A note on global existence for the chemotaxis-Stokes model with nonlinear diffusion

Renjun Duan[1] * and **Zhaoyin Xiang**[2] †

1. *Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong*

2. *School of Mathematical Sciences, University of Electronic Science & Technology of China, Chengdu 610054, China*

Abstract: This note is concerned with the Cauchy problem on the 3D chemotaxis-Stokes equations with nonlinear diffusions and large initial data. We prove the global existence of weak solutions for all adiabatic exponents $m \in [1, +\infty)$. In particular, the result fills up the gap between $m = 1$ by Winkler (Comm. Part. Diff. Eqs., 37(2012), 319-351) and $m \in (\frac{4}{3}, 2]$ by Liu-Lorz (Ann. I. H. Poincaré-AN, 28(2011), 643-652). A similar result also holds for the 2D chemotaxis-Navier-Stokes equations.

Keywords: chemotaxis model, nonlinear diffusion, Stokes equations

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1 Introduction

It is well-known that the chemotaxis is a biological process in which cells (e.g., bacteria) move towards a chemically more favorable environment. For example, bacteria often swim towards higher concentration of oxygen to survive. A typical model describing chemotaxis are the Keller-Segel equations derived by Keller and Segel [10] which have become one of the best-studied models in mathematical biology.

Considering that in nature cells often live in a viscous fluid so that cells and chemical substrates are also transported with fluid, and meanwhile the motion of the fluid is under the influence of gravitational forcing generated by aggregation of cells, the authors in [19] proposed the following model:

$$\begin{cases} n_t + u \cdot \nabla n = \delta \Delta n^m - \nabla \cdot (n\chi(c)\nabla c), & \text{in} \quad \Omega \times (0, +\infty), \\ c_t + u \cdot \nabla c = \mu \Delta c - n\kappa(c), & \text{in} \quad \Omega \times (0, +\infty), \\ u_t + u \cdot \nabla u + \nabla P = \nu \Delta u - n\nabla\phi, & \text{in} \quad \Omega \times (0, +\infty), \\ \nabla \cdot u = 0, & \text{in} \quad \Omega \times (0, +\infty) \end{cases} \tag{1.1}$$

with $m = 1$ for the unknown bacterial density n, the oxygen concentration c, the fluid velocity field u and the associated pressure P in the physical domain $\Omega \subset \mathbb{R}^N$, $(N = 2, 3)$. The nonnegative constants δ, μ and ν are the corresponding diffusion coefficients for the cells, substrate and fluid. The function $\chi(c)$ measures the chemotactic sensitivity, $\kappa(c)$ is the consumption rate of the substrate by the cells and $\phi = \phi(x)$ is a given gravitational potential function. We remark that the sign in front of the n-term in the equation $(1.1)_2$ on c is different from the one used in the Keller-Segel model and thus we are interested in the case that the chemical substrate can be consumed by the cells.

The key mathematical problem is to establish global existence of solutions to (1.1) for general data. Although related to a different phenomenon, the Keller-Segel model offers a good paradigm on the possible singular behavior of solutions. The problem of global existence or blow-up has been completely solved for the elliptic-parabolic Keller-Segel model in $\mathbb{R}^2$. Precisely, there is a critical mass M, below M we have global existence and above M we have finite-time blow-up (see [1, 2, 4, 5] and references therein). Also, several authors in the chemotaxis literature have recently addressed the prevention of finite-time blow-up (overcrowding, from the modeling viewpoint) by assuming e.g. that, due to the finite size of the bacteria, the random mobility increases for large densities. This leads to a nonlinear porous-medium-like diffusion instead of a linear one (see e. g. [3, 11]).

*E-mail: rjduan@math.cuhk.edu.hk

†Corresponding author. E-mail: zxiang@uestc.edu.cn

For system (1.1) and related systems, Lorz [15] recently proved the local existence of solutions. Then for $m = 1$, Duan-Lorz-Markowich [6] established the global existence of classical solutions near constant states in $\mathbb{R}^2$ and global existence of weak solutions in $\mathbb{R}^3$ under assumption that the external forcing is weak or the substrate concentration is small. The time-decay rates in L^p-norms for perturbations are also obtained by combining the energy estimates and spectral analysis developed by Duan et al. [7] in the study of the nonlinear Boltzmann equation.

In case $\Omega = \mathbb{R}^2$, Francesco-Lorz-Markowich [8] proved that system (1.1) with $m \in (\frac{3}{2}, 2]$ features global in time solutions for large data. Then Liu-Lorz [14] and Winkler [21] showed that the same result holds for $m = 1$. More recently, under the assumption that $\Omega \subset \mathbb{R}^2$ is bounded, Tao-Winkler [17] established the global existence of weak solution to the initial-boundary value problem for large data and arbitrary $m > 1$, which in particular fills the gap $(1, \frac{3}{2}]$.

In case $\Omega = \mathbb{R}^3$, considering that the fluid flow is slow, Francesco-Lorz-Markowich [8] used the *chemotaxis-Stokes system* instead of the chemotaxis-Navier-Stokes system and proved the global existence of weak solutions for large data if $m \in (m^*, 2]$ with $m^* > \frac{3}{2}$. Then Liu-Lorz [14] and Winkler [21] obtained the same results for $m = \frac{4}{3}$ and $m = 1$, respectively. Very recently, Tao-Winkler [18] established the the global existence of locally bounded solutions for $m > \frac{8}{7}$ under the assumption that $\chi(c) \equiv 1$ and $\kappa(c) = c$. As Winkler [21] and Tao-Winkler [18] pointed out, in the case $\Omega \subset \mathbb{R}^3$, a complete classification of all $m \geq 1$ which allow for global solutions is still lacking.

Hence, filling the above gap $(1, \frac{3}{2})$ for $\Omega \subset \mathbb{R}^3$ is one of our motivations. We are mainly interested in $\Omega = \mathbb{R}^3$ but that the method works also on bounded domains. We will show that for all $m \geq 1$, there exist global solutions for large initial data. To see this, we will first consider the following three dimensional chemotaxis-Stokes fluid model

$$\begin{cases} n_t + u \cdot \nabla n = \delta \Delta n^m - \nabla \cdot (n\chi(c)\nabla c), & \text{in} \quad \mathbb{R}^3 \times (0, +\infty), \\ c_t + u \cdot \nabla c = \mu \Delta c - n\kappa(c), & \text{in} \quad \mathbb{R}^3 \times (0, +\infty), \\ u_t + \nabla P = \nu \Delta u - n\nabla\phi, & \text{in} \quad \mathbb{R}^3 \times (0, +\infty), \\ \nabla \cdot u = 0, & \text{in} \quad \mathbb{R}^3 \times (0, +\infty). \end{cases} \tag{1.2}$$

We state a set of assumptions on (1.2) required throughout this paper. First, we assume

$$\textbf{(A1)} \quad \delta > 0,\ \mu > 0,\ \nu > 0,$$

which means that all the diffusions of the cells, substrate and fluid are taken into account.

Since $\chi(c)$ measures the chemotactic sensitivity and we are interested in the case the chemical substrate can be consumed by the cells, it is reasonable to suppose that the consumption rate $\kappa(c)$ is an increasing function of the chemical density c. That is, we will assume that

$$\textbf{(A2)} \quad \kappa(\cdot),\ \chi(\cdot) \text{ are smooth with } \kappa(0) = 0 \text{ and } \kappa(s),\ \kappa'(s),\ \chi(s) \geq 0 \text{ for all } s \in \mathbb{R}.$$

Then we further assume that

$$\textbf{(A3)} \quad \chi(s) > 0,\ \frac{\mathrm{d}(\kappa(s)\chi(s))}{\mathrm{d}s} > 0,\ \frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\frac{\kappa(s)}{\chi(s)}\right) < 0.$$

A simple model case can be obtained upon the choices $\kappa(s) = s$ and $\chi(s) = \chi_0 + s$ for a positive constant χ_0. However, such a choice does not cover the case $\kappa(s) = s$ and $\chi(s) = \chi_0$ used in [21, 18], which satisfy our all assumptions except for holding $\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\frac{\kappa(s)}{\chi(s)}\right) = 0$ only.

