)$, we define the instant energy functional $\mathcal{E}_l(t)$ and the instant high-order energy functional $\mathcal{E}_l^h(t)$, for $l \geq 0$, to be functionals satisfying the equivalent relations

$$\mathcal{E}_l(t) \sim \sum_{|\alpha|\leq K} \|\partial^\alpha E(t)\|^2 + \sum_{|\alpha|+|\beta|\leq K} \left\|w_l(\alpha,\beta)\partial_\beta^\alpha f(t)\right\|^2, \tag{1.13}$$

$$\mathcal{E}_l^h(t) \sim \sum_{|\alpha|\leq K} \|\partial^\alpha E(t)\|^2 + \sum_{1\leq|\alpha|\leq K} \|\partial^\alpha \mathbf{P} f\|^2 + \sum_{|\alpha|+|\beta|\leq K} \left\|w_l(\alpha,\beta)\partial_\beta^\alpha (\mathbf{I}-\mathbf{P}) f\right\|^2, \tag{1.14}$$

and define the dissipation rate functional $\mathcal{D}_l(t)$ by

$$\mathcal{D}_l(t) = \sum_{|\alpha|\leq K-1} \|\partial^\alpha E(t)\|^2 + \sum_{1\leq|\alpha|\leq K} \|\partial^\alpha \mathbf{P} f(t)\|^2 + \sum_{|\alpha|+|\beta|\leq K} \left\|w_l(\alpha,\beta)\partial_\beta^\alpha (\mathbf{I}-\mathbf{P}) f(t)\right\|_{N_\gamma^s}^2. \tag{1.15}$$

Here, $E = E(t,x)$ is understood to be determined by $f(t,x,v)$ in terms of

$$E = -\nabla_x \phi, \quad \phi = \frac{1}{4\pi|x|} *_x \rho_f, \quad \rho_f = \int_{\mathbf{R}^3} (f_+ - f_-)\sqrt{\mu(v)}\, dv.$$

The main result of the paper is stated as follows.

Theorem 1.1. *Let $-3 < \gamma < -2s$, $1/2 \leq s < 1$. Fix $l_0 \geq 0$, $1/2 < p < 1$, the integer $K \geq 8$, and define*

$$l_1 = \frac{5}{4(1-p)} \frac{\gamma+2s}{\gamma}.$$

Let $f_0(x,v) = [f_{0,+}(x,v), f_{0,-}(x,v)]$ satisfy $F_\pm(0,x,v) = \mu(v) + \sqrt{\mu(v)} f_{0,\pm}(x,v) \geq 0$, and

$$\int_{\mathbf{R}^3} \rho_{f_0}\, dx = 0, \quad \int_{\mathbf{R}^3} (1+|x|)|\rho_{f_0}|\, dx < \infty. \tag{1.16}$$

Then, there are functionals $\mathcal{E}_l(t)$ and $\mathcal{E}_l^h(t)$ in the sense of (1.13) and (1.14) such that the following thing holds true. If

$$\epsilon_0 = \sqrt{\mathcal{E}_{l_0+l_1}(0)} + \left\|w^{l_2} f_0\right\|_{Z_1} + \|(1+|x|)\rho_0\|_{L^1}, \tag{1.17}$$

is sufficiently small, where $l_2 > \frac{5(\gamma+2s)}{4\gamma}$ is a constant, then there exists a unique global solution $f(t,x,v)$ to the Cauchy problem (1.4)-(1.6) of the Vlasov-Poisson-Boltzmann system such that $F_\pm(t,x,v) = \mu(v) + \sqrt{\mu(v)} f_\pm(t,x,v) \geq 0$ and

$$\mathcal{E}_{l_0+l_1}(t) \lesssim \epsilon_0^2, \tag{1.18}$$

$$\mathcal{E}_{l_0}(t) \lesssim \epsilon_0^2 (1+t)^{-\frac{3}{2}}, \tag{1.19}$$

$$\mathcal{E}_{l_0}^h(t) \lesssim \epsilon_0^2 (1+t)^{-\frac{3}{2}-p}, \tag{1.20}$$

for any $t \geq 0$.

Some remarks are given as follows. Notice that l_0 can take zero while the choice of other parameters l_1, l_2, K and p could not be optimal in order for initial data to have the weaker regularity and velocity moments. The restriction of those parameters is related to obtaining the following closed a priori estimate

$$X(t) \lesssim \epsilon_0^2 + X^{\frac{3}{2}}(t) + X^2(t)$$

with respect to the time-weighted energy norm $X(t)$ defined by

$$X(t) = \sup_{0\leq\tau\leq t} \mathcal{E}_{l_0+l_1}(\tau) + \sup_{0\leq\tau\leq t} (1+\tau)^{\frac{3}{2}} \mathcal{E}_{l_0}(\tau) + \sup_{0\leq\tau\leq t} (1+\tau)^{\frac{3}{2}+p} \mathcal{E}_{l_0}^h(\tau). \tag{1.21}$$

Here, the parameter p is introduced to take care the time-decay of the high-order energy functional $\mathcal{E}_{l_0}^h(t)$. The time decay rate $(1+t)^{-\frac{3}{2}-p}$ for $\mathcal{E}_{l_0}^h(t)$ in (1.20) is not optimal compared to the linearized system; see Theorem 4.1 given later on. It is then of interest to upgrade it to $(1+t)^{-5/2}$, that is to prove (1.20) in the case of $p=1$, and we will make a further discussion of possibility at the end of the paper. Moreover, similar results in the hard potential case $\gamma+2s \geq 0$ with $0 < s < 1$ could be considered in the simpler way, but for the soft potential case $-3<\gamma<-2s$ in Theorem 1.1, the restriction $s\geq 1/2$ is necessary in our proof due to the technique of the approach; we will clarify this point later. The assumptions concerning (1.16) and (1.17) arise from the linearized analysis of the Vlasov-Poisson-Boltzmann system in order to assure the strong enough time-decay rate of the linearized solution operator; see (4.25) in the proof of Theorem 4.1.

In what follows let us point out several key technical points in the proof of Theorem 1.1. First of all, we emphasize the role of the velocity weight $w_l(\alpha,\beta)$ in (1.12). Such weight was firstly introduced in [30, 31] to deal with the time decay of the Boltzmann equation for soft potentials on torus, and it was also used recently in [21] to investigate the Vlasov-Poisson-Landau system for Coulomb potentials. In fact, the linearized non cutoff Boltzmann operator enjoys the anisotropic dissipation norm

$$\|f\|_{N^s_\gamma}^2 \gtrsim \|\langle v\rangle^{\frac{\gamma+2s}{2}} f\|^2+\|\langle v\rangle^{\frac{\gamma}{2}}(1-\Delta_v)^{\frac{s}{2}} f\|^2. \tag{1.22}$$

Then, in terms of this dissipation norm, the choice of $w_l(\alpha,\beta)$ should depend on the weighted estimates on the linear term $v\cdot\nabla_x f$ and two nonlinear terms $\nabla_x\phi\cdot\nabla_v f$ and $v\cdot\nabla_x\phi f$. For $v\cdot\nabla_x f$, one has to bound

$$\left(\partial^{\alpha+e_i}_{\beta-e_i} f, w_l^2(\alpha,\beta)\partial^\alpha_\beta f\right) \tag{1.23}$$

with $|\alpha|+|\beta|\leq K$ and $|\beta|\geq 1$. For simplicity of presentation, let f be purely microscopic, i.e. $f=(\mathbf{I}-\mathbf{P})f$. Since $|\beta|\geq 1$, the trick is to do the splitting $\partial^\alpha_\beta f=\partial_{e_i}\partial^\alpha_{\beta-e_i} f$ in the inner product term above, write in a rough way the first-order velocity differentiation as $\partial_{e_i}=\partial_{e_i}^{1/2}\partial_{e_i}^{1/2}$, and further use $w_l(\alpha,\beta)=w_l(\alpha,\beta-e_i)\langle v\rangle^\gamma$ to gain the degenerate velocity weight, so that (1.23) can be bounded by the dissipation norm

$$\|\langle v\rangle^{\frac{\gamma}{2}}(1-\Delta_v)^{\frac{s}{2}}[w_l(\alpha,\beta-e_i)\partial^\alpha_{\beta-e_i} f]\|^2+\|\langle v\rangle^{\frac{\gamma}{2}}(1-\Delta_v)^{\frac{s}{2}}[w_l(\alpha+e_i,\beta-e_i)\partial^{\alpha+e_i}_{\beta-e_i} f]\|^2$$

up to some other controllable terms, where $s\geq 1/2$ was used. Notice that two terms in the sum above correspond to the second term on the right-hand side of (1.22), and they also have velocity differentiation whose order is $|\beta-e_i|$ less than $|\beta|$. For $\nabla_x\phi\cdot\nabla_v f$, one has to meet with the estimate on the trilinear inner product term

$$\left(\partial^{\alpha_1+e_i}\phi\,\partial^{\alpha-\alpha_1}_{\beta+e_i} f, w_l^2(\alpha,\beta)\partial^\alpha_\beta f\right)$$

with $|\alpha|+|\beta|\leq K$ and $0<\alpha_1\leq\alpha$. For this term, we use the same trick to make estimates as for (1.23) by writing $\partial^{\alpha-\alpha_1}_{\beta+e_i} f=\partial_{e_i}\partial^{\alpha-\alpha_1}_\beta f$ and $\partial_{e_i}=\partial_{e_i}^{1/2}\partial_{e_i}^{1/2}$. For $v\cdot\nabla_x\phi f$, one has to bound

$$\left(v_i\partial^{\alpha_1+e_i}\phi\,\partial^{\alpha-\alpha_1}_\beta f, w_l^2(\alpha,\beta)\partial^\alpha_\beta f\right)$$

with $|\alpha|+|\beta|\leq K$ and $0<\alpha_1\leq\alpha$. Since $\partial^{\alpha-\alpha_1} f$ losses at least one space differentiation compared to $\partial^\alpha f$, one can gain the velocity moment $\langle v\rangle^\gamma$ from $w_l(\alpha,\beta)$ so that

$$|v_i|w_l(\alpha,\beta)\leq\langle v\rangle^{\gamma+2s}w_l(\alpha-\alpha_1,\beta)$$

due to $s\geq 1/2$ once again. Therefore, one can use the first part on the right-hand side of (1.22) from the dissipation norm to bound

$$\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}|\partial^{\alpha_1+e_i}\phi|\cdot|\langle v\rangle^{\frac{\gamma+2s}{2}}w_l(\alpha-\alpha_1,\beta)\partial^{\alpha-\alpha_1}_\beta f|\cdot|\langle v\rangle^{\frac{\gamma+2s}{2}}w_l(\alpha,\beta)\partial^\alpha_\beta f|\,dxdv.$$

The second technical point concerns the $w_l(\alpha,\beta)$-weighted estimate on the nonlinear term $\Gamma(f,f)$. The corresponding results obtained in [13] can not be directly applied here, since the velocity weight function depends on the total order of space-velocity differentiation. We will make some slight modifications involving the distribution of weights. Essentially, whenever $|v'|^2\sim|v|^2+|u|^2$, instead of using $w^{2\ell}(v')\lesssim w^{2\ell}(u)w^{2\ell}(v)$, we estimate $w^{2\ell}(v')$ as

$$w^{2\ell}(v')\lesssim w^{2\ell}(u)+w^{2\ell}(v).$$

The third point is to design the time-weighted norm $X(t)$ in (1.21) to capture the dispersive property of the Vlasov-Poisson-Boltzmann system in the case of the whole space $\mathbf{R}^3$. This trick has be been used in [9] and the recent work [10, 11, 12], where the key issue is to apply the time-decay property of solutions to close all the nonlinear energy estimates. Specifically, due to the technique of the approach, one has to deal with a term in the form of

$$\|\partial_t\phi\|_{L^\infty}\mathcal{E}_l(t)$$

which can not be absorbed by the energy dissipation norm even under the smallness assumption on the solution itself. Observe from the later proof that $\|\partial_t\phi\|_{L^\infty}$ is bounded by the high-order energy functional $\mathcal{E}_l^h(t)$ and hence is time integrable as shown in [21]. Thus, it is natural to use a kind of time-weighted norm to close the a priori estimates. On the other hand, notice that the study of the time-decay property for the linearized system with or without the self-consistent forces is now well established; see [7, 8, 36] for hard potentials and [29, 11] for soft potentials; see also [38] in terms of the energy method only.

The rest of the paper is arranged as follows. In Section 2, we list basic lemmas concerning the properties of L and Γ in the functional framework of [13]. In Section 3, we present series of the $w_l(\alpha,\beta)$-weighted estimates on all the nonlinear terms. Section 4 and Section 5 are concerned with the linearized analysis for the time-decay property and the energy estimates to gain the macroscopic dissipation, respectively. In Section 6 we make series of the a priori estimates through the energy method, and in Section 7 we complete the proof Theorem 1.1.

2. Preliminaries

In this section, we list two basic lemmas which will be used in the later proof. Recall $w=w(v)=\langle v\rangle^{-\gamma}$ in (1.11), and we always suppose $-3<\gamma<-2s$ and $1/2\leq s<1$. The first lemma concerns the coercivity estimate on the linearized collision operators L; its proof can be found in [13, pp.783, Lemma 2.6 and pp.829, Theorem 8.1].

Lemma 2.1. *(i) It holds that*

$$(Lg,g)\gtrsim\|(\mathbf{I}-\mathbf{P})g\|_{N^s_\gamma}^2. \tag{2.1}$$

(ii) Let $\ell\in\mathbf{R}$. There is $C>0$ such that

$$\left(w^{2\ell}Lg,g\right)\gtrsim\left\|w^\ell g\right\|_{N^s_\gamma}^2-C\left\|g\right\|_{L^2(B_C)}^2.$$

(iii) Let $\beta>0$, $\ell\in\mathbf{R}$. For any $\eta>0$, there are $C_\eta>0$, $C>0$ such that

$$\left(w^{2\ell}\partial_\beta Lg,\partial_\beta g\right)\gtrsim\left\|w^\ell\partial_\beta g\right\|_{N^s_\gamma}^2-\eta\sum_{\beta_1\leq\beta}\left\|w^\ell\partial_{\beta_1}g\right\|_{N^s_\gamma}^2-C_\eta\left\|g\right\|_{L^2(B_C)}^2.$$

The second lemma concerns the estimates on the nonlinear collision operator Γ. We point out that the corresponding results obtained in [13, pp.817, Lemma 6.1] can not be directly applied here. One has to make some slight modifications involving the distribution of weights. In fact, for the weighted estimate on the triple inner product term $\langle w^{2\ell}\partial^\alpha_\beta\Gamma(g_1,g_2),\partial^\alpha_\beta g_3\rangle$, the weight $w^{2\ell}$ was assigned to every one of the three functions g_1,g_2 and g_3 in [13]. When the integration with respect to space variable is further taken, Sobolev's inequality must be used to control the L^∞_x-norm of either g_1 or g_2 so that the order of x-derivatives should be lifted. However, the lifting is dangerous in the case when the weight $w_l(\alpha,\beta)$ is used for soft potentials. This can be seen by noticing that ℓ depends also on the order of x-derivatives and hence the higher order x-derivatives are associated with the weaker weight.

Lemma 2.2. *Let $g_i=[g_{i,+},g_{i,-}]\in C_0^\infty(\mathbf{R}^3,\mathbf{R}^2)$, $1\leq i\leq 3$, and let $|\alpha|+|\beta|\leq K$ with $\alpha=\alpha_1+\alpha_2$ and $(\beta_1,\beta_2)\leq\beta$. Then for any $\ell\geq 0$ and $m\geq 0$,* *(i) when* $|\alpha_1|+|\beta_1|\le K/2$,

$$\begin{aligned}
&\left|\left(w^{2\ell}\partial^\alpha_\beta\Gamma_\pm(g_1,g_2),\partial^\alpha_\beta g_{3,\pm}\right)\right|\\
&\lesssim \sum_{\substack{\alpha_1+\alpha_2=\alpha\\(\beta_1,\beta_2)\le\beta}}\int_{\mathbf{R}^3}\left|\partial^{\alpha_1}_{\beta_1}g_1\right|_2\left|w^\ell\partial^{\alpha_2}_{\beta_2}g_2\right|_{N^s_\gamma}\left|w^\ell\partial^\alpha_\beta g_3\right|_{N^s_\gamma}dx\\
&\quad+\sum_{\substack{\alpha_1+\alpha_2=\alpha\\(\beta_1,\beta_2)\le\beta}}\int_{\mathbf{R}^3}\left|w^\ell\partial^{\alpha_1}_{\beta_1}g_1\right|_2\left|\partial^{\alpha_2}_{\beta_2}g_2\right|_{N^s_\gamma}\left|w^\ell\partial^\alpha_\beta g_3\right|_{N^s_\gamma}dx\\
&\quad+\sum_{\substack{\alpha_1+\alpha_2=\alpha\\(\beta_1,\beta_2)\le\beta}}\int_{\mathbf{R}^3}\left|w^{-m}\partial^{\alpha_1}_{\beta_1}g_1\right|_{H^2}\left|w^\ell\partial^{\alpha_2}_{\beta_2}g_2\right|_{N^s_\gamma}\left|w^\ell\partial^\alpha_\beta g_3\right|_{N^s_\gamma}dx,
\end{aligned}\tag{2.2}$$

(ii) when $|\alpha_1|+|\beta_1|\ge K/2$,

$$\begin{aligned}
&\left|\left(w^{2\ell}\partial^\alpha_\beta\Gamma_\pm(g_1,g_2),\partial^\alpha_\beta g_{3,\pm}\right)\right|\\
&\lesssim \sum_{\substack{\alpha_1+\alpha_2=\alpha\\(\beta_1,\beta_2)\le\beta}}\int_{\mathbf{R}^3}\left|\partial^{\alpha_1}_{\beta_1}g_1\right|_2\left|w^\ell\partial^{\alpha_2}_{\beta_2}g_2\right|_{N^s_\gamma}\left|w^\ell\partial^\alpha_\beta g_3\right|_{N^s_\gamma}dx\\
&\quad+\sum_{\substack{\alpha_1+\alpha_2=\alpha\\(\beta_1,\beta_2)\le\beta}}\int_{\mathbf{R}^3}\left|w^\ell\partial^{\alpha_1}_{\beta_1}g_1\right|_2\left|\partial^{\alpha_2}_{\beta_2}g_2\right|_{N^s_\gamma}\left|w^\ell\partial^\alpha_\beta g_3\right|_{N^s_\gamma}dx\\
&\quad+\sum_{\substack{\alpha_1+\alpha_2=\alpha\\(\beta_1,\beta_2)\le\beta}}\int_{\mathbf{R}^3}\left|w^{-m}\partial^{\alpha_1}_{\beta_1}g_1\right|_{2}\left|w^\ell\partial^{\alpha_2}_{\beta_2}g_2\right|_{N^s_{\gamma,2}}\left|w^\ell\partial^\alpha_\beta g_3\right|_{N^s_\gamma}dx.
\end{aligned}\tag{2.3}$$

