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STABILITY OF RAREFACTION WAVE AND BOUNDARY LAYER FOR OUTFLOW PROBLEM ON THE TWO-FLUID NAVIER-STOKES-POISSON EQUATIONS

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Abstract. In this paper, we are concerned with the initial boundary value problem on the two-fluid Navier-Stokes-Poisson system in the half-line $\mathbb{R}_+$. We establish the global-in-time asymptotic stability of the rarefaction wave and the boundary layer both for the outflow problem under the smallness assumption on initial perturbation, where the strength of the rarefaction wave is not necessarily small while the strength of the boundary layer is additionally supposed to be small. Here, the large initial data with densities far from vacuum is also allowed in the case of the non-degenerate boundary layer. The results show that the large-time behavior of solutions coincides with the one for the single Navier-Stokes system in the absence of the electric field. The proof is based on the classical energy method. The main difficulty in the analysis comes from the slower time-decay rate of the system caused by the appearance of the electric field. To overcome it, we use the coupling property of the two-fluid equations to capture the dissipation of the electric field interacting with the nontrivial asymptotic profile.

1. **Introduction.** The two-fluid Navier-Stokes-Poisson (called NSP in the sequel for simplicity) system is a model used to simulate the transport of charged particles (e.g., electrons and ions). It consists of the compressible Navier-Stokes equations of two-fluid under the influence of the electrostatic potential force governed by the self-consistent Poisson equation. In this paper, we are concerned with the two-fluid NSP system in half-line $\mathbb{R}_+ =: [0, +\infty]$, taking the form of

$$\begin{cases} \partial_t \rho_i + \partial_x(\rho_i u_i) = 0, \\ \partial_t(\rho_i u_i) + \partial_x(\rho_i u_i^2 + P_i(\rho_i)) = \mu_i (u_i)_{xx} + \rho_i E, \\ \partial_t \rho_e + \partial_x(\rho_e u_e) = 0, \\ \partial_t(\rho_e u_e) + \partial_x(\rho_e u_e^2 + P_e(\rho_e)) = \mu_e (u_e)_{xx} - \rho_e E, \\ E = E(x,t) = -\int_x^\infty (\rho_i - \rho_e)(y,t)\, dy, \end{cases} \tag{1}$$

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with appropriate initial and boundary data that we shall clarify later on. Here $\rho_{i,e} = \rho_{i,e}(x,t) \geq 0$ and $u_{i,e} = u_{i,e}(x,t) \in \mathbb{R}$ with $x \geq 0$ and $t \geq 0$ are the density and velocity of ions and electrons, respectively. $P_{i,e}(\cdot)$ is the pressure depending only on the density, $\mu_{i,e}$ is the viscosity coefficient, and E is the electric field satisfying

$$E_x = \rho_i - \rho_e.$$

Through this paper, we assume that two fluids of electrons and ions have the same equation of state $P_i(\cdot) = P_e(\cdot) = P(\cdot)$ with $P(\rho) = a\rho^\gamma$ for constants $a > 0$ and $\gamma \geq 1$, and also they have the same viscosity coefficients $\mu_i = \mu_e = 1$.

There have been extensive mathematical studies of the NSP system. Here we mention some of them. The global existence and the optimal time convergence rates of the classical solution around a constant state were obtained in [9, 18, 19, 30]. Later, the pointwise estimate of the solution was established in [29]. The global well-posedness in the Besov type space for the NSP system was also proved in [8]. Green's function and large time behavior were considered in [6] even for the case when the magnetic field is included and thus the Maxwell equations are taken into account. From those work, a common feature shows that the momentum of the NSP system decays at the slower rate than that of the compressible Navier-Stokes system in the absence of the electric field, which thus implies that the electric field could affect the large time behavior of the solution and produce some additional difficulties of analysis. In addition, we also mention that the quasi-neutral limit and some related asymptotic limits were considered in [5, 28], and the global existence and nonexistence were discussed in [4, 2].

In order to study the large time behavior of solutions to the NSP system, we notice that in the simplified case $E = 0$, the problem is reduced to consider the single quasineutral Navier-Stokes system in the form of

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x\left(\rho u^2 + P(\rho)\right) = u_{xx}. \end{cases} \tag{2}$$

The large-time behavior of solutions to the Cauchy problem on the system (2) with initial data $(\rho_0, u_0)(x)$ that admits the limit $(\rho_\pm, u_\pm)$ at $\pm\infty$ is basically described by the superposition of viscous shock waves, viscous contact discontinuities and rarefaction waves. We refer to [10, 12, 14, 15, 20, 22, 23, 27] and references therein. In the case of the initial boundary value problem for (2) over $\mathbb{R}_+$, the boundary data is prescribed by

$$u(0,t) = u_b, \ (\rho, u)(x,0) = (\rho_0, u_0)(x) \to (\rho_+, u_+) \text{ as } x \to +\infty. \tag{3}$$

Here $\rho_+ > 0$, u_+ and u_b are constants. In the case $u_b < 0$, the particles flow away from the boundary $\{x = 0\}$, and thus the problem in such case is called an outflow problem. The case of $u_b = 0$ and $u_b > 0$ is called the impermeable wall problem and the inflow problem, respectively. Notice that for the inflow problem, there should been an additional boundary condition on the density. The large-time behavior of solutions to (2)-(3) is much more complicated than that for the Cauchy problem, cf. [11, 16, 17, 21, 24, 25] and references therein.

In this paper, we will focus on the outflow problem in the case of $u_b < 0$. In 1999, Matsumura and Nishihara [24] gave the classification of the large-time behavior of solutions to the problem (2)-(3) in terms of (ρ_+, u_+) and $u_b < 0$. In particular, Kawashima et al. [16] considered the stability of the boundary layer under the assumptions that the strength of the boundary layer is small and initial data is a small perturbation around the corresponding boundary layer. Recently, Huang and Qin [11] studied the stability of the boundary layer and rarefaction waves by removing the smallness of the initial perturbations. In what follows, let us recall some basic facts concerning the study of the outflow problem. The characteristic speeds of the hyperbolic Euler system corresponding to (2) are

$$\lambda_1 = u - S(\rho),\ \lambda_2 = u + S(\rho), \tag{4}$$

where $S(\rho) = \sqrt{P'(\rho)} = \sqrt{\gamma a}\rho^{\frac{\gamma-1}{2}}$ is the local sound speed. We also denote the Mach number by $M = |u|/S(\rho)$ and the specific volume by $v = 1/\rho$, where their values at infinity are given by $M_+ = |u_+|/S(\rho_+)$ and $v_+ = 1/\rho_+$. The phase plane $\mathbb{R}_+ \times \mathbb{R}$ of (v, u) can be divided into three subsets:

$$\begin{aligned}
\Omega_{sub} &:= \{(v,u) \in \mathbb{R}_+ \times \mathbb{R}; |u| < S(1/v)\},\\
\Gamma_{trans} &:= \{(v,u) \in \mathbb{R}_+ \times \mathbb{R}; |u| = S(1/v)\},\\
\Omega_{super} &:= \{(v,u) \in \mathbb{R}_+ \times \mathbb{R}; |u| > S(1/v)\},
\end{aligned}$$

where Ω_{sub}, Γ_{trans} and Ω_{super} are called the subsonic, transonic and supersonic regions, respectively. In the phase plane, we denote the curves through a point (v_1, u_1):

$$\begin{aligned}
&BL(v_1,u_1) = \{(v,u) \in \mathbb{R}_+ \times \mathbb{R} : \frac{u}{v} = \frac{u_1}{v_1}\},\\
&R_2(v_1,u_1) = \{(v,u) \in \mathbb{R}_+ \times \mathbb{R}; u = u_1 - \sqrt{a\gamma}\int_{v_1}^{v} s^{-\frac{\gamma+1}{2}}ds,\ v > v_1\},\\
&S_2(v_1,u_1) = \{(v,u) \in \mathbb{R}_+ \times \mathbb{R}; u = u_1 - \sqrt{(v_1 - v)\Big[P(\frac{1}{v}) - P(\frac{1}{v_1})\Big]},\ v < v_1\},
\end{aligned}$$

to be the boundary layer line, the 2-rarefaction wave and the 2-shock wave curves, respectively. Then, the large-time behavior of solutions for the compressible isentropic Navier-Stokes system can be considered in the following two cases:

Case I: $(v_+, u_+) \in \Omega_{sub} \cap \{u_+ < 0\}$ and $u_b < \min\{0, u_+\}$. Let (v_*, u_*) be the intersection point of $R_2(v_+, u_+)$ and Γ_{trans}, i.e.,

$$u_* = -\sqrt{a\gamma}v_*^{-\frac{\gamma-1}{2}} = u_+ - \sqrt{a\gamma}\int_{v_+}^{v_*} s^{-\frac{\gamma+1}{2}}ds. \tag{5}$$

Furthermore, if $u_* \le u_b < \min\{0, u_+\}$, then there exists a unique v_b such that $(v_b, u_b) \in R_2(v_+, u_+)$, and the time-asymptotic state of the solution is a 2-rarefaction wave $(\bar{v}, \bar{u})(\frac{x}{t})$, which connects (v_b, u_b) and (v_+, u_+), to the corresponding Riemann problem, see Figure 1.

Case II: $(v_+, u_+) \in \Omega_{super}$ and $u_b < u_*$. Here (v_*, u_*) is an intersection point of $BL(v_+, u_+)$ and $S_2(v_+, u_+)$, i.e.

$$u_+ = \frac{u_+}{v_+}v_* - \sqrt{(v_+ - v_*)\Big[P(\frac{1}{v_*}) - P(\frac{1}{v_+})\Big]},\ u_* = \frac{u_+}{v_+}v_*. \tag{6}$$

Then, there exists a unique v_b such that $(v_b, u_b) \in BL(v_+, u_+)$, and the time-asymptotic state of the solution is a boundary layer $(\tilde{\rho}, \tilde{u})(x)$ which connects (v_b, u_b) and (v_+, u_+), see Figure 2.

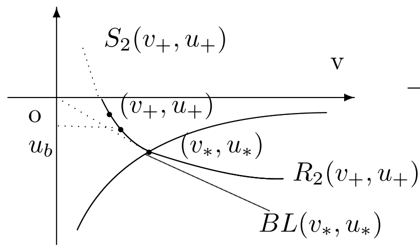

Figure. 1

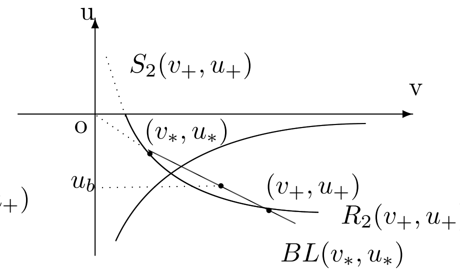

Figure. 2

Now, our goal in this paper is to study the large time behavior of global solutions to the IBVP on the two-fluid NSP system (1). Compared with the case of the single quasineutral Navier-Stokes system (2), we mainly investigate the influence of the electric field on the time asymptotic stability of some nontrivial profiles. Precisely, we consider (1) in the half-line $\mathbb{R}_+$ with initial data

$$(\rho_i, u_i, \rho_e, u_e)(x,0) = (\rho_{i0}, u_{i0}, \rho_{e0}, u_{e0})(x) \to (\rho_+, u_+, \rho_+, u_+) \text{ as } x \to +\infty, \tag{7}$$

and the boundary data

$$u_i(0,t) = u_e(0,t) = u_b < 0. \tag{8}$$

Here, we suppose $\inf_{x\in\mathbb{R}_+} \rho_{i0} > 0$, $\inf_{x\in\mathbb{R}_+} \rho_{e0} > 0$ and further the compatibility condition $u_b = u_{i0}(0) = u_{e0}(0)$.

We shall establish the global-in-time asymptotic stability of the rarefaction wave and the boundary layer both for the outflow problem under the smallness assumption on initial perturbation, where the strength of the rarefaction wave is not necessarily small while the strength of the boundary layer is additionally supposed to be small. Here, the large initial data with densities far from vacuum is also allowed in the case of the non-degenerate boundary layer.

The results that we have obtained in this paper, cf. Theorem 2.3, Theorem 3.3 and Theorem 3.4, show that the large-time behavior of the solutions coincides with the one for the single Navier-Stokes system in the absence of the electric field. To the best of our knowledge, it is the first result concerning the asymptotic behavior of solutions tending to a rarefaction wave or boundary layer for the two-fluid NSP system. We remark that the approach developed here can be adapted to the stability of the superposition of rarefaction wave and boundary layer and also to the study in the case of the whole line.

The proof of those main results is based on the classical energy method. As we mentioned before, the main difficulty in the analysis comes from the slower time-decay rate of the system caused by the appearance of the electric field. In fact, generally, the electric field $E(x,t)$ can not be time-space integrable, that is, the integral

$$\int_0^\infty \int_0^\infty |E(x,t)|^2 dxdt$$

could diverge. Thus, when the asymptotic stability of a nontrivial profile $(\rho_\infty(x,t), u_\infty(x,t))$ is considered, it is impossible to use the above quantity to control the following type of term

$$\frac{1}{2}\int_0^\infty \int_0^\infty \partial_x u_\infty(x,t)|E(x,t)|^2 dxdt \tag{9}$$

that one has to meet in the proof of zero-order energy estimates. One way for estimating (9) is to make use of the time-space integrability of the first-order space derivative of E in terms of the possible space-decay property of $u_\infty(x,t)$ at infinity together with the Hardy inequality. It is the case in the study of the stability of the non-degenerate boundary layer, see the proof of Theorem 3.4. However, due to technical difficulties, the same trick fails for the consideration of both the rarefaction wave and the degenerate boundary layer. To overcome this, we instead use the coupling property of the two-fluid equations to control (9) and hence capture the dissipation of the electric field interacting with the nontrivial asymptotic profile; see also [7] for a similar observation. Specifically, the difference of two momentum equations in (1) gives

$$\begin{aligned} 2E \;&=\; \partial_t(u_i-u_e)+\frac{1}{2}\partial_x(u_i^2-u_e^2)\\ &\quad +\Big[\frac{P'(\rho_i)\partial_x\rho_i}{\rho_i}-\frac{P'(\rho_e)\partial_x\rho_e}{\rho_e}\Big]-\Big[\frac{(u_i)_{xx}}{\rho_i}-\frac{(u_e)_{xx}}{\rho_e}\Big]. \end{aligned}$$

By applying this equation, it formally holds that

$$\frac{1}{2}\partial_x u_\infty|E|^2=\frac{1}{4}\frac{\partial}{\partial t}\left\{\partial_x u_\infty(u_i-u_e)E\right\}+\frac{\rho_i+\rho_e}{8}\partial_x u_\infty|u_i-u_e|^2+(h.o.t.),$$

where $(h.o.t.)$ indeed denotes the high-order terms and the first two terms on the right-hand side can be bounded in terms of the basic energy inequality obtained in the usual way. Notice that u_i and u_e have the same asymptotic profile u_∞ in time. Therefore, if u_∞ is nondecreasing in space variable, the term $(\partial_x u_\infty)^{1/2}E$ which denotes the interaction of the electric field E with the nontrivial asymptotic profile can be time-space integrable.

Finally, it should be pointed out that system (1) in the non-dimensional form depends generally on the ratios of masses, charges and temperatures of two fluids and also on the Debye length, cf. [26] and [1]. In such case, the two-fluid plasma system exhibits more complex coupling structure and the corresponding analysis of the large time behavior of solutions becomes more complicated [3]. The general physical situation is left for the future study.

The rest of this paper is organized as follows. In Section 2, we prove Theorem 2.3 for the stability of the rarefaction wave. In Section 3, we prove Theorem 3.3 and Theorem 3.4 for the stability of the boundary layer in the degenerate and non-degenerate cases, respectively.

Notations. Throughout this paper, the generic positive constants are denoted by c or C. $C(\cdot)$ stands for a generic function depending only on the argument. $L^p(\Omega)$ denotes the usual Lebesgue L^p space on $\Omega\subset\mathbb{R}$, while $W^{k,p}(\Omega)$ denotes the usual k^{th}-order Sobolev space. For simplicity, we denote $H^k(\Omega):=W^{k,2}(\Omega)$, $H_0(\Omega)=L^2(\Omega)$. The corresponding norm is denoted by $\|\cdot\|_k=\|\cdot\|_{H^k(\Omega)}$ and $\|\cdot\|=\|\cdot\|_{L^2(\Omega)}$. The domain Ω will be often abbreviated without confusion.