For the gravitational potential function ϕ, we assume that

$$\textbf{(A4)} \quad \nabla\phi \in L^\infty.$$

One prototypical example is the choice $\phi = ax_1$ for a constant $a \in \mathbb{R}$ depending on the ratio of the fluid mass density to the cell density and the gravity acceleration. Such an assumption also includes the case used by the experimentalists in [19].

As usual, in order for the system (1.2) to be well-posed, it should be supplemented with some initial conditions

$$(n, c, u)\big|_{t=0} = (n_0(x), c_0(x), u_0(x)), \qquad x \in \mathbb{R}^3. \tag{1.3}$$

And we will assume the initial data satisfies

$$
(\mathbf{A5}) \quad \begin{cases} n_0 \geq 0, \quad 0 \leq c_0 \leq c_M < \infty, \quad \nabla \cdot u_0 = 0, \\ n_0(1 + |x| + |\ln n_0|) \in L^1, \quad \left|\nabla n_0^{\frac{m}{2}}\right| \in L^2, \\ c_0 \in L^1, \quad \nabla c_0 \in L^2, \quad \nabla \Psi(c_0) \in L^2, \quad u_0 \in L^2, \end{cases}
$$

where

$$
\Psi(c) = \int_0^c \sqrt{\frac{\chi(s)}{\kappa(s)}} \mathrm{d}s.
$$

From a mathematical point of view, the function Ψ plays a similar role of c and thus gives the dissipation of c (see (1.5)).

We remark that, since $m > 1$, the equation $(1.2)_1$ may be degenerate at $n = 0$ and in general does not allow for classical solvability as the well-known porous medium equations [20]. Thus, we introduce the following definition of weak solution.

Definition 1.1 *By a global weak solution of the chemotaxis-Stokes equation* (1.2) *with initial data* (1.3)*, we mean a triple* (n, c, u) *of functions satisfying*
(i) $n(t, x) \geq 0$, $c(t, x) \geq 0$ *for* $t \geq 0, x \in \mathbb{R}^3$*, and for any* $T > 0$*,*

$$
\begin{cases} n \in L^\infty(0, T; L^1(\mathbb{R}^3)), \quad \nabla n^{\frac{m}{2}} \in L^2(0, T; L^2(\mathbb{R}^3)), \\ c \in L^\infty(0, T; L^\infty(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)), \\ \sqrt{n}\,|\nabla c| \in L^2(0, T; L^2(\mathbb{R}^3)), \\ u \in L^\infty(0, T; L^2(\mathbb{R}^3, \mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3, \mathbb{R}^3)); \end{cases}
$$

and (ii)

$$
\begin{cases} \displaystyle\int_0^T \int_{\mathbb{R}^3} n\left(\varphi_{1t} + u \cdot \nabla \varphi_1 + \delta n^{m-1} \Delta \varphi_1 + \nabla \varphi_1 \cdot (\chi(c) \nabla c)\right) \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^3} n_0 \varphi_1(x, 0) \mathrm{d}x = 0, \\ \displaystyle\int_0^T \int_{\mathbb{R}^3} c\,(\varphi_{2t} + u \cdot \nabla \varphi_1 + \mu \Delta \varphi_2) - n\kappa(c)\varphi_2 \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^3} c_0 \varphi_2(x, 0) \mathrm{d}x = 0, \\ \displaystyle\int_0^T \int_{\mathbb{R}^3} u \cdot (\varphi_t + \nu \Delta \varphi) - n \nabla \psi \cdot \varphi \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^3} u_0 \cdot \varphi(x, 0) \mathrm{d}x = 0, \\ \displaystyle\int_0^T \int_{\mathbb{R}^3} u \cdot \nabla \varphi_3 \mathrm{d}x\mathrm{d}t = 0 \end{cases}
$$

hold for all $\varphi_1, \varphi_2, \varphi_3 \in C^\infty([0, T] \times \mathbb{R}^3)$ *and all* $\varphi \in C^\infty([0, T] \times \mathbb{R}^3, \mathbb{R}^3)$ *with* $\nabla \cdot \varphi = 0$*, where* $\varphi_1, \varphi_2, \varphi_3$ *and* φ *have compact support in* x *and vanish on* $t = T$*.*

For simplicity, we set $\langle x \rangle = \sqrt{1 + |x|^2}$ throughout this paper. We will also use $C(m, \delta, \|n_0\|_1, \cdots)$ to denote positive constants depending on $m, \delta, \|n_0\|_1$ etc. With these assumptions at hand, we can state the result as follows.

Theorem 1.1 *Under the assumptions* $(\mathbf{A1}) - (\mathbf{A5})$*, the Cauchy Problem* (1.2)-(1.3) *with* $m \geq 1$ *admits a global-in-time weak solution* (n, c, u) *satisfying* $n(|x| + |\ln n|) \in L^\infty(0, T; L^1(\mathbb{R}^3))$ *for any* $T > 0$*. Moreover, there exists an entropy functional which grows in time at most linearly, that is, one has the entropy estimate*

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + \mathcal{D}(t) \leq C(m, \delta, \mu, \nu, \lambda_1, \lambda_2, c_M, \|\nabla \phi\|_\infty, \|n_0\|_1) \tag{1.4}
$$

with

$$
\mathcal{E}(t) := \int_{\mathbb{R}^3} \left(n \ln n + n \langle x \rangle + \frac{1}{2} |\nabla \Psi|^2 + \frac{1}{2\lambda_1 \mu \nu} |u|^2\right) \mathrm{d}x \tag{1.5}
$$

and

$$\begin{aligned}\mathcal{D}(t) :=& \frac{2\delta}{m}\int_{\mathbb{R}^3}\left|\nabla n^{\frac{m}{2}}\right|^2 \mathrm{d}x + \frac{1}{2\lambda_1\mu}\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x + \mu\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\left(\frac{\partial^2\Psi}{\partial x_i\partial x_j} - \frac{\mathrm{d}}{\mathrm{d}c}\sqrt{\frac{\kappa}{\chi}}\frac{\partial\Psi}{\partial x_i}\frac{\partial\Psi}{\partial x_j}\right)^2\mathrm{d}x \\ &+ \lambda_1\mu\int_{\mathbb{R}^3}|\nabla\Psi|^4\mathrm{d}x + \lambda_2\int_{\mathbb{R}^3} n|\nabla\Psi|^2\mathrm{d}x,\end{aligned} \tag{1.6}$$

where λ_1 and λ_2 are two positive constants defined in (2.5).

We also establish a similar result for $\mathbb{R}^2$. In this case, we can consider the following chemotaxis-Navier-Stokes model

$$\begin{cases} n_t + u\cdot\nabla n = \delta\Delta n^m - \nabla\cdot(n\chi(c)\nabla c), & \text{in} \quad \mathbb{R}^2\times(0,+\infty),\\ c_t + u\cdot\nabla c = \mu\Delta c - n\kappa(c), & \text{in} \quad \mathbb{R}^2\times(0,+\infty),\\ u_t + u\cdot\nabla u + \nabla P = \nu\Delta u - n\nabla\phi, & \text{in} \quad \mathbb{R}^2\times(0,+\infty),\\ \nabla\cdot u = 0, & \text{in} \quad \mathbb{R}^2\times(0,+\infty),\end{cases} \tag{1.7}$$

with initial data

$$(n,c,u)\big|_{t=0} = (n_0(x), c_0(x), u_0(x)), \qquad x\in\mathbb{R}^2. \tag{1.8}$$

We here remark that it is not clear whether the strategy of obtaining the Leray-Hopf weak solution for the pure incompressible Navier-Stokes equations can apply to the coupled model in three dimensions as considered in Theorem 1.1.