To prove the lemma above, we need to make some preparations by recalling some notations used in [13, pp.791–792]. Notice (1.7) for the definition of $\mathscr{T}$. Consider the following inner product

$$\left\langle w^{2\ell}\partial^\alpha_\beta\mathscr{T}(h_1,h_2),\partial^\alpha_\beta h_3\right\rangle=\sum_{\beta_1+\beta_2+\beta_\mu=\beta}\sum_{\alpha_1+\alpha_1=\alpha}C^{\beta,\beta_1,\beta_\mu}_{\alpha,\alpha_1}\left\langle w^{2\ell}\mathscr{T}_{\beta_\mu}(\partial^{\alpha_1}_{\beta_1}h_1,\partial^{\alpha_2}_{\beta_2}h_2),\partial^\alpha_\beta h_3\right\rangle,$$

where

$$\mathscr{T}_{\beta_\mu}(h_1,h_2)=\int_{\mathbf{R}^3}du\int_{\mathbf{S}^2}d\sigma\,B(v-u,\sigma)\left[\partial_{\beta_\mu}\mu^{1/2}(u)\right][h_1(u')h_2(v')-h_1(u)h_2(v)]\,.$$

Let $\{\eta_\kappa\}_{\kappa=-\infty}^{\kappa=+\infty}$ be a partition on unity on $(0,\infty)$ such that $|\eta_\kappa|_\infty\le 1$ and $\operatorname{supp}(\eta_\kappa)\subset\left[2^{-\kappa-1},2^{-\kappa}\right]$. For each κ we use the notation

$$B_\kappa=B(v-u,\sigma)\mathbb{b}\left(\left\langle\frac{v-u}{|v-u|},\sigma\right\rangle\right)\eta_\kappa(|v-v'|).\tag{2.4}$$

We now denote

$$\begin{aligned}
\mathscr{T}^{\kappa,\ell}_+(h_1,h_2,h_3)&=\int_{\mathbf{R}^3}du\int_{\mathbf{R}^3}dv\int_{\mathbf{S}^2}d\sigma\,B_\kappa(v-u,\sigma)h_1(u)h_2(v)\\
&\qquad\times\left[\partial_{\beta_\mu}\mu^{1/2}(u')\right]h_3(v')w^{2\ell}(v'),\\
\mathscr{T}^{\kappa,\ell}_-(h_1,h_2,h_3)&=\int_{\mathbf{R}^3}du\int_{\mathbf{R}^3}dv\int_{\mathbf{S}^2}d\sigma\,B_\kappa(v-u,\sigma)h_1(u)h_2(v)\\
&\qquad\times\left[\partial_{\beta_\mu}\mu^{1/2}(u)\right]h_3(v)w^{2\ell}(v).
\end{aligned}$$

On the other hand, we express the collision operator (1.7) using its dual formulation as in [13, A1]. In fact, after a transformation, one can put cancellations on the function h_2 as follows

$$\begin{aligned}
&\left\langle w^{2\ell}\mathscr{T}(h_1,h_2),h_3\right\rangle\\
&=\int_{\mathbf{R}^3}du\int_{\mathbf{R}^3}dv'\int_{E^{v'}_u}d\pi_v\,\tilde{B}(v-u,\sigma)h_1(u)h_3(v')w^{2\ell}(v')\left[\mu^{1/2}(u')h_2(v)-\mu^{1/2}(u)h_2(v')\right]\\
&\quad+\mathscr{T}^\ell_*(h_1,h_2,h_3),
\end{aligned}\tag{2.5}$$

where $u' = u + v - v'$, the kernel $\tilde{B}$ is given by

$$\tilde{B} = 4\frac{B\left(v-u, \frac{2v'-v-u}{|2v'-v-u|}\right)}{|v'-u||v-u|},$$

while the corresponding dual operator $\mathscr{T}_*^\ell$ given by

$$\begin{aligned}\mathscr{T}_*^\ell(h_1,h_2,h_3) = &\int_{\mathbf{R}^3} dv'\, h_2(v')h_3(v')w^{2\ell}(v') \int_{\mathbf{R}^3} du\, h_1(u)\left[\partial_{\beta_\mu}\mu^{1/2}(u)\right]\\ &\times \int_{E_u^{v'}} d\pi_v\, \tilde{B}\left(1 - \frac{|v'-u|^{3+\gamma}}{|v-u|^{3+\gamma}}\right).\end{aligned}$$

is not differential at longer. In those integrals, $d\pi_v$ means the Lebesgue measure on the 2-dimensional hyperplane $E_u^{v'}$ defined by $E_u^{v'} = \left\{v \in \mathbf{R}^3 : \langle u - v', v - v'\rangle = 0\right\}$, and v is the variable of integration. Note that in (2.5) we use $\mathscr{T}_*^\ell$ with $\beta_\mu = 0$.

With the observation above, one can use the following alternative representations for $\mathscr{T}_+^{\kappa,\ell}$ as well as $\mathscr{T}_*^{\kappa,\ell}$:

$$\begin{aligned}\mathscr{T}_+^{\kappa,\ell}(h_1,h_2,h_3) = &\int_{\mathbf{R}^3} du \int_{\mathbf{R}^3} dv' \int_{E_u^{v'}} d\pi_v\, \tilde{B}_\kappa(v-u,\sigma)\\ &\times h_1(u)h_2(v)\left[\partial_{\beta_\mu}\mu^{1/2}(u')\right] h_3(v')w^{2\ell}(v'),\end{aligned}$$

$$\begin{aligned}\mathscr{T}_*^{\kappa,\ell}(h_1,h_2,h_3) = &\int_{\mathbf{R}^3} du \int_{\mathbf{R}^3} dv' \int_{E_u^{v'}} d\pi_v\, \tilde{B}_\kappa(v-u,\sigma)\\ &\times h_1(u)h_2(v')\left[\partial_{\beta_\mu}\mu^{1/2}(u)\right] h_3(v')w^{2\ell}(v').\end{aligned}$$

Then for $h_1, h_2, h_3 \in \mathcal{S}(\mathbf{R}^3)$, the pre-post collisional change of variables, the dual representation, and the previous calculation guarantee that

$$\begin{aligned}\left\langle w^{2\ell}\mathscr{T}_{\beta'}(h_1,h_2),h_3\right\rangle = &\sum_{\kappa=-\infty}^{+\infty}\left\{\mathscr{T}_+^{\kappa,\ell}(h_1,h_2,h_3) - \mathscr{T}_-^{\kappa,\ell}(h_1,h_2,h_3)\right\}\\ =&\mathscr{T}_*^{\ell}(h_1,h_2,h_3) + \sum_{\kappa=-\infty}^{+\infty}\left\{\mathscr{T}_+^{\kappa,\ell}(h_1,h_2,h_3) - \mathscr{T}_*^{\kappa,\ell}(h_1,h_2,h_3)\right\}.\end{aligned} \tag{2.6}$$

To the end $\zeta(v)$ denotes an arbitrary smooth function satisfying for some positive constant λ that

$$|\zeta(v)| \sim e^{-\lambda|v|^2}. \tag{2.7}$$

Now we collect estimates for the operators $\mathscr{T}_+^{\kappa,\ell}$, $\mathscr{T}_-^{\kappa,\ell}$ and $\mathscr{T}_*^{\kappa,\ell}$ appearing in (2.6), which can be used to prove Lemma 2.2. Notice that only the soft potential case $-3 < \gamma < -2s$ with $1/2 \le s < 1$ is considered here.

Proposition 2.1. *Let κ be an integer, $m \ge 0$, $\ell \in \mathbf{R}$ and ζ defined by (2.7).*

(i)

$$\left|\mathscr{T}_-^{\kappa,\ell}(h_1,h_2,h_3)\right| \lesssim 2^{2s\kappa}\left|w^{-m}h_1\right|_{H^2}\left|w^\ell h_2\right|_{L^2_{\gamma+2s}}\left|w^\ell h_3\right|_{L^2_{\gamma+2s}},$$

$$\left|\mathscr{T}_-^{\kappa,\ell}(h_1,h_2,h_3)\right| \lesssim 2^{2s\kappa}\left|w^{-m}h_1\right|_{L^2}\left|w^\ell h_2\right|_{H^2_{\gamma+2s}}\left|w^\ell h_3\right|_{L^2_{\gamma+2s}},$$

and

$$\left|\mathscr{T}_-^{\kappa,\ell}(h_1,h_2,\zeta)\right| + \left|\mathscr{T}_-^{\kappa,\ell}(h_1,\zeta,h_2)\right| \lesssim 2^{2s\kappa}\left|w^{-m}h_1\right|_{L^2}\left|w^{-m}h_2\right|_{L^2}.$$

(ii)

$$\left|\mathscr{T}_*^{\kappa,\ell}(h_1,h_2,h_3)\right| \lesssim 2^{2s\kappa}\left|w^{-m}h_1\right|_{H^2}\left|w^\ell h_2\right|_{L^2_{\gamma}}\left|w^\ell h_3\right|_{L^2_{\gamma}},$$

$$\left|\mathscr{T}_*^{\kappa,\ell}(h_1,h_2,h_3)\right| \lesssim 2^{2s\kappa}\left|w^{-m}h_1\right|_{L^2}\left|w^\ell h_2\right|_{H^2_{\gamma}}\left|w^\ell h_3\right|_{L^2_{\gamma}},$$

and

$$\left|\mathscr{T}_*^{\kappa,\ell}(h_1,h_2,\zeta)\right| + \left|\mathscr{T}_*^{\kappa,\ell}(h_1,\zeta,h_2)\right| \lesssim 2^{2s\kappa}\left|w^{-m}h_1\right|_{L^2}\left|w^{-m}h_2\right|_{L^2}.$$

(iii)

$$
\begin{aligned}
\left|\mathscr{T}_+^{\kappa,\ell}(h_1,h_2,h_3)\right| \lesssim & 2^{2s\kappa}\left|h_1\right|_{L^2}\left|w^\ell h_2\right|_{L^2_{\gamma+2s}}\left|w^\ell h_3\right|_{L^2_{\gamma+2s}} \\
& + 2^{2s\kappa}\left|w^\ell h_1\right|_{L^2}\left|h_2\right|_{L^2_{\gamma+2s}}\left|w^\ell h_3\right|_{L^2_{\gamma+2s}} \\
& + 2^{2s\kappa}\left|w^{-m} h_1\right|_{H^2}\left|w^\ell h_2\right|_{L^2_{\gamma+2s}}\left|w^\ell h_3\right|_{L^2_{\gamma+2s}},
\end{aligned} \tag{2.8}
$$

$$
\begin{aligned}
\left|\mathscr{T}_+^{\kappa,\ell}(h_1,h_2,h_3)\right| \lesssim & 2^{2s\kappa}\left|h_1\right|_{L^2}\left|w^\ell h_2\right|_{L^2_{\gamma+2s}}\left|w^\ell h_3\right|_{L^2_{\gamma+2s}} \\
& + 2^{2s\kappa}\left|w^\ell h_1\right|_{L^2}\left|h_2\right|_{L^2_{\gamma+2s}}\left|w^\ell h_3\right|_{L^2_{\gamma+2s}} \\
& + 2^{2s\kappa}\left|w^{-m} h_1\right|_{L^2}\left|w^\ell h_2\right|_{H^2_{\gamma+2s}}\left|w^\ell h_3\right|_{L^2_{\gamma+2s}},
\end{aligned} \tag{2.9}
$$

and

$$
\left|\mathscr{T}_+^{\kappa,\ell}(h_1,h_2,\zeta)\right| \lesssim 2^{s\kappa}\left|w^{-m} h_1\right|_{L^2}\left|w^{-m} h_2\right|_{L^2}. \tag{2.10}
$$

Proof. First of all, notice that (i), (ii) and (2.10) in (iii) are the same as in [13, pp.803–804, Proposition 4.1, 4.2, 4.4, and pp.808, Proposition 4.8], and only (2.8) and (2.9) are different. For brevity, we prove (2.8) only. The key point is to assign the velocity weight to h_1 and h_2 in a better way. The following inequalities will be frequently used in the later proof:

$$
\int_{\mathbf{S}^2} B_\kappa(v-u,\sigma)\,d\sigma \lesssim |v-u|^\gamma \int_{2^{-\kappa-1}|v-u|^{-1}}^{2^{-\kappa}|v-u|^{-1}} \theta^{-1-2s} d\theta \lesssim 2^{2s\kappa}|v-u|^{\gamma+2s}, \tag{2.11}
$$

where we recall (2.4) and the geometric relation $|v'-v| = |v-u|\sin\frac{\theta}{2}$.

On the region $|v-u|\geq 1$, the singularity of $|v-u|^{\gamma+2s}$ is absent. Thus, by Cauchy-Schwarz's inequality and (2.11), we have

$$
\begin{aligned}
\left|\mathscr{T}_+^{\kappa,\ell}(h_1,h_2,h_3)\right| \lesssim & 2^{2s\kappa}\left(\int_{\mathbf{R}^3} du \int_{\mathbf{R}^3} dv\, |v-u|^{2(\gamma+2s)}\langle v'\rangle^{-\gamma-2s}\mu^{1/4}(u')h_3(v')w^{2\ell}(v')\right)^{\frac12} \\
& \times\left(\int_{\mathbf{R}^3} du \int_{\mathbf{R}^3} dv\, h_1(u)h_2(v)\langle v'\rangle^{\gamma+2s}\mu^{1/4}(u')w^{2\ell}(v')\right)^{\frac12} \\
\lesssim & 2^{2s\kappa}\left(\int_{\mathbf{R}^3} dv'\,\langle v'\rangle^{\gamma+2s}|h_3(v')|^2 w^{2\ell}(v')\right)^{\frac12} \\
& \times\left(\int_{\mathbf{R}^3} du \int_{\mathbf{R}^3} dv\, |h_1(u)|^2|h_2(v)|^2\langle v'\rangle^{\gamma+2s}\mu^{1/4}(u')w^{2\ell}(v')\right)^{\frac12}.
\end{aligned} \tag{2.12}
$$

The first factor on the right-hand side of (2.12) is bounded by $\left|w^\ell h_3\right|_{L^2_{\gamma+2s}}$. For the second factor, in the case when $|v'|^2 \leq \frac12(|v|^2+|u|^2)$, since the collisional conversation laws imply $\mu^{\frac14}(u')\leq \mu^{\frac18}(v)\mu^{\frac18}(u)$, it follows that $\langle v'\rangle^{\gamma+2s}\mu^{1/4}(u')w^{2\ell}(v')\lesssim \langle v\rangle^{-m}\langle u\rangle^{-m}$ for any nonnegative constant m. In the case when $|v'|^2\geq\frac12(|v|^2+|u|^2)$ which implies $|v'|^2\sim|v|^2+|u|^2$, we have for $\ell\geq 0$,

$$
w^{2\ell}(v')\lesssim w^{2\ell}(u)+w^{2\ell}(v),
$$

and similarly $\langle v'\rangle^{\gamma+2s}\lesssim\langle v\rangle^{\gamma+2s}$, so that

$$
\langle v'\rangle^{\gamma+2s}\mu^{1/4}(u')w^{2\ell}(v')\lesssim\langle v\rangle^{\gamma+2s}\left(w^{2\ell}(u)+w^{2\ell}(v)\right).
$$

Therefore, in both cases, the second factor on the right-hand side of (2.12) is bounded by

$$
\left|h_1\right|_{L^2}\left|w^\ell h_2\right|_{L^2_{\gamma+2s}}+\left|w^\ell h_1\right|_{L^2}\left|h_2\right|_{L^2_{\gamma+2s}}.
$$

On the remaining region $|v-u|\leq 1$, where $\mu^{1/4}(u')w^{2\ell}(v')\lesssim\mu^\delta(u)\mu^\delta(v)$ for some $0<\delta<1$, it follows that

$$
\begin{aligned}
\left|\mathscr{T}_+^{\kappa,\ell}(h_1,h_2,h_3)\right| \lesssim & 2^{2s\kappa}\int_{\mathbf{R}^3} du\int_{\mathbf{R}^3} dv\,|v-u|^{\gamma+2s}|h_1(u)h_2(v)||h_3(v')| \\
& \times\left(\mu(u')\mu(v')\mu(u)\mu(v)\right)^{\frac{\delta}{2}},
\end{aligned}
$$

which from Cauchy-Schwarz and Sobolev inequalities, is further bounded by

$$
\begin{aligned}
&2^{2s\kappa}\left(\int_{\mathbf{R}^3} dv'\,\langle v'\rangle^{\gamma+2s}|h_3(v')|^2\mu^\delta(v')\right)^{\frac12}\\
&\qquad\times\left|h_1(u)\mu^{\frac{\delta}{2}}(u)\right|_\infty\left(\int_{\mathbf{R}^3} du\int_{\mathbf{R}^3} dv\,\langle v\rangle^{\gamma+2s}|h_2(v)|^2\mu^\delta(v)\right)^{\frac12}\\
&\lesssim 2^{2s\kappa}\left|w^{-m}h_1\right|_{H^2}\left|w^\ell h_2\right|_{L^2_{\gamma+2s}}\left|w^\ell h_3\right|_{L^2_{\gamma+2s}}.
\end{aligned}
$$

This completes the proof of Proposition 2.1. □

Proof of Lemma 2.2. In terms of series of estimates obtained in Propositions 2.1, by applying the cancellation inequalities constructed in [13, Propositions 4.5-4.7] and carrying out the similar procedure as that of [13, Lemma 6.1], one can prove (2.2) and (2.3) and the details are omitted for brevity. This completes the proof of Lemma 2.2. □

3. Nonlinear estimates

The goal of this section is to make the weighted energy estimates on those nonlinear terms in (1.4). The following decomposition will be frequently used in the later proofs:

$$
\begin{aligned}
\Gamma(f,f)=&\Gamma(\mathbf{P}f,\mathbf{P}f)+\Gamma(\mathbf{P}f,\{\mathbf{I}-\mathbf{P}\}f)+\Gamma(\{\mathbf{I}-\mathbf{P}\}f,\mathbf{P}f)\\
&+\Gamma(\{\mathbf{I}-\mathbf{P}\}f,\{\mathbf{I}-\mathbf{P}\}f).
\end{aligned}
\tag{3.1}
$$

Recall (1.12) for $w_l(\alpha,\beta)$, and recall (1.13) and (1.15) for $\mathcal{E}_l(t)$ and $\mathcal{D}_l(t)$, respectively. To the end we always suppose $-3<\gamma<-2s$, $1/2\le s<1$, and $K\ge 8$. The first three lemmas of this section concern the estimates on the nonlinear term $\Gamma(f,f)$.

Lemma 3.1. *Let $l\ge 0$, $|\alpha|+|\beta|\le K$. It holds that*

$$
|(\Gamma_\pm(f,f),f_\pm)|\lesssim \mathcal{E}_l^{\frac12}(t)\mathcal{D}_l(t),\tag{3.2}
$$

$$
\left|\left(w_l^2(\alpha,\beta)\partial^\alpha_\beta\Gamma_\pm(f,f),\partial^\alpha_\beta(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right)\right|\lesssim \mathcal{E}_l^{\frac12}(t)\mathcal{D}_l(t),\tag{3.3}
$$

$$
\left|\left(w_l^2(\alpha,0)\partial^\alpha\Gamma_\pm(f,f),\partial^\alpha f_\pm\right)\right|\lesssim \mathcal{E}_l^{\frac12}(t)\mathcal{D}_l(t).\tag{3.4}
$$

Here and hereafter, we denote $I_\pm f=f_\pm$.

