HYPOCOERCIVITY FOR THE LINEAR BOLTZMANN EQUATION WITH CONFINING FORCES

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ABSTRACT. This paper is concerned with the hypercoercivity property of solutions to the Cauchy problem on the linear Boltzmann equation with a confining potential force. We obtain the exponential time rate of solutions converging to the steady state under some conditions on both initial data and the potential function. Specifically, initial data is properly chosen such that the conservation laws of mass, total energy and possible partial angular momentums are satisfied for all nonnegative time, and a large class of potentials including some polynomials are allowed. The result also extends the case of parabolic forces considered in [7] to the non-parabolic general case here.

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1. INTRODUCTION

Consider the following Cauchy problem on the linear Boltzmann equation with a stationary potential force

$$\partial_t F + \xi \cdot \nabla_x F - \nabla_x \phi \cdot \nabla_\xi F = Q(F, M), \tag{1.1}$$

$$F(0, x, \xi) = F_0(x, \xi). \tag{1.2}$$

Here, $F = F(t, x, \xi) \geq 0$ stands for the density distribution function of particles which have position $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and velocity $\xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n$ at time $t \geq 0$, and the initial data at time $t = 0$ is given by $F_0 = F_0(x, \xi)$. The integer $n \geq 2$ denotes the space dimension. The potential $\phi = \phi(x)$ of the external force is confining in the sense of

$$\int_{\mathbb{R}^n} e^{-\phi(x)}\, dx = 1, \tag{1.3}$$

and M is a normalized global Maxwellian in the form of

$$M = (2\pi)^{-n/2} e^{-|\xi|^2/2}.$$

Q is a bilinear symmetric operator given by

$$Q(F, G) = \int_{\mathbb{R}^n \times \mathbb{S}^{n-1}} |\xi - \xi_*|^\gamma q(\theta)(F' G'_* + F'_* G' - F G_* - F_* G)\, d\omega d\xi_*. \tag{1.4}$$

Here the usual notions $F' = F(t, x, \xi')$, $F_* = F(t, x, \xi_*)$, $F'_* = F(t, x, \xi'_*)$ were used and likewise for G. (ξ, ξ_*) and (ξ', ξ'_*), denoting velocities of two particles before and after their collisions respectively, satisfy

$$\xi' = \xi - [(\xi - \xi_*) \cdot \omega]\omega, \quad \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \omega]\omega, \quad \omega \in \mathbb{S}^{n-1},$$

by the conservation of momentum and energy

$$\xi + \xi_* = \xi' + \xi'_*, \quad |\xi|^2 + |\xi_*|^2 = |\xi'|^2 + |\xi'_*|^2.$$

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The function $|\xi-\xi_*|^\gamma q(\theta)$ in (1.4) is the cross-section depending only on $|\xi-\xi_*|$ and $\cos\theta=(\xi-\xi_*)\cdot\omega/|\xi-\xi_*|$, and it is supposed to satisfy $\gamma\geq 0$ and the Grad's angular cutoff assumption $0\leq q(\theta)\leq q_0|\cos\theta|$ for a positive constant q_0; see [2].

It is well known that Q has $n+2$ collision invariants 1, ξ_1, ξ_2, $\cdots$, ξ_n and $|\xi|^2$. Then, the linear Boltzmann equation (1.1) has the conservation laws of mass and total energy

$$\frac{d}{dt}\iint_{\mathbb{R}^n\times\mathbb{R}^n} F(t,x,\xi)\,dxd\xi=0, \tag{1.5}$$

$$\frac{d}{dt}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \left(\frac{1}{2}|\xi|^2+\phi(x)\right) F(t,x,\xi)\,dxd\xi=0, \tag{1.6}$$

for any $t\geq 0$. The balance laws of momentum and angular momentum are given by

$$\frac{d}{dt}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \xi F(t,x,\xi)\,dxd\xi=-\int_{\mathbb{R}^n}\nabla_x\phi(x)\left[\int_{\mathbb{R}^n}F(t,x,\xi)\,d\xi\right]dx,$$

$$\frac{d}{dt}\iint_{\mathbb{R}^n\times\mathbb{R}^n} x\times\xi F(t,x,\xi)\,dxd\xi=-\int_{\mathbb{R}^n}x\times\nabla_x\phi(x)\left[\int_{\mathbb{R}^n}F(t,x,\xi)\,d\xi\right]dx.$$

Here we used that for $i\neq j$,

$$\nabla_x\{(x\times\xi)_{ij}\}\cdot\xi=0,\quad \nabla_\xi\{(x\times\xi)_{ij}\}\cdot\nabla_x\phi(x)=(x\times\nabla_x\phi(x))_{ij},$$

where for two vectors A and B the tensor product $A\times B$ is defined by $(A\times B)_{ij}=A_iB_j-A_jB_i$ for $i\neq j$ and zero otherwise. Since $\phi(x)$ is confining, it cannot be independent of some coordinate variable of $(x_1,\cdots,x_n)$ so that the conservation of momentum generally does not hold. However the conservation of angular momentum is possible. For that, given $\phi(x)$, we denote an index set S_ϕ by

$$S_\phi=\{(i,j);1\leq i\neq j\leq n,\ x_i\partial_j\phi(x)-x_j\partial_i\phi(x)=0,\ \forall\,x\in\mathbb{R}^n\}.$$

Here and in the sequel, for brevity, ∂_i denotes the spatial derivative ∂_{x_i}. It is now straightforward to check that (1.1) also has the following conservation laws of angular momentum

$$\frac{d}{dt}\iint_{\mathbb{R}^n\times\mathbb{R}^n}(x\times\xi)_{ij}F(t,x,\xi)\,dxd\xi=0,\quad\forall\,(i,j)\in S_\phi, \tag{1.7}$$

for any $t\geq 0$.