The weak solution of system (1.7) and (1.8) can be defined similar to Definition 1.1. Then we have the following global existence result.

Theorem 1.2 *Under the assumptions* (**A1**) − (**A5**)*, the Cauchy Problem* (1.7)-(1.8) *with $m\geq 1$ admits a global-in-time weak solution (n,c,u) satisfying $n(|x| + |\ln n|)\in L^\infty(0,T;L^1(\mathbb{R}^2))$ for any $T>0$. Moreover, there exists an entropy functional which grows in time at most exponentially, that is, one has the entropy estimate*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + \mathcal{D}(t) \leq C(m,\delta,\|\nabla\phi\|_\infty)\mathcal{E}(t) + C(m,\delta,\mu,\nu,\lambda_1,\lambda_2,c_M,\|\nabla\phi\|_\infty,\|n_0\|_1),$$

where $\mathcal{E}(t)$ and $\mathcal{D}(t)$ are the same as (1.5) *and* (1.6) *with $\mathbb{R}^2$ instead of $\mathbb{R}^3$, respectively.*

Remark 1.1 *When $\Omega\subset\mathbb{R}^2$ is bounded, Tao-Winkler [17] has established the global existence for any $m>1$, but it seems that the method used in [17] could not be generalized to $\mathbb{R}^2$ due to their entropy-type estimates depending on the size of the domain Ω. Moreover, we avoid the assumption $\phi\in L^\infty$ used in [17], which excludes the case of gravity $\phi = ax_1$ in $\mathbb{R}^2$.*

Remark 1.2 *Theorem 1.1 and Theorem 1.2 hold for bounded domains with no-flux conditions for c and n and with zero Dirichlet condition for u since our a priori estimates are independent of the domain. Moreover, since the stationary problem under appropriate boundary conditions is important, it would be interesting to generalize the current results to such situation.*

Our proofs for Theorem 1.1 and Theorem 1.2 are based on an improved analysis of the entropy functional $\mathcal{E}(t)$ already used in [6] and on an application of the generalized Galiardo-Nirbenberg-Sobolev interpolation with one of the summation exponents less than one. These two techniques allow us to obtain that $\mathcal{E}(t)$ can at most growing linearly in 3D (and exponentially in 2D) for any $m\geq 1$, which leads to the global existence of the weak solution.

In the whole paper, we will use the standard notion $\|v\|_p = \left(\int_{\mathbb{R}^N}|v(x)|^p\mathrm{d}x\right)^{\frac{1}{p}}$. And in general, the inessential uniform constants C or $C(m,\delta,\|n_0\|_1,\cdots)$ may change from line to line.

The rest of this paper is organized as follows. We establish the global existence of weak solutions to 3D chemotaxis-Stokes equations and prove Theorem 1.1 next section. Then we give a sketch of the proof of Theorem 1.2 in Section 3.

2 Chemotaxis-Stokes model: 3D case

In this section, we investigate the global existence of weak solutions to 3D chemotaxis-Stokes model. We first introduce a generalized result on Galiardo-Nirbenberg-Sobolev interpolation with *one of the summation exponents less than one*, which was strictly proved by Kiselev-Ryzhik [9] recently. Such a result is important for us to deal with the case $m > 2$, although from a mathematical point of view large values of m seem to enhance the tendency towards global solvability.

Lemma 2.1 *Let $v \in C_0^\infty(\mathbb{R}^N)$ with $N \geq 2$. Then*

$$\|v\|_q \leq C(q,N)\|v\|_r^{1-\theta}\|\nabla v\|_2^\theta, \qquad \theta = \frac{\frac{1}{r}-\frac{1}{q}}{\frac{1}{N}-\frac{1}{2}+\frac{1}{r}}.$$

Then inequality holds for all $q, r > 0$ such that $q > r$ and $\frac{1}{N}-\frac{1}{2}+\frac{1}{r} > 0$.

We now turn to the proof of Theorem 1.1, which is divided into several steps.

2.1 Uniform *a priori* estimates

In this subsection, our main purpose is to establish the following entropy estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + \mathcal{D}(t) \leq C(m,\delta,\mu,\nu,\lambda_1,\lambda_2,c_M,\|\nabla\phi\|_\infty,\|n_0\|_1) \tag{2.1}$$

where $\mathcal{E}(t)$ and $\mathcal{D}(t)$ are given by (1.5) and (1.6), respectively.

Since $\kappa(0) = 0$ and $\kappa'(s) \geq 0$, by parabolic comparison we have that n and c preserve the nonnegativity of the initial data as well as the inequality $c \leq \|c_0\|_\infty \leq c_M$. Moreover, a direct integration of the first equation in $(1.2)_1$ yields the mass conservation for n: $\int_{\mathbb{R}^3} n(t,x)\mathrm{d}x = \int_{\mathbb{R}^3} n_0(x)\mathrm{d}x$.

Multiplying equations $(1.2)_1$ with $1+\ln n$ and integrating give

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} n\ln n\mathrm{d}x + \frac{4\delta}{m}\int_{\mathbb{R}^3}\left|\nabla n^{\frac{m}{2}}\right|^2\mathrm{d}x = \int_{\mathbb{R}^3}\chi\nabla n\cdot\nabla c\mathrm{d}x. \tag{2.2}$$

Moreover, in the same way as in [6], multiplying the equation $(1.2)_2$ with Ψ', then with $\Delta\Psi$ and integrating over $\mathbb{R}^3$ give

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\nabla\Psi|^2\,\mathrm{d}x + \mu\int_{\mathbb{R}^3}(\Delta\Psi)^2\mathrm{d}x\\
=&\mu\int_{\mathbb{R}^3}\Psi''|\nabla c|^2\Delta\Psi\mathrm{d}x + \int_{\mathbb{R}^3}\Delta\Psi u\cdot\nabla\Psi + \sqrt{\kappa\chi}n\Delta\Psi\mathrm{d}x\\
=&-\mu\int_{\mathbb{R}^3}\frac{\mathrm{d}}{\mathrm{d}c}\sqrt{\frac{\kappa}{\chi}}|\nabla\Psi|^2\Delta\Psi\mathrm{d}x + \int_{\mathbb{R}^3}\Delta\Psi u\cdot\nabla\Psi - n\frac{\mathrm{d}\sqrt{\kappa\chi}}{\mathrm{d}c}\nabla c\cdot\nabla\Psi - \sqrt{\kappa\chi}\nabla n\cdot\nabla\Psi\mathrm{d}x\\
=&-\mu\int_{\mathbb{R}^3}\frac{\mathrm{d}}{\mathrm{d}c}\sqrt{\frac{\kappa}{\chi}}|\nabla\Psi|^2\Delta\Psi\mathrm{d}x + \int_{\mathbb{R}^3}\Delta\Psi u\cdot\nabla\Psi - n\frac{1}{\Psi'}\frac{\mathrm{d}\sqrt{\kappa\chi}}{\mathrm{d}c}|\nabla\Psi|^2 - \chi\nabla n\cdot\nabla c\mathrm{d}x.
\end{aligned} \tag{2.3}$$

It follows from integrating by parts that

$$\int_{\mathbb{R}^3}\frac{\mathrm{d}}{\mathrm{d}c}\sqrt{\frac{\kappa}{\chi}}|\nabla\Psi|^2\Delta\Psi\mathrm{d}x = -\int_{\mathbb{R}^3}\frac{\mathrm{d}^2}{\mathrm{d}c^2}\sqrt{\frac{\kappa}{\chi}}\sqrt{\frac{\kappa}{\chi}}|\nabla\Psi|^4\mathrm{d}x - 2\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\frac{\mathrm{d}}{\mathrm{d}c}\sqrt{\frac{\kappa}{\chi}}\frac{\partial\Psi}{\partial x_i}\frac{\partial\Psi}{\partial x_j}\frac{\partial^2\Psi}{\partial x_i x_j}\mathrm{d}x.$$