2. **The stability of rarefaction waves.**

In this section we are concerned with the asymptotic stability of the rarefaction wave for the outflow problem (1), (7) and (8) on the two-fluid NSP system. In this case, there exists a unique ρ_b such that $(\rho_b,u_b)\in R_2(\rho_+,u_+)$ and the time-asymptotic state of solution is indeed a 2-rarefaction wave $(\rho^R,u^R)(\frac{x}{t})$ which connects two constant states (ρ_b,u_b) and (ρ_+,u_+) at $x=0$ and $x=\infty$, respectively.

We firstly construct a smooth approximation for the rarefaction wave as follows. Consider the Riemann problem on the Burger's equation

$$\begin{cases} w_t + ww_x = 0, \\ w(0,x) = w_0(x) = \begin{cases} w_-, & x < 0, \\ w_+, & x > 0, \end{cases} \end{cases} \tag{10}$$

for $w_- < w_+$. It is well known that (10) has a continuous weak solution $w^R(x/t)$ connecting w_- and w_+, taking the form of

$$w^R\Big(\frac{x}{t}\Big) = \begin{cases} w_-, & x \le w_- t, \\ \frac{x}{t}, & w_- t < x < w_+ t, \\ w_+, & w_+ t < x. \end{cases}$$

Let $(\rho^R,\ u^R)(\frac{x}{t})$ be defined by

$$w^R\Big(\frac{x}{t}\Big) := u^R + S(\rho^R),\ \frac{du^R}{d\rho^R} = \frac{S(\rho^R)}{\rho^R},$$

with $w_- = u_b + S(\rho_b)$ and $w_+ = u_+ + S(\rho_+)$, where $S(\cdot)$ is defined in (4). Then by a direct calculation, $(\rho^R, u^R)(\frac{x}{t})$ satisfies the following Riemann problem on the Euler equations

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = 0, \\ (\rho, u)(x,0) = (\rho_0, u_0)(x) = \begin{cases} (\rho_b, u_b), & x < 0, \\ (\rho_+, u_+), & x > 0. \end{cases} \end{cases}$$

As usual, $w^R(\frac{x}{t})$ can be approximated by a smooth function $\overline{w}(x,t)$ to be constructed as follows. Consider the Cauchy problem on the Burger's equation:

$$\begin{cases} w_t + ww_x = 0, \\ w(0,x) = w_0(x) = \begin{cases} w_-, & x < 0, \\ w_- + C_q\delta_r \int_0^{\varepsilon x} y^q e^{-y} dy, & x > 0, \end{cases} \end{cases} \tag{11}$$

where $\delta_r := w_+ - w_-$, $q \ge 10$ is a constant, C_q is a constant such that $C_q \int_0^\infty y^q e^{-y} dy = 1$, and $\varepsilon \le 1$ is a positive constant to be determined later. Then, we have

Lemma 2.1. *Let $0 < w_- < w_+$, then the problem (11) has a unique smooth solution $\overline{w}(x,t)$ which satisfies the following properties*

i) $w_- \le \overline{w}(x,t) \le w_+$, $\overline{w}_x \ge 0$ for $x \ge 0,\ t > 0$.

ii) For any p $(1 \le p \le +\infty)$, there exists a constant $C_{p,q}$ such that for $t \ge 0$,

$$\begin{aligned} &||\overline{w}_x(t)||_{L^p} \le C_{p,q} \min\{\delta_r \varepsilon^{1-1/p},\ \delta_r^{1/p} t^{-1+1/p}\}, \\ &||\overline{w}_{xx}(t)||_{L^p} \le C_{p,q} \min\{\delta_r \varepsilon^{2-1/p},\ \delta_r^{1/q} \varepsilon^{1-1/p+1/q} t^{-1+1/q}\}. \end{aligned}$$

iii) When $x \le w_- t$, $\partial_x^k(\overline{w} - w_-) = 0$ for $k = 0, 1, 2$.

iv) $\lim\limits_{t\to+\infty} \sup\limits_{x\in\mathbb{R}} |\overline{w}(x,t) - w^R(x/t)| = 0$.

We now define the approximate solution of $(\rho^R,\ u^R)$ by $(\overline{\rho}, \overline{u})$ in terms of

$$\overline{w}(x, t+1) := \overline{u}(x,t) + S(\overline{\rho}(x,t)),\ \frac{d\overline{u}}{d\overline{\rho}} = \frac{S(\overline{\rho})}{\overline{\rho}} \tag{12}$$

together with $u_b + S(\rho_b) = w_-$ and $u_+ + S(\rho_+) = w_+$. Then, it holds that

$$\begin{cases} \overline{\rho}_t + (\overline{\rho}\,\overline{u})_x = 0, \\ (\overline{\rho}\,\overline{u})_t + (\overline{\rho}\,\overline{u}^2 + P(\overline{\rho}))_x = 0. \end{cases}$$

Let $\delta_r = |\rho_+ - \rho_b| + |u_+ - u_-|$. From the properties of $\overline{w}$ stated in Lemma 2.1, one has the following lemma concerning $(\overline{\rho}, \overline{u})$.

Lemma 2.2. *The approximate solution $(\overline{\rho}, \overline{u})$ given by (12) satisfies*

i) $\overline{\rho}_x \geq 0,\ \overline{u}_x \geq 0$ and $\rho_b \leq \overline{\rho} \leq \rho_+,\ u_b \leq \overline{u} \leq u_+$ for $x \geq 0,\ t > 0$.

ii) For any p $(1 \leq p \leq +\infty)$, there exists a constant C_p such that for $t \geq 0$,

$$
\begin{aligned}
&||(\overline{\rho}_x,\ \overline{u}_x(t)||_{L^p} \leq C_{p,q} \min\{\delta_r \varepsilon^{1-1/p},\ \delta_r^{1/p}(1+t)^{-1+1/p}\},\\
&||(\overline{\rho}_{xx}, \overline{u}_{xx})(t)||_{L^p} \leq C_{p,q} \min\{\delta_r \varepsilon^{2-1/p},\ \delta_r^{1/q}\varepsilon^{1-1/p+1/q}(1+t)^{-1+1/q}\}.
\end{aligned}
$$

iii) When $x \leq 0$, $\partial_x^k(\overline{u} - u_-) = 0$ for $k = 0, 1, 2$.

iv) $\lim\limits_{t\to+\infty} \sup\limits_{x\in\mathbb{R}} |(\overline{\rho}, \overline{u})(x,t) - (\rho^R, u^R)(x/t)| = 0.$

Now, in order to consider the stability of the approximate rarefaction wave $(\overline{\rho}, \overline{u})$ for the IBVP (1), (7) and (8), let us define the perturbation

$$
(\phi_{i,e}, \psi_{i,e}) = (\rho_{i,e} - \overline{\rho}, u_{i,e} - \overline{u}).
$$

Then, $(\phi_{i,e},\ \psi_{i,e})$ satisfies

$$
\begin{cases}
\partial_t \phi_i + u_i \partial_x \phi_i + \rho_i \partial_x \psi_i = -f_i,\\
\rho_i \partial_t \psi_i + P'(\rho_i)\partial_x \phi_i + \rho_i u_i \partial_x \psi_i = (\psi_i)_{xx} - g_i + \rho_i E,\\
\partial_t \phi_e + u_e \partial_x \phi_e + \rho_e \partial_x \psi_e = -f_e,\\
\rho_e \partial_t \psi_e + P'(\rho_e)\partial_x \phi_e + \rho_e u_e \partial_x \psi_e = (\psi_e)_{xx} - g_e - \rho_e E,\\
E(x,t) = -\int_x^\infty (\phi_i - \phi_e)(y,t)\,dy, \quad x \in \mathbb{R}_+, t > 0,\\
(\phi_i, \psi_i)(t, 0) = 0,\\
(\phi_i, \psi_i, \phi_e, \psi_e)(x, 0) := (\phi_{i0}, \psi_{i0}, \phi_{e0}, \psi_{e0})(x) \to 0,\ \textit{as}\ x \to \infty,
\end{cases}
\tag{13}
$$

where $f_{i,e}, g_{i,e}$ are the nonlinear terms, given by

$$
\begin{cases}
f_{i,e} = \overline{u}_x \phi_{i,e} + \overline{\rho}_x \psi_{i,e},\\
g_{i,e} = -\overline{u}_{xx} + \overline{u}_x \rho_{i,e} \psi_{i,e} + \overline{\rho}_x [P'(\rho_{i,e}) - \frac{\rho_{i,e}}{\overline{\rho}} P'(\overline{\rho})].
\end{cases}
$$

The solution of (13) is sought in a set of functions $X_r(0, +\infty)$, where for given $0 < T \leq +\infty$, $X_r(0,T)$ denotes

$$
\begin{aligned}
X_r(0,T) = \Big\{(\phi_{i,e}, \psi_{i,e}, E)\Big|\ &(\phi_{i,e}, \psi_{i,e}) \in C(0,T;H^1),\ (\phi_{i,e})_x \in L^2(0,T;L^2),\\
&(\psi_{i,e})_x \in L^2(0,T;H^1), E \in C(0,T;L^2),\ \psi_{i,e}(t,0) = 0\ (0 \leq t \leq T)\Big\}.
\end{aligned}
$$

The main result of this section is stated as follows.

Theorem 2.3. *Assume that $(v_+, u_+) \in \Omega_{sub} \cap \{u_+ < 0\}$ and $u_* \leq u_b < \min\{0, u_+\}$, where u_* is given in (5). There exist positive constants ε_0 and C_0 such that if*

$$
||E_0|| + ||(\phi_{i0,e0}, \psi_{i0,e0})||_1 + \varepsilon \leq \varepsilon_0,
$$

where ε is the parameter in (11), then the outflow problem (13) of the two-fluid NSP system has a unique global solution $(\phi_{i,e}, \psi_{i,e}, E) \in X_r(0, +\infty)$ satisfying

$$
\begin{aligned}
\sup_{t\geq 0} \Big(||E(t)||^2 + ||(\phi_{i,e}, \psi_{i,e})(t)||_1^2\Big) &+ \int_0^{+\infty} |(E, \phi_{i,e}, (\phi_{i,e})_x)(0,\tau)|^2 d\tau\\
&+ \int_0^\infty \Big[||\sqrt{\overline{u}_x}(\phi_{i,e}, \psi_{i,e})(\tau)||^2 + ||(E_x, (\phi_{i,e})_x, (\psi_{i,e})_x, (\psi_{i,e})_{xx})(\tau)||^2\Big] d\tau\\
&\leq C_0\Big[||E_0||^2 + ||(\phi_{i0,e0}, \psi_{i0,e0})||_1^2 + \varepsilon^{1/q}\Big],
\end{aligned}
$$

and

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}_+}\left|\Big(\rho_{i,e}(x,t)-\rho^R(\frac{x}{t}),u_{i,e}(x,t)-u^R(\frac{x}{t}),E(x,t)\Big)\right|=0.$$

The local existence of the solution $(\phi_{i,e},\psi_{i,e},E)$ to the IBVP (13) is proved by the standard iteration method. As to the proof of Theorem 2.3, it suffices to show the following a prior estimate.

Proposition 1. *(A priori estimates). Suppose that all assumptions in Theorem 2.3 are satisfied. Let $(\phi_{i,e},\psi_{i,e})\in X_r(0,T)$ be a solution to the IBVP (13) for some positive T. Then there exist constants $\varepsilon_1>0$ and C_1 such that if*

$$\sup_{0\le t\le T}\Big(||E(t)||+||(\phi_{i,e},\psi_{i,e})(t)||_1\Big)+\varepsilon\le\varepsilon_1,$$

then the solution $(\phi_{i,e},\psi_{i,e})$ satisfies

$$\begin{aligned}&\sup_{0\le t\le T}\Big(||E(t)||^2+||(\phi_{i,e},\psi_{i,e})(t)||_1^2\Big)+\int_0^T|(E,\phi_{i,e},(\phi_{i,e})_x)(0,\tau)|^2d\tau\\&+\int_0^T\Big[||\sqrt{\overline{u}_x}(\phi_{i,e},\psi_{i,e})(\tau)||^2+||(E_x,(\phi_{i,e})_x,(\psi_{i,e})_x,(\psi_{i,e})_{xx})(\tau)||^2\Big]d\tau\\&\qquad\le C_1\Big[||E_0||^2+||(\phi_{i0,e0},\psi_{i0,e0})||_1^2+\varepsilon^{1/q}\Big].\end{aligned}\tag{14}$$

Proof. Before giving the proof of this proposition, we list two notations

$$N^2(t):=\sup_{0\le\tau\le t}\Big(||E(\tau)||^2+||(\phi_{i,e},\psi_{i,e})(\tau)||_1^2\Big),$$

and

$$\begin{aligned}M^2(t)\quad:=\quad&\int_0^t\Big\{|(E,\phi_{i,e},(\phi_{i,e})_x)(\tau,0)|^2+||\sqrt{\overline{u}_x}(\phi_{i,e},\psi_{i,e})(\tau)||^2\\&+||(E_x,(\phi_{i,e})_x,(\psi_{i,e})_x,(\psi_{i,e})_{xx})(\tau)||^2\Big\}d\tau.\end{aligned}$$

By the Sobolev inequality

$$\sup_{x\in\mathbb{R}_+}|f(x)|\le\sqrt{2}||f||^{1/2}||f_x||^{1/2}\text{ for any }f\in H^1,$$

we have that, for sufficiently small ε_1,

$$\rho_b/2\le\rho_b-\sqrt{2}\varepsilon_1\le\overline{\rho}+\phi_{i,e}=\rho_{i,e}\le\rho_++C\varepsilon_1\le2\rho_+,\tag{15}$$

$$||\psi_{i,e}(t)||_{L^\infty}\le\sqrt{2}||\psi_{i,e}||^{1/2}\ ||(\psi_{i,e})_x||^{1/2}\le\sqrt{2}N(t)\le C\varepsilon_1,\tag{16}$$

and

$$||E(t)||_{L^\infty}\le\sqrt{2}||E||^{1/2}||E_x||^{1/2}=\sqrt{2}||E||^{1/2}||\phi_i-\phi_e||^{1/2}\le2\sqrt{2}N(t)\le C\varepsilon_1.$$

Now, we divide the proof of Proposition 1 by three steps.

Step 1. The zero-order energy estimates.