It is easy to see that the local Maxwellian

$$\mathcal{M}=Me^{-\phi(x)}=(2\pi)^{-n/2}e^{-\left(|\xi|^2/2+\phi(x)\right)}$$

is a time-independent steady solution to the equation (1.1). We suppose that initial data $F_0(x,\xi)$ in (1.2) has the same mass, angular momentum and total energy with $\mathcal{M}$, namely

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n}[F_0(x,\xi)-\mathcal{M}]\,dxd\xi=0, \tag{1.8}$$

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n}(x\times\xi)_{ij}[F_0(x,\xi)-\mathcal{M}]\,dxd\xi=0,\ (i,j)\in S_\phi, \tag{1.9}$$

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n}\left(\frac{1}{2}|\xi|^2+\phi(x)\right)[F_0(x,\xi)-\mathcal{M}]\,dxd\xi=0. \tag{1.10}$$

Due to the conservation laws (1.5), (1.7) and (1.6), the equations above still hold true for the solution $F(t,x,\xi)$ to the Cauchy problem (1.1)-(1.2) at all positive time $t>0$. Therefore, it could be expected that the solution $F(t,x,\xi)$ tends to the steady state $\mathcal{M}$ exponentially in time as in the periodic domain [28]. In general it is not the case for the whole space. However, we shall see that it can be achieved under some additional conditions on the potential function $\phi(x)$. Notice that the collision operator Q is degenerate along $n+2$ number of directions. Even though this, the interplay between Q and the linear transport operator $\partial_t+\xi\cdot\nabla_x-\nabla_x\phi\cdot\nabla_\xi$ involved in a confining force can indeed yield the convergence of solutions to the steady state $\mathcal{M}$ in an exponential rate. The property is now called hypercoercivity; see [29].

To the end, set

$$f(t,x,\xi)=\frac{F(t,x,\xi)-\mathcal{M}}{\mathcal{M}^{1/2}},\quad f_0(x,\xi)=\frac{F_0(x,\xi)-\mathcal{M}}{\mathcal{M}^{1/2}}.$$

Then the Cauchy problem (1.1)-(1.2) is reformulated as

$$\partial_t f + \xi \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_\xi f + Lf = 0, \quad f(0,x,\xi) = f_0(x,\xi), \tag{1.11}$$

where L is the usual linearized self-adjoint operator, defined by

$$Lf = -\frac{1}{M^{1/2}} Q(M^{1/2} f, M).$$

Recall that the null space of L is given by $\ker L = \{M^{1/2}, \xi_i M^{1/2}, 1 \leq i \leq n, |\xi|^2 M^{1/2}\}$, $L = -\nu + K$ with $\nu = \nu(\xi) \sim (1+|\xi|)^\gamma$ and $Ku = \int_{\mathbb{R}^3} K(\xi,\xi_*) u(\xi_*)\, d\xi_*$ for a real symmetric integral kernel $K(\xi,\xi_*)$, cf. [12], and moreover, L satisfies the coercivity estimate:

$$\int_{\mathbb{R}^n} f L f \, d\xi \geq \lambda_0 \int_{\mathbb{R}^n} \nu(\xi) |\{\mathbf{I} - \mathbf{P}\} f|^2 \, d\xi$$

for a constant $\lambda_0 > 0$, where $\mathbf{I}$ is the identity operator and $\mathbf{P}$ stands for the projection operator on $\ker L$ in L^2_ξ; see [2] for more details.

Before stating our main result, let's give the definition of weak solutions of the Cauchy problem (1.11).

Definition 1.1. *For an initial datum $f_0 \in L^2(\mathbb{R}^{2n})$ such that the conservation laws (1.8)-(1.10) hold for $F_0 = \mathcal{M} + \mathcal{M}^{1/2} f_0$, we say that f is a weak solution of the Cauchy problem (1.11) if the following conditions are fulfilled:*

(1) $f(t,x,\xi) \in C\left(\mathbb{R}^+;\ \mathcal{D}'(\mathbb{R}^{2n})\right) \bigcap L^\infty\left(\mathbb{R}^+;\ L^2(\mathbb{R}^{2n})\right)$;
(2) $f(0,x,\xi) = f_0(x,\xi)$;
(3) *for any $t \geq 0$, we have the conservation laws:*

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} f(t,x,\xi) \mathcal{M}^{1/2} \, dx d\xi = 0,$$

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} (x \times \xi)_{ij} f(t,x,\xi) \mathcal{M}^{1/2} \, dx d\xi = 0, \ (i,j) \in S_\phi,$$

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \left(\frac{1}{2}|\xi|^2 + \phi(x)\right) f(t,x,\xi) \mathcal{M}^{1/2} \, dx d\xi = 0;$$

(4) *for any test function $\varphi(t,x,\xi) \in C^1\left(\mathbb{R}^+;\ C_0^\infty(\mathbb{R}^{2n})\right)$ we have*

$$\begin{aligned} &\iint f(t,x,\xi)\varphi(t,x,\xi) dx\, d\xi - \iint f_0(x,\xi)\varphi(0,x,\xi) dx\, d\xi \\ &- \int_0^t d\tau \iint f(\tau,x,\xi) \big[\left(\partial_\tau + \xi \cdot \nabla_x - \nabla_x \phi \cdot \nabla_\xi\right) \varphi(\tau,x,\xi) \big] dx\, d\xi \\ &= - \int_0^t d\tau \iint f(\tau,x,\xi) L\varphi(\tau,x,\xi) dx\, d\xi. \end{aligned}$$

The main result of the paper concerning the hypercoercivity property of solutions to the Cauchy problem (1.11) is stated as follows.

Theorem 1.1. *Assume that $\phi(x) \in C^1(\mathbb{R}^n)$ is a confining potential in the sense of* (1.3) *such that $|x|^2 e^{-\phi} \in L^1(\mathbb{R}^n_x)$ and that $F_0 = \mathcal{M} + \mathcal{M}^{1/2} f_0$ satisfies the conservation laws* (1.8), (1.9) *and* (1.10). *Additionally let the following assumptions on $\phi(x)$ hold:*

(i) For any $(i,j) \in S_\phi$, $\phi(x)$ is even in (x_i, x_j), i.e.,

$$\phi(\cdots, -x_i, \cdots, -x_j, \cdots) = \phi(\cdots, x_i, \cdots, x_j, \cdots), \quad x \in \mathbb{R}^n.$$

(ii) The functions 1, $\{x_i\}_{1 \leq i \leq n}$, $\{\partial_i \phi(x)\}_{1 \leq i \leq n}$ and $\{x_i \partial_j \phi(x) - x_j \partial_i \phi(x); 1 \leq i < j \leq n, (i,j) \in I \setminus S_\phi\}$, where $I = \{(i,j); 1 \leq i \neq j \leq n\}$, are independent in the sense that if there are constants b_0, $\{b_{1i}\}_{1 \leq i \leq n}$, $\{B_{1i}\}_{1 \leq i \leq n}$ and $\{B_{2ij}; 1 \leq i < j \leq n, (i,j) \in I \setminus S_\phi\}$ such that

$$b_0 + \sum_{i=1}^n b_{1i} x_i + \sum_{i=1}^n B_{1i} \partial_i \phi(x) + \sum_{1 \leq i < j \leq n, (i,j) \in I \setminus S_\phi} B_{2ij} [x_i \partial_j \phi(x) - x_j \partial_i \phi(x)] = 0$$

for any $x \in \mathbb{R}^n$ then all coefficients must be zero, i.e.,

$$b_0 = 0;\ b_{1i} = 0, 1 \leq i \leq n;\ B_{1i} = 0, 1 \leq i \leq n;\ B_{2ij} = 0, 1 \leq i < j \leq n, (i,j) \in I \setminus S_\phi.$$