Substituting it into (2.3), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\nabla\Psi|^2\,\mathrm{d}x + \mu\int_{\mathbb{R}^3}\left((\Delta\Psi)^2 - 2\sum_{i,j=1}^{3}\frac{\mathrm{d}}{\mathrm{d}c}\sqrt{\frac{\kappa}{\chi}}\frac{\partial\Psi}{\partial x_i}\frac{\partial\Psi}{\partial x_j}\frac{\partial^2\Psi}{\partial x_i x_j}\right)\mathrm{d}x\\
=&\mu\int_{\mathbb{R}^3}\frac{\mathrm{d}^2}{\mathrm{d}c^2}\sqrt{\frac{\kappa}{\chi}}\sqrt{\frac{\kappa}{\chi}}|\nabla\Psi|^4\mathrm{d}x + \int_{\mathbb{R}^3}\Delta\Psi u\cdot\nabla\Psi - n\frac{1}{\Psi'}\frac{\mathrm{d}\sqrt{\kappa\chi}}{\mathrm{d}c}|\nabla\Psi|^2 - \chi\nabla n\cdot\nabla c\mathrm{d}x.
\end{aligned}$$

Noticing that

$$\int_{\mathbb{R}^3}(\Delta\Psi)^2\mathrm{d}x=\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\frac{\partial^2\Psi}{\partial x_i^2}\frac{\partial^2\Psi}{\partial x_j^2}\mathrm{d}x=\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\left(\frac{\partial^2\Psi}{\partial x_i\partial x_j}\right)^2\mathrm{d}x$$

by integrating by parts, we can deduce that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\nabla\Psi|^2\,\mathrm{d}x+\mu\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\left(\frac{\partial^2\Psi}{\partial x_i\partial x_j}-\frac{\mathrm{d}}{\mathrm{d}c}\sqrt{\frac{\kappa}{\chi}}\frac{\partial\Psi}{\partial x_i}\frac{\partial\Psi}{\partial x_j}\right)^2\mathrm{d}x\\&=\mu\int_{\mathbb{R}^3}\left(\left(\frac{\mathrm{d}}{\mathrm{d}c}\sqrt{\frac{\kappa}{\chi}}\right)^2+\frac{\mathrm{d}^2}{\mathrm{d}c^2}\sqrt{\frac{\kappa}{\chi}}\sqrt{\frac{\kappa}{\chi}}\right)|\nabla\Psi|^4\mathrm{d}x+\int_{\mathbb{R}^3}\Delta\Psi u\cdot\nabla\Psi-n\frac{1}{\Psi'}\frac{\mathrm{d}\sqrt{\kappa\chi}}{\mathrm{d}c}|\nabla\Psi|^2-\chi\nabla n\cdot\nabla c\mathrm{d}x.\end{aligned}$$

Then it follows from the identity

$$\left(\frac{\mathrm{d}}{\mathrm{d}c}\sqrt{\frac{\kappa}{\chi}}\right)^2+\frac{\mathrm{d}^2}{\mathrm{d}c^2}\sqrt{\frac{\kappa}{\chi}}\sqrt{\frac{\kappa}{\chi}}=\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}c^2}\left(\frac{\kappa}{\chi}\right)$$

that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\nabla\Psi|^2\,\mathrm{d}x+\mu\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\left(\frac{\partial^2\Psi}{\partial x_i\partial x_j}-\frac{\mathrm{d}}{\mathrm{d}c}\sqrt{\frac{\kappa}{\chi}}\frac{\partial\Psi}{\partial x_i}\frac{\partial\Psi}{\partial x_j}\right)^2\mathrm{d}x\\&=\frac{\mu}{2}\int_{\mathbb{R}^3}\frac{\mathrm{d}^2}{\mathrm{d}c^2}\left(\frac{\kappa}{\chi}\right)|\nabla\Psi|^4\mathrm{d}x+\int_{\mathbb{R}^3}\Delta\Psi u\cdot\nabla\Psi-n\frac{1}{\Psi'}\frac{\mathrm{d}\sqrt{\kappa\chi}}{\mathrm{d}c}|\nabla\Psi|^2-\chi\nabla n\cdot\nabla c\mathrm{d}x\\&=\frac{\mu}{2}\int_{\mathbb{R}^3}\frac{\mathrm{d}^2}{\mathrm{d}c^2}\left(\frac{\kappa}{\chi}\right)|\nabla\Psi|^4\mathrm{d}x-\int_{\mathbb{R}^3}\left(\sum_{i,j=1}^{3}\frac{\partial\Psi}{\partial x_i}\frac{\partial u_j}{\partial x_i}\frac{\partial\Psi}{\partial x_j}+n\frac{1}{\Psi'}\frac{\mathrm{d}\sqrt{\kappa\chi}}{\mathrm{d}c}|\nabla\Psi|^2+\chi\nabla n\cdot\nabla c\right)\mathrm{d}x.\end{aligned}\tag{2.4}$$

Combining (2.2) and (2.4), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\left(n\ln n+\frac{1}{2}|\nabla\Psi|^2\right)\mathrm{d}x+\frac{4\delta}{m}\int_{\mathbb{R}^3}\left|\nabla n^{\frac{m}{2}}\right|^2\mathrm{d}x+\mu\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\left(\frac{\partial^2\Psi}{\partial x_i\partial x_j}-\frac{\mathrm{d}}{\mathrm{d}c}\sqrt{\frac{\kappa}{\chi}}\frac{\partial\Psi}{\partial x_i}\frac{\partial\Psi}{\partial x_j}\right)^2\mathrm{d}x\\&-\frac{\mu}{2}\int_{\mathbb{R}^3}\frac{\mathrm{d}^2}{\mathrm{d}c^2}\left(\frac{\kappa}{\chi}\right)|\nabla\Psi|^4\mathrm{d}x+\int_{\mathbb{R}^3}n\frac{1}{\Psi'}\frac{\mathrm{d}\sqrt{\kappa\chi}}{\mathrm{d}c}|\nabla\Psi|^2\mathrm{d}x=-\int_{\mathbb{R}^3}\sum_{i,j=1}^{3}\frac{\partial\Psi}{\partial x_i}\frac{\partial u_j}{\partial x_i}\frac{\partial\Psi}{\partial x_j}\mathrm{d}x.\end{aligned}$$

For brevity, we set

$$2\lambda_1:=\frac{1}{2}\min_{0\le c\le c_M}-\frac{\mathrm{d}^2}{\mathrm{d}c^2}\left(\frac{\kappa}{\chi}\right)>0,\quad 2\lambda_2:=\min_{0\le c\le c_M}\frac{1}{\Psi'}\frac{\mathrm{d}\sqrt{\kappa\chi}}{\mathrm{d}c}=\min_{0\le c\le c_M}\frac{1}{2\chi}\frac{\mathrm{d}(\kappa\chi)}{\mathrm{d}c}>0.\tag{2.5}$$