We define the following functionals (see [11] and [16]),

$$\mathcal{E}(\rho,u):=\Phi(\rho,\overline{\rho})+\frac{1}{2}|u-\overline{u}|^2,$$

where

$$\Phi(\rho,\tilde{\rho}):=\int_{\overline{\rho}}^{\rho}\frac{P(s)-P(\overline{\rho})}{s^2}ds=\frac{a}{(\gamma-1)\rho}\Big[\rho^\gamma-\overline{\rho}^\gamma-\gamma\overline{\rho}^{\gamma-1}(\rho-\overline{\rho})\Big].$$

Notice that $\Phi(\rho,\overline{\rho})$ is equivalent to $|\rho-\overline{\rho}|^2$ for $|\rho-\overline{\rho}|<C$. Hence, the energy form is equivalent to $|(\rho-\overline{\rho},u-\overline{u})|^2$, namely, there exist positive constants c_1 and C_1 such that

$$c_1|(\rho-\overline{\rho},u-\overline{u})|^2 \le \mathcal{E}(\rho,u) \le C_1|(\rho-\overline{\rho},u-\overline{u})|^2.$$

For brevity of presentation, we denote

$$\mathcal{E}_{i,e} := \mathcal{E}(\rho_{i,e},u_{i,e}),\ \ \Phi_{i,e} := \Phi(\rho_{i,e},\overline{\rho}),$$

and for any function h, we set

$$h_\alpha := h_i + h_e.$$

From (13), a direct computation yields

$$\begin{aligned}&\Big\{\rho_\alpha\mathcal{E}_\alpha+\frac{1}{2}E^2\Big\}_t+\Big\{\Big[\rho_\alpha u_\alpha\mathcal{E}_\alpha+[P(\rho_\alpha)-P(\overline{\rho})]\psi_\alpha-\psi_\alpha(\psi_\alpha)_x\Big]+\frac{\overline{u}}{2}E^2\Big\}_x\\&\qquad+\overline{u}_x\Big\{\Big[P(\rho_\alpha)-P(\overline{\rho})-P'(\overline{\rho})\phi_\alpha\Big]+\rho_\alpha\psi_\alpha^2-\frac{E^2}{2}\Big\}+[(\psi_\alpha)_x]^2=\overline{u}_{xx}\psi_\alpha.\end{aligned}\tag{17}$$

The main difficulty comes from the term $\overline{u}_xE^2$ on the left-hand side of (17). We estimate it as follows. From the second and fourth equations of (1), one can compute

$$\begin{aligned}2E=\partial_t(u_i-u_e)+\frac{1}{2}\partial_x(u_i^2-u_e^2)&\\+\Big[\frac{P'(\rho_i)\partial_x\rho_i}{\rho_i}-\frac{P'(\rho_e)\partial_x\rho_e}{\rho_e}\Big]-\Big[\frac{(u_i)_{xx}}{\rho_i}-\frac{(u_e)_{xx}}{\rho_e}\Big].&\end{aligned}\tag{18}$$

Multiplying both sides of the above equation by $\overline{u}_xE/4$ and using $E_t=-\rho_iu_i+\rho_eu_e=-\frac{\rho_i+\rho_e}{2}(u_i-u_e)-\frac{(\rho_i-\rho_e)(u_i+u_e)}{2}$, one has

$$\begin{aligned}\frac{\overline{u}_x}{2}E^2&=\Big(\frac{\overline{u}_x}{4}(u_i-u_e)E\Big)_t-\frac{\overline{u}_{xt}}{4}(u_i-u_e)E\\&\quad-\frac{\overline{u}_x}{4}(u_i-u_e)E_t+\frac{\overline{u}_x}{8}E\partial_x(u_i^2-u_e^2)\\&\quad+\frac{A\gamma\overline{u}_x}{4(\gamma-1)}(\rho_i^{\gamma-1}-\rho_e^{\gamma-1})_xE+\frac{\overline{u}_x}{4}E\Big[\frac{(u_i)_{xx}}{\rho_i}-\frac{(u_e)_{xx}}{\rho_e}\Big]\\&=\Big(\frac{\overline{u}_x}{4}(\psi_i-\psi_e)E\Big)_t-\Big(\frac{\overline{u}_t}{4}(\psi_i-\psi_e)E-\frac{\overline{u}_x(\psi_i^2-\psi_e^2)}{8}\Big)_x\\&\quad+\frac{\overline{u}_x}{8}(\rho_i+\rho_e)(\psi_i-\psi_e)^2+\frac{\overline{u}_t+\overline{u}\,\overline{u}_x}{4}\Big((\psi_i-\psi_e)E\Big)_x\\&\quad+\frac{\overline{u}_x^2}{4}E(\psi_i-\psi_e)-\frac{\overline{u}_{xx}}{8}E(\psi_i^2-\psi_e^2)+\frac{A\gamma\overline{u}_x}{4(\gamma-1)}(\rho_i^{\gamma-1}-\rho_e^{\gamma-1})_xE\\&\quad-\frac{\overline{u}_x\overline{u}_{xx}}{4}\frac{EE_x}{\rho_i\rho_e}+\frac{\overline{u}_x}{4}\Big[\frac{(\psi_i)_{xx}}{\rho_i}-\frac{(\psi_e)_{xx}}{\rho_e}\Big]E.\end{aligned}\tag{19}$$

Combining (17)-(19), we arrive at the following equality

$$\begin{aligned}&\Big\{\rho_\alpha\mathcal{E}_\alpha-\frac{\overline{u}_x}{4}(\psi_i-\psi_e)E+\frac{1}{2}E^2\Big\}_t+\Big\{\rho_\alpha u_\alpha\mathcal{E}_\alpha+[P(\rho_\alpha)-P(\overline{\rho})]\psi_\alpha\\&\quad-\psi_\alpha(\psi_\alpha)_x+\frac{\overline{u}_t}{4}(\psi_i-\psi_e)E-\frac{\overline{u}_x(\psi_i^2-\psi_e^2)}{8}+\frac{\overline{u}}{2}E^2\Big\}_x\\&\quad+\overline{u}_x\Big[P(\rho_\alpha)-P(\overline{\rho})-P'(\overline{\rho})\phi_\alpha+\rho_\alpha\psi_\alpha^2-\frac{\rho_\alpha}{8}(\psi_i-\psi_e)^2\Big]+[(\psi_\alpha)_x]^2\\&=\overline{u}_{xx}\psi_\alpha+(H.O.T.),\end{aligned}\tag{20}$$

where

$$
\begin{aligned}
(H.O.T.) = & -\frac{P(\overline{\rho})_x}{4\overline{\rho}}\Big((\psi_i-\psi_e)_x E + (\psi_i-\psi_e)E_x\Big) \\
& +\frac{A\gamma\overline{u}_x}{4}\Big(\rho_i^{\gamma-2}(\phi_i)_x - \rho_e^{\gamma-2}(\phi_e)_x\Big)E - \frac{\overline{u}_x\overline{u}_{xx}}{4}\frac{EE_x}{\rho_i\rho_e} \\
& +\Big[\frac{\overline{u}_x^2}{4}(\psi_i-\psi_e) + \frac{A\gamma\overline{u}_x\overline{\rho}_x}{4(\gamma-1)}(\rho_i^{\gamma-2}-\rho_e^{\gamma-2})\Big]E \\
& -\frac{\overline{u}_{xx}}{8}(\psi_i^2-\psi_e^2)E + \frac{\overline{u}_x}{4}\Big[\frac{(\psi_i)_{xx}}{\rho_i} - \frac{(\psi_e)_{xx}}{\rho_e}\Big]E \\
:= & R_1+R_2+R_3+R_4+R_5+R_6.
\end{aligned} \tag{21}
$$

Therefore, after integrating (20) over $[0,t]\times\mathbb{R}_+$ and noticing $u_b<0$, (20) gives

$$
\begin{aligned}
& \int_{\mathbb{R}_+}\Big\{\rho_\alpha\mathcal{E}_\alpha - \frac{\overline{u}_x}{4}(\psi_i-\psi_e)E + \frac{1}{2}E^2\Big\}dx \\
& \quad + \int_0^t |u_b|\Big[\rho_\alpha\Phi_\alpha(0,\tau) + \frac{1}{2}E^2(0,\tau)\Big]d\tau \\
& \quad + \int_0^t\int_{\mathbb{R}_+}\Big\{\overline{u}_x\Big[\big(P(\rho_\alpha)-P(\overline{\rho})-P'(\overline{\rho})\phi_\alpha\big) \\
& \qquad\qquad + \rho_\alpha\psi_\alpha^2 - \frac{\rho_\alpha}{8}(\psi_i-\psi_e)^2\Big] + |(\psi_\alpha)_x|^2\Big\}dxd\tau \\
= & \int_{\mathbb{R}_+}\Big\{\rho_{\alpha0}\mathcal{E}_{\alpha0} - \frac{\overline{u}_x(x,0)}{4}(\psi_{i0}-\psi_{e0})E_0 + \frac{1}{2}E_0^2\Big\}dx \\
& \qquad + \int_0^t\int_{\mathbb{R}_+}\Big[\overline{u}_{xx}\psi_\alpha + (H.O.T.)\Big]dxd\tau.
\end{aligned} \tag{22}
$$

For the estimates of the terms on the right hand side of (22), by the Sobolev inequality and the Hölader inequality, we have

$$
\begin{aligned}
& \int_0^t\int_{\mathbb{R}_+}\overline{u}_{xx}|\psi_\alpha|dxd\tau \le C\int_0^t ||\overline{u}_{xx}||_{L^1}||\psi_\alpha||^{\frac{1}{2}}||(\psi_\alpha)_x||^{\frac{1}{2}}d\tau \\
\le & C(\delta_r\varepsilon)^{\frac{1}{q}}\int_0^t(1+\tau)^{-1+\frac{1}{q}}||(\psi_\alpha)_x||^{\frac{1}{2}}||\psi_\alpha||^{\frac{1}{2}}d\tau \\
\le & C(\delta_r\varepsilon)^{\frac{1}{q}}\Big[\int_0^t||(\psi_\alpha)_x||^2d\tau + \int_0^t(1+\tau)^{\frac{4(1-q)}{3q}}||\psi_\alpha||^{\frac{2}{3}}d\tau\Big] \\
\le & C(\delta_r\varepsilon)^{\frac{1}{q}}\Big(M^2(t)+N^{\frac{2}{3}}(t)\Big) \le C(\delta_r\varepsilon)^{\frac{1}{q}}\Big(M^2(t)+N^2(t)+1\Big),
\end{aligned} \tag{23}
$$

where we choose $q > 10$ such that $\frac{5(1-q)}{4q} < -1$. For other terms, we use an estimate which reads that for all $f, g \in H^1$,

$$
\begin{aligned}
\Big|\int_0^t\int_{\mathbb{R}_+}\tilde{u}_x f g_x dx d\tau\Big| &\leq \int_0^t\int_{\mathbb{R}_+}|\tilde{u}_x|^{\frac{7}{8}}|f|\ |\tilde{u}_x|^{\frac{1}{8}}|g_x| dx d\tau \\
&\leq \int_0^t ||\tilde{u}_x(\tau)||_{L^\infty}^{\frac{1}{4}}\int_{\mathbb{R}_+}\Big(|\tilde{u}_x|^{\frac{3}{2}}|f|^2+|g_x|^2\Big)dx d\tau \\
&\leq C(\delta_r\varepsilon)^{\frac{1}{4}}\int_0^t\int_{\mathbb{R}_+}\Big((1+\tau)^{-\frac{3}{2}}|f|^2+|g_x|^2\Big)dx d\tau \\
&\leq C(\delta_r\varepsilon)^{\frac{1}{4}}\Big(\sup_{0\leq\tau\leq t}||f(\tau)||+\int_0^t||g_x||^2 d\tau\Big).
\end{aligned}
$$

It should be pointed out that the above estimate also holds true when $\tilde{u}_x$ is replaced by $\tilde{\rho}_x$. Basing on the above inequality, we will give the estimates of the time-space integral of $(H.O.T.)$ in (22) corresponding to (21) term by term. Recalling i) in Lemma 2.1, (15) and (16), we get by Cauchy-Schwarz

$$
\begin{aligned}
&\Big|\int_0^t\int_{\mathbb{R}_+}R_1 dx d\tau\Big| \\
&\leq C(\delta_r\varepsilon)^{\frac{1}{4}}\Big(\sup_{0\leq\tau\leq t}(||E||^2+||\psi_i-\psi_e||^2\Big)+\int_0^t||E_x||^2+||(\psi_i-\psi_e)_x||^2 d\tau\Big) \\
&\leq C(\delta_r\varepsilon)^{\frac{1}{4}}(N^2(t)+M^2(t)). \qquad (24)
\end{aligned}
$$

In a similar way, for R_2 and R_3, we estimate them by

$$
\Big|\int_0^t\int_{\mathbb{R}_+}R_2 dx d\tau\Big| \leq C(\delta_r\varepsilon)^{\frac{1}{4}}(N^2(t)+M^2(t)), \qquad (25)
$$

$$
\Big|\int_0^t\int_{\mathbb{R}_+}R_3 dx d\tau\Big| \leq C\delta_r\varepsilon^2(\delta_r\varepsilon)^{\frac{1}{4}}(N^2(t)+M^2(t)), \qquad (26)
$$

and moreover, for R_4 and R_5, we obtain

$$
\begin{aligned}
&\Big|\int_0^t\int_{\mathbb{R}_+}R_4 dx d\tau\Big| \leq C\int_0^t\int_{\mathbb{R}_+}\Big(|\overline{u}_x^2|+|\overline{\rho}_x\overline{u}_x|\Big)|E| dx d\tau \\
&\leq C(\delta_r\varepsilon)^{\frac{1}{2}}\int_0^t(1+\tau)^{-\frac{4}{3}}||E|| d\tau \leq C(\delta_r\varepsilon)^{\frac{1}{2}}(N^2(t)+1) \qquad (27)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\int_0^t\int_{\mathbb{R}_+}R_5 dx d\tau\Big| \leq C\int_0^t\int_{\mathbb{R}_+}|\overline{u}_{xx}||E| dx d\tau \\
&\leq C(\delta_r\varepsilon)^{\frac{1}{q}}\Big(M^2(t)+N^{\frac{2}{3}}(t)\Big) \leq C(\delta_r\varepsilon)^{\frac{1}{q}}\Big(M^2(t)+N^2(t)+1\Big). \qquad (28)
\end{aligned}
$$

For R_6, since $\overline{u}_x(t,0)=0$, it follows from integrating by parts that

$$
\begin{aligned}
\int_0^t\int_{\mathbb{R}_+} R_6 dxd\tau &= \frac{1}{4}\int_0^t\int_{\mathbb{R}_+} \overline{u}_x E\Big[\frac{(\psi_i)_{xx}}{\rho_i}-\frac{(\psi_e)_{xx}}{\rho_e}\Big]dxd\tau\\
&= -\frac{1}{4}\int_0^t\int_{\mathbb{R}_+} \overline{u}_x E_x\Big[\frac{(\psi_i)_{x}}{\rho_i}-\frac{(\psi_e)_{x}}{\rho_e}\Big]dxd\tau\\
&\quad -\frac{1}{4}\int_0^t\int_{\mathbb{R}_+} \Big(\overline{u}_{xx}+\overline{u}_x\overline{\rho}_x\Big) E\Big[\frac{(\psi_i)_{x}}{\rho_i^2}-\frac{(\psi_e)_{x}}{\rho_e^2}\Big]dxd\tau\\
&\quad -\frac{1}{4}\int_0^t\int_{\mathbb{R}_+} \overline{u}_x E\Big[\frac{(\phi_i)_x(\psi_i)_{x}}{\rho_i^2}-\frac{(\phi_e)_x(\psi_e)_{x}}{\rho_e^2}\Big]dxd\tau\\
&:= R_6^1+R_6^2+R_6^3,
\end{aligned}
$$

where R_6^1, R_6^2, R_6^3 can be estimated as

$$
\begin{aligned}
|R_6^1| &\le C(\delta_r\varepsilon)\int_0^t ||(E_x,(\psi_i)_x,(\psi_e)_x||^2 d\tau \le C(\delta_r\varepsilon)M^2(t),\\
|R_6^2| &\le C\Big(\delta_r\varepsilon^{1+1/q}+\delta_r^{1+1/p}\varepsilon\Big)\int_0^t\Big[(1+\tau)^{-2+\frac{2}{q}}||E||^2+||((\psi_i)_x,(\psi_e)_x)||^2\Big]d\tau\\
&\le C\Big(\delta_r\varepsilon^{1+1/q}+\delta_r^{1+1/p}\varepsilon\Big)(N^2(t)+M^2(t)),\\
|R_6^3| &\le C\delta_r\varepsilon\sup_{0\le\tau\le t}||E(\tau)||_{L^\infty}\int_0^t\Big(||((\phi_i)_x,(\phi_e)_x)||^2+||((\psi_i)_x,(\psi_e)_x)||^2\Big)d\tau,\\
&\le C(\delta_r\varepsilon)(N^2(t)+M^2(t)).
\end{aligned}
$$

The above estimates imply

$$
\Big|\int_0^t\int_{\mathbb{R}_+} R_6 dxd\tau\Big| \le C\delta_r\varepsilon(N^2(t)+M^2(t)). \tag{29}
$$

We now choose ε sufficiently small such that $\delta_r\varepsilon\le 1$, so (24), (25), (26), (27), (28) and (29) yield

$$
\Big|\int_0^t\int_{\mathbb{R}_+}(H.O.T.)dxd\tau\Big| \le C(\delta_r\varepsilon)^{1/q}\Big[N^2(t)+M^2(t)\Big]. \tag{30}
$$