(iii) If $\phi(x)$ is even, i.e., $\phi(-x)=\phi(x)$ for any $x\in\mathbb{R}^n$, then the function

$$\Lambda_\phi\Big(2\phi(x)+x\cdot\nabla_x\phi(x)\Big)+\frac{|x|^2}{2}$$

with Λ_ϕ a constant to be given in the later proof, is not constant-valued; if ϕ is a general potential which may not be even, then there exists a sequence $\{x^\ell\}_{\ell\geq 1}$, with $|x^\ell|\to+\infty$ as $\ell\to+\infty$, such that

$$\lim_{\ell\to+\infty}\frac{2\phi(x^\ell)+x^\ell\cdot\nabla_x\phi(x^\ell)+\left|\nabla_x\phi(x^\ell)\right|}{|x^\ell|^2}=0 \quad\text{or}\quad \lim_{\ell\to+\infty}\frac{|x^\ell|^2+\left|\nabla_x\phi(x^\ell)\right|}{2\phi(x^\ell)+x^\ell\cdot\nabla_x\phi(x^\ell)}=0. \tag{1.12}$$

Then, for any weak solution $f(t,x,\xi)\in L^\infty\left(\mathbb{R}^+;\ L^2(\mathbb{R}^{2n})\right)\bigcap L^2_{loc}\left(\mathbb{R}^+;\ L^2_\nu(\mathbb{R}^{2n})\right)$ to the Cauchy problem (1.11)*, with $L^2_\nu(\mathbb{R}^{2n})$ denoting the weighted space $L^2(\nu(\xi)dx\,d\xi)$, there are constants $\sigma>0$, C such that*

$$\|f(t)\|_{L^2_{x,\xi}}\leq Ce^{-\sigma t}\|f_0\|_{L^2_{x,\xi}} \tag{1.13}$$

for any $t\geq 0$.

A large class of potential functions satisfying conditions in Theorem 1.1 can be allowed by just considering the polynomials in the form of

$$P_N(x)=\sum_{|\alpha|\leq N}C_\alpha x^\alpha,$$

where $\alpha=(\alpha_1,\cdots,\alpha_n)$ is a multi-index, $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ and C_α are constants. Thus, the conditions on $\phi(x)=P_N(x)$ above are equivalent with ones by the proper choice of both α and C_α. In what follows we shall give some simple examples. The first one is the radial function

$$\phi_1(x)=\beta\langle x\rangle^\alpha+C_{\alpha\beta},$$

where $\langle x\rangle:=(1+|x|^2)^{1/2}$, $\beta>0$ and $0<\alpha\neq 2$ are constants, and $C_{\alpha\beta}$ is a constant chosen such that the integral in (1.3) can be normalized to be unit. Notice that as long as the potential $\phi(x)$ is radial, $x\times\nabla_x\phi(x)\equiv 0$ and thus $S_\phi=\{(i,j);1\leq i\neq j\leq n\}$, which implies that all angular momentums are conservative due to (1.7).

Remark 1.1. *The hypercoercivity property for $\phi=\phi_1(x)$ in the case when $\alpha=2$ was discussed in* [7]*, where the proof is based on the approach of constructing the Lyapunov functional through the analysis of the macroscopic system with the help of the macro-micro decomposition as well as the investigation of the Korn-type inequalities, and more conservative quantities were assumed and the weighted $H^1_{x,\xi}$ norm instead of $L^2_{x,\xi}$ was used.*

The second example is the following non-radial but even function

$$\phi_2(x)=\sum_{i=1}^n\beta_i\langle x_i\rangle^{\alpha_i}+C_{\alpha\beta},$$

where $0<\alpha_i\neq 2$ and $\beta_i>0$ for $1\leq i\leq n$, and $C_{\alpha\beta}$ depending on $\alpha=(\alpha_1,\cdots,\alpha_n)$ and $\beta=(\beta_1,\cdots,\beta_n)$ is chosen as in $\phi_1(x)$. Notice that different from the case of $\phi_1(x)$, it is easy to see that S_{ϕ_2} is empty. For the non-radial and non-even potential, a typical example is given by

$$\phi_3(x)=\sum_{i=1}^n\alpha_i x_i^{2(i+1)}+\sum_{i=1}^n\beta_i x_i+C_{\alpha\beta}$$

for $\alpha_i>0$, $\beta_i\neq 0$, $1\leq i\leq n$. One can also verify that S_{ϕ_3} is empty so that condition (i) in Theorem 1.1 is automatically satisfied. Condition (ii) can be justified by comparing the coefficients of each monomial of different orders, and the second limit in (1.12) in condition (iii) also holds since the dominator grows strictly faster than the numerator at infinity. All details of verification are omitted for brevity.

Remark 1.2. *From the later proof of Theorem 1.1, particularly basing on the proof of Lemma 2.1, the conclusion* (1.13) *is still true if both conditions (ii) and (iii) in Theorem 1.1 are combined together* *and replaced by the following new condition (ii)': If there are constants* b_0, $\{b_{1i}\}_{1\le i\le n}$, b_2, $\{B_{1i}\}_{1\le i\le n}$, $\{B_{2ij}\}_{(i,j)\in\tilde{S}_\phi}$ *and* B_3 *such that*

$$b_0+\sum_{i=1}^n b_{1i}x_i+b_2|x|^2+\sum_{i=1}^n B_{1i}\partial_i\phi(x)+\sum_{(i,j)\in\tilde{S}_\phi}B_{2ij}[x_i\partial_j\phi(x)-x_j\partial_i\phi(x)]$$

$$+B_3[2\phi(x)+x\cdot\nabla_x\phi(x)]=0$$

for any $x\in\mathbb{R}^n$ *then all coefficients must be zero, i.e.,*

$$b_0=b_{1i}=b_2=B_{1i}=B_3=0, 1\le i\le n;\ B_{2ij}=0, (i,j)\in\tilde{S}_\phi.$$

Here, $\tilde{S}_\phi=\{1\le i<j\le n,(i,j)\in I\setminus S_\phi\}$.