Then we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\left(n\ln n+\frac{1}{2}|\nabla\Psi|^2\right)\mathrm{d}x+\frac{4\delta}{m}\int_{\mathbb{R}^3}\left|\nabla n^{\frac{m}{2}}\right|^2\mathrm{d}x+\mu\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\left(\frac{\partial^2\Psi}{\partial x_i\partial x_j}-\frac{\mathrm{d}}{\mathrm{d}c}\sqrt{\frac{\kappa}{\chi}}\frac{\partial\Psi}{\partial x_i}\frac{\partial\Psi}{\partial x_j}\right)^2\mathrm{d}x\\&\quad+2\lambda_1\mu\int_{\mathbb{R}^3}|\nabla\Psi|^4\mathrm{d}x+2\lambda_2\int_{\mathbb{R}^3}n|\nabla\Psi|^2\mathrm{d}x\\&\le-\int_{\mathbb{R}^3}\sum_{i,j=1}^{3}\frac{\partial\Psi}{\partial x_i}\frac{\partial u_j}{\partial x_i}\frac{\partial\Psi}{\partial x_j}\mathrm{d}x\le\int_{\mathbb{R}^3}|\nabla\Psi|^2\,|\nabla u|\,\mathrm{d}x\\&\le\lambda_1\mu\int_{\mathbb{R}^3}|\nabla\Psi|^4\mathrm{d}x+\frac{1}{4\lambda_1\mu}\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x.\end{aligned}$$

That is,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\left(n\ln n+\frac{1}{2}|\nabla\Psi|^2\right)\mathrm{d}x+\frac{4\delta}{m}\int_{\mathbb{R}^3}\left|\nabla n^{\frac{m}{2}}\right|^2\mathrm{d}x+\lambda_1\mu\int_{\mathbb{R}^3}|\nabla\Psi|^4\mathrm{d}x+2\lambda_2\int_{\mathbb{R}^3}n|\nabla\Psi|^2\mathrm{d}x\\&+\mu\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\left(\frac{\partial^2\Psi}{\partial x_i\partial x_j}-\frac{\mathrm{d}}{\mathrm{d}c}\sqrt{\frac{\kappa}{\chi}}\frac{\partial\Psi}{\partial x_i}\frac{\partial\Psi}{\partial x_j}\right)^2\mathrm{d}x\le\frac{1}{4\lambda_1\mu}\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x.\end{aligned}\tag{2.6}$$

We now establish the dissipation of u. For this purpose, we multiply equation $(1.2)_3$ with u and then integrate to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|u|^2\mathrm{d}x+\nu\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x=-\int_{\mathbb{R}^3}nu\cdot\nabla\phi\mathrm{d}x. \tag{2.7}$$

To estimate the right hand side, we use Hölder's inequality and Sobolev's embedding $\dot{H}^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$ and then obtain

$$-\int_{\mathbb{R}^3}nu\cdot\nabla\phi\mathrm{d}x\le\|\nabla\phi\|_\infty\|n\|_{\frac{6}{5}}\|u\|_6\le C\|\nabla\phi\|_\infty\|n\|_{\frac{6}{5}}\|\nabla u\|_2\,. \tag{2.8}$$

To bound the $L^{\frac{6}{5}}$ norm of n, we use the Gagliardo-Nirenberg-Sobolev interpolation inequality and deduce that

$$\begin{aligned}\|n\|_{\frac{6}{5}}=\left\|n^{\frac{m}{2}}\right\|_{\frac{12}{5m}}^{\frac{2}{m}}&\le C(m)\left(\left\|n^{\frac{m}{2}}\right\|_{\frac{2}{m}}^{\frac{5m-2}{2(3m-1)}}\left\|\nabla n^{\frac{m}{2}}\right\|_2^{\frac{m}{2(3m-1)}}\right)^{\frac{2}{m}}\\&=C(m)\|n\|_1^{\frac{5m-2}{2(3m-1)}}\left\|\nabla n^{\frac{m}{2}}\right\|_2^{\frac{1}{3m-1}}=C(m)\|n_0\|_1^{\frac{5m-2}{2(3m-1)}}\left\|\nabla n^{\frac{m}{2}}\right\|_2^{\frac{1}{3m-1}},\end{aligned} \tag{2.9}$$

where we have used Lemma 2.1 to deal with the case $m>2$ and the mass conservation for n. Then we insert (2.8)-(2.9) into (2.7) and use Young's inequality to obtain

$$\begin{aligned}&\frac{1}{2\lambda_1\mu\nu}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}u^2\mathrm{d}x+\frac{1}{\lambda_1\mu}\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\\&\le C(m,\mu,\nu,\lambda_1)\|\nabla\phi\|_\infty\|n_0\|_1^{\frac{5m-2}{2(3m-1)}}\left\|\nabla n^{\frac{m}{2}}\right\|_2^{\frac{1}{3m-1}}\|\nabla u\|_2\\&\le\frac{1}{8\lambda_1\mu}\|\nabla u\|_2^2+C(m,\mu,\nu,\lambda_1)\|\nabla\phi\|_\infty^2\|n_0\|_1^{\frac{5m-2}{3m-1}}\left\|\nabla n^{\frac{m}{2}}\right\|_2^{\frac{2}{3m-1}}\\&\le\frac{1}{8\lambda_1\mu}\|\nabla u\|_2^2+\frac{\delta}{m}\left\|\nabla n^{\frac{m}{2}}\right\|_2^2+C(m,\delta,\mu,\nu,\lambda_1,\|\nabla\phi\|_\infty,\|n_0\|_1).\end{aligned} \tag{2.10}$$

Finally, we control the first-order spatial moment of $n(t,x)$. For this purpose, we multiply $(1.2)_1$ by $\langle x\rangle$ and take integration to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\langle x\rangle n\mathrm{d}x=\int_{\mathbb{R}^3}nu\cdot\nabla\langle x\rangle\,\mathrm{d}x+\delta\int_{\mathbb{R}^3}n^m\Delta\langle x\rangle\,\mathrm{d}x+\int_{\mathbb{R}^3}\sqrt{\kappa\chi}\,n\nabla\Psi\cdot\nabla\langle x\rangle\,\mathrm{d}x. \tag{2.11}$$

We estimate each term on the right hand side of the above identity. For the first term, we take similar calculation as (2.7) and use Hölder's inequality, Sobolev's embedding, the Gagliardo-Nirenberg-Sobolev interpolation inequality and Young's inequality to obtain

$$\begin{aligned}\int_{\mathbb{R}^3}nu\cdot\nabla\langle x\rangle\,\mathrm{d}x&\le\|\nabla\langle x\rangle\|_\infty\|n\|_{\frac{6}{5}}\|u\|_6\le C(m)\|n\|_{\frac{6}{5}}\|\nabla u\|_2\\&\le C\left\|\nabla n^{\frac{m}{2}}\right\|_2^{\frac{1}{3m-1}}\|\nabla u\|_2\le\frac{1}{4\lambda_1\mu}\|\nabla u\|_2^2+C(m,\mu,\lambda_1)\left\|\nabla n^{\frac{m}{2}}\right\|_2^{\frac{2}{3m-1}}\\&\le\frac{1}{4\lambda_1\mu}\|\nabla u\|_2^2+\frac{\delta}{m}\left\|\nabla n^{\frac{m}{2}}\right\|_2^2+C(m,\delta,\mu,\lambda_1).\end{aligned} \tag{2.12}$$