Then, by using (23) and (30), it follows from (22) that

$$
\begin{aligned}
&\int_{\mathbb{R}_+}\Big\{\rho_\alpha\mathcal{E}_\alpha-\frac{\overline{u}_x}{4}(\psi_i-\psi_e)E+\frac{1}{2}E^2\Big\}dx\\
&\quad+\int_0^t|u_b|\Big[\rho_\alpha\Phi_\alpha(0,\tau)+\frac{1}{2}E^2(0,\tau)\Big]d\tau\\
&\quad+\int_0^t\int_{\mathbb{R}_+}\Big\{\overline{u}_x\Big[\big(P(\rho_\alpha)-P(\overline{\rho})-P'(\overline{\rho})\phi_\alpha\big)\\
&\qquad\qquad+\rho_\alpha\psi_\alpha^2-\frac{\rho_\alpha}{8}(\psi_i-\psi_e)^2\Big]+|(\psi_\alpha)_x|^2\Big\}dxd\tau\\
&\le C\Big(||(\phi_{\alpha0},\psi_{\alpha0})||^2+||E_0||^2+(\delta_r\varepsilon)^{\frac{1}{q}}\Big)+C(\delta_r\varepsilon)^{\frac{1}{q}}(N^2(t)+M^2(t)).
\end{aligned} \tag{31}
$$

Here, notice that

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}_+} \overline{u}_x \Big[\rho_\alpha \psi_\alpha^2 - \frac{\rho_\alpha}{8}(\psi_i-\psi_e)^2\Big] dxd\tau \\
&= \int_0^t \int_{\mathbb{R}_+} \overline{u}_x \overline{\rho}\Big[\psi_i^2+\psi_e^2 - \frac{(\psi_i-\psi_e)^2}{4}\Big] + \overline{u}_x\Big[\phi_\alpha \psi_\alpha^2 - \frac{\phi_\alpha}{8}(\psi_i-\psi_e)^2\Big] dxd\tau \\
&\geq \frac{1}{2}\int_0^t \int_{\mathbb{R}_+} \overline{u}_x \overline{\rho}\Big[\psi_i^2+\psi_e^2\Big] - CN(t)M^2(t). \qquad (32)
\end{aligned}
$$

Recall that (15) implies that the densities $\rho_{i,e}$ are bounded below. Therefore, by choosing ε further small enough, we have from (31) and (32)

$$
\begin{aligned}
&||\phi_\alpha(t)||^2 + ||\psi_\alpha(t)||^2 + ||E(t)||^2 + \int_0^t \Big[|\phi_\alpha(0,\tau)|^2 + E^2(0,\tau)\Big] d\tau \\
&\quad + \int_0^t \int_{\mathbb{R}_+} \Big\{\overline{u}_x\Big[|\phi_\alpha|^2 + |\psi_\alpha|^2 + E^2\Big] + |(\psi_\alpha)_x|^2\Big\} dxd\tau \\
&\leq C\Big(||(\phi_{\alpha 0})||^2 + ||E_0||^2 + (\delta_r\varepsilon)^{\frac{1}{q}}\Big) + C(\delta_r\varepsilon)^{\frac{1}{q}}(N^2(t)+M^2(t)), \qquad (33)
\end{aligned}
$$

which is the desired zero-order energy estimate.

Step 2. The estimate of $(\phi_\alpha)_x$.

By differentiating the first and third equations of (13) in x and then multiplying them by $\frac{(\phi_i)_x}{\rho_i^3}$ and $\frac{(\phi_e)_x}{\rho_e^3}$, respectively, one has

$$
\begin{aligned}
-(f_i)_x \frac{(\phi_i)_x}{\rho_i^3} &= \frac{(\phi_i)_{xt}(\phi_i)_x}{(\rho_i)^3} + \frac{u_i(\phi_i)_{xx}(\phi_i)_x}{(\rho_i)^3} + \frac{(u_i)_x(\phi_i)_x^2}{(\rho_i)^3} + \frac{(\phi_i)_x(\psi_i)_{xx}}{(\rho_i)^2} \\
&= \Big(\frac{[(\phi_i)_x]^2}{2(\rho_i)^3}\Big)_t + \Big(\frac{u_i[(\phi_i)_x]^2}{2(\rho_i)^3}\Big)_x - \overline{u}_x\frac{[(\phi_i)_x]^2}{\rho_i^3} \\
&\quad + \overline{\rho}_x\frac{(\phi_i)_x(\phi_i)_x}{(\rho_i)^3} + \frac{(\phi_i)_x(\psi_i)_{xx}}{(\rho_i)^2}, \qquad (34)
\end{aligned}
$$

and

$$
\begin{aligned}
-(f_e)_x \frac{(\phi_e)_x}{\rho_e^3} &= \Big(\frac{[(\phi_e)_x]^2}{2(\rho_e)^3}\Big)_t + \Big(\frac{u_e[(\phi_e)_x]^2}{2(\rho_e)^3}\Big)_x - \overline{u}_x\frac{[(\phi_e)_x]^2}{\rho_e^3} \\
&\quad + \overline{\rho}_x\frac{(\phi_e)_x(\phi_e)_x}{(\rho_e)^3} + \frac{(\phi_e)_x(\psi_e)_{xx}}{(\rho_e)^2}. \qquad (35)
\end{aligned}
$$

Moreover, multiplying the second and fourth equations of (13) by $\frac{(\phi_i)_x}{\rho_i^2}$ and $\frac{(\phi_e)_x}{\rho_e^2}$, respectively, gives

$$
\begin{aligned}
-g_i\frac{(\phi_i)_x}{\rho_i^2} &= \frac{(\phi_i)_x(\psi_i)_t}{\rho_i} + \frac{u_i(\phi_i)_x(\psi_i)_x}{\rho_i} + \frac{P'(\rho_i)}{\rho_i^2}[(\phi_i)_x]^2 - \frac{(\phi_i)_x(\psi_i)_{xx}}{(\rho_i)^2} + \frac{E(\phi_i)_x}{\rho_i} \\
&= \Big(\frac{(\phi_i)_x\psi_i}{\rho_i}\Big)_t - \Big(\frac{(\phi_i)_t\psi_i}{\rho_i} + \overline{\rho}_x\frac{(\psi_i)^2}{\rho_i}\Big)_x - [(\psi_i)_x]^2 + \frac{P'(\rho_i)}{\rho_i^2}[(\phi_i)_x]^2 \\
&\quad + \overline{u}_x\frac{\phi_i(\psi_i)_x}{\rho_i} + \overline{\rho}_{xx}\frac{(\psi_i)^2}{\rho_i} + 2\overline{\rho}_x\frac{\psi_i(\psi_i)_x}{\rho_i} + \overline{u}_x\frac{(\overline{\rho}_x\phi_i - \overline{\rho}(\phi_i)_x)\psi_i}{\rho_i^2} \\
&\quad - \frac{(\phi_i)_x(\psi_i)_{xx}}{(\rho_i)^2} + \frac{E(\phi_i)_x}{\rho_i}, \qquad (36)
\end{aligned}
$$

and

$$
\begin{aligned}
-g_e\frac{(\phi_e)_x}{\rho_e^2} \quad = \quad & \Big(\frac{(\phi_e)_x\psi_e}{\rho_e}\Big)_t-\Big(\frac{(\phi_e)_t\psi_e}{\rho_e}+\overline{\rho}_x\frac{(\psi_e)^2}{\rho_e}\Big)_x-[(\psi_e)_x]^2+\frac{P'(\rho_e)}{\rho_e^2}[(\phi_e)_x]^2\\
&+\overline{u}_x\frac{\phi_e(\psi_e)_x}{\rho_e}+\overline{\rho}_{xx}\frac{(\psi_e)^2}{\rho_e}+2\overline{\rho}_x\frac{\psi_e(\psi_e)_x}{\rho_e}+\overline{u}_x\frac{(\overline{\rho}_x\phi_e-\overline{\rho}(\phi_e)_x)\psi_e}{\rho_e^2}\\
&-\frac{(\phi_e)_x(\psi_e)_{xx}}{(\rho_e)^2}-\frac{E(\phi_e)_x}{\rho_e}. \qquad (37)
\end{aligned}
$$

Now, adding (34), (35), (36) and (37) together, we get

$$
\begin{aligned}
&\Big(\frac{[(\phi_\alpha)_x]^2}{2\rho_\alpha^3}+\frac{(\phi_\alpha)_x\psi_\alpha}{\rho_\alpha}\Big)_t\\
&\quad+\Big(\frac{u_\alpha[(\phi_\alpha)_x]^2}{2(\rho_\alpha)^3}-\frac{(\phi_\alpha)_t\psi_\alpha}{\rho_\alpha}-\frac{\overline{\rho}_x(\psi_\alpha)^2}{\rho_\alpha}\Big)_x+\frac{P'(\rho_\alpha)[(\phi_\alpha)_x]^2}{\rho_\alpha^2}\\
&=[(\psi_\alpha)_x]^2+\frac{\overline{u}_x\phi_\alpha(\psi_\alpha)_x}{\rho_\alpha}-\frac{\overline{\rho}_{xx}(\psi_\alpha)^2}{\rho_\alpha}-\frac{2\overline{\rho}_x\psi_\alpha(\psi_\alpha)_x}{\rho_\alpha}\\
&\quad+\frac{\overline{u}_x(\overline{\rho}_x\phi_\alpha-\overline{\rho}(\phi_\alpha)_x)\psi_\alpha}{\rho_\alpha^2}-\frac{\overline{u}_x[(\phi_\alpha)_x]^2}{\rho_\alpha^3}+\frac{\overline{\rho}_x(\phi_\alpha)_x(\phi_\alpha)_x}{(\rho_\alpha)^3}\\
&\quad-\frac{(f_\alpha)_x(\phi_\alpha)_x}{\rho_\alpha^3}-\frac{g_\alpha(\phi_\alpha)_x}{\rho_\alpha^2}+E\Big(\frac{(\phi_i)_x}{\rho_i}-\frac{(\phi_e)_x}{\rho_e}\Big). \qquad (38)
\end{aligned}
$$

The last term in the above equation can be further written as

$$
\begin{aligned}
&E\Big(\frac{(\phi_i)_x}{\rho_i}-\frac{(\phi_e)_x}{\rho_e}\Big)=\Big[(\ln\rho_i-\ln\rho_e)_x\Big]E-\frac{(\rho_i-\rho_e)E\overline{\rho}_x}{\rho_i\rho_e}\\
&=\Big[(\ln\rho_i-\ln\rho_e)E\Big]_x-(\ln\rho_i-\ln\rho_e)E_x+\frac{E_xE\overline{\rho}_x}{\rho_i\rho_e}.
\end{aligned}
$$

By integrating (38) over $[0,t]\times\mathbb{R}_+$, similar to Step 1, it is straightforward to check that

$$
\begin{aligned}
&\int_{\mathbb{R}_+}\frac{[(\phi_\alpha)_x]^2}{\rho_\alpha^3}+\frac{(\phi_\alpha)_x\psi_\alpha}{\rho_\alpha}dx+\int_0^t\frac{|u_b|[(\phi_\alpha)_x]^2}{(\rho_\alpha)^3}(0,\tau)d\tau\\
&\quad+\int_0^t\int_{\mathbb{R}_+}\Big\{\frac{P'(\rho_\alpha)[(\phi_\alpha)_x]^2}{\rho_\alpha^2}+(\ln\rho_i-\ln\rho_e)E_x\Big\}dxd\tau\\
&\le C(||(\phi_{\alpha0},\psi_{\alpha0})||^2+||(\phi_{\alpha0})_x||^2+(\delta_r\varepsilon)^{1/q})\\
&\quad+\int_0^t(\ln\rho_i-\ln\rho_e)E(0,\tau)d\tau+C(\delta_r\varepsilon)^{1/q}(N^2(t)+M^2(t)). \qquad (39)
\end{aligned}
$$

Now, let us further consider the estimate of the electric field E appearing in (39). Notice that $(\ln x-\ln y)(x-y)\ge(\sqrt{x}-\sqrt{y})^2$ for all $x,y>0$. Under the assumption (15) on the densities, we have

$$
\int_0^t\int_{\mathbb{R}_+}(\ln\rho_i-\ln\rho_e)E_xdxd\tau\ge\int_0^t\int_{\mathbb{R}_+}(\sqrt{\rho_i}-\sqrt{\rho_e})^2dxd\tau\ge c\int_0^t\int_{\mathbb{R}_+}E_x^2(x,\tau)dxd\tau.
$$

Furthermore, since $\ln(1+x)\le x$ for all $x\ge 0$, $|\ln x-\ln y|\le\frac{|x-y|}{\min\{x,y\}}$ holds true and we hence have

$$
\int_0^t\Big((\ln\rho_i-\ln\rho_e)E(0,\tau)\Big)d\tau\le C\int_0^t[\phi_\alpha^2+E^2](0,\tau)d\tau.
$$

Multiplying (39) by a suitably small positive constant λ and adding the resulting inequality to (33), we then obtain

$$\begin{aligned}&||(\phi_\alpha, (\phi_\alpha)_x, \psi_\alpha)(t)||^2 + ||E(t)||^2 + \int_0^t |(\phi_\alpha, (\phi_\alpha)_x, E)(0,\tau)|^2 d\tau \\ &+ \int_0^t \int_{\mathbb{R}_+} \Big\{\overline{u}_x \Big[|(\phi_\alpha, \psi_\alpha, E)|^2\Big] + |((\phi_\alpha)_x, (\psi_\alpha)_x, E_x)|^2\Big\} dx d\tau \\ &\leq C\Big(N(0) + (\delta_r \varepsilon)^{\frac{1}{q}}\Big) + C(\delta_r\varepsilon)^{\frac{1}{q}}(N^2(t) + M^2(t)). \end{aligned} \tag{40}$$

Step 3. The estimate of $(\psi_\alpha)_x$.