Proof. For brevity, we only prove (3.3) in the case when $\alpha=\beta=0$, and the other two estimates (3.2) and (3.4) can be proved in the similar way. For this, we set

$$
J_1=\left|\left(w_l^2(0,0)\Gamma_\pm(f,f),(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right)\right|,
$$

and denote $J_{1,1},J_{1,2},J_{1,3},J_{1,4}$ to be the terms corresponding to the decomposition (3.1). Now we turn to estimate these terms one by one. First, for $J_{1,1}$, recalling (1.10) and applying Lemma 2.2 with $\ell=l+K$, one has

$$
J_{1,1}\lesssim\|(a_\pm,b,c)\|_{H^1}\|\nabla_x(a_\pm,b,c)\|\,\|w_l(0,0)(\mathbf{I}_\pm-\mathbf{P}_\pm)f\|_{N^s_\gamma},
$$

where we have used Sobolev's inequalities

$$
\|(a_\pm,b,c)\|_{L^3}\lesssim\|(a_\pm,b,c)\|_{H^1},\quad \|(a_\pm,b,c)\|_{L^6}\lesssim\|\nabla_x(a_\pm,b,c)\|.
$$

For $J_{1,2}$, by using (2.2) in Lemma 2.2 with $\ell=l+K$, it follows that

$$
J_{1,2}\lesssim\|(a_\pm,b,c)\|_{L^\infty}\|w_l(0,0)(\mathbf{I}_\pm-\mathbf{P}_\pm)f\|^2_{N^s_\gamma}\lesssim\|\nabla_x(a_\pm,b,c)\|_{H^1}\|w_l(0,0)(\mathbf{I}_\pm-\mathbf{P}_\pm)f\|^2_{N^s_\gamma}.
$$

In the same way, $J_{1,3}$ has the same bound as $J_{1,2}$. Finally, for $J_{1,4}$, due to Lemma 2.2 and Sobolev's inequality, we obtain

$$\begin{aligned}
J_{1,4} \lesssim & \, \| |(\mathbf{I}_\pm - \mathbf{P}_\pm) f|_{L^2} \|_{L^\infty_x} \|w_l(0,0)(\mathbf{I}_\pm - \mathbf{P}_\pm) f\|^2_{N^s_\gamma} \\
& + \|w_l(0,0)(\mathbf{I}_\pm - \mathbf{P}_\pm) f\| \left\| |(\mathbf{I}_\pm - \mathbf{P}_\pm) f|_{N^s_\gamma} \right\|_{L^\infty_x} \|w_l(0,0)(\mathbf{I}_\pm - \mathbf{P}_\pm) f\|_{N^s_\gamma} \\
& + \left\| |w^{-m}(\mathbf{I}_\pm - \mathbf{P}_\pm) f|_{H^2} \right\|_{L^\infty_x} \|w_l(0,0)(\mathbf{I}_\pm - \mathbf{P}_\pm) f\|^2_{N^s_\gamma} \\
\lesssim & \sum_{|\alpha|\le 2} \|\partial^\alpha (\mathbf{I}_\pm - \mathbf{P}_\pm) f\| \, \|w_l(0,0)(\mathbf{I}_\pm - \mathbf{P}_\pm) f\|^2_{N^s_\gamma} \\
& + \|w_l(0,0)(\mathbf{I}_\pm - \mathbf{P}_\pm) f\| \sum_{|\alpha|\le 2} \|\partial^\alpha (\mathbf{I}_\pm - \mathbf{P}_\pm) f\|_{N^s_\gamma} \|w_l(0,0)(\mathbf{I}_\pm - \mathbf{P}_\pm) f\|_{N^s_\gamma} \\
& + \sum_{|\alpha_1|\le 2, |\beta_1|\le 2} \left\| w_l(\alpha_1,\beta_1) \partial^{\alpha_1}_{\beta_1} (\mathbf{I}_\pm - \mathbf{P}_\pm) f \right\| \|w_l(0,0)(\mathbf{I}_\pm - \mathbf{P}_\pm) f\|^2_{N^s_\gamma} .
\end{aligned}$$

Now the desired estimate (3.3) in the case when $\alpha = \beta = 0$ holds by combing all the above estimates. This completes the proof of Lemma 3.1. □

Lemma 3.2. *Let $\zeta(v)$ be a smooth function satisfying (2.7), and let $|\alpha| \le K$. Writing*

$$\partial^\alpha \Gamma(f,f) = \sum_{\alpha_1+\alpha_2=\alpha} \Gamma(\partial^{\alpha_1} f, \partial^{\alpha_2} f),$$

one has

$$\left\| \int \Gamma(\partial^{\alpha_1} f, \partial^{\alpha_2} f) \zeta(v)\, dv \right\| \lesssim \mathcal{E}_l^{\frac12}(t) \mathcal{D}_l^{\frac12}(t). \tag{3.5}$$

Proof. With Lemma 2.2 in hand, (3.5) can be verified directly by applying Sobolev's embedding inequalities, and details are omitted for brevity. □

Lemma 3.3. *Let $\ell \ge 0$. It holds that*

$$\sum_{|\alpha|\le 1} \left| w^\ell \partial^\alpha \Gamma(f,f) \right|_{L^2} \lesssim \sum_{|\alpha|\le 1} \left| w^\ell \partial^\alpha f \right|_{L^2_{\gamma+2s}} \sum_{|\alpha|\le 1} \left| w^\ell \partial^\alpha f \right|_{H^4_{\gamma+2s}} . \tag{3.6}$$

Moreover,

$$\left\| w^\ell \Gamma(f,f) \right\|_{H^1_x} + \left\| w^\ell \Gamma(f,f) \right\|_{Z_1} \lesssim \sum_{|\alpha|+|\beta|\le 5} \left\| w^{\ell - \frac{\gamma+2s}{2\gamma}} \partial^\alpha_\beta f \right\|^2 . \tag{3.7}$$

Proof. First of all, (3.6) has been proved in [29, pp.21, Proposition 3.1]. Then applying (3.6) and Sobolev's inequality, we have

$$\begin{aligned}
\left\| w^\ell \Gamma(f,f) \right\|^2_{H^1_x} & \lesssim \int_{\mathbf{R}^3} \sum_{|\alpha|\le 1} \left| w^\ell \partial^\alpha f \right|^2_{L^2_{\gamma+2s}} \sum_{|\alpha|\le 1} \left| w^\ell \partial^\alpha f \right|^2_{H^4_{\gamma+2s}} dx \\
& \lesssim \sup_x \sum_{|\alpha|\le 1} \left| w^\ell \partial^\alpha f \right|^2_{L^2_{\gamma+2s}} \int_{\mathbf{R}^3} \sum_{|\alpha|\le 1} \left| w^\ell \partial^\alpha f \right|^2_{H^4_{\gamma+2s}} dx \\
& \lesssim \sum_{|\alpha'|\le 3} \left\| w^\ell \partial^{\alpha'} f \right\|^2_{L^2_{\gamma+2s}} \sum_{|\alpha|\le 1} \left\| w^\ell \partial^\alpha f \right\|^2_{H^4_{\gamma+2s}} .
\end{aligned}$$

Therefore $\left\| w^\ell \Gamma(f,f) \right\|_{H^1_x}$ is bounded by the right-hand term of (3.7). As in [29], $\left\| w^\ell \Gamma(f,f) \right\|_{Z_1}$ can be estimated in the completely same way as for $\left\| w^\ell \Gamma(f,f) \right\|_{H^1_x}$, and details are omitted for brevity. This completes the proof of Lemma 3.3. □

The following two lemmas concern the estimates on $v \cdot \nabla_x \phi f_\pm$. As in [21], for simplicity, we use e_i to denote the multi-index with the ith element unit and the rest ones zeros.

Lemma 3.4. *Let $1\leq |\alpha|\leq K$, and $l\geq 0$. Suppose $\sqrt{\mathcal{E}_l(t)}<\delta$ for a constant $\delta>0$. Then, it holds that*

$$\sum_{1\leq|\alpha_1|\leq|\alpha|}\big(v_i\partial^{\alpha_1+e_i}\phi\partial^{\alpha-\alpha_1}f_\pm,e^{\pm\phi}w_l^2(\alpha,0)\partial^\alpha f_\pm\big)\lesssim \mathcal{E}_l^{\frac12}(t)\mathcal{D}_l(t). \tag{3.8}$$

Proof. In terms of $f_\pm=(\mathbf{I}_\pm-\mathbf{P}_\pm)f+\mathbf{P}_\pm f$, set

$$\begin{aligned}
&\sum_{1\leq|\alpha_1|\leq|\alpha|}\big(v_i\partial^{\alpha_1+e_i}\phi\partial^{\alpha-\alpha_1}\mathbf{P}_\pm f,e^{\pm\phi}w_l^2(\alpha,0)\partial^\alpha f_\pm\big)\\
&\qquad+\sum_{1\leq|\alpha_1|\leq|\alpha|}\big(v_i\partial^{\alpha_1+e_i}\phi\partial^{\alpha-\alpha_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f,e^{\pm\phi}w_l^2(\alpha,0)\partial^\alpha \mathbf{P}_\pm f\big)\\
&\qquad+\sum_{1\leq|\alpha_1|\leq|\alpha|}\big(v_i\partial^{\alpha_1+e_i}\phi\partial^{\alpha-\alpha_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f,e^{\pm\phi}w_l^2(\alpha,0)\partial^\alpha (\mathbf{I}_\pm-\mathbf{P}_\pm)f\big)\\
&=J_{2,1}+J_{2,2}+J_{2,3}.
\end{aligned}$$

Notice by Sobolev inequality that

$$|\phi|_{L^\infty}\lesssim\|\nabla_x\phi\|_{H^1}\lesssim\sqrt{\mathcal{E}_l(t)}<\delta. \tag{3.9}$$

For $J_{2,1}$, one has

$$|J_{2,1}|\lesssim\sum_{1\leq|\alpha_1|\leq|\alpha|}\int_{\mathbf{R}^3}|\partial^{\alpha_1+e_i}\phi||\partial^{\alpha-\alpha_1}(a_\pm,b,c)|\partial^\alpha f_\pm|_{L^2_\gamma}dx,$$

where for $|\alpha_1|\leq[K/2]$, the integrand is bounded by

$$\sup_x|\partial^{\alpha_1+e_i}\phi|\cdot\|\partial^{\alpha-\alpha_1}(a_\pm,b,c)\|\cdot\|\partial^\alpha f_\pm\|_{L^2_\gamma},$$

while for $|\alpha_1|\geq[K/2]+1$, it is bounded by

$$\sup_x|\partial^{\alpha-\alpha_1}(a_\pm,b,c)|\cdot\|\partial^{\alpha_1+e_i}\phi\|\cdot\|\partial^\alpha f_\pm\|_{L^2_\gamma}.$$

Thus, by Sobolev inequality,

$$|J_{2,1}|\lesssim\|\nabla_x^2\phi\|_{H^{K-1}}\sum_{1\leq|\alpha|\leq K}\Big\{\|\partial^\alpha(a_\pm,b,c)\|\|\partial^\alpha f_\pm\|_{L^2_\gamma}\Big\}.$$

Similarly, for $J_{2,2}$, one has

$$\begin{aligned}
|J_{2,2}|\lesssim&\sum_{1\leq|\alpha_1|\leq|\alpha|}\int_{\mathbf{R}^3}|\partial^{\alpha_1+e_i}\phi|\,\big|\partial^{\alpha-\alpha_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\big|_{L^2_\gamma}|\partial^\alpha(a_\pm,b,c)|dx\\
\lesssim&\sum_{|\alpha_1|\leq[K/2]}\sup_x|\partial^{\alpha_1+e_i}\phi|\cdot\|\partial^{\alpha-\alpha_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\|_{L^2_\gamma}\cdot\|\partial^\alpha(a_\pm,b,c)\|\\
&+\sum_{|\alpha_1|\geq[K/2]+1}\sup_x|\partial^{\alpha-\alpha_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f|_{L^2_\gamma}\cdot\|\partial^{\alpha_1+e_i}\phi\|\cdot\|\partial^\alpha(a_\pm,b,c)\|\\
\lesssim&\|\nabla_x^2\phi\|_{H^{K-1}}\sum_{1\leq|\alpha|\leq K}\|\partial^\alpha(a_\pm,b,c)\|\sum_{|\alpha|\leq K}\|\partial^\alpha(\mathbf{I}_\pm-\mathbf{P}_\pm)f\|_{L^2_\gamma}.
\end{aligned}$$

Finally, for $J_{2,3}$, since $1/2\leq s<1$, it is straightforward to see

$$|w_l(\alpha,0)v_i|\lesssim w_l(\alpha-e_i,0)\langle v\rangle^{\gamma+2s}.$$

Therefore, one has

$$
\begin{aligned}
|J_{2,3}| \lesssim & \sum_{1\le|\alpha_1|\le|\alpha|} \int_{\mathbf{R}^3} |\partial^{\alpha_1+e_i}\phi| \cdot \left|w_l(\alpha-e_i,0)\partial^{\alpha-\alpha_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right|_{L^2_{\gamma+2s}} \\
& \times |w_l(\alpha,0)\partial^{\alpha}(\mathbf{I}_\pm-\mathbf{P}_\pm)f|_{L^2_{\gamma+2s}}\,dx \\
\lesssim & \sum_{1\le|\alpha_1|\le[K/2]} \sup_x |\partial^{\alpha_1+e_i}\phi| \cdot \left\|w_l(\alpha-e_i,0)\partial^{\alpha-\alpha_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|_{L^2_{\gamma+2s}} \\
& \qquad \times \|w_l(\alpha,0)\partial^{\alpha}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\|_{L^2_{\gamma+2s}} \\
& + \sum_{|\alpha|\ge|\alpha_1|\ge[K/2]+1} \sup_x \left|w_l(\alpha-e_i,0)\partial^{\alpha-\alpha_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right|_{L^2_{\gamma+2s}} \cdot \|\partial^{\alpha_1+e_i}\phi\| \\
& \qquad \times \|w_l(\alpha,0)\partial^{\alpha}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\|_{L^2_{\gamma+2s}} \\
\lesssim & \|\nabla_x^2\phi\|_{H^{K-1}} \sum_{|\alpha|\le K} \|w_l(\alpha,0)\partial^{\alpha}(\mathbf{I}_\pm-\mathbf{P}_\pm)f_\pm\|^2_{L^2_{\gamma+2s}},
\end{aligned}
$$

where we have used $|\alpha-\alpha_1|\le|\alpha-e_i|$ in the case when $1\le|\alpha_1|\le[K/2]$, and $|\alpha-\alpha_1|+2\le|\alpha-e_i|$ in the case when $|\alpha|\ge|\alpha_1|\ge[K/2]+1$. Collecting all the estimates, (3.8) follows. This then completes the proof of Lemma 3.4. □

Lemma 3.5. *Let $1\le|\alpha|+|\beta|\le K$ with $|\beta|\ge1$, and $l\ge0$. Suppose $\sqrt{\mathcal{E}_l(t)}<\delta$ for a constant $\delta>0$. Then, it holds that*

$$
\sum_{\substack{|\alpha_1|+|\beta_1|\ge1\\ |\alpha_1|\le|\alpha|,|\beta_1|\le1}} \left(\partial_{\beta_1}v_i\partial^{\alpha_1+e_i}\phi\partial^{\alpha-\alpha_1}_{\beta-\beta_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f, e^{\pm\phi}w_l^2(\alpha,\beta)\partial^{\alpha}_{\beta}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right) \lesssim \mathcal{E}_l^{\frac12}(t)\mathcal{D}_l(t). \tag{3.10}
$$

Proof. For brevity, we denote the left-hand term of (3.10) as J_3. As in Lemma 3.4, we prove (3.10) by considering the following two cases. For the case $|\alpha_1|\le[K/2]$, from (3.9), one has

$$
\begin{aligned}
|J_3| \lesssim & \sup_x |\partial^{\alpha_1+e_i}\phi| \cdot \left\|w_l(\alpha-\alpha_1,\beta-\beta_1)\partial^{\alpha-\alpha_1}_{\beta-\beta_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|_{L^2_{\gamma+2s}} \\
& \times \left\|w_l(\alpha,\beta)\partial^{\alpha}_{\beta}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|_{L^2_{\gamma+2s}},
\end{aligned}
$$

which is further bounded by the right-hand term of (3.10). For the case $|\alpha_1|\ge[K/2]+1$, it is similar to verify

$$
\begin{aligned}
|J_3| \lesssim & \sup_x \left|w_l(\alpha-\alpha_1,\beta-\beta_1)\partial^{\alpha-\alpha_1}_{\beta-\beta_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right|_{L^2_{\gamma+2s}} \|\partial^{\alpha_1+e_i}\phi\| \\
& \times \left\|w_l(\alpha,\beta)\partial^{\alpha}_{\beta}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|_{L^2_{\gamma+2s}},
\end{aligned}
$$

which is also bounded by the right-hand term of (3.10). This completes the proof of Lemma 3.5. □

The following two lemmas concern the estimates on $\nabla_x\phi\cdot\nabla_v f_\pm$.

Lemma 3.6. *Let $1\le|\alpha|\le K$, and $l\ge0$. Suppose $\sqrt{\mathcal{E}_l(t)}<\delta$ for a constant $\delta>0$. Then, it holds that*

$$
\sum_{|\alpha_1|\le|\alpha|} \left(\partial^{\alpha_1+e_i}\phi\partial^{\alpha-\alpha_1}_{e_i}f_\pm, e^{\pm\phi}w_l^2(\alpha,0)\partial^{\alpha}f_\pm\right) \lesssim \mathcal{E}_l^{\frac12}(t)\mathcal{D}_l(t). \tag{3.11}
$$

Proof. Denote the left-hand term of (3.11) by J_4. When $\alpha_1=0$, by taking integration by parts with respect to v_i, one has

$$
\begin{aligned}
J_4 \lesssim & \sup_x |\partial^{e_i}\phi| \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} |\partial^\alpha f_\pm|^2 \left|\partial_{e_i}w_l^2(\alpha,0)\right| dxdv \\
\lesssim & \|\nabla_x\phi\|_{H^2} \|w_l(\alpha,0)\partial^\alpha f_\pm\| \|w_l(\alpha,0)\partial^\alpha f_\pm\|_{L^2_{\gamma+2s}} \lesssim \mathcal{E}_l^{\frac12}(t)\mathcal{D}_l(t),
\end{aligned}
$$

where the inequality $|\partial_{e_i}w_l^2(\alpha,0)|\lesssim\langle v\rangle^{\frac{\gamma+2s}{2}}w_l^2(\alpha,0)$ has been used due to $-3<\gamma<-2s$ and $1/2\le s<1$.