Here we used the boundedness of $\|\nabla\langle x\rangle\|_\infty$ and the mass conservation for n. The second term can be straightforwardly estimated as

$$\begin{aligned}\delta\int_{\mathbb{R}^3}n^m\Delta\langle x\rangle\,\mathrm{d}x&\le\delta\|\Delta\langle x\rangle\|_\infty\|n^m\|_1=\delta\|\Delta\langle x\rangle\|_\infty\|n^{\frac{m}{2}}\|_2^2\\&\le C\delta\|\Delta\langle x\rangle\|_\infty\left(\|n^{\frac{m}{2}}\|_{\frac{2}{m}}^{\frac{2}{3(m-1)+2}}\|\nabla n^{\frac{m}{2}}\|_2^{\frac{3(m-1)}{3(m-1)+2}}\right)^2=C\delta\|\Delta\langle x\rangle\|_\infty\|n\|_1^{\frac{2m}{3(m-1)+2}}\|\nabla n^{\frac{m}{2}}\|_2^{\frac{6(m-1)}{3(m-1)+2}}\\&=C\delta\|\Delta\langle x\rangle\|_\infty\|n_0\|_1^{\frac{2m}{3(m-1)+2}}\|\nabla n^{\frac{m}{2}}\|_2^{\frac{6(m-1)}{3(m-1)+2}}\\&\le\frac{\delta}{m}\|\nabla n^{\frac{m}{2}}\|_2^2+C(m,\delta,\|n_0\|_1)\end{aligned} \tag{2.13}$$

by the boundedness of $\|\Delta\langle x\rangle\|_\infty$ and the mass conservation for n. For the third term, it follows from Young's inequality and and the mass conservation for n that

$$\begin{aligned}&\int_{\mathbb{R}^3}\sqrt{\kappa(c)\chi(c)}\,n\nabla\Psi\cdot\nabla\langle x\rangle\,\mathrm{d}x\le\lambda_2\int_{\mathbb{R}^3}n|\nabla\Psi|^2\mathrm{d}x+C(\lambda_2)\int_{\mathbb{R}^3}\kappa(c)\chi(c)n\mathrm{d}x\\&\le\lambda_2\int_{\mathbb{R}^3}n|\nabla\Psi|^2\mathrm{d}x+C(\lambda_2)\left(\sup_{0\le c\le c_M}\kappa(c)\chi(c)\right)\int_{\mathbb{R}^3}n_0\mathrm{d}x\le\lambda_2\int_{\mathbb{R}^3}n|\nabla\Psi|^2\mathrm{d}x+C(\lambda_2,c_M,\|n_0\|_1).\end{aligned}\tag{2.14}$$

Combining the above estimates (2.11)-(2.14) yields that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\langle x\rangle n\mathrm{d}x\le\frac{\delta}{m}\left\|\nabla n^{\frac{m}{2}}\right\|_2^2+\frac{1}{4\lambda_1\mu}\|\nabla u\|_2^2+\lambda_2\int_{\mathbb{R}^3}n|\nabla\Psi|^2\mathrm{d}x+C(m,\delta,\mu,\lambda_2,c_M,\|n_0\|_1).\tag{2.15}$$

By combining the inequalities (2.6), (2.10) and (2.15), we see that the desired entropy estimate (2.1) holds.

2.2 Approximate solutions

For $\epsilon\in(0,1)$, we consider the approximate problem given by

$$\begin{cases}n_{\epsilon t}+u_\epsilon\cdot\nabla n_\epsilon=\delta\nabla\cdot\left(D_\epsilon(n_\epsilon)\nabla n_\epsilon\right)-\nabla\cdot\left(n_\epsilon\left((\chi(c_\epsilon)\nabla c_\epsilon)*\sigma_\epsilon\right)\right), & \text{in } \mathbb{R}^3\times(0,+\infty),\\ c_{\epsilon t}+u_\epsilon\cdot\nabla c_\epsilon=\mu\Delta c_\epsilon-\kappa(c_\epsilon)\left(n_\epsilon*\sigma_\epsilon\right), & \text{in } \mathbb{R}^3\times(0,+\infty),\\ u_{\epsilon t}+\nabla P_\epsilon=\nu\Delta u_\epsilon-(n_\epsilon\nabla\phi)*\sigma_\epsilon, & \text{in } \mathbb{R}^3\times(0,+\infty),\\ \nabla\cdot u_\epsilon=0, & \text{in } \mathbb{R}^3\times(0,+\infty)\end{cases}\tag{2.16}$$

with initial data

$$(n_\epsilon,c_\epsilon,u_\epsilon)\big|_{t=0}=\left(n_0*\sigma_\epsilon(x),c_0*\sigma_\epsilon(x),u_0*\sigma_\epsilon(x)\right),\qquad x\in\mathbb{R}^3,$$

where $D_\epsilon(s)=m(s+\epsilon)^{m-1}$ for $s\ge0$ and σ_ϵ is a mollifier.

For any given $\epsilon\in(0,1)$, the global existence of solutions to the approximate problem above can be proven by freezing the velocity field u_ϵ in the equations for c_ϵ and n_ϵ and by solving them globally in time using the standard procedure as in [13]. That is, we first consider the initial-value problem

$$\begin{cases}n_{\epsilon t}+u_\epsilon\cdot\nabla n_\epsilon=\delta\nabla\cdot\left(D_\epsilon(n_\epsilon)\nabla n_\epsilon\right)-\nabla\cdot\left(n_\epsilon\left((\chi(c_\epsilon)\nabla c_\epsilon)*\sigma_\epsilon\right)\right), & \text{in } \mathbb{R}^3\times(0,+\infty),\\ c_{\epsilon t}+u_\epsilon\cdot\nabla c_\epsilon=\mu\Delta c_\epsilon-\kappa(c_\epsilon)\left(n_\epsilon*\sigma_\epsilon\right), & \text{in } \mathbb{R}^3\times(0,+\infty),\\ (n_\epsilon,c_\epsilon)\big|_{t=0}=\left(n_0*\sigma_\epsilon(x),c_0*\sigma_\epsilon(x)\right), & \text{in } \mathbb{R}^3\end{cases}$$

for the frozen divergence free vector field u_ϵ. The estimates on n_ϵ and c_ϵ are independent of u_ϵ because of the incompressibility condition on u_ϵ. Then, one solves the initial-value problem for the equation u_ϵ

$$\begin{cases}u_{\epsilon t}+\nabla P_\epsilon=\nu\Delta u_\epsilon-(n_\epsilon\nabla\phi)*\sigma_\epsilon, & \text{in } \mathbb{R}^3\times(0,+\infty),\\ \nabla\cdot u_\epsilon=0, & \text{in } \mathbb{R}^3\times(0,+\infty)\\ u_\epsilon\big|_{t=0}=u_0*\sigma_\epsilon(x), & \text{in } \mathbb{R}^3,\end{cases}$$

by following the usual theory in [12]. A simple bootstrap argument can be easily adopted to provide a strong solution in the limit. As some of the needed estimates will be reproduced in the sequel, we shall omit the details of the above mentioned procedure.

Then for the regularized system, we can take similar calculations as the *a priori* estimates to obtain the entropy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\epsilon(t)+\mathcal{D}_\epsilon(t)\le C(m,\delta,\mu,\nu,\lambda_1,\lambda_2,c_M,\|\nabla\phi\|_\infty,\|n_0\|_1)$$

with

$$\mathcal{E}_\epsilon(t):=\int_{\mathbb{R}^3}\left(n_\epsilon\ln n_\epsilon+n_\epsilon\langle x\rangle+\frac{1}{2}|\nabla\Psi_\epsilon|^2+\frac{1}{2\lambda_1\mu\nu}|u_\epsilon|^2\right)\mathrm{d}x$$

and

$$\mathcal{D}_\epsilon(t) := \frac{2\delta}{m}\int_{\mathbb{R}^3}\left|\nabla n_\epsilon^{\frac{m}{2}}\right|^2 \mathrm{d}x + \frac{1}{2\lambda_1\mu}\int_{\mathbb{R}^3}|\nabla u_\epsilon|^2\,\mathrm{d}x + \lambda_1\mu\int_{\mathbb{R}^3}|\nabla\Psi_\epsilon|^4\mathrm{d}x + \lambda_2\int_{\mathbb{R}^3} n_\epsilon|\nabla\Psi_\epsilon|^2\mathrm{d}x$$
$$+\mu\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\left(\frac{\partial^2\Psi_\epsilon}{\partial x_i\partial x_j} - \frac{\mathrm{d}}{\mathrm{d}c}\sqrt{\frac{\kappa(c_\epsilon)}{\chi(c_\epsilon)}}\frac{\partial\Psi_\epsilon}{\partial x_i}\frac{\partial\Psi_\epsilon}{\partial x_j}\right)^2\mathrm{d}x.$$