There has been an extensive literature in the study of hypocoercivity for different kinetic models with relaxations under different physical situations, for instance, Fokker-Planck equations, Boltzmann equations and Landau equations, with or without external forces, in the whole space or on the bounded domain; in terms of different approaches, we mention the relative entropy [1, 3, 4, 5, 9], Lyapunov functionals [6, 25, 29], applications of the hypoellipticity theory [17, 18, 19], the solution semigroup [20, 28], Green's functions [23, 24, 31] and the robust energy method together with the macro-micro decomposition [13, 15, 16]. Interested readers may refer to Villani [29] for the abstract result on hypocoercivity and also for more references therein.

In what follows we mention in more details some known work related to the result and approach of the paper. First of all, we should point out that one of motivations of the paper is to generalize the recent abstract result Dolbeault-Mouhot-Schmeiser [6] for the relaxation kinetic equation conserving only the mass with the confining force to the case when several conservation laws occur by considering the linear Boltzmann equation as an example. In fact, when there is no external force, the solution to the linearized Boltzmann equation (1.11) in the whole space decays with the polynomial time rate, due to Ukai [28] using the spectral analysis. The polynomial rate was also obtained by Kawashima [20] through constructing the compensation function on the basis of the Grad's thirteen moments method [11]; the first author of the paper generalized in [7] this method to the case when there is a harmonic confining potential force; see Remark 1.1.

When the whole space as the spatial domain is changed to the torus, the approach of [28] and [20] still works under the additional assumptions on the conservation laws and then yields the exponential time-decay rate of solutions. See also [4, 13, 16, 27] for the study of the nonlinear Boltzmann equation.

As far as a potential force is concerned, some constructive methods based on the hypoelliptic techniques, such as pseudo-differential calculus and the celebrated Kohn's method, and functional calculus have been developed to establish the explicit exponential convergence to equilibrium. This was initiated by Hérau-Nier [18] to study the spectral gap and the resulting exponential convergence, with general applications to the Fokker-Planck equation and the relaxation Boltzmann equation with the confining force in the whole space [17, 19]. A more general abstract theorem, making crucial use of the commutator techniques, has been developed by Villani [29]. We here mention again the relative works [25] and [6].

Unfortunately we are unable to apply those constructive methods above to the Boltzmann operator considered here, due to the absence of some structure properties enjoyed for hypoelliptic equations. But an alternative non-constructive method basing on the compactness arguments by Guo [16, 13] still works, although it fails to give the explicit rate. There actually has been several related works in this direction. For the nonlinear Boltzmann equation on the torus, if a potential force is present, the exponential rate for the solution trending to the steady state $\mathcal{M}=Me^{-\phi(x)}$ was recently obtained in [8] under some symmetry conditions on both the potential and initial data, and it was later extent in [21] to the case without any symmetry restriction on the basis of the L^∞-L^2 estimate developed by Guo [13]. Notice that the smallness of the potential force is needed in [8] but it is not necessary in [21] since the former deals with perturbations in the high-order Sobolev space and the latter does so in L^∞ space. We here point out that it remains a difficult problem to generalize the results of those works [7, 21] as well as this paper to prove the stability of the steady state $\mathcal{M}=Me^{-\phi(x)}$ for the nonlinear Boltzmann equation with the potential force in the whole space in the case when the potential is confining; this is because the steady state is continuously connected to vacuum at spatial infinity such that the usual energy method as in [16, 14] does not work; see also the nice review paper [30] by Yang for more details.

The current work is also concerned with the non-symmetry potential, and our method, as to be seen later on, depends heavily on the conservation laws. We hope it would give insights on the hypocoercivity properties that may be expected for the general potential force. Finally we remark that once the Boltzmann equation without angular cutoff is concerned, it's likely that the hypoelliptic technique mentioned above will be applicable to derive the explicit convergence rate, thanks to the intrinsic hypoelliptic structure arising from the angular singularity. We hope to have the opportunity to address this problem in a near future.

We conclude this section with explaining the idea in the proof of Theorem 1.1. In fact, in order to prove Theorem 1.1, we need to obtain a criterion of zero solutions to the transport equation satisfying extra conservation laws; see Lemma 2.1. Indeed, those conditions postulated in Theorem 1.1 on the confining potential can assure the uniqueness of zero solutions. The proof of Lemma 2.1 is based on the analysis of the macroscopic equations (2.7)-(2.11) under the constraints (2.12)-(2.14). Once the crucial Lemma 2.1 is established, it is a standard procedure as in [13] and [21] to prove the refined coercivity estimate (3.1) and hence derive the exponential time-decay rate in $L^2_{x,\xi}$.

2. Criterion for zero solution

This section is devoted to the proof of the following

Lemma 2.1. *Let all conditions on $\phi(x)$ in Theorem 1.1 hold, and let the function $z = z(t,x,\xi) \in L^2_t((0,1); L^2_{x,\xi})$ admit the form of*

$$z(t,x,\xi) = \{a(t,x) + b(t,x)\cdot\xi + c(t,x)|\xi|^2\}\mathcal{M}^{1/2}, \tag{2.1}$$

with $e^{-\phi(x)/2}a$, $e^{-\phi(x)/2}b$, $e^{-\phi(x)/2}c \in L^2((0,1)\times\mathbb{R}^n_x)$. If $z(t,x,\xi)$ satisfies in the sense of distribution the transport equation

$$\partial_t z + \xi\cdot\nabla_x z - \nabla_x\phi\cdot\nabla_\xi z = 0 \tag{2.2}$$

with the integrated conservation laws

$$\int_0^1 \rho(t)\iint_{\mathbb{R}^n\times\mathbb{R}^n} z(t,x,\xi)\mathcal{M}^{1/2}\,dx\,d\xi\,dt = 0, \tag{2.3}$$

$$\int_0^1 \rho(t)\iint_{\mathbb{R}^n\times\mathbb{R}^n} (x\times\xi)_{ij} z(t,x,\xi)\mathcal{M}^{1/2}\,dx\,d\xi\,dt = 0,\ (i,j)\in S_\phi, \tag{2.4}$$

$$\int_0^1 \rho(t)\iint_{\mathbb{R}^n\times\mathbb{R}^n} \left(\frac{1}{2}|\xi|^2 + \phi(x)\right) z(t,x,\xi)\mathcal{M}^{1/2}\,dx\,d\xi\,dt = 0, \tag{2.5}$$

where $\rho(t) \in L^2([0,1])$ is an arbitrary function, then

$$z(t,x,\xi) \equiv 0 \ \text{in } L^2((0,1)\times\mathbb{R}^n\times\mathbb{R}^n). \tag{2.6}$$

Lemma 2.1 implies that if a linearized local Maxwellian satisfies the free transport equation with the confining force and also satisfies all the possible conservation laws, then it must be trivial. Our main idea for the proof is to derive the evolution equations of the fluid components a, b and c in terms of the Grad's moment method [10, 11], and further argue that the only possible solution is zero due to the additional constraint conditions from conservation laws. For the issue of using the moment method to study the fluid-dynamic behavior of the macro components, we only mention the related work [22] and [26].