Whenever $\alpha_1>0$, one can write

$$
J_4 = J_{4,1}+J_{4,2},
$$

with

$$J_{4,1}=\sum_{1\le|\alpha_1|\le|\alpha|}\left(\partial^{\alpha_1+e_i}\phi\partial^{\alpha-\alpha_1}_{e_i}\mathbf{P}_\pm f,e^{\pm\phi}w_l^2(\alpha,0)\partial^\alpha f_\pm\right),$$

$$J_{4,2}=\sum_{1\le|\alpha_1|\le|\alpha|}\left(\partial^{\alpha_1+e_i}\phi\partial^{\alpha-\alpha_1}_{e_i}(\mathbf{I}_\pm-\mathbf{P}_\pm)f,e^{\pm\phi}w_l^2(\alpha,0)\partial^\alpha f_\pm\right).$$

We now estimate $J_{4,1}$ and $J_{4,2}$ as follows. By using the similar argument as for estimating $J_{2,1}$, we deduce that

$$J_{4,1}\lesssim\|\nabla_x^2\phi\|_{H^{K-1}}\sum_{1\le|\alpha|\le K}\left\{\|\partial^\alpha(a_\pm,b,c)\|\cdot\|\partial^\alpha f_\pm\|_{L^2_\gamma}\right\}.$$

To estimate $J_{4,2}$, we use the trick as in [21]. First of all, notice that $J_{4,2}$ can be written as

$$\begin{aligned}J_{4,2}=&\sum_{1\le|\alpha_1|\le|\alpha|}\left\{\left(\partial^{\alpha_1+e_i}\phi\partial_{e_i}\left[w_l(\alpha-\alpha_1,0)w^{-\frac{|\alpha_1|}{2}}\partial^{\alpha-\alpha_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right],e^{\pm\phi}w_l(\alpha,0)w^{-\frac{|\alpha_1|}{2}}\partial^\alpha f_\pm\right)\right.\\&\left.-\left(\partial^{\alpha_1+e_i}\phi\partial_{e_i}\left[w_l(\alpha-\alpha_1,0)w^{-\frac{|\alpha_1|}{2}}\right]w^{-\frac{|\alpha_1|}{2}}\partial^{\alpha-\alpha_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f,e^{\pm\phi}w_l(\alpha,0)\partial^\alpha f_\pm\right)\right\}\\=&J_{4,2}^{(1)}+J_{4,2}^{(2)}.\end{aligned}$$

For the term $J_{4,2}^{(2)}$, it is straightforward to estimate it by

$$\begin{aligned}&C\int_{\mathbf{R}^3}\left|\partial^{\alpha_1+e_i}\phi\right|\left|w_l(\alpha-\alpha_1,0)\partial^{\alpha-\alpha_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right|_{L^2_{\gamma+2s}}|w_l(\alpha,0)\partial^\alpha f_\pm|_{L^2_{\gamma+2s}}\,dx\\&\lesssim\|\nabla_x^2\phi\|_{H^{K-1}}\sum_{|\alpha|\le K}\|w_l(\alpha,0)\partial^\alpha(\mathbf{I}_\pm-\mathbf{P}_\pm)f\|^2_{L^2_{\gamma+2s}}\sum_{1\le|\alpha|\le K}\|w_l(\alpha,0)\partial^\alpha f_\pm\|^2_{L^2_{\gamma+2s}}.\end{aligned}$$

For $J_{4,2}^{(1)}$, noticing $1/2\le s<1$, by the Parseval identity, one has

$$\begin{aligned}J_{4,2}^{(1)}\lesssim&\int_{\mathbf{R}^3}\left|\int_{\mathbf{R}^3}\mathbb{i}\xi_i\mathcal{F}_v\left[w_l(\alpha-\alpha_1,0)w^{-\frac{|\alpha_1|}{2}}\partial^{\alpha-\alpha_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right]\overline{\mathcal{F}_v\left[w_l(\alpha,0)w^{-\frac{|\alpha_1|}{2}}\partial^\alpha f_\pm\right]}d\xi\right|\\&\times\left|\partial^{\alpha_1+e_i}\phi\right|dx\\\lesssim&\int_{\mathbf{R}^3}\left|\langle\xi\rangle^{\frac12}\mathcal{F}_v\left[w_l(\alpha-\alpha_1,0)w^{-\frac{|\alpha_1|}{2}}\partial^{\alpha-\alpha_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right]\right|_{L^2_\xi}\left|\langle\xi\rangle^{\frac12}\mathcal{F}_v\left[w_l(\alpha,0)w^{-\frac{|\alpha_1|}{2}}\partial^\alpha f_\pm\right]\right|_{L^2_\xi}\\&\times\left|\partial^{\alpha_1+e_i}\phi\right|dx\\\lesssim&\int_{\mathbf{R}^3}\left|w_l(\alpha-\alpha_1,0)w^{-\frac{|\alpha_1|}{2}}\partial^{\alpha-\alpha_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right|_{H^s_v}\left|w_l(\alpha,0)w^{-\frac{|\alpha_1|}{2}}\partial^\alpha f_\pm\right|_{H^s_v}\\&\times\left|\partial^{\alpha_1+e_i}\phi\right|dx\\\lesssim&\|\nabla_x^2\phi\|_{H^{K-1}}\sum_{|\alpha|\le K}\|w_l(\alpha,0)\partial^\alpha(\mathbf{I}_\pm-\mathbf{P}_\pm)f\|^2_{H^s_\gamma}\sum_{1\le|\alpha|\le K}\|w_l(\alpha,0)\partial^\alpha f_\pm\|^2_{H^s_\gamma},\end{aligned}$$

where $\mathcal{F}_v$ is the Fourier transform with respect to v-variable, ξ denotes the corresponding frequency variable, $\bar{\cdot}$ denotes the complex conjugate, and $\mathbb{i}=\sqrt{-1}\in\mathbb{C}$ is the pure imaginary unit. Collecting the estimates above, this completes the proof of Lemma 3.6. ☐

Lemma 3.7. *Let $1\le|\alpha|+|\beta|\le K$ with $|\beta|\ge1$, and $l\ge0$. Suppose $\sqrt{\mathcal{E}_l(t)}<\delta$ for a constant $\delta>0$. Then, it holds that*

$$\sum_{\alpha_1\le\alpha}\left(\partial^{\alpha_1+e_i}\phi\partial^{\alpha-\alpha_1}_{\beta+e_i}(\mathbf{I}_\pm-\mathbf{P}_\pm)f,e^{\pm\phi}w_l^2(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right)\lesssim\mathcal{E}_l^{\frac12}(t)\mathcal{D}_l(t).\tag{3.12}$$

Proof. (3.12) can be proved by using the same argument as for Lemma 3.6, and the details are omitted for brevity. ☐

4. The linearized system and time-decay

In this section, we consider the Cauchy problem on the linearized system with a nonhomogeneous source $h = [h_+, h_-]$:

$$\begin{cases} \partial_t f_\pm + v\cdot\nabla_x f_\pm \pm \sqrt{\mu} v\cdot\nabla_x\phi + Lf_\pm = h_\pm, \\ -\Delta_x\phi = \rho_f = \displaystyle\int_{\mathbf{R}^3}(f_+ - f_-)\sqrt{\mu}\,dv, \quad \phi\to 0 \text{ as } |x|\to\infty, \\ f_\pm|_{t=0} = f_{0,\pm}, \end{cases} \tag{4.1}$$

where $h_\pm(t,x,v)$, $f_{0,\pm} = f_{0,\pm}(x,v)$ are given. Notice that for the nonlinear Vlasov-Poisson-Boltzmann system (1.4) and (1.5), the nonhomogeneous source takes the form of

$$h_\pm = \pm\nabla_x\phi\cdot\nabla_v f_\pm \mp \frac{1}{2}\nabla_x\phi\cdot v f_\pm + \Gamma_\pm(f,f), \tag{4.2}$$

which satisfies the mass conservation laws

$$\langle h_\pm, \sqrt{\mu}\rangle = 0. \tag{4.3}$$

Whenever the linearized system is homogeneous, i.e. $h=0$, the formal solution to the Cauchy problem (4.1) can be written as the mild form

$$f = S(t)f_0, \tag{4.4}$$

where $S(t)$ denotes the solution operator for the Cauchy problem on the linearized system without any source. For an integer $k\geq 0$, set the index σ_k of the time-decay rate by

$$\sigma_k = \frac{3}{4} + \frac{k}{2}.$$

The time-decay property of $S(t)$ is stated in the following

Theorem 4.1. *Recall $w = w(v) = \langle v\rangle^{-\gamma}$. Let $-3<\gamma<-2s$, $1/2\leq s<1$, $l\geq 0$ and $k\geq 0$ be an integer, and let $l_* > \sigma_k(\gamma+2s)/\gamma$. Assume that*

$$\int_{\mathbf{R}^3}\rho_{f_0}\,dx = 0, \quad \int_{\mathbf{R}^3}(1+|x|)|\rho_{f_0}|\,dx < \infty,$$

where

$$\rho_{f_0} = \int_{\mathbf{R}^3}(f_{0,+} - f_{0,-})\sqrt{\mu}\,dv.$$

Then, for $f(t) = S(t)f_0$, it holds that

$$\left\|w^l\nabla_x^k f(t)\right\| + \|\nabla_x^k E(t)\| \lesssim (1+t)^{-\sigma_k}\left(\left\|w^{l+l_*}f_0\right\|_{Z_1} + \|(1+|x|)\rho_{f_0}\|_{L^1} + \left\|w^{l+l_*}\nabla_x^k f_0\right\|\right), \tag{4.5}$$

for any $t\geq 0$.

Before proving Theorem 4.1, we make some preparations as follows. Firstly, for the later use in the general situation, let us consider the linearized system (4.1) with the nonhomogeneous source h satisfying (4.3). As in [20], one can derive the corresponding local conservation laws. In fact, from multiplying the first equation of (4.1) by $\sqrt{\mu}$, $v_j\sqrt{\mu}$ $(j=1,2,3)$ and $\frac{1}{6}\left(|v|^2-3\right)\sqrt{\mu}$ and then integrating them over $v\in\mathbf{R}^3$, one has

$$\partial_t a_\pm + \nabla_x\cdot b = -\nabla_x\cdot\langle v\sqrt{\mu}, (\mathbf{I}_\pm - \mathbf{P}_\pm)f\rangle, \tag{4.6}$$

$$\begin{aligned} &\partial_t\left\{b_j + \langle v_j\sqrt{\mu}, (\mathbf{I}_\pm-\mathbf{P}_\pm)f\rangle\right\} + \partial_j(a_\pm + 2c) \mp E_j \\ &\qquad = -\langle v\cdot\nabla_x(\mathbf{I}_\pm-\mathbf{P}_\pm)f, v_j\sqrt{\mu}\rangle + \langle h_\pm - L_\pm, v_j\sqrt{\mu}\rangle, \end{aligned} \tag{4.7}$$

$$\begin{aligned} \partial_t\left\{c + \frac{1}{6}\left\langle\left(|v|^2-3\right)\sqrt{\mu}, (\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\rangle\right\} + \frac{1}{3}\nabla_x\cdot b = &-\frac{1}{6}\left\langle v\cdot\nabla_x(\mathbf{I}_\pm-\mathbf{P}_\pm)f, (|v|^2-3)\sqrt{\mu}\right\rangle \\ &+ \frac{1}{6}\left\langle h_\pm - L_\pm, (|v|^2-3)\sqrt{\mu}\right\rangle. \end{aligned} \tag{4.8}$$

Moreover, we need the equations of high-order moments. For that, as in [6, 7], we define the high-order moment functions $\Xi(f_\pm) = (\Xi_{jk}(f_\pm))_{3\times 3}$ and $\Pi(f_\pm) = (\Pi_1(f_\pm), \Pi_2(f_\pm), \Pi_3(f_\pm))$ by

$$\Xi_{jk}(f_\pm) = \langle(v_jv_k - 1)\sqrt{\mu}, f_\pm\rangle, \quad \Pi_j(f_\pm) = \frac{1}{10}\langle(|v|^2-5)v_j\sqrt{\mu}, f_\pm\rangle.$$

Then, by taking the velocity integrations of the first equation of (4.1) with respect to the above high-order moments, one has

$$\partial_t\{\Xi_{jj}(\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f)+2c\}+2\partial_j b_j=\Xi_{jj}(r_\pm+h_\pm), \tag{4.9}$$

$$\begin{aligned}\partial_t\Xi_{jk}(\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f)+\partial_j b_k+&\partial_k b_j+\nabla_x\cdot\langle v\sqrt{\mu},(\mathbf{I}_\pm-\mathbf{P}_\pm)f\rangle\\ &=\Xi_{jk}(r_\pm+h_\pm)+\langle h_\pm,\sqrt{\mu}\rangle\,,\quad j\neq k,\end{aligned} \tag{4.10}$$

$$\partial_t\Pi_j(\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f)+\partial_j c=\Pi_j(r_\pm+h_\pm), \tag{4.11}$$

with $r_\pm=-v\cdot\nabla_x\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f+L_\pm f$. Here, notice that we have used (4.6) to derive (4.10).

Consequently, as in [8], by taking the mean value of every two equations with $\pm$ sign for (4.6)-(4.8) and noticing $\langle h_\pm,\sqrt{\mu}\rangle=0$, one has

$$\begin{cases}\partial_t\left(\dfrac{a_++a_-}{2}\right)+\nabla_x\cdot b=0,\\[2mm] \partial_t b_j+\partial_j\left(\dfrac{a_++a_-}{2}+2c\right)+\dfrac{1}{2}\partial_j\Xi_{jk}(\{\mathbf{I}-\mathbf{P}\}f\cdot[1,1])=\dfrac{1}{2}\langle h_++h_-,v_j\sqrt{\mu}\rangle,\\[2mm] \partial_t c+\dfrac{1}{3}\nabla_x\cdot b+\dfrac{5}{6}\displaystyle\sum_{j=1}^3\partial_j\Pi_j(\{\mathbf{I}-\mathbf{P}\}f\cdot[1,1])=\dfrac{1}{12}\left\langle h_++h_-,(|v|^2-3)\sqrt{\mu}\right\rangle,\end{cases} \tag{4.12}$$

for $1\leq j\leq 3$, and similarly, it follows from (4.9)-(4.11) that

$$\begin{cases}\partial_t\left\{\dfrac{1}{2}\Xi_{jk}(\{\mathbf{I}-\mathbf{P}\}f\cdot[1,1])+2c\delta_{jk}\right\}+\partial_j b_k+\partial_k b_j=\dfrac{1}{2}\Xi_{jk}(r_++r_-+h_++h_-),\\[2mm] \dfrac{1}{2}\partial_t\Pi_j(\{\mathbf{I}-\mathbf{P}\}f\cdot[1,1])+\partial_j c=\dfrac{1}{2}\Pi_j(r_++r_-+h_++h_-),\end{cases}$$

for $1\leq j,k\leq 3$, and δ_{jk} denoted the Kronecker delta. Moreover, in order to obtain the dissipation rate of the electric field E, by taking the difference of two equations with $\pm$ sign for (4.6) and (4.7), one has

$$\partial_t(a_+-a_-)+\nabla_x\cdot G=0, \tag{4.13}$$

$$\partial_t G+\nabla(a_+-a_-)-2E+\nabla_x\cdot\Xi((\mathbf{I}-\mathbf{P})f\cdot q_1)=\langle h-Lf,[v\sqrt{\mu},-v\sqrt{\mu}]\rangle, \tag{4.14}$$

where

$$G=\langle v\sqrt{\mu},(\mathbf{I}-\mathbf{P})f\cdot q_1\rangle. \tag{4.15}$$

Notice that the second equation of (4.1) gives

$$\nabla_x\cdot E=a_+-a_-. \tag{4.16}$$

Now, we will prove Theorem 4.1 by using as in [8] the following lemma whose proof is omitted for brevity. To state it, in what follows we let h be identical to zero and let $f(t)=S(t)f_0$ be the solution to the Cauchy problem (4.1) with $h\equiv 0$. Some more notations are given as follows. For two complex vectors $z_1,z_2\in\mathbf{C}^3$, $(z_1|z_2)=z_1\cdot\overline{z_2}$ denotes the dot product in the complex field $\mathbf{C}$, where $\overline{z_2}$ is the complex conjugate of z_2. $\hat{g}(\xi)$ denotes the Fourier transform $\mathcal{F}_x g$ with respect to the variable x, where for notational simplicity we still use ξ to denote the corresponding frequency variable. In fact, one has

Lemma 4.1. *(i) For any $t\geq 0$ and $\xi\in\mathbf{R}^3$, it holds that*

$$\partial_t\left\{|\hat{f}|^2+|\hat{E}|^2\right\}+\lambda|(\mathbf{I}-\mathbf{P})\hat{f}|^2_{N^s_\gamma}\leq 0.$$

(ii) There is a time-frequency interactive functional $\mathcal{E}^{(0)}(t,\xi)$ defined by

$$\begin{aligned}\mathcal{E}^{(0)}_{int}(t,\xi)=&\sum_{j=1}^3\frac{1}{2}\left(\mathrm{i}\xi_j\hat{c}|\Pi_j(\{\mathbf{I}-\mathbf{P}\}\hat{f}\cdot[1,1])\right)\\ &+\kappa_1\sum_{j,k=1}^3\left(\mathrm{i}\xi_j\hat{b}_k+\mathrm{i}\xi_k\hat{b}_j|\frac{1}{2}\Xi_{jk}(\{\mathbf{I}-\mathbf{P}\}\hat{f}\cdot[1,1])+2\hat{c}\delta_{jk}\right)\\ &+\kappa_2\sum_{j=1}^3\left(\mathrm{i}\xi_j\frac{\hat{a}_++\hat{a}_-}{2}|\hat{b}_j\right),\end{aligned}$$

with two properly chosen constants $0<\kappa_2\ll\kappa_1$*, such that for any* $t\geq 0$ *and* $\xi\in\mathbf{R}^3$,

$$\partial_t\Re\frac{\mathcal{E}^{(0)}_{int}(t,\xi)}{1+|\xi|^2}+\frac{\lambda|\xi|^2}{1+|\xi|^2}\left(|\hat{a}_++\hat{a}_-|^2+|\hat{b}|^2+|\hat{c}|^2\right)\lesssim|\{\mathbf{I}-\mathbf{P}\}\hat{f}|^2_{N^s_\gamma}.$$

(iii) There is $0<\kappa_3\ll 1$ *such that for any* $t\geq 0$ *and* $\xi\in\mathbf{R}^3$,

$$\begin{aligned}\frac{\partial_t\Re(\hat{G}|\mathrm{i}\xi(\hat{a}_+-\hat{a}_-))}{1+|\xi|^2}&-\kappa_3\partial_t\Re(\hat{G}|\hat{E})+\lambda|\hat{a}_+-\hat{a}_-|^2+\lambda|\hat{E}|^2\\ &\lesssim|\{\mathbf{I}-\mathbf{P}\}\hat{f}|^2_{N^s_\gamma}.\end{aligned}$$

(iv) Let $[f,E]$ *be the solution to the Cauchy problem* (4.1) *with* $h=0$*. Then there is a time-frequency interactive functional* $\mathcal{E}^{(1)}_{int}(t,\xi)$ *such that*

$$\mathcal{E}^{(1)}_{int}(t,\xi)\sim\left|\widehat{f}\right|^2+|\widehat{E}|^2,$$

and for any $t\geq 0$ *and* $\xi\in\mathbf{R}^3$,

$$\partial_t\mathcal{E}^{(1)}_{int}(t,\xi)+\frac{\lambda|\xi|^2}{1+|\xi|^2}\left\{\left|\widehat{f}\right|^2_{L^2_{\gamma+2s}}+|\widehat{E}|^2\right\}\leq 0. \tag{4.17}$$

The proof of Theorem 4.1. It follows from the first equation of (4.1) that

$$\begin{aligned}&\partial_t(\mathbf{I}-\mathbf{P})\widehat{f}+\mathrm{i}\xi\cdot v(\mathbf{I}-\mathbf{P})\widehat{f}+L(\mathbf{I}-\mathbf{P})\widehat{f}\\ &\quad=-(\mathbf{I}-\mathbf{P})(\widehat{E}\cdot v)\sqrt{\mu}q_1-(\mathbf{I}-\mathbf{P})\left[\mathrm{i}\xi\cdot v\mathbf{P}\widehat{f}\right]+\mathbf{P}\left[\mathrm{i}\xi\cdot v(\mathbf{I}-\mathbf{P})\widehat{f}\right],\end{aligned}$$

which yields that for any $l\geq 0$,

$$\begin{aligned}&\frac{1}{2}\partial_t\left|w^l(\mathbf{I}-\mathbf{P})\widehat{f}\right|_2^2+\Re\left\langle L(\mathbf{I}-\mathbf{P})\widehat{f},w^{2l}(\mathbf{I}-\mathbf{P})\widehat{f}\right\rangle\\ &\quad=-\Re\left\langle(\mathbf{I}-\mathbf{P})(\widehat{E}\cdot v)\sqrt{\mu}q_1+(\mathbf{I}-\mathbf{P})\left[\mathrm{i}\xi\cdot v\mathbf{P}\widehat{f}\right]-\mathbf{P}\left[\mathrm{i}\xi\cdot v(\mathbf{I}-\mathbf{P})\widehat{f}\right],w^{2l}(\mathbf{I}-\mathbf{P})\widehat{f}\right\rangle.\end{aligned} \tag{4.18}$$

Notice that from Lemma 2.1,

$$\mathcal{R}\left\langle L(\mathbf{I}-\mathbf{P})\widehat{f},w^{2l}(\mathbf{I}-\mathbf{P})\widehat{f}\right\rangle\geq\lambda\left|w^l(\mathbf{I}-\mathbf{P})\widehat{f}\right|^2_{L^2_{\gamma+2s}}-C\left|(\mathbf{I}-\mathbf{P})\widehat{f}\right|^2_{L^2_{B_C}},$$

and also the right-hand term of (4.18) can be bounded by

$$C|\widehat{E}|^2+C|\xi|^2\left|\widehat{f}\right|^2_{L^2_{\gamma+2s}}+\eta\left|w^l(\mathbf{I}-\mathbf{P})\widehat{f}\right|^2_{L^2_{\gamma+2s}},$$

for small $\eta>0$.