Multiplying the second and fourth equation of (13) by $-(\psi_i)_{xx}/\rho_i$ and $-(\psi_e)_{xx}/\rho_e$, respectively, it follows that

$$\begin{aligned}&\Big(\frac{[(\psi_i)_x]^2}{2}\Big)_t - \Big((\psi_i)_t(\psi_i)_x + \frac{u_i[(\psi_i)_x]^2}{2}\Big)_x + \frac{[(\psi_i)_{xx}]^2}{\rho_i} \\ &= -\frac{[(\psi_i)_x]^3}{2} - \frac{\overline{u}_x[(\psi_i)_x]^2}{2} + \frac{P'(\rho_i)(\phi_i)_x(\psi_i)_{xx}}{\rho_i} + \frac{g_i(\psi_i)_{xx}}{\rho_i} - E(\psi_i)_{xx}, \end{aligned} \tag{41}$$

and

$$\begin{aligned}&\Big(\frac{[(\psi_e)_x]^2}{2}\Big)_t - \Big((\psi_e)_t(\psi_e)_x + \frac{u_e[(\psi_e)_x]^2}{2}\Big)_x + \frac{[(\psi_e)_{xx}]^2}{\rho_e} \\ &= -\frac{[(\psi_e)_x]^3}{2} - \frac{\overline{u}_x[(\psi_e)_x]^2}{2} + \frac{P'(\rho_e)(\phi_e)_x(\psi_e)_{xx}}{\rho_e} + \frac{g_e(\psi_e)_{xx}}{\rho_e} + E(\psi_e)_{xx}. \end{aligned} \tag{42}$$

Notice that

$$E(\psi_i - \psi_e)_{xx} = [E(\psi_i - \psi_e)_x]_x - E_x(\psi_i - \psi_e)_x.$$

Then, integrating (41) and (42) over $[0,t] \times \mathbb{R}_+$, adding them together, and using Cauchy's inequality, we obtain

$$\begin{aligned}&||(\psi_\alpha)_x(t)||^2 + \int_0^t \int_{\mathbb{R}_+} \overline{u}_x|(\psi_\alpha)_x|^2 + |(\psi_\alpha)_{xx}(\tau)|^2 dx d\tau \\ &\leq C\Big\{||(\phi_{\alpha 0}, \psi_{\alpha 0})||_1 + \int_0^t |(E, (\psi_i)_x, (\psi_e)_x)(0,\tau)|^2 d\tau \\ &+ \int_0^t \int_{\mathbb{R}_+} \Big(E_x^2 + [(\psi_\alpha)_x]^2 + (|(\phi_\alpha)_x| + |g_\alpha|)|(\psi_\alpha)_{xx}| + |(\psi_\alpha)_x|^3\Big) dx d\tau\Big\}. \end{aligned} \tag{43}$$

We now estimate those terms on the right-hand side of (43). By Sobolev inequality,

$$\begin{aligned}\int_0^t |(\psi_\alpha)_x(0,\tau)|^2 d\tau &\leq \int_0^t ||(\psi_\alpha)_x||^2_{L^\infty} d\tau \\ &\leq 8C \int_0^t ||(\psi_\alpha)_x||^2_{L^2} d\tau + \frac{1}{8C}\int_0^t ||(\psi_\alpha)_{xx}||^2_{L^2} d\tau, \end{aligned} \tag{44}$$

and

$$\int_0^t \int_{\mathbb{R}_+} |(\phi_\alpha)_x(\psi_\alpha)_{xx}| dx d\tau \leq 8C \int_0^t ||((\phi_\alpha)_x||^2_{L^2} d\tau + \frac{1}{8C}\int_0^t ||(\psi_\alpha)_{xx}||^2_{L^2} d\tau. \tag{45}$$

Because $g_{i,e} = \overline{u}_{xx} + \overline{u}_x \rho_{i,e}\psi_{i,e} + \overline{\rho}_x[P'(\rho_{i,e}) - P'(\overline{\rho}) - \frac{P'(\overline{\rho})}{\overline{\rho}}\phi_{i,e}]$, then we get that

$$\int_0^t \int_{\mathbb{R}_+} |g_\alpha(\psi_\alpha)_{xx}| dx d\tau \leq C\Big(\varepsilon(\delta_r\varepsilon)^{2/q} + \delta_r \varepsilon M^2(t)\Big) + \frac{1}{8C}\int_0^t ||(\psi_\alpha)_{xx}||^2_{L^2} d\tau. \tag{46}$$

Furthermore, we have

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}_+} |(\psi_\alpha)_x|^3 dx d\tau \le C \int_0^t ||(\psi_\alpha)_x||^{\frac{5}{2}} ||(\psi_\alpha)_{xx}||^{\frac{1}{2}} d\tau \\
&\le \int_0^t \Big[8C ||(\psi_\alpha)_x||^{\frac{10}{3}} + \frac{1}{8C} ||(\psi_\alpha)_{xx}||^2 \Big] d\tau \\
&\le 8C \sup_{0\le\tau\le t} ||(\psi_\alpha)_x||^{\frac{4}{3}} \int_0^t ||(\psi_\alpha)_x||^2 d\tau + \frac{1}{8C} \int_0^t ||(\psi_\alpha)_{xx}||^2 d\tau \\
&\le 8CN^{\frac{4}{3}}(t) M^2(t) + \frac{1}{8C} \int_0^t ||(\psi_\alpha)_{xx}||^2 d\tau. \qquad (47)
\end{aligned}$$

Thus, by plugging (44), (45), (46) and (47) into (43), we arrive at

$$\begin{aligned}
||(\psi_\alpha)_x(t)||^2 + \int_0^t \int_{\mathbb{R}_+} \overline{u}_x |(\psi_\alpha)_x|^2 + |(\psi_\alpha)_{xx}(\tau)|^2 d\tau \le C \Big\{ ||(\phi_{\alpha 0}, \psi_{\alpha 0})||_1 \\
+\varepsilon(\delta_r \varepsilon)^{2/q} + \int_0^t |E(0,\tau)|^2 d\tau + M^2(t) + N^{\frac{4}{3}}(t) M^2(t) \Big\}. \quad (48)
\end{aligned}$$

Combing (48) with the inequality (40) obtained in Step 2, we have

$$\begin{aligned}
&||(E, \phi_\alpha, \psi_\alpha, (\phi_\alpha)_x, (\psi_\alpha)_x)(t)||^2 + \int_0^t |(E, \phi_\alpha, (\phi_\alpha)_x)(0,\tau)|^2 d\tau \\
&\quad + \int_0^t \int_{\mathbb{R}_+} \Big\{ \overline{u}_x \Big[|(E, \phi_\alpha, \psi_\alpha, (\psi_\alpha)_x)|^2 \Big] + |(E_x, (\phi_\alpha)_x, (\psi_\alpha)_x, (\psi_\alpha)_{xx})|^2 \Big\} dx d\tau \\
&\le C\Big(N(0) + (\delta_r \varepsilon)^{\frac{1}{q}} \Big) + C(\delta_r \varepsilon)^{\frac{1}{q}} (N^2(t) + M^2(t)) + CN^{4/3}(t) M^2(t). \qquad (49)
\end{aligned}$$

From (49), it follows that there exists a constant ε_0 such that if $N(t)+\varepsilon \le \varepsilon_0$, then, for all $t \ge 0$, it holds that

$$N^2(t) + M^2(t) \le C[N^2(0) + (\delta_r \varepsilon)^{1/q}],$$

which implies (15). The proof of Proposition 1 is completed. □

Proof of Theorem 2.3: The existence of the solution follows from the standard continuation argument based on the local existence and the a priori estimates in Proposition 1. Therefore, it suffices to show the large time behavior of the solution as $t \to \infty$. First of all, we prove that

$$\lim_{t\to+\infty} ||((\phi_\alpha)_x, (\psi_\alpha)_x, E_x)(t)|| = 0. \qquad (50)$$

In fact, it is direct to check that

$$\begin{aligned}
E_{xt} &= (\rho_i u_i - \rho_e u_e)_x \\
&= \overline{\rho}_x (\psi_i - \psi_e) + \overline{\rho} (\psi_i - \psi_e)_x + \overline{u}_x E_x + \overline{u} E_{xx} + (\phi_i \psi_i - \psi_i \psi_e)_x.
\end{aligned}$$

Thus, one has

$$\begin{aligned}
&\Big|\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}_+} E_x^2 dx\Big| = \Big|\int_{\mathbb{R}_+} E_x E_{xt} dx\Big| \\
&= \Big|\int_{\mathbb{R}_+} E_x\Big[\overline{\rho}_x(\psi_i-\psi_e)+\overline{\rho}(\psi_i-\psi_e)_x+\overline{u}_x E_x+\overline{u}E_{xx}+(\phi_i\psi_i-\psi_i\psi_e)_x\Big]dx\Big| \\
&\le \int_{\mathbb{R}_+}|\overline{\rho}_x E_x(\psi_i-\psi_e)|dx+\int_{\mathbb{R}_+}|\overline{\rho}(\psi_i-\psi_e)_x|dx \\
&\qquad +\frac{|u_b|E_x^2(0,t)}{2}+\frac{1}{2}\int_{\mathbb{R}_+}\overline{u}_x E_x^2 dx+\int_{\mathbb{R}_+}|E_x(\phi_i\psi_i-\psi_i\psi_e)_x|dx,
\end{aligned}$$

which after further taking the time integration, implies that for all $t \ge 0$,

$$\int_0^t \Big|\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}_+} E_x^2 dx\Big| d\tau \le C(N^2(t)+M^2(t)) < +\infty. \tag{51}$$

From Proposition 1, the inequalities (38), (41), (43) and (51) yield

$$\int_0^\infty \Big[||(E_x,(\phi_\alpha)_x,(\psi_\alpha)_x)||^2+\Big|\frac{d}{dt}||(E_x,(\phi_\alpha)_x,(\psi_\alpha)_x)||^2\Big|\Big]dt < +\infty.$$

It implies that (50) holds true. Hence, by Sobolev inequality,

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}_+}|(\rho_{i,e}(x,t)-\overline{\rho}(x,t),\ u_{i,e}(x,t)-\overline{u}(x,t),\ E(x,t))| = 0.$$

Furthermore, by the construction of the smooth approximation function of the rarefaction wave, in terms of iv) in Lemma 2.2, we have the desired asymptotic behavior of the solution

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}_+}|(\rho_{i,e}(x,t)-\rho^R(\frac{x}{t}),\ u_{i,e}-u^R(\frac{x}{t}),\ E(x,t))| = 0.$$

This completes the proof of Theorem 2.3. □

3. **Stability of the boundary layer.** In this section, we are concerned with the time asymptotic stability of boundary layers for the IBVP (1), (7) and (8) of the two-fluid NSP system. Here, a boundary layer $(\tilde{\rho},\tilde{u})(x)$ is defined to be the stationary solution to

$$\begin{cases} (\tilde{\rho}\,\tilde{u})_x = 0, & x\in\mathbb{R}_+, \\ (\tilde{\rho}\,\tilde{u}^2+p(\tilde{\rho}))_x = \tilde{u}_{xx}, & x\in\mathbb{R}_+, \\ \tilde{u}(0)=u_b,\ (\tilde{\rho},\tilde{u})(\infty)=(\rho_+,u_+), & \inf_{x\in\mathbb{R}_+}\tilde{\rho}(x)>0. \end{cases} \tag{52}$$

In what follows let us present the existence and some known properties of the boundary layer $(\tilde{\rho},\tilde{u})(x)$ connecting (ρ_b,u_b) with (ρ_+,u_+) for the above single isentropic Navier-Stokes system in $\mathbb{R}_+$ in the absence of the electric field. First of all, integrating the first equation of (52) over $[x,\infty)$ for $x>0$ yields

$$\tilde{\rho}(x) = \rho_+ u_+(\tilde{u}(x))^{-1},$$

which implies by letting $x \to 0+$

$$\rho_b := \tilde{\rho}(0) = \rho_+ u_+ (u_b)^{-1}.$$

Thus, the condition $u_+ < 0$ has to be assumed whenever the outflow problem, i.e., the case $u_b < 0$, is considered. Moreover, let the strength of the boundary layer $(\tilde{\rho},\tilde{u})(x)$ be measured by

$$\tilde{\delta} := |u_+ - u_b|.$$

Then, one has the following

Lemma 3.1. *(See [16].) Assume that $(v_+,u_+)\in\Omega_{supp}$ and $u_b<0$. Then the stationary problem (52) has a smooth solution $(\tilde\rho,\tilde u)$, if and only if $M^+\le 1$ and $u_b<u_*$, where u_* is given in (6). The solution $(\tilde\rho,\tilde u)(x)$ is monotonic, that is, $\tilde u_x\lesseqgtr 0$ and $\tilde\rho_x\lesseqgtr 0$ if $u_b\gtreqless u_+$. If $M_+=1$, $\tilde u(x)$ is monotonically increasing and converges to u_+ algebraically as x tends to infinity, precisely, there exists a positive constant C such that*

$$|\partial_x^k(\tilde u-u_+)|\le\frac{C\tilde\delta^{k+1}}{(1+\tilde\delta x)^{k+1}}\ \textit{for}\ k=0,1,2,\cdots.$$

If $M^+>1$, $\tilde u(x)$ converges to u_+ exponentially as x tends to infinity, precisely, there exist two positive constants c,C such that

$$|\partial_x^k(\tilde u(x)-u_+)|\le C\tilde\delta e^{-cx}\ \textit{for}\ k=0,1,2,\cdots.$$

Remark 1. (i) For the existence of the boundary layer, the strength $\tilde\delta$ is not necessarily small.

(ii) It is easy to see that $\tilde\rho(x)$ satisfies the same properties as $\tilde u(x)$ above.

(iii) In the case $M_+=1$, the boundary layer is degenerate. For this case, $\tilde\rho(x)$ and $\tilde u(x)$ are increasing functions with

$$\rho_b\le\tilde\rho(x)\le\rho_+.$$

For later use, we also need the Poincaré type inequality obtained in the following

Lemma 3.2. *(i) Let $M_+=1$. For any function h and $k+j>2$, there is a positive constant C such that,*

$$\int_0^t\int_0^\infty|\partial_x^k\tilde u|^j|h|^2dxd\tau\le C\tilde\delta^{(k+1)j-2}\int_0^t\tilde\delta|h(0,\tau)|^2+||h_x(\tau)||^2d\tau,\tag{53}$$

provided that the right hand side of (53) is bounded.

(ii) Let $M_+>1$. For any function h and $k+j\ge 2$, there is a positive constant C such that

$$\int_0^t\int_0^\infty|\partial_x^k\tilde u|^j|h|^2dxd\tau\le C\tilde\delta^j\int_0^t|h(0,\tau)|^2+||h_x(\tau)||^2d\tau,\tag{54}$$

provided that the right hand side of (54) is bounded.

Proof. We only prove (53) in the degenerate case. The proof in the non-degenerate case is similar and thus is omitted for brevity. In fact, for the case $M_+=1$, whenever $k+j>2$,

$$\begin{aligned}&\int_0^t\int_0^\infty|\partial_x^k\tilde u|^j|h|^2dxd\tau\\&\le\int_0^t\int_0^\infty\frac{C\tilde\delta^{(k+1)j}}{(1+\tilde\delta x)^{(k+1)j}}\Big|h(0,\tau)+\int_0^x h_y(y,\tau)dy\Big|^2dxd\tau\\&\le\int_0^t\int_0^\infty\frac{C\tilde\delta^{(k+1)j}}{(1+\tilde\delta x)^{(k+1)j}}\Big[|h(0,\tau)|^2+x||h_x(\cdot,\tau)||^2\Big]dxd\tau\\&\le C\tilde\delta^{j(k+1)-2}\int_0^t\tilde\delta|h(0,\tau)|^2+||h_x(\tau)||^2d\tau.\end{aligned}$$

Thus, (53) holds true. The proof of Lemma 3.2 is complete. □

In order to study the stability of the boundary layer, as in the previous section, we also define the perturbation $(\phi_{i,e},\ \psi_{i,e})$ by

$$(\phi_{i,e},\ \psi_{i,e}) = (\rho_{i,e}(x,t) - \tilde{\rho}(x),\ u_{i,e}(x,t) - \tilde{u}(x)),$$

where $(\tilde{\rho},\tilde{u})$ is the boundary layer defined by (52). Then, the IBVP (1), (7) and (8) of the two-fluid NSP system is reformulated as

$$\begin{cases} \partial_t\phi_i + u_i\partial_x\phi_i + \rho_i\partial_x\psi_i = -f_i, \\ \rho_i\partial_t\psi_i + P'(\rho_i)\partial_x\phi_i + \rho_iu_i\partial_x\psi_i = (\psi_i)_{xx} - g_i + \rho_iE, \\ \partial_t\phi_e + u_e\partial_x\phi_e + \rho_e\partial_x\psi_e = -f_e, \\ \rho_e\partial_t\psi_e + P'(\rho_e)\partial_x\phi_e + \rho_eu_e\partial_x\psi_e = (\psi_e)_{xx} - g_e - \rho_eE, \\ E = -\int_x^\infty(\phi_i - \phi_e)(y,t)\,dy, \quad x,t\in\mathbb{R}_+, \\ (\phi_i,\psi_i)(t,0) = 0, \\ (\phi_i,\psi_i,\phi_e,\psi_e)(x,0) := (\phi_{i0},\psi_{i0},\phi_{e0},\psi_{e0})(x) \to 0,\ \textit{as } x\to\infty, \end{cases} \tag{55}$$

where the nonlinear terms $f_{i,e}, g_{i,e}$ are given by

$$\begin{cases} f_{i,e} = \tilde{u}_x\phi_{i,e} + \tilde{\rho}_x\psi_{i,e}, \\ g_{i,e} = \tilde{u}_x(\tilde{u}\phi_{i,e} + \rho_{i,e}\psi_{i,e}) + \tilde{\rho}_x[P'(\rho_{i,e}) - P'(\tilde{\rho})]. \end{cases}$$

Notice that for brevity, we have used the same notations as in (13) without any confusion.

In the following two subsections, we will discuss the global existence and large time behavior of solutions to the above IBVP (55) by two cases: the degenerate case $M_+ = 1$ and the non-degenerate case $M_+ > 1$.