Now we can summarize the following property for the regularized system. Firstly, by maximum principle, we have $n_\epsilon \geq 0$, $0 \leq c_\epsilon \leq c_M$. Also, it follows from the entropy inequality that

$$u_\epsilon \in L^\infty((0,T);L^2(\mathbb{R}^3,\mathbb{R}^3)) \cap L^2((0,T);H^1(\mathbb{R}^3,\mathbb{R}^3)), \quad \left|\nabla n_\epsilon^{\frac{m}{2}}\right| \in L^2((0,T);L^2(\mathbb{R}^3)).$$

Then we claim that $n_\epsilon|\ln n_\epsilon| \in L^\infty((0,T);L^1(\mathbb{R}^3))$. Indeed, this follows from

$$\int_{\mathbb{R}^3} n_\epsilon|\ln n_\epsilon|\mathrm{d}x = \int_{\mathbb{R}^3} n_\epsilon \ln n_\epsilon \mathrm{d}x + 2\int_{\mathbb{R}^3} n_\epsilon \ln\frac{1}{n_\epsilon}\mathbf{I}_{\{x\,|\,n_\epsilon(x)\leq 1\}}\mathrm{d}x$$

and

$$0 \leq \int_{\mathbb{R}^3} n_\epsilon \ln\frac{1}{n_\epsilon}\mathbf{I}_{\{x\,|\,n_\epsilon(x)\leq 1\}}\mathrm{d}x = \int_{\mathbb{R}^3} n_\epsilon \ln\frac{1}{n_\epsilon}\mathbf{I}_{\{x\,|\,e^{-\langle x\rangle}<n_\epsilon(x)\leq 1\}}\mathrm{d}x + \int_{\mathbb{R}^3} n_\epsilon \ln\frac{1}{n_\epsilon}\mathbf{I}_{\{x\,|\,n_\epsilon(x)\leq e^{-\langle x\rangle}\}}\mathrm{d}x \leq \int_{\mathbb{R}^3}\langle x\rangle\, n_\epsilon\mathrm{d}x + C,$$

where $\mathbf{I}_A$ denotes the indicator function of a subset $A \subset \mathbb{R}$.

Next, by using Gagliardo-Nirenberg-Sobolev interpolation inequality and the mass conservation for n we have

$$\|n_\epsilon\|_2 = \left\|n_\epsilon^{\frac{m}{2}}\right\|_{\frac{4}{m}}^{\frac{2}{m}} \leq C\left(\left\|n_\epsilon^{\frac{m}{2}}\right\|_{\frac{2}{m}}^{\frac{3m-2}{2(3m-1)}}\left\|\nabla n_\epsilon^{\frac{m}{2}}\right\|_2^{\frac{3m}{2(3m-1)}}\right)^{\frac{2}{m}} = C\,\|n_\epsilon\|_1^{\frac{3m-2}{2(3m-1)}}\left\|\nabla n_\epsilon^{\frac{m}{2}}\right\|_2^{\frac{3}{3m-1}} \leq C\left\|\nabla n_\epsilon^{\frac{m}{2}}\right\|_2^{\frac{3}{3m-1}},$$

which implies that $n_\epsilon \in L^{p_0}((0,T);L^2(\mathbb{R}^3))$ with $p_0 := \frac{2(3m-1)}{3} > 1$, and

$$\left\|n_\epsilon^{\frac{m}{2}}\right\|_2^2 \leq C\left(\left\|n_\epsilon^{\frac{m}{2}}\right\|_{\frac{2}{m}}^{\frac{2}{3m-1}}\left\|\nabla n_\epsilon^{\frac{m}{2}}\right\|_2^{\frac{3(m-1)}{3m-1}}\right)^2 = C\,\|n_\epsilon\|_1^{\frac{2m}{3m-1}}\left\|\nabla n_\epsilon^{\frac{m}{2}}\right\|_2^{\frac{6(m-1)}{3m-1}} \leq C\left\|\nabla n_\epsilon^{\frac{m}{2}}\right\|_2^{\frac{6(m-1)}{3m-1}}.$$

This together with $|\nabla n_\epsilon^{\frac{m}{2}}| \in L^2((0,T);L^2(\mathbb{R}^3))$ implies that $n_\epsilon^{\frac{m}{2}} \in L^2((0,T);H^1(\mathbb{R}^3))$ for any given $T > 0$. By the standard regularity for Stokes equations [12, 13], one can show that $u_{\epsilon t} \in L^{p_0}((0,T);H^{-1}(\mathbb{R}^3,\mathbb{R}^3))$. Then using the parabolic regularity for the c_ϵ-equation together with the space-time bound on $n_\epsilon\kappa(c_\epsilon)$ and the regularity of u_ϵ, we easily deduce that $c_\epsilon \in L^\infty((0,T);H^1(\mathbb{R}^3)) \cap L^{p_0}((0,T);H^2(\mathbb{R}^3))$ and $c_{\epsilon t} \in L^{p_0}((0,T);L^2(\mathbb{R}^3))$.

Finally, we have $n_{\epsilon t} \in L^p((0,T);H^{-3}(\mathbb{R}^3))$. Indeed, it is a direct consequence of the estimate

$$\begin{aligned}
< n_{\epsilon t},\varphi > =& < n_\epsilon u_\epsilon,\nabla\varphi > + \delta < n_\epsilon^m,\Delta\varphi > + < n_\epsilon\left((\chi(c_\epsilon)\nabla c_\epsilon) * \sigma_\epsilon\right),\nabla\varphi > \\
\leq\ & \|n_\epsilon\|_2\|u_\epsilon\|_2\|\nabla\varphi\|_\infty + \delta\left\|n_\epsilon^m\right\|_2\|\Delta\varphi\|_2 + C\|n_\epsilon\|_2\|\nabla c_\epsilon\|_2\|\nabla\varphi\|_\infty \\
\leq\ & C\|n_\epsilon\|_2\|\varphi\|_{H^3} + \delta\left\|n_\epsilon^m\right\|_2\|\varphi\|_{H^3} + C\|n_\epsilon\|_2\|\varphi\|_{H^3},
\end{aligned}$$

where we used the previous estimates $u_\epsilon \in L^\infty((0,T);L^2(\mathbb{R}^3,\mathbb{R}^3))$, $c_\epsilon \in L^\infty((0,T);H^1(\mathbb{R}^3))$ and the Sobolev's embedding $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ in the last inequality.

2.3 Passing to the limit

Now we can use the above *a priori* estimates to complete our arguments by means of the following Aubin-Lions compactness lemma.

Lemma 2.2 (Aubin-Lions) *[16] Let X_0, X and X_1 be three Banach spaces with $X_0 \subset X \subset X_1$. Suppose that X_0 and X_1 are reflexive. For $1 < p, q < +\infty$, set*

$$W = \left\{u \in L^p([0,T];X_0)\,\middle|\,\dot{u} \in L^q([0,T];X_1)\right\}.$$

If X_0 is compactly embedded in X and X is continuously embedded in X_1, then the embedding of W into $L^p([0,T];X)$ is also compact.