Plugging the above estimates into (4.18), one has

$$\partial_t\left|w^l(\mathbf{I}-\mathbf{P})\widehat{f}\right|_{L^2}+\lambda\left|w^l(\mathbf{I}-\mathbf{P})\widehat{f}\right|^2_{L^2_{\gamma+2s}}\lesssim|\widehat{E}|^2+|\xi|^2\left|\widehat{f}\right|^2_{L^2_{\gamma+2s}}+\left|(\mathbf{I}-\mathbf{P})\widehat{f}\right|^2_{L^2_{B_C}}. \tag{4.19}$$

In a similar way, starting with the first equation of (4.1), the direct weighted estimates also give

$$\frac{1}{2}\partial_t\left|w^l\widehat{f}\right|^2_{L^2}+\lambda\left|w^l\widehat{f}\right|^2_{L^2_{\gamma+2s}}\lesssim|\widehat{E}|^2+\left|\widehat{f}\right|^2_{L^2_{B_C}}. \tag{4.20}$$

In the case when $|\xi|\geq 1$ which implies $|\xi|^2/(1+|\xi|^2)\geq 1/2$, one can combine (4.17) and (4.20) to obtain

$$\partial_t\mathscr{E}^1_l(t,\xi)+\lambda\left\{\left|w^l\widehat{f}\right|^2_{L^2_{\gamma+2s}}+|\widehat{E}|^2\right\}\chi_{|\xi|\geq 1}\leq 0, \tag{4.21}$$

where $\mathscr{E}^1_l(t,\xi)$ is defined by

$$\mathscr{E}^1_l(t,\xi)=\left\{\mathcal{E}^{(1)}_{int}(t,\xi)+\kappa_4\left|w^l\widehat{f}\right|_2^2\right\}\chi_{|\xi|\geq 1}$$

for a constant $\kappa_4>0$ small enough. In the case when $|\xi|\le 1$ which implies $|\xi|^2/(1+|\xi|^2)\ge|\xi|^2/2$, one can combine (4.17) and (4.19) on $|\xi|\le 1$ to obtain

$$\partial_t \mathscr{E}_l^0(t,\xi)+\lambda|\xi|^2\left\{\left|w^l\widehat{f}\right|^2_{L^2_{\gamma+2s}}+|\widehat{E}|^2\right\}\chi_{|\xi|\le 1}\le 0, \tag{4.22}$$

where $\mathscr{E}_l^0(t,\xi)$ is defined by

$$\mathscr{E}_l^0(t,\xi)=\left\{\mathcal{E}^{(1)}_{int}(t,\xi)+\kappa_5\left|w^l(\mathbf{I}-\mathbf{P})\widehat{f}\right|_2^2\right\}\chi_{|\xi|\le 1}$$

for a constant $\kappa_5>0$ small enough. Therefore, for $l\ge 0$, by introducing

$$\mathscr{E}_l(t,\xi)=\mathscr{E}_l^0(t,\xi)+\mathscr{E}_l^1(t,\xi)\sim\left|w^l\hat{f}(t,\xi,v)\right|_2^2+|\widehat{E}|^2,$$

it follows from (4.21) and (4.22) that

$$\partial_t\mathscr{E}_l(t,\xi)+\lambda\frac{|\xi|^2}{1+|\xi|^2}\left\{\left|w^l\widehat{f}\right|^2_{L^2_{\gamma+2s}}+|\widehat{E}|^2\right\}\le 0,$$

that is,

$$\partial_t\mathscr{E}_l(t,\xi)+\lambda\frac{|\xi|^2}{1+|\xi|^2}\left\{\left|w^{l-\frac{\gamma+2s}{2\gamma}}\widehat{f}\right|^2_{L^2}+|\widehat{E}|^2\right\}\le 0. \tag{4.23}$$

Now, basing on the above estimate (4.23) for any $l\ge 0$, one can use the trick in either [29] or [11] to deduce the desired estimate (4.5). We here use the trick in [29] to deal with the velocity degeneration. In fact, for $j>0$, it follows from the Hölder inequality that

$$\mathscr{E}_l(t,\xi)\lesssim\mathscr{E}^{j/(j+1)}_{l-\frac{\gamma+2s}{2\gamma}}(t,\xi)\mathscr{E}^{1/(j+1)}_{l+j\frac{\gamma+2s}{2\gamma}}(t,\xi)\lesssim\left\{\left|w^{l-\frac{\gamma+2s}{2\gamma}}\widehat{f}\right|^2_{L^2}+|\widehat{E}|^2\right\}^{j/(j+1)}\mathscr{E}^{1/(j+1)}_{l+j\frac{\gamma+2s}{2\gamma}}(t,\xi),$$

which implies

$$\mathscr{E}_l^{(j+1)/j}(t,\xi)\lesssim\left\{\left|w^{l-\frac{\gamma+2s}{2\gamma}}\widehat{f}\right|^2_{L^2}+|\widehat{E}|^2\right\}\mathscr{E}^{1/j}_{l+j\frac{\gamma+2s}{2\gamma}}(t,\xi)\lesssim\left\{\left|w^{l-\frac{\gamma+2s}{2\gamma}}\widehat{f}\right|^2_{L^2}+|\widehat{E}|^2\right\}\mathscr{E}^{1/j}_{l+j\frac{\gamma+2s}{2\gamma}}(0,\xi).$$

Then, (4.23) together with the above estimate give

$$\partial_t\mathscr{E}_l(t,\xi)+\lambda\frac{|\xi|^2}{1+|\xi|^2}\mathscr{E}_l^{(j+1)/j}(t,\xi)\mathscr{E}^{-1/j}_{l+j\frac{\gamma+2s}{2\gamma}}(0,\xi)\le 0.$$

By solving the inequality, one has

$$j\mathscr{E}_l^{-1/j}(0,\xi)-j\mathscr{E}_l^{-1/j}(t,\xi)\le-\lambda t\frac{|\xi|^2}{1+|\xi|^2}\mathscr{E}^{-1/j}_{l+j\frac{\gamma+2s}{2\gamma}}(0,\xi).$$

Therefore, for any $l\ge 0$ and $j>0$, it holds that

$$\mathscr{E}_l(t,\xi)\lesssim\mathscr{E}_{l+j\frac{\gamma+2s}{2\gamma}}(0,\xi)\left(1+\frac{\lambda t}{j}\cdot\frac{|\xi|^2}{1+|\xi|^2}\right)^{-j}. \tag{4.24}$$

Here we have used the fact that $\mathscr{E}_l(0,\xi)\lesssim\mathscr{E}_{l+j\frac{\gamma+2s}{2\gamma}}(0,\xi)$. By integrating (4.24) over $|\xi|\ge 1$,

$$\begin{aligned}\int_{|\xi|\ge 1}|\xi|^{2k}\mathscr{E}_l(t,\xi)d\xi\lesssim&\left(1+\frac{\lambda t}{2j}\right)^{-j}\int_{|\xi|\ge 1}|\xi|^{2k}\mathscr{E}_{l+j\frac{\gamma+2s}{2\gamma}}(0,\xi)d\xi\\ \lesssim&(1+t)^{-j}\left\|w^{l+j\frac{\gamma+2s}{2\gamma}}\nabla_x^k f_0\right\|^2,\end{aligned}$$

where we have used

$$\int_{|\xi|\ge 1}|\xi|^{2k}|\hat{E}_0|^2d\xi=\int_{|\xi|\ge 1}|\xi|^{2k-2}|\cdot|\widehat{\rho_{f_0}}|^2d\xi\lesssim\int_{|\xi|\ge 1}|\xi|^{2k}|\cdot|\widehat{\rho_{f_0}}|^2d\xi\lesssim\left\|\nabla_x^k f_0\right\|^2.$$

On the other hand, over $|\xi|\leq 1$, one can let $j>2\sigma_k$ so that

$$\begin{aligned}\int_{|\xi|<1}|\xi|^{2k}\mathscr{E}_l(t,\xi)d\xi\leq&\int_{|\xi|<1}\left(1+\frac{\lambda t}{j}\cdot\frac{|\xi|^2}{1+|\xi|^2}\right)^{-j}|\xi|^{2k}\mathscr{E}_{l+j\frac{\gamma+2s}{2\gamma}}(0,\xi)d\xi\\ \lesssim&(1+t)^{-2\sigma_k}\left\|\mathscr{E}_{l+j\frac{\gamma+2s}{2\gamma}}(0,\xi)\right\|_{L^\infty_\xi}\\ \lesssim&(1+t)^{-2\sigma_k}\left\{\left\|w^{l+j\frac{\gamma+2s}{2\gamma}}f_0\right\|_{Z^1}^2+\|(1+|x|)\rho_0\|_{L^1}^2\right\},\end{aligned}$$

where the fact that

$$\|\hat{E}_0\|_{L^\infty_\xi}=\left\||\xi|^{-1}\widehat{\rho_{f_0}}\right\|_{L^\infty_\xi}\lesssim\|(1+|x|)\rho_{f_0}\|_{L^1}\tag{4.25}$$

has been used. Then, the desired estimate (4.5) follows by taking $\ell_*=j\frac{\gamma+2s}{2\gamma}$ with $j>2\sigma_k$. This completes the proof of Theorem 4.1. □

5. Macroscopic dissipation

This section is concerned with the analysis of the macroscopic dissipation basing on the linearized system (4.1) with the nonhomogeneous source h satisfying (4.3), which will be applied in the next section to the energy estimates on the nonlinear Vlasov-Poisson-Boltzmann system (1.4) and (1.5).

Lemma 5.1. *Let $[f,E]$ be the solution to the Cauchy problem (4.1) with the nonhomogenous term h satisfying $\langle h,\sqrt{\mu}\rangle=0$. Let $K\geq 8$ be an integer. Then, there are two time-frequency interaction functionals $\mathscr{E}^K_{int}(t)$ and $\mathscr{E}^{K,h}_{int}(t)$ satisfying*

$$\begin{aligned}|\mathscr{E}^K_{int}(t)|\lesssim&\sum_{|\alpha|\leq K}\|\partial^\alpha f\|^2+\sum_{|\alpha|\leq K-1}\|\partial^\alpha E\|^2,\\ |\mathscr{E}^{K,h}_{int}(t)|\lesssim&\sum_{1\leq|\alpha|\leq K}\|\partial^\alpha\mathbf{P}f\|^2+\sum_{|\alpha|\leq K}\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\|^2+\sum_{|\alpha|\leq K-1}\|\partial^\alpha E\|^2,\end{aligned}\tag{5.1}$$

such that for any $t\geq 0$,

$$\begin{aligned}&\partial_t\mathscr{E}^K_{int}(t)+\lambda\sum_{|\alpha|\leq K-1}\|\partial^\alpha\nabla_x(a_\pm,b,c)\|^2+\lambda\|(a_+-a_-)\|^2+\lambda\|E\|^2\\ &\lesssim\sum_{|\alpha|\leq K}\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\|^2_{N^s_\gamma}+\sum_{|\alpha|\leq K-1}\|\zeta(v)\partial^\alpha h\|^2,\end{aligned}\tag{5.2}$$

and

$$\begin{aligned}&\partial_t\mathscr{E}^{K,h}_{int}(t)+\lambda\sum_{1\leq|\alpha|\leq K-1}\|\partial^\alpha\nabla_x(a_\pm,b,c)\|^2+\lambda\|\nabla_x(a_+-a_-)\|^2+\lambda\sum_{|\alpha|\leq K-1}\|\partial^\alpha E\|^2\\ &\lesssim\sum_{|\alpha|\leq K}\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\|^2_{N^s_\gamma}+\sum_{|\alpha|\leq K-1}\|\zeta(v)\partial^\alpha h\|^2,\end{aligned}\tag{5.3}$$

where $\zeta(v)$ is defined by (2.7).

Proof. Take α with $|\alpha|\leq K$. By denoting

$$\begin{aligned}\mathcal{E}^{(\alpha)}_{int}(t)=&\frac{1}{2}\left(\nabla_x\partial^\alpha c,\Pi(\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\cdot[1,1])\right)\\ &+\kappa_1\sum_{j,k=1}^3\left(\partial_j\partial^\alpha b_k+\partial_k\partial^\alpha b_j,\frac{1}{2}\Xi_{jk}(\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\cdot[1,1])+2\partial^\alpha c\delta_{jk}\right)\\ &+\kappa_2\left(\nabla_x\frac{\partial^\alpha a_++\partial^\alpha a_-}{2},\partial^\alpha b\right),\end{aligned}\tag{5.4}$$

one can use the same argument as in [6, Lemma 4.1] to obtain

$$\frac{d}{dt}\mathcal{E}^{(\alpha)}_{int}(t)+\lambda\|\partial^\alpha\nabla((a_++a_-),b,c)\|^2\lesssim\sum_{|\alpha'|\leq|\alpha|+1}\|\partial^{\alpha'}(\mathbf{I}-\mathbf{P})f\|^2_{N^s_\gamma}+\|\zeta(v)\partial^\alpha h\|^2,\tag{5.5}$$

and similarly, as for obtaining (iii) in Lemma 4.1, one has

$$\begin{aligned}\frac{d}{dt}(\partial^\alpha G, \nabla_x\partial^\alpha(a_+-a_-)) + \lambda \sum_{|\alpha|\le|\alpha'|\le|\alpha|+1}\|\partial^{\alpha'}(a_+-a_-)\|^2 \\ \lesssim \sum_{|\alpha'|\le|\alpha|+1}\|\partial^{\alpha'}(\mathbf{I}-\mathbf{P})f\|^2_{N^s_\gamma} + \|\zeta(v)\partial^\alpha h\|^2 .\end{aligned} \tag{5.6}$$

To include the zero-order term $\|E\|$ in the dissipation, from applying ∂^α to (4.14) and then taking the inner product with $\partial^\alpha E$ over $\mathbf{R}^3\times\mathbf{R}^3$, we have

$$\begin{aligned} 2\|\partial^\alpha E\|^2 \le & (\partial_t\partial^\alpha G, \partial^\alpha E) + |(\partial^\alpha\nabla_x\cdot\Xi((\mathbf{I}-\mathbf{P})f\cdot q_1),\partial^\alpha E)| + (\nabla_x\partial^\alpha(a_+-a_-),\partial^\alpha E)\\ & + |(\partial^\alpha\langle h-Lf,[v\sqrt{\mu},-v\sqrt{\mu}]\rangle,\partial^\alpha E)|\\ \le & \frac{d}{dt}(\partial^\alpha G,\partial^\alpha E) - (\partial^\alpha G,\partial_t\partial^\alpha E) + \epsilon\|\partial^\alpha E\|^2\\ & + \frac{C}{\epsilon}\left\{\|\partial^\alpha\nabla_x(\mathbf{I}-\mathbf{P})f\|^2 + \|\partial^\alpha\nabla_x(a_+-a_-)\|^2 + \|\partial^\alpha\langle h-Lf,[v\sqrt{\mu},-v\sqrt{\mu}]\rangle\|^2\right\},\end{aligned} \tag{5.7}$$

where in the last inequality we have used integrations by part in the t-variable and the Cauchy-Schwarz inequality with ϵ. Notice that from (4.13) and (4.16),

$$-(\partial^\alpha G,\partial_t\partial^\alpha E) = (\partial^\alpha G,\partial^\alpha\nabla_x\Delta_x^{-1}\nabla_x\cdot G) \lesssim \|\partial^\alpha G\|^2 \lesssim \|\partial^\alpha(\mathbf{I}-\mathbf{P})f\|^2_{N^s_\gamma}. \tag{5.8}$$

Therefore, (5.7) together with (5.8) imply

$$-\frac{d}{dt}(\partial^\alpha G,\partial^\alpha E) + \|\partial^\alpha E\|^2 \lesssim \sum_{|\alpha'|\le|\alpha|+1}\|\partial^{\alpha'}(\mathbf{I}-\mathbf{P})f\|^2_{N^s_\gamma} + \|\partial^\alpha\nabla_x(a_+-a_-)\|^2 + \|\zeta(v)\partial^\alpha h\|^2 . \tag{5.9}$$

Now, we define

$$\begin{aligned}\mathscr{E}^{K,h}_{int}(t) =& \frac{1}{2}\sum_{1\le|\alpha|\le K-1}(\nabla_x\partial^\alpha c, \Pi(\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\cdot[1,1]))\\ & + \kappa_1\sum_{1\le|\alpha|\le K-1}\sum_{j,k=1}^3\left(\partial_j\partial^\alpha b_k + \partial_k\partial^\alpha b_j, \frac{1}{2}\Xi_{jk}(\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\cdot[1,1]) + 2\partial^\alpha c\delta_{jk}\right)\\ & + \kappa_2\sum_{1\le|\alpha|\le K-1}\left(\nabla_x\frac{\partial^\alpha a_+ + \partial^\alpha a_-}{2},\partial^\alpha b\right) + \sum_{1\le|\alpha|\le K-1}(\partial^\alpha G,\nabla_x\partial^\alpha(a_+-a_-))\\ & - \kappa_6\sum_{|\alpha|\le K-1}(\partial^\alpha G,\partial^\alpha E),\end{aligned}$$

for a suitably chosen constant $0<\kappa_6\ll 1$. It is straightforward to see that the second equation of (5.1) holds true and also (5.3) follows by taking the sum of (5.5), (5.6) and (5.9) $\times\,\kappa_6$. In a similar way, basing on the obtained estimates, $\mathscr{E}^K_{int}(t)$ can be constructed to satisfy both the first equation of (5.1) and (5.2). This completes the proof of Lemma 5.1. □

6. The a priori estimates

In this section, we are going to deduce the uniform-in-time a priori estimates on the solution to the Cauchy problem (1.4)-(1.6) of the Vlasov-Poisson-Boltzmann system. For this purpose, we define the following time-weighted energy norm $X(t)$ by

$$X(t) = \sup_{0\le\tau\le t}\mathcal{E}_{l_0+l_1}(\tau) + \sup_{0\le\tau\le t}(1+\tau)^{\frac{3}{2}}\mathcal{E}_{l_0}(\tau) + \sup_{0\le\tau\le t}(1+\tau)^{\frac{3}{2}+p}\mathcal{E}^h_{l_0}(\tau), \tag{6.1}$$

with all involved parameters fixed to satisfy

$$-3<\gamma<-2s,\ \frac{1}{2}\le s<1,\ l_0\ge 0,\ K\ge 8,\ \frac{1}{2}<p<1, \tag{6.2}$$

and

$$l_1 = \frac{5}{4(1-p)}\frac{\gamma+2s}{\gamma}, \tag{6.3}$$

where the construction of the temporal energy functionals $\mathcal{E}_l(t)$ and $\mathcal{E}_l^h(t)$ will be given in the following two lemmas. Suppose that the Cauchy problem (1.4)-(1.6) admits a smooth solution $f(t,x,v)$ over $0\leq t\leq T$ for $0<T\leq\infty$, and also the solution $f(t,x,v)$ satisfies

$$\sup_{0\leq t\leq T} X(t)\leq \delta_0, \tag{6.4}$$

where $\delta_0>0$ is a suitably small constant.