3.1. **The stability of boundary layer in degenerate case $M_+ = 1$.** In this case, we look for the solution $(\phi_{i,e},\ \psi_{i,e})$ to (55) in the solution space

$$\begin{aligned} X_{bd}(0,T) = \Big\{(\phi_{i,e},\psi_{i,e},E)\Big|\ &(\phi_{i,e},\psi_{i,e})\in C(0,T;H^1),\ (\phi_{i,e})_x\in L^2(0,T;L^2), \\ &(\psi_{i,e})_x\in L^2(0,T;H^1), E\in C(0,T;L^2),\ \psi_{i,e}(t,0) = 0\ (0\le t<T)\Big\}, \end{aligned}$$

where $0 < T \le \infty$ is the lifespan.

The first result of this section concerning the stability for the boundary layer in the degenerate case is stated as follows.

Theorem 3.3. *Let $M_+ = 1$. For each given constant state $(v_+,u_+)\in\Gamma^-_{trans}$ and $u_b < u_+$, there is a unique v_b such that (52) admits a boundary layer $(\tilde{\rho},\tilde{u})$ connecting (v_b,u_b) and (v_+,u_+). Furthermore, there exist positive constants δ_0 and C_0 such that if*

$$||E_0|| + ||(\phi_{i0,e0},\psi_{i0,e0})||_1 + \tilde{\delta} \le \delta_0,$$

then the reformulated outflow problem (55) of the two-fluid NSP system has a unique global solution $(\phi_{i,e},\psi_{i,e},E)\in X_{bd}(0,+\infty)$, satisfying

$$\begin{aligned} &\sup_{t\ge0}\Big(||E(t)||^2 + ||(\phi_{i,e},\psi_{i,e})(t)||_1^2\Big) + \int_0^\infty\Big\{|(E,\phi_{i,e},(\phi_{i,e})_x)(0,\tau)|^2 \\ &\qquad + ||\sqrt{\tilde{u}_x}(\phi_{i,e},\psi_{i,e})(\tau)||^2 + ||(E_x,(\phi_{i,e})_x,(\psi_{i,e})_x,(\psi_{i,e})_{xx})(\tau)||^2\Big\}d\tau \\ &\le C_0\Big[||E_0||^2 + ||(\phi_{i0,e0},\psi_{i0,e0})||_1^2\Big], \end{aligned}$$

and

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}_+}|\big(\rho_{i,e}(x,t) - \tilde{\rho}(x), u_{i,e}(x,t) - \tilde{u}(x), E(x,t)\big)| = 0.$$

The local existence of the solution $(\phi_{i,e}, \psi_{i,e}, E)$ to the IBVP (55) can be proved by the standard iteration method and the details are omitted here. Therefore, to prove Theorem 3.3, it suffices to consider the uniform-in-time a priori estimates obtained in the following

Proposition 2. *(A priori estimates). Suppose that all conditions in Theorem 3.3 hold. Let $(\phi_{i,e}, \psi_{i,e}) \in X_{bd}(0,T)$ be a solution to the IBVP (55) for $0 < T \leq \infty$. Then there exist positive constants δ_1 and C_1 such that if*

$$\sup_{0\leq t\leq T}\Big(||E|| + ||(\phi_{i,e}, \psi_{i,e})||_1\Big) \leq \delta_1, \tag{56}$$

then the solution $(\phi_{i,e}, \psi_{i,e})$ satisfies

$$\begin{aligned}&\sup_{0\leq t<T}\Big(||E(t)||^2 + ||(\phi_{i,e}, \psi_{i,e})(t)||_1^2\Big) + \int_0^T \Big\{|(E, \phi_{i,e}, (\phi_{i,e})_x)(0,\tau)|^2 \\ &\quad + ||\sqrt{\tilde{u}_x}(\phi_{i,e}, \psi_{i,e})(\tau)||^2 + ||(E_x, (\phi_{i,e})_x, (\psi_{i,e})_x, (\psi_{i,e})_{xx})(\tau)||^2\Big\}d\tau \\ &\leq C_1\Big(||E_0||^2 + ||(\phi_{i0,e0}, \psi_{i0,e0})||_1^2\Big).\end{aligned} \tag{57}$$

Proof. As in the proof of Proposition 1, we denote

$$N^2(t) := \sup_{0\leq\tau\leq t}\Big(||E(\tau)||^2 + ||(\phi_{i,e}, \psi_{i,e})(\tau)||_1^2\Big),$$

and

$$\begin{aligned}M^2(t) \quad := \quad &\int_0^t \Big\{|(E, \phi_{i,e}, (\phi_{i,e})_x)(\tau, 0)|^2 + ||\sqrt{\tilde{u}_x}(\phi_{i,e}, \psi_{i,e})(\tau)||^2 \\ &+||(E_x, (\phi_{i,e})_x, (\psi_{i,e})_x, (\psi_{i,e})_{xx})(\tau)||^2\Big\}d\tau.\end{aligned}$$

By the Sobolev inequality and the assumption (56),

$$\begin{aligned}&||E(t)||_{L^\infty} \leq \sqrt{2}||E||^{1/2}||E_x||^{1/2} = \sqrt{2}||E||^{1/2}||\phi_i - \phi_e||^{1/2} \leq 2\sqrt{2}N(t) \leq C\delta_1, \\ &||(\phi_{i,e}, \psi_{i,e})(t)||_{L^\infty} \leq \sqrt{2}||\psi_{i,e}||^{1/2}\ ||(\psi_{i,e})_x||^{1/2} \leq \sqrt{2}N(t) \leq C\delta_1,\end{aligned} \tag{58}$$

for any $0 \leq t < T$. Therefore, for δ_1 sufficiently small,

$$\frac{1}{2}\rho_b \leq \rho_b - C\delta_1 \leq \tilde{\rho}(x) + \phi_{i,e}(x,t) = \rho_{i,e}(x,t) \leq \rho_+ + C\delta_1 \leq 2\rho_+, \tag{59}$$

for any $x \geq 0$ and $0 \leq t < T$.

We now divide the proof by three steps.

Step1. The zero-order energy estimates.

Similar to Step 1 in the proof of Proposition 1, we define the functionals

$$\mathcal{E}(\rho, u) := \Phi(\rho, \tilde{\rho}) + \frac{1}{2}|u - \tilde{u}|^2, \ \Phi(\rho, \tilde{\rho}) := \int_{\tilde{\rho}}^{\rho} \frac{P(s) - P(\tilde{\rho})}{s^2} ds,$$

where the energy form $\mathcal{E}(\rho, u)$ is equivalent to $|(\rho - \tilde{\rho}, u - \tilde{u})|^2$. For brevity, we also denote

$$\mathcal{E}_{i,e} := \mathcal{E}(\rho_{i,e}, u_{i,e}), \quad \Phi_{i,e} := \Phi(\rho_{i,e}, \tilde{\rho}), \quad h_\alpha := h_i + h_e, \tag{60}$$

where h is an arbitrary function taking different forms at different places. From (55), a direct computation yields

$$\begin{aligned}&\Big\{\rho_\alpha\mathcal{E}_\alpha+\frac{1}{2}E^2\Big\}_t+\Big\{\Big[\rho_\alpha u_\alpha\mathcal{E}_\alpha+[p(\rho_\alpha)-p(\tilde\rho)]\psi_\alpha-\psi_\alpha(\psi_\alpha)_x\Big]+\frac{\tilde u}{2}E^2\Big\}_x\\&\quad+\tilde u_x\Big\{\Big[P(\rho_\alpha)-P(\tilde\rho)-P'(\tilde\rho)\phi_\alpha+\rho_\alpha\psi_\alpha^2\Big]-\frac{1}{2}E^2\Big\}+(\psi_\alpha)_x^2\\&=-\frac{\tilde u_{xx}}{\tilde\rho}\Big[\phi_\alpha\psi_\alpha\Big].\end{aligned}\tag{61}$$

Notice that from the second and fourth equations of (1),

$$\begin{aligned}2E=\partial_t(u_i-u_e)+\frac{1}{2}\partial_x(u_i^2-u_e^2)\\+\Big[\frac{P'(\rho_i)\partial_x\rho_i}{\rho_i}-\frac{P'(\rho_e)\partial_x\rho_e}{\rho_e}\Big]-\Big[\frac{(u_i)_{xx}}{\rho_i}-\frac{(u_e)_{xx}}{\rho_e}\Big].\end{aligned}\tag{62}$$

By further multiplying (62) by $\tilde u_xE/4$, one has

$$\begin{aligned}\frac{\tilde u_x}{2}E^2&=\Big(\frac{\tilde u_x}{4}(u_i-u_e)E\Big)_t-\frac{\tilde u_x}{4}(u_i-u_e)E_t\\&\quad+\frac{\tilde u_x}{8}E\partial_x(u_i^2-u_e^2)+\frac{A\gamma\overline{u}_x}{4(\gamma-1)}(\rho_i^{\gamma-1}-\rho_e^{\gamma-1})_xE\\&\quad+\frac{\tilde u_x}{4}E\Big[\frac{(u_i)_{xx}}{\rho_i}-\frac{(u_e)_{xx}}{\rho_e}\Big]\\&=\Big(\frac{\tilde u_x}{4}(u_i-u_e)E\Big)_t+\Big(\frac{\tilde u\tilde u_x}{4}E(\psi_i-\psi_e)\Big)_x\\&\quad+\frac{\tilde u_x}{8}(\rho_i+\rho_e)(\psi_i-\psi_e)^2-\frac{\tilde u\tilde u_{xx}}{4}E(\psi_i-\psi_e)\\&\quad+\frac{\tilde u_x}{8}E_x(\psi_i^2-\psi_e^2)+\frac{A\gamma\overline{u}_x}{4(\gamma-1)}(\rho_i^{\gamma-1}-\rho_e^{\gamma-1})_xE\\&\quad+\frac{\tilde u_x}{4}E\Big(\psi_i(\psi_i)_x-\psi_e(\psi_e)_x\Big)+\frac{\tilde u_x}{4}\Big[\frac{(u_i)_{xx}}{\rho_i}-\frac{(u_e)_{xx}}{\rho_e}\Big]E.\end{aligned}\tag{63}$$

Plugging (63) into (61), we arrive at

$$\begin{aligned}&\Big\{\rho_\alpha\mathcal{E}_\alpha-\frac{\tilde u_x(\psi_i-\psi_e)E}{4}+\frac{E^2}{2}\Big\}_t\\&\quad+\Big\{\Big[\rho_\alpha u_\alpha\mathcal{E}_\alpha+[P(\rho_\alpha)-P(\overline\rho)]\psi_\alpha-(\frac{\psi_\alpha^2}{2})_x\Big]+\frac{\tilde u\tilde u_x(\psi_i-\psi_e)E}{4}+\frac{\tilde uE^2}{2}\Big\}_x\\&\quad+\tilde u_x\Big[P(\rho_\alpha)-P(\overline\rho)-P'(\tilde\rho)\phi_\alpha+\rho_\alpha\psi_\alpha^2-\frac{\rho_\alpha}{8}(\psi_i-\psi_e)^2\Big]+[(\psi_\alpha)_x]^2\\&=-\frac{\tilde u_{xx}}{\tilde\rho}\Big[\phi_\alpha\psi_\alpha\Big]+(H.O.T.)',\end{aligned}\tag{64}$$

where

$$\begin{aligned}(H.O.T.)'&=-\frac{\tilde u\tilde u_{xx}}{4}E(\psi_i-\psi_e)+\frac{\tilde u_x}{8}E_x(\psi_i^2-\psi_e^2)+\frac{\tilde u_xE}{8}\Big(\psi_i^2-\psi_e^2\Big)_x\\&\quad+\frac{A\gamma\tilde u_xE}{4(\gamma-1)}(\rho_i^{\gamma-1}-\rho_e^{\gamma-1})_x+\frac{\tilde u_x}{4}\Big[\frac{(u_i)_{xx}}{\rho_i}-\frac{(u_e)_{xx}}{\rho_e}\Big]E\\&:=R_1'+R_2'+R_3'+R_4'+R_5'.\end{aligned}$$

Integrating (64) over $[0,t]\times\mathbb{R}_+$ and noticing $u_b<0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}_+}\Big\{\rho_\alpha\mathcal{E}_\alpha-\frac{\tilde u_x(\psi_i-\psi_e)E}{4}+\frac{E^2}{2}\Big\}dx+\int_0^t|u_b|\Big[\rho_\alpha\Phi_\alpha(0,\tau)+\frac{E^2(0,\tau)}{2}\Big]d\tau\\
&\quad+\int_0^t\int_{\mathbb{R}_+}\Big\{\tilde u_x\Big[\big(P(\rho_\alpha)-P(\tilde\rho)-P'(\tilde\rho)\phi_\alpha\big)\\
&\qquad\qquad+\rho_\alpha\psi_\alpha^2-\frac{\rho_\alpha}{8}(\psi_i-\psi_e)^2\Big]+|(\psi_\alpha)_x|^2\Big\}dxd\tau\\
&=\int_{\mathbb{R}_+}\Big\{\rho_{\alpha0}\mathcal{E}_{\alpha0}-\frac{\tilde u_x(\psi_{i0}-\psi_{e0})E_0}{4}+\frac{E_0^2}{2}\Big\}dx\\
&\qquad\qquad+\int_0^t\int_{\mathbb{R}_+}\Big[-\frac{\tilde u_{xx}\phi_\alpha\psi_\alpha}{\tilde\rho}+(H.O.T.)'\Big]dxd\tau. \qquad (65)
\end{aligned}$$

For the first term on the right hand side of (65), we estimate it as

$$\begin{aligned}
&\Big|\int_0^t\int_{\mathbb{R}_+}\frac{\tilde u_{xx}\phi_\alpha\psi_\alpha}{\tilde\rho}dxd\tau\Big|\le\Big|\int_0^t\int_{\mathbb{R}_+}\frac{\tilde u_{xx}}{\tilde\rho}\Big[\phi_\alpha^2+\psi_\alpha^2\Big]dxd\tau\Big|\\
&\le C\tilde\delta\int_0^t\tilde\delta|\phi_\alpha(0,\tau)|^2+||((\phi_\alpha)_x,(\psi_\alpha)_x)||^2d\tau\le C\tilde\delta M^2(t).
\end{aligned}$$

Similarly, we get

$$\Big|\int_0^t\int_{\mathbb{R}_+}R_1'dxd\tau\Big|\le C\tilde\delta\int_0^t\tilde\delta|E(0,\tau)|^2+||(E_x,(\psi_\alpha)_x)||^2d\tau\le C\tilde\delta M^2(t).$$

We now estimate the rest terms corresponding to R_2', R_3', R_4', R_5'. In a general way, for given functions f and g,

$$\begin{aligned}
&\Big|\int_0^t\int_{\mathbb{R}_+}\tilde u_x f g_x dxd\tau\Big|\\
&\le\Big|\int_0^t\int_{\mathbb{R}_+}\tilde u_x^{\frac34}f\tilde u_x^{\frac14}g_x dxd\tau\Big|\le C\tilde\delta\int_0^t\int_{\mathbb{R}_+}\Big[\tilde u_x f^2+g_x^2\Big]dxd\tau\\
&\le C\tilde\delta\int_0^t\tilde\delta|f(0,\tau)|^2+||f_x||^2+||g_x||^2\Big]dxd\tau,
\end{aligned}$$

where Lemma 3.1 and Lemma 3.2 have been used. Recalling the inequalities (3.2), (58) and (59), we obtain

$$\int_0^t\int_{\mathbb{R}_+}|R_2'|+|R_3'|dxd\tau\le C\tilde\delta\int_0^t\tilde\delta|E(0,\tau)|^2+||(E_x,(\psi_\alpha)_x)||^2d\tau\le C\tilde\delta M^2(t).$$

Since

$$(\rho_i^{\gamma-1}-\rho_e^{\gamma-1})_x=(\gamma-1)\Big[(\rho_i^{\gamma-2}-\rho_e^{\gamma-2})\tilde\rho_x+\rho_i^{\gamma-2}(\phi_i)_x-\rho_e^{\gamma-2}(\phi_e)_x\Big],$$

we have

$$\int_0^t\int_{\mathbb{R}_+}|R_4'|dxd\tau\le C\tilde\delta\int_0^t\tilde\delta|E(0,\tau)|^2+||(E_x,(\phi_\alpha)_x)||^2d\tau\le C\tilde\delta M^2(t).$$