We now start the proof of Lemma 2.1 by six steps.

Step 1. As in [16, 14], by plugging (2.1) into (2.2) and then collecting the coefficients of moments of different orders, $a(t,x)$, $b(t,x)$ and $c(t,x)$ satisfy in the sense of distribution a system of equations

$$\dot a - \nabla_x\phi\cdot b = 0, \tag{2.7}$$

$$\dot b + \nabla_x a - 2c\nabla_x\phi = 0, \tag{2.8}$$

$$\dot c + \partial_i b_i = 0, \quad 1\le i\le n, \tag{2.9}$$

$$\partial_j b_i + \partial_i b_j = 0, \quad 1\le i\ne j\le n, \tag{2.10}$$

$$\nabla_x c = 0. \tag{2.11}$$

Here and hereafter, for brevity, $\dot{a}$ denotes the time derivative of $a(t,x)$, and likewise for others. Moreover, putting (2.1) into (2.3), (2.4) and (2.5) gives, for any $\rho(t) \in L^2([0,1])$,

$$\int_0^1 \rho(t) \int_{\mathbb{R}^n} (a + A_1 c)\, e^{-\phi}\, dx\, dt = 0, \tag{2.12}$$

$$\int_0^1 \rho(t) \int_{\mathbb{R}^n} (x \times b)_{ij}\, e^{-\phi}\, dx\, dt = 0, \quad (i,j) \in S_\phi, \tag{2.13}$$

$$\int_0^1 \rho(t) \int_{\mathbb{R}^n} \left(\frac{A_1}{2} a + \frac{A_2}{2} c \right) e^{-\phi}\, dx\, dt + \int_0^1 \rho(t) \int (a + A_1 c)\, \phi e^{-\phi}\, dx\, dt = 0, \tag{2.14}$$

where the constants A_1 and A_2 depending only on the dimension n are given by

$$A_1 = \int_{\mathbb{R}^n} |\xi|^2 M\, d\xi, \quad A_2 = \int_{\mathbb{R}^n} |\xi|^4 M\, d\xi.$$

We remark that the above integrals make sense due to the assumptions on a, b, c and the confining condition on the potential ϕ.

Step 2. First of all we determine $c(t,x)$. It is easy to see from (2.11) that

$$c(t,x) = c(t), \tag{2.15}$$

i.e., c is independent of x. Next, we turn to $b(t,x)$. We shall use the same idea as in [7] for the proof of Korn-type inequality; see [7, Step 4, Theorem 5.1]. Then, in terms of (2.9) and (2.10), one has

$$b(t,x) = -\dot{c}(t)x + B_2(t)x + B_1(t), \tag{2.16}$$

where $B_2(t) = (B_{2ij}(t))_{1 \le i,j \le n}$ is an $n \times n$ skew-symmetric matrix and $B_1(t) = (B_{11}(t), \cdots, B_{1n}(t))$ is a vector, both depending only on t. Plugging the expression (2.16) of $b(t,x)$ into (2.8), it holds that

$$-\ddot{c}(t)x + \dot{B}_2(t)x + \dot{B}_1(t) + \nabla_x \{a(t,x) - 2c(t)\phi(x)\} = 0,$$

or equivalently

$$\dot{B}_2(t)x + \nabla_x \left\{ a(t,x) - 2c(t)\phi(x) - \frac{1}{2}\ddot{c}(t)\,|x|^2 + \dot{B}_1(t) \cdot x \right\} = 0. \tag{2.17}$$

This implies that $\nabla_x \times \{\dot{B}_2(t)x\} = 0$, i.e.,

$$\dot{B}_{2ij}(t) - \dot{B}_{2ji}(t) = 0, \quad 1 \le i \ne j \le n.$$

On the other hand, the skew-symmetry of the matrix $B_2(t)$ shows $B_{2ij}(t) + B_{2ji}(t) = 0$ and thus

$$\dot{B}_{2ij}(t) + \dot{B}_{2ji}(t) = 0.$$

Then $\dot{B}_{2ij}(t) = 0$ for any t and all i, j. We have shown

$$\dot{B}_2(t) = 0, \tag{2.18}$$

that is, $B_2(t) = B_2$ is a skew-symmetric matrix with constant entries. As a result, we may rewrite $a(t,x)$ and $b(t,x)$ as

$$a(t,x) = 2c(t)\phi(x) + \frac{1}{2}\ddot{c}(t)\,|x|^2 - \dot{B}_1(t) \cdot x + d(t), \tag{2.19}$$

and

$$b(t,x) = -\dot{c}(t)x + B_2 x + B_1(t), \tag{2.20}$$

due to (2.17) and (2.18), where $d(t)$ is a function depending only on t.

Step 3. Plugging the expressions (2.19) and (2.15) of $a(t,x)$ and $c(t,x)$ into (2.12) and (2.14), and observing the arbitrariness of $\rho(t)$, one can deduce a second order linear system

$$c(t)\lambda_1^1 + \ddot{c}(t)\lambda_2^1 + d(t)\lambda_3^1 - \dot{B}_1(t) \cdot \lambda_4^1 = 0, \tag{2.21}$$

$$c(t)\lambda_1^2 + \ddot{c}(t)\lambda_2^2 + d(t)\lambda_3^2 - \dot{B}_1(t) \cdot \lambda_4^2 = 0, \tag{2.22}$$

where λ_j^i $(i=1,2, j=1,2,3,4)$ are constants depending only on $\phi(x)$, given by

$$\begin{aligned}
\lambda_1^1 &= 2\int_{\mathbb{R}^n}(\frac{A_1}{2}+\phi)e^{-\phi}\,dx, & \lambda_1^2 &= 2\int_{\mathbb{R}^n}(\phi^2+A_1\phi+\frac{A_2}{4})\,e^{-\phi}\,dx,\\
\lambda_2^1 &= \int_{\mathbb{R}^n}\frac{1}{2}|x|^2\,e^{-\phi}\,dx, & \lambda_2^2 &= \int_{\mathbb{R}^n}\frac{1}{2}|x|^2(\frac{A_1}{2}+\phi)\,e^{-\phi}\,dx,\\
\lambda_3^1 &= \int_{\mathbb{R}^n}e^{-\phi}\,dx=1, & \lambda_3^2 &= \int_{\mathbb{R}^n}(\frac{A_1}{2}+\phi)\,e^{-\phi}\,dx,\\
\lambda_4^1 &= \int_{\mathbb{R}^n}xe^{-\phi}\,dx, & \lambda_4^2 &= \int_{\mathbb{R}^n}(\frac{A_1}{2}+\phi)x\,e^{-\phi}\,dx.
\end{aligned}$$