Lemma 6.1. *For any l with $0\leq l\leq l_0+l_1$, there is $\mathcal{E}_l(t)$ satisfying (1.13) such that*

$$\frac{d}{dt}\mathcal{E}_l(t)+\lambda\mathcal{D}_l(t)\lesssim \|\partial_t\phi\|_{L^\infty}\mathcal{E}_l(t) \tag{6.5}$$

holds for any $0\leq t\leq T$, where $\mathcal{D}_l(t)$ is defined by (1.15).

Proof. It is divided by three steps. Notice that (1.4) can be rewritten as

$$\left[\partial_t+v_i\partial^{e_i}\mp\partial^{e_i}\phi\partial_{e_i}\right]f_\pm\pm\frac{1}{2}(\partial^{e_i}\phi v_i)f_\pm\pm(\partial^{e_i}\phi v_i)\sqrt{\mu}+L_\pm f=\Gamma_\pm(f,f). \tag{6.6}$$

Step 1. Energy estimates without any weight: Applying ∂^α with $|\alpha|\leq K$ to (6.6) and taking the inner product with $e^{\pm\phi}\partial^\alpha f_\pm$ over $\mathbf{R}^3\times\mathbf{R}^3$, one has

$$\begin{aligned}
&\left(\partial_t\partial^\alpha f_\pm,e^{\pm\phi}\partial^\alpha f_\pm\right)+\left(v_i\partial^{e_i+\alpha}f_\pm,e^{\pm\phi}\partial^\alpha f_\pm\right)\pm\frac{1}{2}\left((\partial^{e_i}\phi v_i)\partial^\alpha f_\pm,e^{\pm\phi}\partial^\alpha f_\pm\right)\\
&\qquad\pm\left((\partial^{e_i+\alpha}\phi v_i)\sqrt{\mu},\partial^\alpha f_\pm\right)+\left(L_\pm\partial^\alpha f,\partial^\alpha f_\pm\right)\\
&\quad=\mp\chi_{|\alpha|}\sum_{1\leq|\alpha_1|\leq|\alpha|}C_\alpha^{\alpha_1}\frac{1}{2}\left((\partial^{e_i+\alpha_1}\phi v_i)\partial^{\alpha-\alpha_1}f_\pm,e^{\pm\phi}\partial^\alpha f_\pm\right)\\
&\qquad\pm\chi_{|\alpha|}\sum_{1\leq|\alpha_1|\leq|\alpha|}C_\alpha^{\alpha_1}\left((\partial^{e_i+\alpha_1}\phi\partial_{e_i}^{\alpha-\alpha_1}f_\pm,e^{\pm\phi}\partial^\alpha f_\pm\right)\\
&\qquad\mp\left((\partial^{e_i+\alpha}\phi v_i)\sqrt{\mu},\left(1-e^{\pm\phi}\right)\partial^\alpha f_\pm\right)\\
&\qquad+\left(L_\pm\partial^\alpha f,\left(1-e^{\pm\phi}\right)\partial^\alpha f_\pm\right)+\left(\partial^\alpha\Gamma_\pm(f,f),e^{\pm\phi}\partial^\alpha f_\pm\right),
\end{aligned}\tag{6.7}$$

where $\chi_{|\alpha|}=1$ if $|\alpha|>0$, and $\chi_{|\alpha|}=0$ if $|\alpha|=0$. Now we sum the above equations with $\pm$ and estimate the resulting equations term by term.

The first term on the left hand side of (6.7) is just

$$\frac{1}{2}\frac{d}{dt}\sum_\pm\left\|e^{\frac{\pm\phi}{2}}\partial^\alpha f_\pm\right\|\mp\sum_\pm\frac{1}{2}(\partial_t\phi\partial^\alpha f_\pm,\partial^\alpha f_\pm).$$

As in [21], the extra factor $e^{\pm\phi}$ is designed to treat the second term. In fact, from integration by parts,

$$\left(v_i\partial^{e_i+\alpha}f_\pm,e^{\pm\phi}\partial^\alpha f_\pm\right)\pm\frac{1}{2}\left((\partial^{e_i}\phi v_i)\partial^\alpha f_\pm,e^{\pm\phi}\partial^\alpha f_\pm\right)=0.$$

Recalling (1.2), (4.13) and (4.16), one has

$$\sum_\pm\pm\left((\partial^{e_i+\alpha}\phi v_i)\sqrt{\mu},\partial^\alpha f_\pm\right)=\frac{1}{2}\frac{d}{dt}\|\partial^\alpha\nabla_x\phi\|^2.$$

Notice that by (2.1),

$$\sum_\pm(L_\pm\partial^\alpha f,\partial^\alpha f_\pm)\geq\lambda\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\|^2_{N^s_\gamma},$$

and by Lemma 3.4 and Lemma 3.6, the first and second terms on the right hand side of (6.7) are bounded by $C\sqrt{\mathcal{E}_l(t)}\mathcal{D}_l(t)$.

We denote the third term on the right hand side of (6.7) as J_5. Notice

$$\sup_{x\in\mathbf{R}^3}|1-e^{\pm\phi}|\lesssim\|\phi\|_{L^\infty}\lesssim\|\nabla_x\phi\|_{H^1}\lesssim\delta_0. \tag{6.8}$$

For $\alpha=0$, by Cauchy-Schwarz and Sobolev inequalities,

$$J_5\lesssim\|\nabla_x\phi\|^2\|\mu^{\frac{1}{4}}f\|_{H^2}\lesssim\sqrt{\mathcal{E}_l(t)}\mathcal{D}_l(t),$$

while for $|\alpha|\geq 1$,

$$J_5 \lesssim \|\nabla_x\phi\|_{H^2}\sum_{1\leq|\alpha|\leq K}\left\{\|\partial^\alpha\nabla_x\phi\|^2\left\|\mu^{\frac{1}{4}}\partial^\alpha f\right\|\right\}\lesssim\sqrt{\mathcal{E}_l(t)}\mathcal{D}_l(t).$$

For the fourth term on the right hand side of (6.7), recalling (1.10) and (1.8), and then applying Lemma 3.1, one has

$$\begin{aligned}\left|\left(L_\pm\partial^\alpha f,\left(1-e^{\pm\phi}\right)\partial^\alpha f_\pm\right)\right| &= \left|\left(L_\pm\partial^\alpha(\mathbf{I}_\pm-\mathbf{P}_\pm)f,\left(1-e^{\pm\phi}\right)\partial^\alpha f_\pm\right)\right|\\ &\lesssim\int_{\mathbf{R}^3}|\partial^\alpha(\mathbf{I}-\mathbf{P})f|_{N^s_\gamma}|\partial^\alpha f|_{N^s_\gamma}|\nabla_x\phi|dx\\ &\lesssim\|\nabla_x\phi\|_{H^2}\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\|_{N^s_\gamma}\|\partial^\alpha f\|_{N^s_\gamma}\lesssim\sqrt{\mathcal{E}_l(t)}\mathcal{D}_l(t).\end{aligned}\tag{6.9}$$

For the fifth term on the right hand side of (6.7), by Lemma 3.1,

$$\sum_\pm\left(\partial^\alpha\Gamma_\pm(f,f),e^{\pm\phi}\partial^\alpha f_\pm\right)\lesssim\sqrt{\mathcal{E}_l(t)}\mathcal{D}_l(t).$$

Furthermore, one can apply the estimate (5.2) in Lemma 5.1, where for the term involving h defined by (4.2), through splitting f into $\mathbf{P}f+(\mathbf{I}-\mathbf{P})f$ and using Lemma 3.2, it can be bounded as

$$\sum_{|\alpha|\leq K-1}\|\zeta(v)\partial^\alpha h\|^2\lesssim\mathcal{E}_l(t)\mathcal{D}_l(t).\tag{6.10}$$

Therefore, by combing (5.2) and all the estimates above, we conclude in this step that

$$\begin{aligned}&\frac{d}{dt}\left\{\sum_{|\alpha|\leq K}\sum_\pm\left\|e^{\frac{\pm\phi}{2}}\partial^\alpha f_\pm\right\|^2+\sum_{|\alpha|\leq K}\|\partial^\alpha\nabla_x\phi\|^2+\kappa\mathscr{E}^K_{int}(t)\right\}\\ &\quad+\lambda\sum_{|\alpha|\leq K-1}\|\partial^\alpha\nabla(a_\pm,b,c)\|^2+\lambda\|(a_+-a_-)\|^2+\lambda\|\nabla_x\phi\|^2+\lambda\sum_{|\alpha|\leq K}\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\|^2_{N^s_\gamma}\\ &\lesssim\|\partial_t\phi\|_{L^\infty}\sum_{|\alpha|\leq K}\sum_\pm\|\partial^\alpha f_\pm\|^2+\left(\sqrt{\mathcal{E}_l(t)}+\mathcal{E}_l(t)\right)\mathcal{D}_l(t),\end{aligned}\tag{6.11}$$

where the constant $\kappa>0$ is small enough.

Step 2. Energy estimates with weight function $w_l(\alpha,\beta)$:

Step 2.1 One can rewrite (6.6) as

$$\begin{aligned}&\left[\partial_t+v_i\partial^{e_i}\mp\partial^{e_i}\phi\partial_{e_i}\right](\mathbf{I}_\pm-\mathbf{P}_\pm)f\pm\frac{1}{2}(\partial^{e_i}\phi v_i)(\mathbf{I}_\pm-\mathbf{P}_\pm)f+L_\pm(\mathbf{I}_\pm-\mathbf{P}_\pm)f\\ &=-\left[\partial_t+v_i\partial^{e_i}\mp\partial^{e_i}\phi\partial_{e_i}\right]\mathbf{P}_\pm f\mp\frac{1}{2}(\partial^{e_i}\phi v_i)\mathbf{P}_\pm f\mp(\partial^{e_i}\phi v_i)\sqrt{\mu}+\Gamma_\pm(f,f).\end{aligned}\tag{6.12}$$

By multiplying (6.12) by $e^{\pm\phi}w_l^2(0,0)(\mathbf{I}_\pm - \mathbf{P}_\pm)f$ and taking the integration over $\mathbf{R}^3\times\mathbf{R}^3$, one has

$$
\begin{aligned}
&\frac{1}{2}\sum_{\pm}\frac{d}{dt}\left\|e^{\frac{\pm\phi}{2}}w_l(0,0)(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|^2+\sum_{\pm}\left(L_\pm(\mathbf{I}_\pm-\mathbf{P}_\pm)f,w_l^2(0,0)(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right)\\
&\qquad-\frac{1}{2}\sum_{\pm}\left(\pm\partial_t\phi(\mathbf{I}_\pm-\mathbf{P}_\pm)f,e^{\pm\phi}w_l^2(0,0)(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right)\\
&\quad=\sum_{\pm}\left(L_\pm(\mathbf{I}_\pm-\mathbf{P}_\pm)f,\left(1-e^{\pm\phi}\right)w_l^2(0,0)(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right)\\
&\qquad+\sum_{\pm}\left(\pm\partial^{e_i}\phi\partial_{e_i}(\mathbf{I}_\pm-\mathbf{P}_\pm)f,e^{\pm\phi}w_l^2(0,0)(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right)\\
&\qquad\mp\left(\frac{1}{2}(\partial^{e_i}\phi v_i)\mathbf{P}_\pm f,e^{\pm\phi}w_l^2(0,0)(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right)\\
&\qquad\mp\sum_{\pm}\left((\partial^{e_i}\phi v_i)\sqrt{\mu},e^{\pm\phi}w_l^2(0,0)(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right)\\
&\qquad-\sum_{\pm}\left(\left[\partial_t+v_i\partial^{e_i}\mp\partial^{e_i}\phi\partial_{e_i}\right]\mathbf{P}_\pm f,e^{\pm\phi}w_l^2(0,0)(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right)\\
&\qquad+\sum_{\pm}\left(\Gamma_\pm(f,f),e^{\pm\phi}w_l^2(0,0)(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right),
\end{aligned}
\tag{6.13}
$$

where we have used the identity

$$
\left(v_i\partial^{e_i}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\pm\frac{1}{2}(\partial^{e_i}\phi v_i)(\mathbf{I}_\pm-\mathbf{P}_\pm)f,e^{\pm\phi}w_l^2(0,0)(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right)=0.
$$

Now we further estimate (6.13) term by term. The third term on the left hand side of (6.13) is bounded by

$$
\|\partial_t\phi\|_{L^\infty}\left\|e^{\frac{\pm\phi}{2}}w_l(0,0)(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|^2.
$$

For the left-hand second term, it follows from Lemma 2.1 that

$$
\sum_{\pm}\left(L_\pm(\mathbf{I}_\pm-\mathbf{P}_\pm)f,w_l^2(0,0)(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right)\geq\lambda\left\|w_l(0,0)(\mathbf{I}-\mathbf{P})f\right\|^2_{N^s_\gamma}-C\|(\mathbf{I}-\mathbf{P})f\|_{L^2(B_C)}.
$$

Similar to estimating (6.9), the first term on the right hand side of (6.13) is bounded by

$$
\|\nabla_x\phi\|_{H^2}\left\|w_l(0,0)(\mathbf{I}-\mathbf{P})f\right\|^2_{N^s_\gamma}.
$$

The right-hand second and third terms of (6.13) are bounded by $C\sqrt{\mathcal{E}_l(t)}\mathcal{D}_l(t)$, where we have used the velocity integration by parts for the second term and (1.10) for the third term, as well as the a priori assumption (6.4). From Cauchy-Schwarz inequality with η and the local conservation laws (4.12) and (4.13), the right-hand fourth and fifth terms are bounded by

$$
\begin{aligned}
&\eta\left\|w_l(0,0)(\mathbf{I}-\mathbf{P})f\right\|^2_{N^s_\gamma}\\
&\quad+C\left\{\|\nabla_x\phi\|^2+\|\nabla_x(a_\pm,b,c)\|^2+\sum_{|\alpha|\leq1}\|(\mathbf{I}-\mathbf{P})f\|^2_{N^s_\gamma}+\mathcal{E}_l(t)\mathcal{D}_l(t)\right\}.
\end{aligned}
$$

For the nonlinear term $\Gamma_\pm$, Lemma 3.1 implies

$$
\sum_{\pm}\left(\Gamma_\pm(f,f),e^{\pm\phi}w_l^2(0,0)(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right)\lesssim\sqrt{\mathcal{E}_l(t)}\mathcal{D}_l(t).
$$

Plugging all the above estimates into (6.13) and fixing a properly small constant $\eta > 0$ yield

$$
\begin{aligned}
&\sum_{\pm}\frac{d}{dt}\left\|e^{\frac{\pm\phi}{2}}w_l(0,0)(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|^2+\lambda\left\|w_l(0,0)(\mathbf{I}-\mathbf{P})f\right\|^2_{N^s_\gamma}\\
&\lesssim\|\partial_t\phi\|_{L^\infty}\sum_{\pm}\left\|e^{\frac{\pm\phi}{2}}w_l(0,0)(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|^2\\
&\quad+\|\nabla_x\phi\|^2+\|\nabla_x(a_\pm,b,c)\|^2+\sum_{|\alpha|\le 1}\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\|^2_{N^s_\gamma}+\left\{\sqrt{\mathcal{E}_l(t)}+\mathcal{E}_l(t)\right\}\mathcal{D}_l(t).
\end{aligned}
\tag{6.14}
$$

Step 2.2. For the weighted estimate on the spatial derivatives, we start with (6.6). In fact, take $1 \le |\alpha| \le N$. By applying ∂^α to (6.6) and taking the inner product with $e^{\pm\phi}w_l^2(\alpha,0)\partial^\alpha f_\pm$ over $\mathbf{R}^3\times\mathbf{R}^3$, one has

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{\pm}\left\|e^{\frac{\pm\phi}{2}}w_l(\alpha,0)\partial^\alpha f_\pm\right\|+\lambda\left\|w_l(\alpha,0)\partial^\alpha f\right\|^2_{N^s_\gamma}-C\|\partial^\alpha f\|^2_{L^2_{B_C}}\\
&\lesssim\|\partial_t\phi\|_{L^\infty}\sum_{\pm}\left\|e^{\frac{\pm\phi}{2}}w_l(\alpha,0)\partial^\alpha f_\pm\right\|^2+\sum_{\pm}\sum_{1\le|\alpha_1|\le|\alpha|}\frac{1}{2}C_\alpha^{\alpha_1}\left(\mp(\partial^{e_i+\alpha_1}\phi v_i)\partial^{\alpha-\alpha_1}f_\pm,e^{\pm\phi}w_l^2(\alpha,0)\partial^\alpha f_\pm\right)\\
&\quad+\sum_{\pm}\sum_{|\alpha_1|\le|\alpha|}C_\alpha^{\alpha_1}\left(\pm(\partial^{e_i+\alpha_1}\phi\partial^{\alpha-\alpha_1}_{e_i}f_\pm,e^{\pm\phi}w_l^2(\alpha,0)\partial^\alpha f_\pm\right)+\sum_{\pm}\left(\mp(\partial^{e_i+\alpha}\phi v_i)\sqrt{\mu},e^{\pm\phi}w_l^2(\alpha,0)\partial^\alpha f_\pm\right)\\
&\quad+\sum_{\pm}\left(L_\pm\partial^\alpha f,\left(1-e^{\pm\phi}\right)w_l^2(\alpha,0)\partial^\alpha f_\pm\right)+\left(\partial^\alpha\Gamma_\pm(f,f),e^{\pm\phi}w_l^2(\alpha,0)\partial^\alpha f_\pm\right),
\end{aligned}
\tag{6.15}
$$

where we have used Lemma 3.1 and the fact that

$$
\left(v_i\partial^{e_i+\alpha}f_\pm,e^{\pm\phi}w_l^2(\alpha,0)\partial^\alpha f_\pm\right)\pm\frac{1}{2}\left((\partial^{e_i}\phi v_i)\partial^\alpha f_\pm,e^{\pm\phi}w_l^2(\alpha,0)\partial^\alpha f_\pm\right)=0.
$$