Similar to estimating R_6 as in the proof of Proposition 1, a direct computation yields

$$
\begin{aligned}
R_5' = & \Big[\frac{\tilde{u}_x E}{4}\Big(\frac{(\psi_i)_x}{\rho_i} - \frac{(\psi_e)_x}{\rho_e}\Big)\Big]_x + \frac{(\tilde{u}_{xx} + \tilde{u}_x\tilde{\rho}_x)E}{4}\Big(\frac{(\psi_i)_x}{\rho_i^2} - \frac{(\psi_e)_x}{\rho_e^2}\Big) \\
& + \frac{\tilde{u}_x E_x}{4}\Big(\frac{(\psi_i)_x}{\rho_i} - \frac{(\psi_e)_x}{\rho_e}\Big) + \frac{\tilde{u}_x E}{4}\Big(\frac{(\phi_i)_x(\psi_i)_x}{\rho_i^2} - \frac{(\phi_e)_x(\psi_e)_x}{\rho_e^2}\Big) \\
& + \frac{\tilde{u}_x \tilde{u}_{xx}}{4\rho_i\rho_e} E E_x,
\end{aligned}
$$

which further implies

$$
\begin{aligned}
& \int_0^t \int_{\mathbb{R}_+} |R_5'| dx d\tau \\
\le\ & C\tilde{\delta}\int_0^t \Big[|(E, (\psi_\alpha)_x)(\tau, 0)|^2 + ||(E_x, (\phi_\alpha)_x, (\psi_\alpha)_x)(\tau)||^2\Big] d\tau \le C\tilde{\delta} M^2(t).
\end{aligned}
$$

Then, by collecting the previous estimates, it follows from (65) that

$$
\begin{aligned}
& \int_{\mathbb{R}_+} \Big\{\rho_\alpha \mathcal{E}_\alpha - \frac{\tilde{u}_x}{4}(\psi_i - \psi_e)E + \frac{1}{2}E^2\Big\} dx \\
& \quad + \int_0^t |u_b|\Big[\rho_\alpha \Phi_\alpha(0, \tau) + \frac{1}{2}E^2(0, \tau)\Big] d\tau \\
& \quad + \int_0^t \int_{\mathbb{R}_+} \Big\{\tilde{u}_x\Big[\big(P(\rho_\alpha) - P(\overline{\rho}) - P'(\overline{\rho})\phi_\alpha\big) \\
& \qquad\qquad\qquad + \rho_\alpha\psi_\alpha^2 - \frac{\rho_\alpha}{8}(\psi_i - \psi_e)^2\Big] + |(\psi_\alpha)_x|^2\Big\} dx d\tau \\
\le\ & C\Big(||(\phi_{\alpha 0}, \psi_{\alpha 0})||^2 + ||E_0||^2\Big) + C\tilde{\delta} M^2(t).
\end{aligned}
$$

Here, further notice that

$$
\begin{aligned}
& \int_0^t \int_{\mathbb{R}_+} \overline{u}_x\Big[\rho_\alpha\psi_\alpha^2 - \frac{\rho_\alpha}{8}(\psi_i - \psi_e)^2\Big] dx d\tau \\
=\ & \int_0^t \int_{\mathbb{R}_+} \overline{u}_x\overline{\rho}\Big[\psi_i^2 + \psi_e^2 - \frac{(\psi_i - \psi_e)^2}{4}\Big] + \overline{u}_x\Big[\phi_\alpha\psi_\alpha^2 - \frac{\phi_\alpha}{8}(\psi_i - \psi_e)^2\Big] dx d\tau \\
\ge\ & \frac{1}{2}\int_0^t \int_{\mathbb{R}_+} \overline{u}_x\overline{\rho}\Big[\psi_i^2 + \psi_e^2\Big] dx d\tau - CN(t)M^2(t).
\end{aligned}
$$

Therefore, by choosing $\tilde{\delta}_1$ small enough, we obtain

$$
\begin{aligned}
& ||\phi_\alpha(t)||^2 + ||\psi_\alpha(t)||^2 + ||E(t)||^2 + \int_0^t \Big[|\phi_\alpha(0, \tau)|^2 + E^2(0, \tau)\Big] d\tau \\
& \quad + \int_0^t \int_{\mathbb{R}_+} \Big\{\overline{u}_x\Big[|\phi_\alpha|^2 + |\psi_\alpha|^2 + E^2\Big] + ||(\psi_\alpha)_x(\tau)||^2\Big\} dx \\
\le\ & C\Big(||(\phi_{\alpha 0}, \psi_{\alpha 0})||^2 + ||E_0||^2\Big) + C\tilde{\delta} M^2(t). \qquad (66)
\end{aligned}
$$

Step 2. The estimates of $(\phi_\alpha)_x$.

Similar to obtaining (34)-(37), we also get

$$\begin{aligned}
&\Big(\frac{[(\phi_\alpha)_x]^2}{2\rho_\alpha^3}+\frac{(\phi_\alpha)_x\psi_\alpha}{\rho_\alpha}\Big)_t\\
&\quad+\Big(\frac{u_\alpha[(\phi_\alpha)_x]^2}{2(\rho_\alpha)^3}-\frac{(\phi_\alpha)_t\psi_\alpha}{\rho_\alpha}-\frac{\tilde\rho_x(\psi_\alpha)^2}{\rho_\alpha}\Big)_x+\frac{P'(\rho_\alpha)[(\phi_\alpha)_x]^2}{\rho_\alpha^2}\\
&=[(\psi_\alpha)_x]^2+\frac{\tilde u_x\phi_\alpha(\psi_\alpha)_x}{\rho_\alpha}-\frac{\tilde\rho_{xx}(\psi_\alpha)^2}{\rho_\alpha}-\frac{2\tilde\rho_x\psi_\alpha(\psi_\alpha)_x}{\rho_\alpha}\\
&\quad+\frac{\tilde u_x(\tilde\rho_x\phi_\alpha-\tilde\rho(\phi_\alpha)_x)\psi_\alpha}{\rho_\alpha^2}-\frac{\tilde u_x[(\phi_\alpha)_x]^2}{\rho_\alpha^3}+\frac{\tilde\rho_x(\phi_\alpha)_x(\phi_\alpha)_x}{(\rho_\alpha)^3}\\
&\quad-\frac{(f_\alpha)_x(\phi_\alpha)_x}{\rho_\alpha^3}-\frac{g_\alpha(\phi_\alpha)_x}{\rho_\alpha^2}+E\Big(\frac{(\phi_i)_x}{\rho_i}-\frac{(\phi_e)_x}{\rho_e}\Big).
\end{aligned}\tag{67}$$

The last term in the above equation can be rewritten as

$$\begin{aligned}
&E\Big(\tfrac{(\phi_i)_x}{\rho_i}-\tfrac{(\phi_e)_x}{\rho_e}\Big)=\Big[(\ln\rho_i-\ln\rho_e)_x\Big]E-\tfrac{(\rho_i-\rho_e)E\tilde\rho_x}{\rho_i\rho_e}\\
&=\Big[(\ln\rho_i-\ln\rho_e)E\Big]_x-(\ln\rho_i-\ln\rho_e)E_x+\tfrac{E_xE\tilde\rho_x}{\rho_i\rho_e}.
\end{aligned}$$

Integrating (67) over $[0,t]\times\mathbb{R}_+$, similar to Step 1, it is straightforward to check that

$$\begin{aligned}
&\int_{\mathbb{R}_+}\frac{[(\phi_\alpha)_x]^2}{\rho_\alpha^3}+\frac{(\phi_\alpha)_x\psi_\alpha}{\rho_\alpha}dx+\int_0^t\frac{|u_b|[(\phi_\alpha)_x]^2}{(\rho_\alpha)^3}(0,\tau)d\tau\\
&\quad+\int_0^t\int_{\mathbb{R}_+}\Big\{\frac{P'(\rho_\alpha)[(\phi_\alpha)_x]^2}{\rho_\alpha^2}+(\ln\rho_i-\ln\rho_e)E_x\Big\}dxd\tau\\
&\le C\Big(||(\psi_{\alpha0},(\phi_{\alpha0})_x)||^2\Big)+\int_0^t(\ln\rho_i-\ln\rho_e)E(0,\tau)d\tau+C\tilde\delta M^2(t).
\end{aligned}\tag{68}$$

Under the assumption (59) on the densities, it holds that

$$\int_0^t\int_{\mathbb{R}_+}(\ln\rho_i-\ln\rho_e)E_xdxd\tau\ge c\int_0^t\int_{\mathbb{R}_+}E_x^2(x,\tau)dxd\tau,$$

and

$$\int_0^t\Big((\ln\rho_i-\ln\rho_e)E(0,\tau)\Big)d\tau\le C\int_0^t[\phi_\alpha^2+E^2](0,\tau)d\tau.$$

Then, using the above two estimates in (68), multiplying it by a suitably small positive constant λ and adding the resulting inequality to (66), we obtain

$$\begin{aligned}
&||(\phi_\alpha,(\phi_\alpha)_x,\psi_\alpha)(t)||^2+||E(t)||^2+\int_0^t|(\phi_\alpha,(\phi_\alpha)_x,E)(0,\tau)|^2d\tau\\
&\quad+\int_0^t\int_{\mathbb{R}_+}\Big\{\tilde u_x|(\phi_\alpha,\psi_\alpha,E)|^2+|((\phi_\alpha)_x,(\psi_\alpha)_x,E_x)|^2\Big\}dxd\tau\\
&\le CN(0)+C\tilde\delta M^2(t).
\end{aligned}\tag{69}$$

Step 3. The estimate of $(\psi_\alpha)_x$.

After multiplying the second and fourth equations of (55) by $-(\psi_i)_{xx}/\rho_i$ and $-(\psi_e)_{xx}/\rho_e$ respectively, adding the resulting equations together, and then integrating them over $[0,t]\times\mathbb{R}_+$, it follows that

$$\begin{aligned}
&||(\psi_\alpha)_x(t)||^2+\int_0^t\int_{\mathbb{R}_+}|(\psi_\alpha)_{xx}(\tau)|^2d\tau\\
&\le C\Big\{||(\phi_{\alpha 0},\psi_{\alpha 0})||_1+\int_0^t|(E,(\psi_\alpha)_x)(0,\tau)|^2d\tau+\int_0^t\int_{\mathbb{R}_+}\Big(\overline{u}_x|(\psi_\alpha)_x|^2\\
&\qquad+E_x^2+[(\psi_\alpha)_x]^2+|(\phi_\alpha)_x(\psi_\alpha)_{xx}|+|g_\alpha(\psi_\alpha)_{xx}|+|(\psi_\alpha)_x|^3\Big)dxd\tau\Big\},
\end{aligned}\tag{70}$$

which further implies

$$\begin{aligned}
&||(\psi_\alpha)_x(t)||^2+\int_0^t\int_{\mathbb{R}_+}|(\psi_\alpha)_{xx}(\tau)|^2d\tau\\
&\qquad\qquad\le C\Big\{||(\phi_{\alpha 0},\psi_{\alpha 0})||_1+M^2(t)+N^{\frac{4}{3}}(t)M^2(t)\Big\}.
\end{aligned}\tag{71}$$

We now combing the estimate (71) with the inequality (69) so as to obtain

$$\begin{aligned}
&||(E,\phi_\alpha,\psi_\alpha,(\phi_\alpha)_x,(\psi_\alpha)_x)(t)||^2+\int_0^t|(E,\phi_\alpha,(\phi_\alpha)_x)(0,\tau)|^2d\tau\\
&+\int_0^t\int_{\mathbb{R}_+}\Big\{\overline{u}_x|(E,\phi_\alpha,\psi_\alpha,(\psi_\alpha)_x)|^2+|(E_x,(\phi_\alpha)_x,(\psi_\alpha)_x,(\psi_\alpha)_{xx})|^2\Big\}dxd\tau\\
&\le C\Big(N(0)+N^{4/3}(t)M^2(t)\Big).
\end{aligned}\tag{72}$$

From (72), it follows that there exists a constant δ_1 such that if $N(t)+\delta\le\delta_1$, then, for all $0\le t<T$,

$$N^2(t)+M^2(t)\le CN^2(0).$$

This implies (57). The proof of Proposition 2 is completed. □

Proof of Theorem 3.3. The proof is similar to that for Theorem 2.3, and the details are omitted for brevity. □

3.2. **The stability of boundary layer in non-degenerate case $M_+>1$.** In this case, we look for the solution $(\phi_{i,e},\ \psi_{i,e})$ to the IBVP (55) in the solution space

$$\begin{aligned}
&X_{bnd}(0,T)\\
&\ =\Big\{(\phi_{i,e},\psi_{i,e},E)\Big|(\phi_{i,e},\psi_{i,e})\in C(0,T;H^1),\ (\phi_{i,e})_x\in L^2(0,T;L^2),\\
&\qquad(\psi_{i,e})_x\in L^2(0,T;H^1),E\in C(0,T;L^2),\ \psi_{i,e}(t,0)=0\ (0\le t<T),\\
&\qquad\frac{1}{2M_0}\le\tilde\rho(x)+\phi_{i,e}(x,t)\le 2M_0,\ \forall\,x\in\mathbb{R}_+,0\le t<T\Big\},
\end{aligned}\tag{73}$$

where $0<T\le\infty$ is the lifespan and $M_0>0$ is a given constant to be chosen later.

The second result of this section concerning the asymptotic stability of the boundary layer in the non-degenerate case is stated as follows.

Theorem 3.4. *Let $M_+>1$. For each given constant state $(v_+,u_+)\in\Gamma^-_{supp}$ and $u_b<u_*$ with u_* given in (6), there is a unique v_b such that (52) admits a boundary*

layer solution $(\tilde{\rho},\tilde{u})$ connecting (v_b,u_b) and (v_+,u_+). Furthermore, assume that initial data satisfies

$$E_0\in L^2(\mathbb{R}_+),\ (\phi_{i0,e0},\psi_{i0,e0})\in H^1(\mathbb{R}^+),\ \inf_{x\in\mathbb{R}^+}(\rho_{i0,e0})>0.$$

Then, there exists a positive constant $\delta_2<1$ depending only on initial data such that if $\tilde{\delta}=|u_+-u_b|\le\delta_2$, the reformulated outflow problem (55) of the two-fluid NSP system has a unique global solution $(\phi_{i,e},\psi_{i,e},E)\in X_{bnd}(0,\infty)$ for an approximate constant $M_0>0$. Moreover, it holds that

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}_+}|\big(\rho_{i,e}(x,t)-\tilde{\rho}(x),u_{i,e}(x,t)-\tilde{u}(x),E(x,t)\big)|=0.$$

To prove the above theorem, similarly as before, it suffices to establish the a priori estimates on the solution in the following proposition with the rest of this subsection further devoted to its proof.

Proposition 3. *(A priori estimates). Suppose that all conditions in Theorem 3.4 hold. Let $(\phi_{i,e},\psi_{i,e},E)\in X_{bnd}(0,T)$ be a solution to the IBVP (55) for some $0<T\le\infty$. Then there exist positive constants $\delta_3>0$ and $C_1>0$ such that if $\tilde{\delta}=|u_+-u_b|\le\delta_3$, the solution $(\phi_{i,e},\psi_{i,e},E)$ satisfies*

$$\begin{aligned}\sup_{0\le t<T}&\Big(||E(t)||^2+||(\phi_{i,e},\psi_{i,e})(t)||_1^2\Big)+\int_0^T\Big\{|(E,\phi_{i,e},(\phi_{i,e})_x)(0,\tau)|^2\\&+||\sqrt{\tilde{u}_x}(\phi_{i,e},\psi_{i,e})(\tau)||^2+||(E_x,(\phi_{i,e})_x,(\psi_{i,e})_x,(\psi_{i,e})_{xx})(\tau)||^2\Big\}d\tau\\&\le C_1\Big(||E_0||^2+||(\phi_{i0,e0},\psi_{i0,e0})||_1^2\Big).\end{aligned}\tag{74}$$

Proof. We divide the proof by the following three steps.