Moreover, note $\lambda_3^1=1>0$. In order to cancel $d(t)$, by multiplying (2.21) by λ_3^2 and then taking difference with (2.22), it follows that

$$c(t)(\lambda_1^2-\lambda_1^1\lambda_3^2)+\ddot{c}(t)(\lambda_2^2-\lambda_2^1\lambda_3^2)-\dot{B}_1(t)\cdot\left(\lambda_4^2-\lambda_4^1\lambda_3^2\right)=0, \tag{2.23}$$

where one can compute

$$\lambda_1^2-\lambda_1^1\lambda_3^2 = 2\left\{\int_{\mathbb{R}^n}\phi^2\,d\mu-\left[\int_{\mathbb{R}^n}\phi\,d\mu\right]^2+\frac{A_2-A_1^2}{4}\right\}, \tag{2.24}$$

$$\lambda_2^2-\lambda_2^1\lambda_3^2 = \frac{1}{2}\left\{\int_{\mathbb{R}^n}|x|^2\phi\,d\mu-\int_{\mathbb{R}^n}|x|^2\,d\mu\int_{\mathbb{R}^n}\phi\,d\mu\right\}, \tag{2.25}$$

$$\lambda_4^2-\lambda_4^1\lambda_3^2 = \int_{\mathbb{R}^n}x\phi\,d\mu-\int_{\mathbb{R}^n}x\,d\mu\int_{\mathbb{R}^n}\phi\,d\mu. \tag{2.26}$$

Here and in the sequel, for brevity, we have used the notation $d\mu=e^{-\phi(x)}\,dx$.

Lemma 2.2. *It holds that*

$$\lambda_1^2-\lambda_1^1\lambda_3^2>0, \tag{2.27}$$

and

$$\lambda_2^2-\lambda_2^1\lambda_3^2=\frac{1}{4}\iint_{\mathbb{R}^n\times\mathbb{R}^n}[\phi(x)-\phi(y)](|x|^2-|y|^2)e^{-\phi(x)-\phi(y)}\,dxdy, \tag{2.28}$$

$$\lambda_4^2-\lambda_4^1\lambda_3^2=\frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n}(\phi(x)-\phi(y))(x-y)e^{-\phi(x)-\phi(y)}\,dxdy. \tag{2.29}$$

Proof. We verify (2.27) only. (2.28) and (2.29) can be proved from (2.25) and (2.26) in a similar way. In fact, it is straightforward to compute

$$\begin{aligned}
&\int_{\mathbb{R}^n}\phi^2\,d\mu-\left[\int_{\mathbb{R}^n}\phi\,d\mu\right]^2\\
&=\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\phi^2(x)+\phi^2(y)}{2}e^{-\phi(x)-\phi(y)}\,dxdy-\iint_{\mathbb{R}^n\times\mathbb{R}^n}\phi(x)\phi(y)\,e^{-\phi(x)-\phi(y)}\,dxdy\\
&=\frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n}[\phi(x)-\phi(y)]^2\,e^{-\phi(x)-\phi(y)}\,dxdy,
\end{aligned}$$

and

$$\begin{aligned}
&A_2-A_1^2\\
&=\int_{\mathbb{R}^n}|\xi|^4M\,d\xi-\left[\int_{\mathbb{R}^n}|\xi|^2M\,d\xi\right]^2\\
&=\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|\xi|^4+|\xi_*|^4}{2}MM_*\,d\xi d\xi_*-\iint_{\mathbb{R}^n\times\mathbb{R}^n}|\xi|^2|\xi_*|^2MM_*\,d\xi d\xi_*\\
&=\frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n}[|\xi|^2-|\xi_*|^2]^2MM_*\,d\xi d\xi_*.
\end{aligned}$$

Therefore, (2.27) follows in terms of (2.24). This completes the proof of Lemma 2.2. $\square$

Since $\lambda_1^2 - \lambda_1^1\lambda_3^2 > 0$, from (2.23), one can write

$$c(t) = \Lambda_\phi\, \ddot{c}(t) + V_\phi \cdot \dot{B}_1(t), \tag{2.30}$$

where the constant Λ_ϕ and the constant vector V_ϕ, both depending only on $\phi(x)$, are given by

$$\Lambda_\phi = -\frac{\lambda_2^2 - \lambda_2^1\lambda_3^2}{\lambda_1^2 - \lambda_1^1\lambda_3^2}, \quad V_\phi = \frac{\lambda_4^2 - \lambda_4^1\lambda_3^2}{\lambda_1^2 - \lambda_1^1\lambda_3^2}.$$

Step 4. Plugging the expressions (2.19) and (2.20) of $a(t,x)$ and $b(t,x)$ into (2.7),

$$\dot{c}(t)\,(2\phi(x) + x\cdot\nabla_x\phi(x)) + \frac{1}{2}c^{(3)}(t)\,|x|^2 = \nabla_x\phi(x)\cdot B_2 x + \nabla_x\phi(x)\cdot B_1(t) + \ddot{B}_1(t)\cdot x - \dot{d}(t). \tag{2.31}$$

Throughout this step, we consider the simplified case when the potential function $\phi(x)$ is even. In such case, $\lambda_4^1 = \lambda_4^2 = 0$ and hence $V_\phi = 0$. Then, (2.30) is reduced to

$$c(t) = \Lambda_\phi\, \ddot{c}(t). \tag{2.32}$$

It follows from (2.31) that

$$\underbrace{\dot{c}(t)\,(2\phi(x) + x\cdot\nabla_x\phi(x)) + \frac{1}{2}c^{(3)}(t)\,|x|^2 - \nabla_x\phi(x)\cdot B_2 x + \dot{d}(t)}_{\text{even function with respect to } x} = 0, \tag{2.33}$$

and

$$\underbrace{\nabla_x\phi(x)\cdot B_1(t) + \ddot{B}_1(t)\cdot x}_{\text{odd function with respect to } x} = 0. \tag{2.34}$$