From Lemma 3.4 and Lemma 3.6, the right-hand second and third terms of (6.15) are bounded by $\sqrt{\mathcal{E}_l(t)}\mathcal{D}_l(t)$. For the right-hand fourth term of (6.15), by Cauchy-Schwarz inequality with η, one has

$$
\sum_{\pm}\left((\partial^{e_i+\alpha}\phi v_i)\sqrt{\mu},e^{\pm\phi}w_l^2(\alpha,0)\partial^\alpha f_\pm\right)\lesssim\eta\left\|w_l(\alpha,0)\partial^\alpha f\right\|^2_{N^s_\gamma}+C\left\|\partial^{e_i+\alpha}\phi\right\|^2.
$$

From Lemma 3.1, the right-hand fifth and sixth terms of (6.15) are also bounded by $\sqrt{\mathcal{E}_l(t)}\mathcal{D}_l(t)$. Putting those estimates into (6.15), taking summation over $1\le|\alpha|\le K$ and fixing a small constant $\eta>0$ give

$$
\begin{aligned}
&\frac{d}{dt}\sum_{\pm}\sum_{1\le|\alpha|\le K}\left\|e^{\frac{\pm\phi}{2}}w_l(\alpha,0)\partial^\alpha f_\pm\right\|^2+\lambda\sum_{1\le|\alpha|\le K}\|w_l(\alpha,0)\partial^\alpha f\|^2_{N^s_\gamma}\\
&\lesssim\|\partial_t\phi\|_{L^\infty}\sum_{1\le|\alpha|\le K}\sum_{\pm}\left\|e^{\frac{\pm\phi}{2}}w_l(\alpha,0)\partial^\alpha f_\pm\right\|^2+\sum_{1\le|\alpha|\le K}\|\nabla_x\partial^\alpha\phi\|^2\\
&\quad+\sum_{1\le|\alpha|\le K}\|\partial^\alpha f\|^2_{N^s_\gamma}+\sqrt{\mathcal{E}_l(t)}\mathcal{D}_l(t).
\end{aligned}
\tag{6.16}
$$

Step 2.3. For the weighted estimate on the mixed derivatives, we start with (6.12). Let $1\le m\le K$. By applying ∂^α_β with $|\beta|=m$ and $|\alpha|+|\beta|\le K$ to (6.12), taking the inner product with $e^{\pm\phi}w_l^2(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}_\pm-$

$\mathbf{P}_\pm)f$ over $\mathbf{R}^3\times\mathbf{R}^3$, one obtains

$$\begin{aligned}
&\frac{1}{2}\sum_{\pm}\frac{d}{dt}\left\|e^{\frac{\pm\phi}{2}}w_l(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|^2+\underbrace{\sum_\pm\left(\partial^\alpha_\beta L_\pm(\mathbf{I}_\pm-\mathbf{P}_\pm)f,w_l^2(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right)}_{J_6}\\
&\qquad\underbrace{-\frac{1}{2}\sum_\pm\left(\pm\partial_t\phi\partial^\alpha_\beta(\mathbf{I}_\pm-\mathbf{P}_\pm)f,e^{\pm\phi}w_l^2(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right)}_{J_7}\\
&\quad=\underbrace{\sum_\pm\left(\partial^\alpha_\beta L_\pm(\mathbf{I}_\pm-\mathbf{P}_\pm)f,\left(1-e^{\pm\phi}\right)w_l^2(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right)}_{J_8}+\sum_{m=9}^{18}J_m,
\end{aligned}\tag{6.17}$$

with

$$\begin{aligned}
J_9&=\sum_\pm\sum_{|\alpha_1|\le|\alpha|}C^{\alpha_1}_\alpha\left(\pm\partial^{\alpha_1+e_i}\phi\partial^{\alpha-\alpha_1}_{\beta+e_i}(\mathbf{I}_\pm-\mathbf{P}_\pm)f,e^{\pm\phi}w_l^2(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right),\\
J_{10}&=\sum_\pm\sum_{|\alpha_1|\le|\alpha|}C^{\alpha_1}_\alpha\left(\pm\partial^{\alpha_1+e_i}\phi\partial^{\alpha-\alpha_1}_{\beta+e_i}\mathbf{P}_\pm f,e^{\pm\phi}w_l^2(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right),\\
J_{11}&=-\sum_\pm\left(\pm\frac{1}{2}(\partial^{e_i}\phi v_i)\partial^\alpha_\beta\mathbf{P}_\pm f,e^{\pm\phi}w_l^2(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right),\\
J_{12}&=-\sum_\pm\sum_{\substack{|\alpha_1|+|\beta_1|\ge1\\|\beta_1|\le1}}C^{\alpha_1,\beta_1}_{\alpha,\beta}\left(\pm\frac{1}{2}(\partial^{\alpha_1+e_i}\phi\partial_{\beta_1}v_i)\partial^{\alpha-\alpha_1}_{\beta-\beta_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f,e^{\pm\phi}w_l^2(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right),
\end{aligned}$$

and

$$\begin{aligned}
J_{13}&=-\sum_\pm\sum_{\substack{|\alpha_1|+|\beta_1|\ge1\\|\beta_1|\le1}}C^{\alpha_1,\beta_1}_{\alpha,\beta}\left(\pm\frac{1}{2}(\partial^{\alpha_1+e_i}\phi\partial_{\beta_1}v_i)\partial^{\alpha-\alpha_1}_{\beta-\beta_1}\mathbf{P}_\pm f,e^{\pm\phi}w_l^2(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right),\\
J_{14}&=-\sum_\pm\sum_{|\alpha_1|\le|\alpha|}C^{\alpha_1}_\alpha\left(\pm(\partial^{\alpha_1+e_i}\phi v_i)\sqrt{\mu},e^{\pm\phi}w_l^2(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right),\\
J_{15}&=-\sum_\pm\left(\left[\partial_t+v_i\partial^{e_i}\right]\partial^\alpha_\beta\mathbf{P}_\pm f,e^{\pm\phi}w_l^2(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right),
\end{aligned}$$

and

$$\begin{aligned}
J_{16}&=-\sum_\pm C^{e_i}_\beta\left(\partial^{\alpha+e_i}_{\beta-e_i}(\mathbf{I}_\pm-\mathbf{P}_\pm)f,e^{\pm\phi}w_l^2(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right),\\
J_{17}&=-\sum_\pm C^{e_i}_\beta\left(\partial^{\alpha+e_i}_{\beta-e_i}\mathbf{P}_\pm f,e^{\pm\phi}w_l^2(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right),\\
J_{18}&=\sum_\pm\left(\partial^\alpha_\beta\Gamma_\pm(f,f),e^{\pm\phi}w_l^2(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right),
\end{aligned}$$

where as before we have used

$$\left(v_i\partial^{\alpha+e_i}_\beta(\mathbf{I}_\pm-\mathbf{P}_\pm)f\pm\frac{1}{2}(\partial^{e_i}\phi v_i)\partial^\alpha_\beta(\mathbf{I}_\pm-\mathbf{P}_\pm)f,e^{\pm\phi}w_l^2(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right)=0.$$

Now we turn to estimate J_i $(6\le i\le 18)$ term by term. Lemma 2.1 yields

$$J_6\gtrsim\left\|w_l(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\right\|^2_{N^s_\gamma}-\eta\sum_{\beta_1\le\beta}\left\|w^l\partial^\alpha_{\beta_1}(\mathbf{I}-\mathbf{P})f\right\|^2_{N^s_\gamma}-C_\eta\left\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\right\|^2_{L^2(B_C)}.$$

It is straightforward to see that J_7 is bounded by

$$C\|\partial_t\phi\|_{L^\infty}\left\|w_l(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\right\|^2.$$

From Lemma 3.1, J_8 and J_{18} have the same upper bound $\sqrt{\mathcal{E}_l(t)}\mathcal{D}_l(t)$, and so do J_9 and J_{12} by Lemma 3.7 and Lemma 3.5. Moreover, since there is the exponential decay in v in the terms J_{10}, J_{11} and J_{13}, one can verify that they all have the upper bound $\sqrt{\mathcal{E}_l(t)}\mathcal{D}_l(t)$.

Next, by Cauchy-Schwartz's inequality with η and local conservation laws (4.12) and (4.13), we obtain

$$J_{14} \lesssim C \sum_{|\alpha|\leq K-1} \|\partial^\alpha \nabla_x \phi\|^2 + \eta \left\|w_l(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\right\|^2_{N^s_\gamma},$$

$$\begin{aligned} J_{15} \lesssim&\eta \left\|w_l(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\right\|^2_{N^s_\gamma} \\ &+C\left\{\sum_{|\alpha|\leq K-1}\|\partial^\alpha\nabla_x\phi\|^2+\sum_{|\alpha|\leq K-1}\|\partial^\alpha\nabla_x(a_\pm,b,c)\|^2+\sum_{|\alpha|\leq K}\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\|^2_{N^s_\gamma}+\mathcal{E}_l(t)\mathcal{D}_l(t)\right\}, \end{aligned}$$

$$J_{17} \lesssim C \sum_{|\alpha|\leq K-1} \|\partial^\alpha \nabla_x (a_\pm,b,c)\|^2 + \eta \left\|w_l(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\right\|^2_{N^s_\gamma}.$$

For the remaining term J_{16}, since $|\beta|\geq 1$, by writing

$$\begin{aligned} &\left(\partial^{\alpha+e_i}_{\beta-e_i}(\mathbf{I}_\pm-\mathbf{P}_\pm)f, e^{\pm\phi}w_l^2(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right)\\ &=\left(\partial^{\alpha+e_i}_{\beta-e_i}(\mathbf{I}_\pm-\mathbf{P}_\pm)f, e^{\pm\phi}w_l^2(\alpha,\beta)\partial_{e_i}\partial^\alpha_{\beta-e_i}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right), \end{aligned}$$

and then doing the similar calculations as $J_{4,2}$, one obtains

$$J_{16} \lesssim \sum_{\substack{|\alpha'|+|\beta'|\leq K\\ |\beta'|=|\beta|-1}} \left\|w_l(\alpha',\beta')\partial^{\alpha'}_{\beta'}(\mathbf{I}-\mathbf{P})f\right\|^2_{N^s_\gamma}.$$

Therefore, by plugging all the estimates above into (6.17), taking the summation over $\{|\beta|=m, |\alpha|+|\beta|\leq K\}$ for each given $1\leq m\leq K$, and then taking the proper linear combination of those K estimates with properly chosen constants $C_m>0$ $(1\leq m\leq K)$ and η small enough, one has

$$\begin{aligned} &\frac{d}{dt}\sum_{m=1}^K C_m \sum_{|\beta|=m,|\alpha|+|\beta|\leq K}\sum_\pm \left\|e^{\frac{\pm\phi}{2}}w_l(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|^2\\ &\quad+\lambda\sum_{|\beta|=m,|\alpha|+|\beta|\leq K}\left\|w_l(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\right\|^2_{N^s_\gamma}\\ &\lesssim|\partial_t\phi|\sum_{|\beta|=m,|\alpha|+|\beta|\leq K}\left\|w_l(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\right\|^2+C\sum_{|\alpha|\leq K}\|w_l(\alpha,0)\partial^\alpha(\mathbf{I}-\mathbf{P})f\|^2_{N^s_\gamma}\\ &\quad+C\sum_{|\alpha|\leq K-1}\|\partial^\alpha\nabla_x(a_\pm,b,c)\|^2+C\sum_{|\alpha|\leq K-1}\|\partial^\alpha\nabla_x\phi\|^2+C\left\{\sqrt{\mathcal{E}_l(t)}+\mathcal{E}_l(t)\right\}\mathcal{D}_l(t). \end{aligned} \tag{6.18}$$

Step 3. We are in a position to prove (6.5) by taking the proper linear combination of those estimates obtained in the previous two steps as follows. The combination $C_2\times[C_1\times(6.11)+(6.14)+(6.16)]+(6.18)$ for $C_1>0$ and $C_2>0$ large enough gives

$$\frac{d}{dt}\mathcal{E}_l(t)+\lambda\mathcal{D}_l(t)\lesssim\|\partial_t\phi\|_{L^\infty}\mathcal{E}_l(t)+\left\{\sqrt{\mathcal{E}_l(t)}+\mathcal{E}_l(t)\right\}\mathcal{D}_l(t), \tag{6.19}$$

where $\mathcal{E}_l(t)$ is given by

$$\begin{aligned} \mathcal{E}_l(t)=&C_2\left[C_1\left\{\sum_{|\alpha|\leq K}\sum_\pm\left\|e^{\frac{\pm\phi}{2}}\partial^\alpha f_\pm\right\|^2+\sum_{|\alpha|\leq K}\|\partial^\alpha\nabla_x\phi\|^2+\kappa\mathscr{E}^K_{int}(t)\right\}\right.\\ &+\left\|e^{\frac{\pm\phi}{2}}w_l(0,0)(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|^2+\sum_{1\leq|\alpha|\leq K}\left\|e^{\frac{\pm\phi}{2}}w_l(\alpha,0)\partial^\alpha f_\pm\right\|^2\\ &\left.+\sum_{m=1}^K C_m\sum_{|\beta|=m,|\alpha|+|\beta|\leq K}\sum_\pm\left\|e^{\frac{\pm\phi}{2}}w_l(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|^2\right]. \end{aligned}$$

Noticing (5.4), it is easy to see

$$\mathcal{E}_l(t) \sim \sum_{|\alpha|\leq K} \|\partial^\alpha \nabla_x \phi\|^2 + \sum_{|\alpha|\leq K} \|\partial^\alpha \mathbf{P} f\|^2 + \sum_{|\alpha|+|\beta|\leq K} \left\|w_l(\alpha,\beta)\partial^\alpha_\beta (\mathbf{I}-\mathbf{P})f\right\|^2.$$

Recalling the a priori assumption (6.4), the desired estimate (6.5) follows directly by (6.19). This completes the proof of Lemma 6.1. □

Lemma 6.2. *For any l with $0 \leq l \leq l_0$, there is $\mathcal{E}^h_l(t)$ satisfying (1.14) such that*

$$\frac{d}{dt}\mathcal{E}^h_l(t) + \lambda \mathcal{D}_l(t) \lesssim \|\nabla_x (a_\pm, b, c)(t)\|^2 + \|\partial_t \phi\|_{L^\infty} \mathcal{E}^h_l(t) \tag{6.20}$$

holds for any $0 \leq t \leq T$, where $\mathcal{D}_l(t)$ is defined by (1.15).

Proof. By letting $|\alpha| \geq 1$ in (6.7), repeating those computations in (6.7)-(6.10) and combing (5.3), one can instead obtain

$$\begin{aligned}
&\frac{d}{dt}\left\{\sum_{1\leq|\alpha|\leq K}\sum_{\pm}\left\|e^{\frac{\pm\phi}{2}}\partial^\alpha f_\pm\right\|^2 + \sum_{1\leq|\alpha|\leq K}\|\partial^\alpha\nabla_x\phi\|^2 + \kappa\mathscr{E}^{K,h}_{int}(t)\right\}\\
&\quad + \lambda \sum_{1\leq|\alpha|\leq K-1} \|\partial^\alpha \nabla(a_\pm,b,c)\|^2 + \lambda\|\nabla_x(a_+ - a_-)\|^2\\
&\quad + \lambda\|\nabla^2_x\phi\|^2 + \lambda\|\nabla_x\phi\|^2 + \lambda\sum_{1\leq|\alpha|\leq K}\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\|^2_{N^s_\gamma}\\
&\lesssim \sum_{1\leq|\alpha|\leq K}\sum_{\pm}\|\partial_t\phi\|_{L^\infty}\|\partial^\alpha f_\pm\|^2 + \|(\mathbf{I}-\mathbf{P})f\|^2_{N^s_\gamma} + \left(\sqrt{\mathcal{E}_l(t)} + \mathcal{E}_l(t)\right)\mathcal{D}_l(t).
\end{aligned} \tag{6.21}$$

Moreover, taking the inner product of (6.12) with $e^{\pm\phi}(\mathbf{I}_\pm - \mathbf{P}_\pm)f$ over $\mathbf{R}^3\times\mathbf{R}^3$ gives

$$\begin{aligned}
&\frac{1}{2}\sum_{\pm}\frac{d}{dt}\left\|e^{\frac{\pm\phi}{2}}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|^2 + \sum_{\pm}(L_\pm(\mathbf{I}_\pm-\mathbf{P}_\pm)f,(\mathbf{I}_\pm-\mathbf{P}_\pm)f)\\
&\quad+\sum_{\pm}(\pm(\partial^{e_i}\phi v_i)\sqrt{\mu},(\mathbf{I}_\pm-\mathbf{P}_\pm)f) - \frac{1}{2}\sum_{\pm}\left(\pm\partial_t\phi(\mathbf{I}_\pm-\mathbf{P}_\pm)f, e^{\pm\phi}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right)\\
&=\sum_{\pm}\left(L_\pm(\mathbf{I}_\pm-\mathbf{P}_\pm)f,\left(1-e^{\pm\phi}\right)(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right) - \sum_{\pm}\left(\frac{1}{2}\pm(\partial^{e_i}\phi v_i)\mathbf{P}_\pm f, e^{\pm\phi}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right)\\
&\quad-\sum_{\pm}\left(\pm(\partial^{e_i}\phi v_i)\sqrt{\mu},\left(e^{\pm\phi}-1\right)(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right) - \sum_{\pm}\left(\partial_t\mathbf{P}_\pm f, e^{\pm\phi}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right)\\
&\quad-\sum_{\pm}\left(v_i\partial^{e_i}\mathbf{P}_\pm f, e^{\pm\phi}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right) - \sum_{\pm}\left(\mp\partial^{e_i}\phi\partial_{e_i}\mathbf{P}_\pm f, e^{\pm\phi}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right)\\
&\quad+\sum_{\pm}\left(\Gamma_\pm(f,f), e^{\pm\phi}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right).
\end{aligned} \tag{6.22}$$

Now we turn to further estimate (6.22) term by term. In light of (4.13), (4.15) and (4.16),

$$\sum_{\pm}(\pm(\partial^{e_i}\phi v_i)\sqrt{\mu},(\mathbf{I}_\pm-\mathbf{P}_\pm)f) = -(\phi,\nabla_x G) = -(\phi,\partial_t\Delta\phi) = \frac{d}{dt}\|\nabla_x\phi\|^2.$$

It is also straightforward to get from Lemma 2.1 that

$$\sum_{\pm}(L_\pm(\mathbf{I}_\pm-\mathbf{P}_\pm)f,(\mathbf{I}_\pm-\mathbf{P}_\pm)f) \geq \lambda\|(\mathbf{I}-\mathbf{P})f\|^2_{N^s_\gamma}.$$