Step1. The basic energy estimates.

We use the same notation $\mathcal{E}_{i,e}$ as in (60). From (55), a direct computation yields

$$\begin{aligned}&\Big\{\rho_\alpha\mathcal{E}_\alpha+\frac{1}{2}E^2\Big\}_t\\&\quad+\Big\{\Big[\rho_\alpha u_\alpha\mathcal{E}_\alpha+[P(\rho_\alpha)-P(\tilde{\rho})]\psi_\alpha-\psi_\alpha(\psi_\alpha)_x\Big]+\frac{\tilde{u}}{2}E^2\Big\}_x+(\psi_\alpha)_x^2\\&=-\tilde{u}_x\Big\{\Big[P(\rho_\alpha)-P(\tilde{\rho})-P'(\tilde{\rho})\phi_\alpha+\rho_\alpha\psi_\alpha^2\Big]-\frac{1}{2}E^2\Big\}-\frac{\tilde{u}_{xx}}{\tilde{\rho}}\Big[\phi_\alpha\psi_\alpha\Big],\end{aligned}$$

which after taking integration over $[0,t]\times\mathbb{R}_+$ and noticing $u_b<0$, implies

$$\begin{aligned}&\int_{\mathbb{R}_+}\Big\{\rho_\alpha\mathcal{E}_\alpha+\frac{1}{2}E^2\Big\}dx\\&\quad+\int_0^t\Big\{|u_b|\Big[\rho_\alpha\Phi_\alpha(0,\tau)+\frac{1}{2}E^2(0,\tau)\Big]+||(\psi_\alpha)_x(\tau)||^2\Big\}d\tau\\&\le C\Big(||(\phi_{\alpha0},\psi_{\alpha0},E_0)||^2\Big)-\int_0^t\int_{\mathbb{R}_+}\Big\{\tilde{u}_x\Big[\Big(P(\rho_\alpha)-P(\tilde{\rho})\\&\qquad-P'(\tilde{\rho})\phi_\alpha+\rho_\alpha\psi_\alpha^2\Big)-\frac{1}{2}E^2\Big]+\frac{\tilde{u}_{xx}}{\tilde{\rho}}[\phi_\alpha\psi_\alpha]\Big\}dxd\tau.\end{aligned}$$

From Lemma 3.2, we have

$$
\begin{aligned}
&\Big|\int_0^t\int_{\mathbb{R}_+}\tilde{u}_x\big(P(\rho_\alpha)-P(\tilde{\rho})-P'(\tilde{\rho})\phi_\alpha+\rho_\alpha\psi_\alpha^2\big)+\frac{\tilde{u}_{xx}}{\tilde{\rho}}[\phi_\alpha\psi_\alpha]dxd\tau\Big|\\
&\le C(M_0)\int_0^t\int_{\mathbb{R}_+}(\tilde{u}_x+\tilde{u}_{xx})\Big[\phi_\alpha^2+\psi_\alpha^2\Big]dxd\tau\\
&\le C(M_0)\tilde{\delta}\int_0^t|u_b|\rho_\alpha\Phi_\alpha(0,\tau)+||\big((\phi_\alpha)_x,(\psi_\alpha)_x\big)(\tau)||^2d\tau,
\end{aligned}
$$

since $\psi(0,\tau)=0$ and $\phi_\alpha^2(0,\tau)\le C(M_0)(\rho_\alpha\Phi_\alpha(0,\tau))$ for all $0\le\tau\le t$. At the same time, from (iii) in Lemma (3.2), we directly have

$$
\int_0^t\int_{\mathbb{R}_+}\tilde{u}_xE^2dxd\tau\le C\tilde{\delta}\int_0^t|E(0,\tau)|^2+||E_x(\tau)||^2d\tau.
$$

The above estimate is different from one in the degenerate case. Here we do not need to give the estimate similar to (63) from the system. By choosing δ_4 suitably small such that

$$
C(M_0)\delta_4\le\frac{1}{2},\ C\delta_4\le\frac{1}{4}|u_b|,
$$

then, we get that, for $\tilde{\delta}\le\delta_4$

$$
\begin{aligned}
&\int_{\mathbb{R}_+}\Big\{\rho_\alpha\mathcal{E}_\alpha+\frac{1}{2}E^2\Big\}dx\\
&\qquad+\int_0^t\Big\{|u_b|\Big[\rho_\alpha\Phi_\alpha(0,\tau)+\frac{1}{2}E^2(0,\tau)\Big]+||(\psi_\alpha)_x(\tau)||^2\Big\}d\tau\\
&\le C\Big(||(E_0,\phi_{\alpha0},\psi_{\alpha0})||^2+\tilde{\delta}\int_0^t||E_x(\tau)||^2d\tau\Big).
\end{aligned}\tag{75}
$$

Similar to obtaining (68), it is straightforward to check that

$$
\begin{aligned}
&\int_{\mathbb{R}_+}\frac{[(\phi_\alpha)_x]^2}{\rho_\alpha^3}+\frac{(\phi_\alpha)_x\psi_\alpha}{\rho_\alpha}dx+\int_0^t\frac{|u_b|[(\phi_\alpha)_x]^2}{(\rho_\alpha)^3}(0,\tau)d\tau\\
&\quad+\int_0^t\int_{\mathbb{R}_+}\Big\{\frac{p'(\rho_\alpha)[(\phi_\alpha)_x]^2}{\rho_\alpha^2}+(\ln\rho_i-\ln\rho_e)E_x\Big\}dxd\tau\\
&\le C(||(\phi_{\alpha0},\psi_{\alpha0})||^2+||(\phi_{\alpha0})_x||^2)+\int_0^t\Big((\ln\rho_i-\ln\rho_e)E(0,\tau)\Big)d\tau\\
&\quad+C(M_0)\tilde{\delta}\int_0^t||\big((\phi_\alpha)_x,(\psi_\alpha)_x\big)(\tau)||^2d\tau\\
&\quad+C(M_0)\int_0^t\int_{\mathbb{R}_+}(\tilde{u}_{xx}+\tilde{u}_x^2)(\phi_\alpha^2+\psi_\alpha^2+E^2)dxd\tau.
\end{aligned}\tag{76}
$$

Here we have used the fact that $|\tilde{\rho}_{xx}|\le C(\tilde{u}_{xx}+\tilde{u}_x^2)$. Similarly as before, we have

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}_+}(\tilde{u}_{xx}+\tilde{u}_x^2)(\phi_\alpha^2+\psi_\alpha^2+E^2)dxd\tau\\
&\le C(M_0)\tilde{\delta}\int_0^t(\rho_\alpha\Phi_\alpha+E^2)(0,\tau)+||\big((\phi_\alpha)_x,(\phi_\alpha),E_x\big)(\tau)||^2d\tau.
\end{aligned}
$$

Moreover, it holds that

$$\int_0^t \int_{\mathbb{R}_+} (\ln \rho_i - \ln \rho_e) E_x dx d\tau \geq \int_0^t \int_{\mathbb{R}_+} (\sqrt{\rho_i} - \sqrt{\rho_e})^2 dx d\tau \\ \geq C(M_0) \int_0^t \int_{\mathbb{R}_+} E_x^2(x,\tau) dx d\tau,$$

and

$$\int_0^t \Big((\ln \rho_i - \ln \rho_e) E(0,\tau) \Big) d\tau \leq C(M_0) \int_0^t \rho_\alpha \Phi_\alpha(0,\tau) d\tau + \int_0^t E^2(0,\tau) d\tau.$$

So, by plugging the above estimates into (76) and further taking summation of (75) and (76) multiplied by a properly small constant $\lambda > 0$ such that

$$0 < \tilde{\delta} \leq \tilde{\delta}_5 \ll \lambda \ll 1, \ C(M_0)\tilde{\delta}_5 < 1/2,$$

one has

$$\begin{aligned} &\int_{\mathbb{R}_+} \Big\{ \rho_\alpha \mathcal{E}_\alpha + \frac{[(\phi_\alpha)_x]^2}{\rho_\alpha^3} + \frac{E^2}{2} \Big\} dx \\ &\quad + \int_0^t \Big(\rho_\alpha \Phi_\alpha + \frac{[(\phi_\alpha)_x]^2}{\rho_\alpha^3} + E^2 \Big)(0,\tau) d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}_+} \Big\{ \frac{p'(\rho_\alpha)[(\phi_\alpha)_x]^2}{\rho_\alpha^2} + [(\psi_\alpha)_x]^2 + E_x^2 \Big\} dx d\tau \\ &\leq C(||(\phi_{\alpha 0}, \psi_{\alpha 0})||^2 + ||(\phi_{\alpha 0})_x||^2). \end{aligned} \tag{77}$$

Step 2. The bounds of the densities $\rho_{i,e}$.

We now use (77) to determine the constant M_0 appearing in the definition of $X_{bnd}(0,T)$ (73). Equivalently one has to consider the lower and upper bounds of ρ_i, ρ_e. In terms of the Kanel's method [13], let us define

$$\Psi(\eta) := \eta - 1 - \int_1^\eta s^{-\gamma} ds, \ \eta \in \mathbb{R}_+, \ \widetilde{\Psi}(\theta) := \int_1^\theta \frac{\sqrt{\Psi(\eta)}}{\eta} d\eta, \ \theta \in \mathbb{R}_+.$$

It is straightforward to see

$$\Psi\Big(\frac{\tilde{\rho}}{\rho}\Big) = \tilde{\rho}\Phi(\rho,\tilde{\rho}), \ \widetilde{\Psi}\Big(\frac{\tilde{\rho}}{\rho}\Big) \to \begin{cases} -\infty, \ \rho \to \infty, \\ \infty, \ \rho \to 0+. \end{cases} \tag{78}$$

On the other hand, setting $\phi = \rho - \tilde{\rho}$, we have

$$\begin{aligned} \Big|\widetilde{\Psi}\Big(\frac{\tilde{\rho}}{\rho}\Big)\Big| &= \Big| \int_\infty^x \widetilde{\Psi}\Big(\frac{\tilde{\rho}}{\rho}\Big)_y dy \Big| = \Big| \int_\infty^x \sqrt{\Psi\Big(\frac{\tilde{\rho}}{\rho}\Big)} \frac{\rho}{\tilde{\rho}} \Big(\frac{\tilde{\rho}}{\rho}\Big)_y dy \Big| \\ &\leq \int_{\mathbb{R}_+} \Big\{ \rho \Psi\Big(\frac{\tilde{\rho}}{\rho}\Big) + \frac{\phi_y^2}{\rho^3} + \frac{\tilde{\rho}_y^2 \phi^2}{\tilde{\rho}^2 \rho^3} \Big\} dy \\ &\leq C \int_{\mathbb{R}_+} \Big\{ \rho\Phi + \frac{\phi_y^2}{\rho^3} + C(M_0)\tilde{\delta}^4 \rho\Phi \Big\} dy. \end{aligned} \tag{79}$$

Now, from (77) and (79), by letting $\tilde{\delta} \leq \delta_4$ and $C(M_0)\delta_4^4 \leq 1$ for $\delta_4 > 0$ small enough, we have

$$\Big|\widetilde{\Psi}\Big(\frac{\tilde{\rho}}{\rho_\alpha}\Big)\Big| \leq C(||(\phi_{\alpha 0}, \psi_{\alpha 0})||^2 + ||(\phi_{\alpha 0})_x||^2). \tag{80}$$

Then, in view of (78) and (80), there exists a positive constant M_1 only depending on the initial data such that

$$M_1^{-1} \leq \rho_{i,e}(x,t) \leq M_1, \quad \forall\, x \in \mathbb{R}_+, 0 \leq t < T. \tag{81}$$

Now, one can choose $M_0 = \frac{1}{2}M_1$.

Step 3. The estimate of $(\psi_{i,e})_x$.

Similar to obtaining (70), we have

$$\begin{aligned} &||(\psi_\alpha)_x(t)||^2 + \int_0^t ||(\psi_\alpha)_{xx}(\tau)||^2 d\tau \\ &\leq C(M_0)\Big\{||(\phi_{\alpha 0}, \psi_{\alpha 0})||_1 + \int_0^t |E(\psi_i - \psi_e)_x(0,\tau)| d\tau + \int_0^t \int_{\mathbb{R}_+} \Big(E_x^2 \\ &\qquad + [(\psi_\alpha)_x]^2 + |(\phi_\alpha)_x(\psi_\alpha)_{xx}| + |g_\alpha(\psi_\alpha)_{xx}| + |(\psi_\alpha)_x|^3\Big) dx d\tau\Big\}. \end{aligned} \tag{82}$$

By Sobolev inequality, it holds that

$$\begin{aligned} &\int_0^t |E(\psi_i - \psi_e)_x(0,\tau)| d\tau \\ &\leq \int_0^t |E(0,\tau)|^2 d\tau + \int_0^t |(\psi_i - \psi_e)_x(0,\tau)|^2 d\tau \\ &\leq \int_0^t |E(0,\tau)|^2 d\tau + \int_0^t ||(\psi_i - \psi_e)_x||_{L^\infty}^2 d\tau \\ &\leq \int_0^t |E(0,\tau)|^2 d\tau + 4C(M_0) \int_0^t ||(\psi_i - \psi_e)_x||_{L^2}^2 d\tau \\ &\qquad + \frac{1}{4C(M_0)} \int_0^t ||(\psi_i - \psi_e)_{xx}||_{L^2}^2 d\tau \\ &\leq 4C(M_0)(||(\phi_{\alpha 0}, \psi_{\alpha 0})||_1) + \frac{1}{4C(M_0)} \int_0^t ||(\psi_i - \psi_e)_{xx}||_{L^2}^2 d\tau. \end{aligned}$$

Moreover, one has

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}_+} |(\phi_\alpha)_x(\psi_\alpha)_{xx}| + |g_\alpha(\psi_\alpha)_{xx}| dx d\tau \\ &\qquad \leq C(M_0) \int_0^t ||\big((\phi_\alpha)_x,\ (\psi_\alpha)_x\big)||^2 d\tau + \frac{1}{4C(M_0)} \int_0^t ||(\psi_\alpha)_{xx}||^2 d\tau, \end{aligned}$$

and

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}_+} |(\psi_\alpha)_x|^3 dx d\tau \leq C \int_0^t ||(\psi_\alpha)_x||^{\frac{5}{2}} ||(\psi_\alpha)_{xx}||^{\frac{1}{2}} d\tau \\ &\leq \int_0^t \Big[C(M_0)||(\psi_\alpha)_x||^{\frac{10}{3}} + \frac{1}{4C(M_0)}||(\psi_\alpha)_{xx}||^2\Big] d\tau \\ &\leq C(M_0) \sup_{0\leq\tau\leq t} ||(\psi_\alpha)_x||^{\frac{4}{3}} \int_0^t ||(\psi_\alpha)_x||^2 d\tau + \frac{1}{4C(M_0)} \int_0^t ||(\psi_\alpha)_{xx}||^2 d\tau \\ &\leq C(M_0) \sup_{0\leq\tau\leq t} ||(\psi_\alpha)_x||^{\frac{4}{3}} + \frac{1}{4C(M_0)} \int_0^t ||(\psi_\alpha)_{xx}||^2 d\tau \\ &\leq C(M_0) + \frac{1}{2} \sup_{0\leq\tau\leq t} ||(\psi_\alpha)_x||^2 + \frac{1}{4C(M_0)} \int_0^t ||(\psi_\alpha)_{xx}||^2 d\tau. \end{aligned}$$

Then, by putting the above estimates into (82) and further taking the proper linear combination with (77), we have

$$||(\psi_\alpha)_x(t)||^2 + \int_0^t ||(\psi_\alpha)_{xx}(\tau)||d\tau \le C(M_0)||(\phi_{\alpha 0}, \psi_{\alpha 0})||_1. \tag{83}$$

Therefore, the combination of (77), (81) and (83) implies (74) and thus completes the proof of Proposition 3. □

Proof of Theorem 3.4. Similar to that for Theorem 2.3, the proof follows by the local existence and Proposition 3, and the details are omitted for brevity. □

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