Due to the condition (ii) in Theorem 1.1, it is direct to obtain from (2.34),

$$\ddot{B}_1(t) = B_1(t) \equiv 0 \tag{2.35}$$

for all $0 \le t \le 1$. Moreover, note that B_2 is a constant matrix, then by (2.33) we have, taking the derivative with respect to t,

$$\ddot{c}(t)\,(2\phi(x) + x\cdot\nabla_x\phi(x)) + \frac{1}{2}c^{(4)}(t)\,|x|^2 + \ddot{d}(t) = 0,$$

which along with (2.32) give

$$\left(\Lambda_\phi\,(2\phi(x) + x\cdot\nabla_x\phi(x)) + \frac{|x|^2}{2}\right)c^{(4)}(t) + \ddot{d}(t) = 0.$$

By using the condition (iii) for the case when $\phi(x)$ is even in Theorem 1.1, we conclude $c^{(4)}(t) \equiv 0$. As a result, in view of (2.32), it follows immediately that $c(t) \equiv 0$, and thus $d(t) \equiv 0$ by (2.21). Then, further recalling (2.19) and (2.20) as well as (2.35), we arrive at

$$a(t,x) \equiv 0, \quad b(t,x) = B_2 x. \tag{2.36}$$

Step 5. In this step we will prove a result similar to (2.36) for the general case when $\phi(x)$ may not be even. In such case we shall use the other condition in (iii) in Theorem 1.1. Note that (2.31) still holds true and B_2 is independent of t. Then, taking the first order derivative with respect to t on both sides of (2.31), we obtain

$$\ddot{c}(t)\,(2\phi(x) + x\cdot\nabla_x\phi(x)) + \frac{1}{2}c^{(4)}(t)\,|x|^2 = \nabla_x\phi(x)\cdot \dot{B}_1(t) + B_1^{(3)}(t)\cdot x + \ddot{d}(t). \tag{2.37}$$

We first claim that $\ddot{c}(t) \equiv 0$ and $\ddot{d}(t) \equiv 0$. Indeed, the first relation in (1.12) implies $c^{(4)}(t) \equiv 0$, while the second one gives $\ddot{c}(t) \equiv 0$. Moreover if $c^{(4)}(t) \equiv 0$ then we have, taking derivatives twice with respect to t on both sides of (2.37),

$$B_1^{(3)}(t)\cdot\nabla_x\phi(x) + B_1^{(5)}(t)\cdot x + d^{(4)}(t) = 0. \tag{2.38}$$

From the condition (ii) in Theorem 1.1, (2.38) implies

$$B_1^{(3)}(t) = B_1^{(5)}(t) \equiv 0, \quad d^{(4)}(t) \equiv 0.$$

Thus it follows from (2.30) that $\ddot{c}(t) \equiv 0$ and thus $\ddot{d}(t) \equiv 0$ due to (2.21).

The equation (2.37) is now reduced to

$$\dot{B}_1(t) \cdot \nabla_x \phi(x) + B_1^{(3)}(t) \cdot x = 0, \quad \forall\, x \in \mathbb{R}^n.$$

Again from the condition (ii) in Theorem 1.1, we conclude

$$\dot{B}_1(t) \equiv 0,$$

which together with (2.30) and $\ddot{c}(t) \equiv 0$ give $c(t) \equiv 0$, and thus $d(t) \equiv 0$ due to (2.21). Therefore, similar to obtaining (2.36), one has

$$a(t,x) \equiv 0, \quad b(t,x) = B_2 x + B_1,$$

where B_1 is constant.

Step 6. We are now going to determine $B_1 = 0$, $B_2 = 0$ and hence $b(t,x) \equiv 0$. In fact, the equation (2.31) in both cases given in previous two steps is reduced to

$$\nabla_x \phi(x) \cdot B_2 x + \nabla_x \phi(x) \cdot B_1 = 0, \quad \forall\, x \in \mathbb{R}^n. \tag{2.39}$$

Noticing that B_2 is skew-symmetric, one can write

$$\begin{aligned}
\nabla_x \phi(x) \cdot B_2 x &= \sum_{ij} B_{2ij} x_j \partial_i \phi(x) = \left\{ \sum_{i<j} + \sum_{i>j} \right\} B_{2ij} x_j \partial_i \phi(x) \\
&= \sum_{1 \le i < j \le n} \{ B_{2ij} x_j \partial_i \phi(x) + B_{2ji} x_i \partial_j \phi(x) \} \\
&= - \sum_{1 \le i < j \le n} B_{2ij} \{ x_i \partial_j \phi(x) - x_j \partial_i \phi(x) \} \\
&= - \sum_{\substack{1 \le i < j \le n \\ (i,j) \in I \setminus S_\phi}} B_{2ij} \{ x_i \partial_j \phi(x) - x_j \partial_i \phi(x) \}.
\end{aligned}$$

Therefore, (2.39) together with the condition (ii) in Theorem 1.1 yields

$$B_1 = (B_{11}, \cdots, B_{1n}) = 0, \quad B_{2ij} = 0 \quad \text{for } 1 \le i < j \le n, (i,j) \in I \setminus S_\phi,$$

which again due to the skew-symmetry of B_2, gives

$$B_{2ij} = 0 \quad \text{for } 1 \le i \ne j \le n, (i,j) \in I \setminus S_\phi. \tag{2.40}$$

For $(i,j) \in S_\phi$, by plugging $b(t,x) = B_2 x$ into (2.13),

$$\begin{aligned}
0 &= \int_{\mathbb{R}^n} \{ x \times B_2 x \}_{ij}\, e^{-\phi(x)}\, dx \\
&= \int_{\mathbb{R}^n} \{ x_i [B_2 x]_j - x_j [B_2 x]_i \}\, e^{-\phi(x)}\, dx \\
&= \int_{\mathbb{R}^n} \left\{ x_i \sum_{\ell=1}^n B_{2j\ell} x_\ell - x_j \sum_{\ell=1}^n B_{2i\ell} x_\ell \right\} e^{-\phi(x)}\, dx \\
&= \int_{\mathbb{R}^n} \left\{ x_i^2 B_{2ji} - x_j^2 B_{2ij} \right\}\, e^{-\phi(x)}\, dx + \int_{\mathbb{R}^n} \left\{ x_i \sum_{\ell \ne i,j} B_{2j\ell} x_\ell - x_j \sum_{\ell \ne i,j} B_{2i\ell} x_\ell \right\} e^{-\phi(x)}\, dx \\
&= B_{2ji} \int_{\mathbb{R}^n} (x_i^2 + x_j^2)\, e^{-\phi(x)}\, dx,
\end{aligned} \tag{2.41}$$