The first term on the right-hand side of (6.22) can be bounded by $\sqrt{\mathcal{E}_l(t)}\mathcal{D}_l(t)$ according to the estimate (6.9). The second, third, sixth and seventh terms on the right-hand side of (6.22) are also dominated by $\sqrt{\mathcal{E}_l(t)}\mathcal{D}_l(t)$, where we have used Sobolev's inequality for the second, third and sixth terms and Lemma 2.2 for the seventh term. For the fourth term, by the definitions of $\mathbf{P}$ and $\mathbf{I}-\mathbf{P}$,

$$\sum_{\pm}\left(\partial_t\mathbf{P}_\pm f, e^{\pm\phi}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right) = \sum_{\pm}\left(\partial_t\mathbf{P}_\pm f,\left[e^{\pm\phi}-1\right](\mathbf{I}_\pm-\mathbf{P}_\pm)f\right).$$

Recalling (6.8) and the local conservation laws (4.12), (4.13), and applying Sobolev's inequality as well as Cauchy-Schwarz's inequality with λ, it follows that

$$\begin{aligned}\left|\sum_{\pm}\left(\partial_t\mathbf{P}_\pm f, e^{\pm\phi}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right)\right| \lesssim &\frac{\lambda}{4}\|(\mathbf{I}-\mathbf{P})f\|_{N^s_\gamma}^2+\left[\|\nabla_x(a_\pm,b,c)\|^2+\|\nabla_x\phi\|^2\right]\|\nabla_x\phi\|_{H^1}^2\\ &+\|\nabla_x\phi\|_{H^1}^2\|\nabla_x(\mathbf{I}-\mathbf{P})f\|_{N^s_\gamma}^2+\mathcal{E}_l(t)\mathcal{D}_l(t)\\ \lesssim&\frac{\lambda}{4}\|(\mathbf{I}-\mathbf{P})f\|_{N^s_\gamma}^2+\mathcal{E}_l(t)\mathcal{D}_l(t).\end{aligned}$$

For the remaining fifth term, by Cauchy-Schwarz's inequality with λ,

$$\left|\sum_{\pm}\left(v_i\partial^{e_i}\mathbf{P}_\pm f, e^{\pm\phi}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right)\right|\lesssim\frac{\lambda}{4}\|(\mathbf{I}-\mathbf{P})f\|_{N^s_\gamma}^2+\|\nabla_x(a_\pm,b,c)\|^2.$$

Therefore, combing all the estimates above, one has

$$\begin{aligned}&\frac{d}{dt}\left\{\sum_{\pm}\left\|e^{\frac{\pm\phi}{2}}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|^2+\|\nabla_x\phi\|^2\right\}+\lambda\|(\mathbf{I}-\mathbf{P})f\|_{N^s_\gamma}^2\\ \lesssim&\|\partial_t\phi\|_{L^\infty}\sum_{\pm}\left\|e^{\frac{\pm\phi}{2}}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|^2+\|\nabla_x(a_\pm,b,c)\|^2+\left\{\sqrt{\mathcal{E}_l(t)}+\mathcal{E}_l(t)\right\}\mathcal{D}_l(t).\end{aligned}\tag{6.23}$$

As in step 3 in the proof of Lemma 6.1, a suitably linear combination of (6.21), (6.23), (6.14), (6.16) and (6.18) yields

$$\frac{d}{dt}\mathcal{E}_l^h(t)+\lambda\mathcal{D}_l(t)\lesssim\|\nabla_x(a_\pm,b,c)\|^2+\|\partial_t\phi\|_{L^\infty}\mathcal{E}_l^h(t)+\left\{\sqrt{\mathcal{E}_l(t)}+\mathcal{E}_l(t)\right\}\mathcal{D}_l(t),$$

which hence implies the desired estimate (6.20) under the a priori assumption (6.4). This completes the proof of Lemma 6.2. $\square$

7. Proof of global existence

In this section we prove Theorem 1.1 through obtaining the closed uniform-in-time estimates on $X(t)$ in the following lemma. Recall (6.1) for the definition of $X(t)$ with (6.2)-(6.3) for the choice of the corresponding parameters.

Lemma 7.1. *Consider the Cauchy problem* (1.4)-(1.6). *Assume that*

$$\int_{\mathbf{R}^3}\rho_{f_0}\,dx=0,\quad\int_{\mathbf{R}^3}(1+|x|)|\rho_{f_0}|\,dx<\infty,$$

where $\rho_{f_0}=\int_{\mathbf{R}^3}(f_{0,+}-f_{0,-})\sqrt{\mu}\,dv$. *Define* ϵ_0 *by*

$$\epsilon_0=\sqrt{\mathcal{E}_{l_0+l_1}(0)}+\left\|w^{l_2}f_0\right\|_{Z_1}+\|(1+|x|)\rho_{f_0}\|_{L^1},$$

where $l_2>\frac{5(\gamma+2s)}{4\gamma}$ *is a constant. Then, under the a priori assumption (6.4) for* $\delta_0>0$ *suitably small, one has*

$$X(t)\lesssim\epsilon_0^2+X^{\frac{3}{2}}(t)+X^2(t),\tag{7.1}$$

for any $0\le t\le T$.

Proof. We divide it by three steps.

Step 1. It follows from (4.13) and (4.16) that

$$\partial_t\phi=-\Delta_x^{-1}\partial_t(a_+-a_-)=\Delta_x^{-1}\nabla_x\cdot G,$$

where G is given by (4.15). Then, by Sobolev and Riesz inequalities,

$$\|\partial_t\phi\|_{L^\infty}^2\lesssim\|\nabla_x\partial_t\phi\|\cdot\|\nabla_x^2\partial_t\phi\|\lesssim\|G\|\cdot\|\nabla_xG\|\lesssim\mathcal{E}_{l_0}^h(t)\lesssim(1+t)^{-\frac{3}{2}-p}X(t).\tag{7.2}$$

Here notice $3/2+p>2$ since $p>1/2$. By applying Lemma 6.1 in the case when $l=l_0+l_1$, one has from (6.5) that

$$\frac{d}{dt}\mathcal{E}_{l_0+l_1}(t)+\lambda\mathcal{D}_{l_0+l_1}(t)\lesssim\|\partial_t\phi\|_{L^\infty}\mathcal{E}_{l_0+l_1}(t),$$

which together with (7.2) and Gronwall's inequality as well as (6.4), imply

$$\mathcal{E}_{l_0+l_1}(t) \lesssim \mathcal{E}_{l_0+l_1}(0) e^{C\int_0^t \|\partial_t\phi(\tau)\|_{L^\infty}\,d\tau} \lesssim \mathcal{E}_{l_0+l_1}(0) \lesssim \epsilon_0^2 \tag{7.3}$$

for any $0 \le t \le T$.

Step 2. We begin with (6.20) with $l = l_0$ in Lemma 6.2, i.e.

$$\frac{d}{dt}\mathcal{E}^h_{l_0}(t) + \lambda \mathcal{D}_{l_0}(t) \lesssim \|\nabla_x(a_\pm, b, c)(t)\|^2 + \|\partial_t\phi\|_{L^\infty}\mathcal{E}^h_{l_0}(t). \tag{7.4}$$

As in [30, 31], at time t, we split the velocity space $\mathbf{R}^3_v$ into

$$\mathbf{E}(t) = \left\{\langle v\rangle^{-\gamma-2s} \le t^{1-p}\right\}, \quad \mathbf{E}^c(t) = \left\{\langle v\rangle^{-\gamma-2s} > t^{1-p}\right\}, \tag{7.5}$$

where recall $\gamma + 2s < 0$ and $1/2 < p < 1$. Corresponding to this splitting, we define $\mathcal{E}^{h,low}_{l_0}(t)$ to be the restriction of $\mathcal{E}^h_{l_0}(t)$ to $\mathbf{E}(t)$ and similarly $\mathcal{E}^{h,high}_{l_0}(t)$ to be the restriction of $\mathcal{E}^h_{l_0}(t)$ to $\mathbf{E}^c(t)$. Then, due to (7.5) and (1.14), it follows from (7.4) that

$$\frac{d}{dt}\mathcal{E}^h_{l_0}(t) + \lambda t^{p-1}\mathcal{E}^h_{l_0}(t) \lesssim \|\nabla_x \mathbf{P} f\|^2 + \|\partial_t\phi\|_{L^\infty}\mathcal{E}^h_{l_0}(t) + t^{p-1}\mathcal{E}^{h,high}_{l_0}(t),$$

which by solving the ODE inequality, gives

$$\mathcal{E}^h_{l_0}(t) \lesssim e^{-\lambda t^p}\mathcal{E}^h_{l_0}(0) + \int_0^t d\tau\, e^{-\lambda(t^p-\tau^p)}\left(\|\nabla_x\mathbf{P}f(\tau)\|^2 + \|\partial_t\phi\|_{L^\infty}\mathcal{E}^h_{l_0}(\tau) + \tau^{p-1}\mathcal{E}^{h,high}_{l_0}(\tau)\right). \tag{7.6}$$

In what follows we estimate those three terms in the time integral on the right-hand side of (7.6). First of all, by (7.2),

$$\|\partial_t\phi\|_{L^\infty}\mathcal{E}^h_{l_0}(t) \lesssim [\mathcal{E}^h_{l_0}(t)]^{\frac{3}{2}} \lesssim (1+t)^{-(\frac{3}{2}+p)\frac{3}{2}} X^{\frac{3}{2}}(t). \tag{7.7}$$

Next, one can prove

$$\mathcal{E}^{h,high}_{l_0}(t) \lesssim \epsilon_0^2(1+t)^{-5/2}. \tag{7.8}$$

In fact, it is straightforward to see from (7.3) that for any $0 \le t \le T$,

$$\mathcal{E}^{h,high}_{l_0}(t) \lesssim \mathcal{E}^h_{l_0}(t) \lesssim \mathcal{E}_{l_0}(t)) \lesssim \mathcal{E}_{l_0+l_1}(t) \lesssim \epsilon_0^2.$$

On the other hand, noticing that $t^{5/2} \le w^{\frac{5}{2(1-p)}\frac{\gamma+2s}{\gamma}}$ on the set $E^c(t)$ and l_1 is given by

$$l_1 = \frac{5}{4(1-p)}\frac{\gamma+2s}{\gamma},$$

one has from (7.3) that

$$\mathcal{E}^{h,high}_{l_0}(t) \lesssim t^{-5/2}\mathcal{E}_{l_0+l_1}(t) \lesssim \epsilon_0^2 t^{-5/2}.$$

Therefore, (7.8) holds true. Finally, one also can prove

$$\|\nabla_x\mathbf{P}f(t)\| + \|\nabla_x^2\phi\| \lesssim (1+t)^{-5/4}\left\{\epsilon_0 + X(t)\right\}. \tag{7.9}$$

In fact, recalling (4.4), the solution $f(t,x,v)$ to (1.4) can be written as

$$f(t) = S(t)f_0 + \int_0^t S(t-\tau)h(\tau)\,d\tau,$$

with

$$h = q\nabla_x\phi\cdot\nabla_v f - \frac{q}{2}v\cdot\nabla_x\phi f + \Gamma(f,f). \tag{7.10}$$

Notice $\langle h_\pm, \sqrt{\mu}\rangle = 0$. Applying Theorem 4.1 in the case when $k = 0$ and $\sigma_1 = 5/4$, it follows that

$$\begin{aligned}\|\nabla_x\mathbf{P}f(t)\| + \|\nabla_x^2\phi\| &\lesssim \|\nabla_x f(t)\| + \|\nabla_x^2\phi\| \\ &\lesssim \epsilon_0(1+t)^{-5/4} + \int_0^t (1+t-\tau)^{-5/4}\left(\left\|w^{l_*}h(\tau)\right\|_{Z_1} + \left\|w^{l_*}\nabla_x h(\tau)\right\|\right)d\tau,\end{aligned}$$

where in the homogeneous part, we choose $l_* = l_2$ with $l_2 > \frac{\sigma_1(\gamma+2s)}{\gamma} = \frac{5(\gamma+2s)}{4\gamma}$, while in the nonhomogeneous part, for later use, we choose l_* such that

$$\frac{5(\gamma+2s)}{4\gamma} < l_* \le K - 5 + \frac{\gamma+2s}{2\gamma}. \tag{7.11}$$

In fact, $K\geq 8$, $-3<\gamma<-2s$ and $1/2\leq s<1$ imply that

$$\frac{\gamma+2s}{\gamma}=\left|\frac{\gamma+2s}{\gamma}\right|\leq\frac{3-2s}{2s}\in[1/2,2],$$

and hence

$$K>5+\frac{3(\gamma+2s)}{4\gamma},\ \text{i.e.}\ \frac{5(\gamma+2s)}{4\gamma}<K-5+\frac{\gamma+2s}{2\gamma},$$

so that it is valid to choose l_* as in (7.11). By using the direct calculations on the collision operator, cf. Lemma 3.3, one can deduce that

$$\left\|w^{l_*}\Gamma(f,f)\right\|_{Z_1}\lesssim\mathcal{E}_{l_0}(t),$$
$$\left\|w^{l_*}q\nabla_x\phi\cdot\nabla_v f\right\|_{Z_1}+\left\|w^{l_*}\frac{q}{2}v\cdot\nabla_x\phi f\right\|_{Z_1}\lesssim\mathcal{E}_{l_0}(t).$$

where we have used

$$\sum_{|\alpha|+|\beta|\leq 5}\left\|w^{l_*-\frac{\gamma+2s}{2\gamma}}\partial^\alpha_\beta f\right\|\lesssim\sum_{|\alpha|+|\beta|\leq 5}\left\|w^{l_0+K-5}\partial^\alpha_\beta f\right\|,$$

which can be guaranteed by $l_0+K-5\geq l_*-\frac{\gamma+2s}{2\gamma}$ due to $l_0\geq 0$ and the choice of l_* as before. Then,

$$\left\|w^{l_*}h(s)\right\|_{Z_1}\lesssim\mathcal{E}_{l_0}(t).$$

Furthermore, in a similar way, one has

$$\left\|w^{l_*}q\nabla_x(\nabla_x\phi\cdot\nabla_v f)\right\|+\left\|w^{l_*}\frac{q}{2}v\cdot\nabla_x\left(\nabla_x\phi f\right)\right\|\lesssim\mathcal{E}_{l_0}(t),$$
$$\left\|w^{l_*}\nabla_x\Gamma(f,f)\right\|\lesssim\mathcal{E}_{l_0}(t),$$

which imply

$$\left\|w^{l_*}h(t)\right\|_{Z_1}+\left\|w^{l_*}\nabla_x h(t)\right\|\lesssim\mathcal{E}_{l_0}(t).$$

With the above estimates and then using $\mathcal{E}_{l_0}(t)\leq(1+t)^{-3/2}X(t)$, one has

$$\begin{aligned}\|\nabla_x\mathbf{P}f(t)\|+\|\nabla_x^2\phi\|\lesssim&\epsilon_0(1+t)^{-5/4}+X(t)\int_0^t(1+t-\tau)^{-5/4}(1+\tau)^{-3/2}d\tau\\ \lesssim&(1+t)^{-5/4}\left(\epsilon_0+X(t)\right),\end{aligned}\tag{7.12}$$

which is the desired estimate (7.9).

Notice that one has the following inequalities

$$\int_0^t e^{-\lambda(t^p-\tau^p)}(1+\tau)^{-(\frac{3}{2}+p)\frac{3}{2}}\,d\tau\lesssim(1+t)^{-(\frac{5}{4}+\frac{5p}{2})},$$
$$\int_0^t e^{-\lambda(t^p-\tau^p)}\tau^{p-1}(1+\tau)^{-\frac{5}{2}}\,d\tau\lesssim(1+t)^{-\frac{5}{2}},$$

and

$$\int_0^t e^{-\lambda(t^p-\tau^p)}(1+\tau)^{-5/2}\,d\tau\lesssim(1+t)^{-\frac{3}{2}-p}.\tag{7.13}$$

Then, plugging (7.7), (7.8) and (7.9) into (7.6) and using $1/2\leq p<1$, one has

$$\mathcal{E}^h_{l_0}(t)\lesssim\left\{\epsilon_0^2+X^{\frac{3}{2}}(t)+X^2(t)\right\}(1+t)^{-\frac{3}{2}-p},\tag{7.14}$$

for any $0\leq t\leq T$.

Step 3. In the same way to prove (7.12) basing on (7.9) and (7.10), one can show

$$\|f\|+\|\nabla_x\phi\|\lesssim(1+t)^{-\frac{3}{4}}\left\{\epsilon_0+X(t)\right\}.\tag{7.15}$$

Noticing $\mathcal{E}_{l_0}(t)\sim\|\mathbf{P}f\|^2+\|\nabla_x\phi\|^2+\mathcal{E}^h_{l_0}(t)$, (7.15) together with (7.14) give

$$\mathcal{E}_{l_0}(t)\lesssim(1+t)^{-\frac{3}{2}}\left\{\epsilon_0^2+X^{\frac{3}{2}}(t)+X^2(t)\right\}.\tag{7.16}$$

Now, recall (6.1). The desired estimate (7.1) follows from (7.3), (7.14) and (7.16). This completes the proof of Lemma 7.1. □

The proof of Theorem 1.1. It is immediate to follow from the a priori estimate (7.1) that $X(t) \lesssim \epsilon_0^2$ holds true for any $0 \le t \le T$, as long as ϵ_0 is sufficiently small. The rest is to prove the local existence and uniqueness of solutions in terms of the energy norm $\mathcal{E}_{l_0+l_1}(t)$ and the non negativity of $F_\pm = \mu + \sqrt{\mu} f$, and the details of the proof are omitted for brevity; see also [21, 32] and [13]. Therefore, the global existence of solutions follows with the help of the continuity argument, and the estimates (1.18), (1.19) and (1.20) hold by the definition of $X(t)$ (6.1). This completes the proof of Theorem 1.1. $\square$

Finally we give a remark on the possibility of upgrading the rate $(1+t)^{-\frac{3}{2}-p}$ in (1.20) to $(1+t)^{-5/2}$ that corresponds to the case $p = 1$. In fact, due to the technique of the paper it seems difficult to achieve such optimal rate for $\mathcal{E}_{l_0}^h(t)$. The main reason is that the energy dissipation rate $\mathcal{D}_{l_0}(t)$ in (7.4) is degenerate at large velocity for soft potentials, and thus, although the first term on the right-hand side of (7.4) decays with the optimal rate $(1+t)^{-5/2}$, it is impossible to deduce from (7.4) the optimal time decay of $\mathcal{E}_{l_0}^h(t)$ because of the inequality (7.13). On the other hand it is still possible to obtain the optimal time decay of only those high-order space differentiations in $\mathcal{E}_{l_0}^h(t)$ by applying the linearized estimate (4.5) to the nonlinear system. Indeed, define

$$\widetilde{\mathcal{E}}_0^h(t) = \sum_{1 \le k \le K_0} \left(\|\nabla_x^k f(t)\|^2 + \|\nabla_x^k E(t)\|^2 \right).$$

Then, with the help of Duhamel's principle as well as Theorem 4.1, similar for obtaining (7.12), it is straightforward to show that

$$\widetilde{\mathcal{E}}_0^h(t) \lesssim \epsilon_0^2 (1+t)^{-\frac{5}{2}}$$

for an appropriate choice of K_0 in terms of K large enough.

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