Firstly, it follows from the *a priori* estimates $n_\epsilon \in L^p((0,T);L^2(\mathbb{R}^3))$ and $n_{\epsilon t} \in L^p((0,T);H^{-3}(\mathbb{R}^3))$ that there exists a subsequence of n_ϵ converging strongly to some function n in $L^p((0,T);L^2_{\rm loc}(\mathbb{R}^3))$. Since $n_\epsilon^{\frac{m}{2}} \in L^2((0,T);H^1(\mathbb{R}^3))$, we can further take the subsequence of n_ϵ such that for a.e. $t \in [0,T]$, $n_\epsilon^{\frac{m}{2}}(t,\cdot)$ converges strongly to some function $v^{\frac{m}{2}}(t,\cdot)$ in $L^2_{\rm loc}(\mathbb{R}^3)$. Then we can conclude that such a subsequence satisfies that n_ϵ converge strongly to v (and thus $v = n$ by the uniqueness of limit) in $L^1((0,T);L^1_{\rm loc}(\mathbb{R}^3))$ and that n_ϵ^m converges strongly to v^m in $L^1((0,T);L^1_{\rm loc}(\mathbb{R}^3))$. Indeed, for any given bounded $\Omega \subset \mathbb{R}^3$, we have

$$\begin{aligned}\int_\Omega \left|n_\epsilon^m - v^m\right| \mathrm{d}x &= \int_\Omega \left|\left(n_\epsilon^{\frac{m}{2}} + v^{\frac{m}{2}}\right)\left(n_\epsilon^{\frac{m}{2}} - v^{\frac{m}{2}}\right)\right| \mathrm{d}x \le \left\|n_\epsilon^{\frac{m}{2}} + v^{\frac{m}{2}}\right\|_{L^2(\Omega)} \left\|n_\epsilon^{\frac{m}{2}} - v^{\frac{m}{2}}\right\|_{L^2(\Omega)} \\ &\le C\left\|n_\epsilon^{\frac{m}{2}} - v^{\frac{m}{2}}\right\|_{L^2(\Omega)} \to 0 \quad \text{as} \quad \epsilon \to 0,\end{aligned}$$

and thus

$$\int_\Omega |n_\epsilon - v| \,\mathrm{d}x = |\Omega|^{1-\frac{1}{m}} \left(\int_\Omega |n_\epsilon - v|^m \,\mathrm{d}x\right)^{\frac{1}{m}} \le |\Omega|^{1-\frac{1}{m}} \left(\int_\Omega \left|n_\epsilon^m - v^m\right| \mathrm{d}x\right)^{\frac{1}{m}} \to 0 \quad \text{as} \quad \epsilon \to 0,$$

which verify our conclusions.

Next, we can carry out the similar process to see that there exists a subsequence of c_ϵ converges strongly to some function c in $L^p((0,T);H^1_{\rm loc}(\mathbb{R}^3))$ and a subsequence of u_ϵ converges strongly to some function u in $L^2((0,T);L^2_{\rm loc}(\mathbb{R}^3))$ since $c_\epsilon \in L^p((0,T);H^2(\mathbb{R}^3))$, $c_{\epsilon t} \in L^p((0,T);L^2(\mathbb{R}^3))$, $u_\epsilon \in L^2((0,T);H^1(\mathbb{R}^3,\mathbb{R}^3))$ and $u_{\epsilon t} \in L^p((0,T);H^{-1}(\mathbb{R}^3,\mathbb{R}^3))$.

Finally $\kappa(c_\epsilon)$ and $\chi(c_\epsilon)$ converge almost everywhere, and thus the product terms $n_\epsilon u_\epsilon$, $n_\epsilon \chi(c_\epsilon)\nabla c_\epsilon$, $c_\epsilon u_\epsilon$ and $n_\epsilon \kappa(c_\epsilon) c_\epsilon$ are weakly convergent in $L^1((0,T);L^1_{\rm loc}(\mathbb{R}^3))$.

Therefore we can pass to the weak limit in (2.16) and thus the limit (n,c,u) is a weak solution of equations (1.2); see more discussions in [8, 17, 18]. Moreover, such a solution satisfies (1.4). This completes the proof of Theorem 1.1. □

3 Chemotaxis-Navier-Stokes model: 2D case

In this section, we study the global existence of global solutions to 2D chemotaxis-Navier-Stokes model (1.7). The proof is similar to the 3D chemotaxis-Stokes model (1.7). The key difference between $\mathbb{R}^2$ and $\mathbb{R}^3$ is to deal with $\|\nabla u\|_2$ due to $\dot{H}^1(\mathbb{R}^2) \not\hookrightarrow L^6(\mathbb{R}^2)$. To overcome this difficulty, we instead use the Sobolev's embedding $W^{1,2}(\mathbb{R}^2) \hookrightarrow L^6(\mathbb{R}^2)$, which results in the entropy functional growing at most exponentially in time for $\mathbb{R}^2$ instead of at most linearly for $\mathbb{R}^3$.

Proof of Theorem 1.2: Repeating the proof of Theorem 1.1, we can obtain the 2D versions of (2.6) and (2.7). We now estimate the right hand side of (2.7) as

$$\begin{aligned}-\int_{\mathbb{R}^2} nu\cdot\nabla\phi \mathrm{d}x &\le \|\nabla\phi\|_\infty \|n\|_{\frac{6}{5}} \|u\|_6 \le C\|\nabla\phi\|_\infty \|n\|_{\frac{6}{5}} (\|\nabla u\|_2 + \|u\|_2) \\ &\le C(m,\|\nabla\phi\|_\infty)\left\|\nabla n^{\frac{m}{2}}\right\|_2^{\frac{1}{3m}} \|\nabla u\|_2 + C(m,\|\nabla\phi\|_\infty)\left\|\nabla n^{\frac{m}{2}}\right\|_2^{\frac{1}{3m}} \|u\|_2 \\ &\le \frac{\delta}{m}\left\|\nabla n^{\frac{m}{2}}\right\|_2^2 + \frac{1}{2\lambda_1\mu}\|\nabla u\|_2^2 + C(m,\delta,\|\nabla\phi\|_\infty)\|u\|_2^2 + C(m,\delta,\mu,\lambda_1,\|\nabla\phi\|_\infty),\end{aligned}$$

where we used the estimates

$$\|n\|_{\frac{6}{5}} = \left\|n^{\frac{m}{2}}\right\|_{\frac{12}{5m}}^{\frac{2}{m}} \le C(m)\left(\left\|n^{\frac{m}{2}}\right\|_{\frac{2}{m}}^{\frac{5}{6}} \left\|\nabla n^{\frac{m}{2}}\right\|_2^{\frac{1}{6}}\right)^{\frac{2}{m}} = C(m)\|n\|_1^{\frac{5}{6}} \left\|\nabla n^{\frac{m}{2}}\right\|_2^{\frac{1}{3m}} = C(m)\|n_0\|_1^{\frac{5}{6}} \left\|\nabla n^{\frac{m}{2}}\right\|_2^{\frac{1}{3m}}$$

by the general Gagliardo-Nirenberg-Sobolev interpolation inequality (see Lemma 2.1).

It then follows that

$$\begin{aligned}&\frac{1}{2\lambda_1\mu\nu}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} u^2 \mathrm{d}x + \frac{1}{2\lambda_1\mu}\int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x \\ &\le \frac{\delta}{m}\left\|\nabla n^{\frac{m}{2}}\right\|_2^2 + C(m,\delta,\|\nabla\phi\|_\infty)\|u\|_2^2 + C(m,\delta,\mu,\nu,\lambda_1,\|\nabla\phi\|_\infty,\|n_0\|_1).\end{aligned}$$

The further similar calculations can be used to estimate the first term on the right side of (2.11). Then we can establish the following entropy estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + \mathcal{D}(t) \leq C(m, \delta, \|\nabla\phi\|_\infty)\mathcal{E}(t) + C(m, \delta, \mu, \nu, \lambda_1, \lambda_2, c_M, \|\nabla\phi\|_\infty, \|n_0\|_1),$$

where $\mathcal{E}(t)$ and $\mathcal{D}(t)$ are the same as (1.5) and (1.6), respectively, for $\mathbb{R}^2$ instead for $\mathbb{R}^3$.

Then similar to $\mathbb{R}^3$ case, we can construct a regularized system for (1.7)-(1.8) and establish the uniform estimates of approximation solutions. The use of Aubin-Lions compactness theorem ensures that we can obtain at least a global weak solution to (1.7)-(1.8). This completes the proof of Theorem 1.2. □

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