where due to the condition (i) in Theorem 1.1, we have used

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \Big\{ x_i \sum_{\ell\neq i,j} B_{2j\ell} x_\ell - x_j \sum_{\ell\neq i,j} B_{2i\ell} x_\ell \Big\} e^{-\phi(x)}\, dx \\
&= \int_{\mathbb{R}^n} \Big\{ (-x_i) \sum_{\ell\neq i,j} B_{2j\ell} x_\ell - (-x_j) \sum_{\ell\neq i,j} B_{2i\ell} x_\ell \Big\} e^{-\phi(\cdots,-x_i,\cdots,-x_j,\cdots)}\, dx \\
&= -\int_{\mathbb{R}^n} \Big\{ x_i \sum_{\ell\neq i,j} B_{2j\ell} x_\ell - x_j \sum_{\ell\neq i,j} B_{2i\ell} x_\ell \Big\} e^{-\phi(\cdots,x_i,\cdots,x_j,\cdots)}\, dx \\
&= 0.
\end{aligned}
$$

Then it follows from (2.41) that

$$B_{2ji} = -B_{2ij} = 0, \quad \forall\, (i,j) \in S_\phi, \tag{2.42}$$

since

$$\int_{\mathbb{R}^n} (x_i^2 + x_j^2)\, e^{-\phi(x)}\, dx > 0.$$

As a result, combining (2.40) and (2.42) implies $B_{2ij} = 0$ for all $1 \le i, j \le n$. Hence $B_2 = 0$ and

$$b(t,x) = B_2 x \equiv 0.$$

Therefore,

$$z(t,x,\xi) = \{a(t,x) + b(t,x)\cdot\xi + c(t,x)|\xi|^2\}\mathcal{M}^{1/2} \equiv 0.$$

This proves (2.6) and hence completes the proof of Lemma 2.1. □

3. Exponential rate

In this section we use $\langle\cdot,\ \cdot\rangle$ and $\|\cdot\|$ to denote respectively the inner product and norm in $L^2(\mathbb{R}^n_x\times\mathbb{R}^n_\xi)$, while use $\|\cdot\|_\nu$ to denote the weighted norm with respect to $\nu = \nu(\xi)$, i.e., $\|\cdot\|_\nu^2 = \langle\nu(\xi)\cdot,\ \cdot\rangle$. To the end, for brevity, L^2_ν also stands for the weighted $L^2(\mathbb{R}^n_x\times\mathbb{R}^n_\xi)$ space with norm $\|\cdot\|_\nu$.

Before proving Theorem 1.1, we give the following result on the basis of Lemma 2.1 and the property of the linearized collision operator L.

Proposition 3.1. *Suppose that the potential function $\phi(x)$ satisfies all the conditions stated in Theorem 1.1. Let $f(t,x,\xi) \in L^2\left((0,1);\ L^2_\nu\right)$ be a weak solution of the linear Boltzmann equation (1.11) with initial data $F_0 = \mathcal{M} + \mathcal{M}^{1/2} f_0$ satisfying the conservation laws* (1.8), (1.9) *and* (1.10). *Then there exists a constant $C > 0$ such that*

$$\int_0^1 \langle Lf(s), f(s)\rangle\, ds \ge C \int_0^1 \|f(s)\|_\nu^2\, ds. \tag{3.1}$$

Proof. We use the contradiction argument as did in [16] and [21] with a little modifications in order to treat the whole spatial space case here instead of torus.

Step 1. Assume that the inequality (3.1) is not true. Then for any $k \ge 1$ there exists a sequence of solutions $z_k(t,x,\xi)$ such that

$$\int_0^1 \langle Lz_k(s), z_k(s)\rangle\, ds < k^{-1} \int_0^1 \|z_k(s)\|_\nu^2\, ds.$$

Dividing both sides by the factor $\int_0^1 \|z_k(s)\|_\nu^2\, ds$, we may assume without loss of generality that

$$\int_0^1 \|z_k(s)\|_\nu^2\, ds = 1 \tag{3.2}$$

and

$$\int_0^1 \langle Lz_k(s), z_k(s)\rangle\, ds \le k^{-1}, \quad \forall\, k \ge 1. \tag{3.3}$$

Note that $\{z_k(t,x,\xi)\}_{k\geq 1}$ are solutions to (1.11), satisfying the conservation laws

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} z_k(t,x,\xi)\mathcal{M}^{1/2}\,dxd\xi = 0,$$
$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} (x\times\xi)_{ij} z_k(t,x,\xi)\mathcal{M}^{1/2}\,dxd\xi = 0,\ (i,j)\in S_\phi,$$
$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \left(\frac{1}{2}|\xi|^2+\phi(x)\right) z_k(t,x,\xi)\mathcal{M}^{1/2}\,dxd\xi = 0,$$

for any $0\leq t\leq 1$. By the weak compactness in the Hilbert space $L^2\left((0,1);L^2_\nu\right)$, we conclude that there exists $z(t,x,\xi)\in L^2\left((0,1);L^2_\nu\right)$ such that, up to a subsequence,

$$z_k \rightharpoonup z \ \text{ weakly in } L^2\left((0,1);L^2_\nu\right), \tag{3.4}$$

and

$$\int_0^1 \left\|z(s)\right\|_\nu^2\,ds\leq 1,$$

due to (3.2). The weak convergence in $L^2\left((0,1);L^2_\nu\right)$ along with the above conservation laws for z_k yields that z satisfies the integrated conservation laws as given in (2.3), (2.4) and (2.5).

Step 2. For each $r>0$, we denote by $B_r\subset\mathbb{R}^n_x$ the ball with center 0 and radius r. In this step we claim that for any fixed r,

$$Kz_k\to Kz \text{ strongly in } L^2((0,1)\times B_r\times\mathbb{R}^n). \tag{3.5}$$

The proof is nearly the same as given in [16] and [21], with B_r instead of the torus there. We omit the details for brevity.

Step 3. We prove

$$1-\int_0^1 \langle Kz(s),z(s)\rangle\,ds = 0. \tag{3.6}$$

To do so, firstly we have, using the strong convergence (3.5) and weak convergence (3.4),

$$\int_0^1\int_{B_r}\int_{\mathbb{R}^n} Kz(s,x,\xi)z(t,x,\xi)\,d\xi\,dx\,ds = \lim_{k\to+\infty}\int_0^1\int_{B_r}\int_{\mathbb{R}^n} Kz_k(s,x,\xi)z_k(t,x,\xi)\,d\xi\,dx\,ds. \tag{3.7}$$

Recall (3.2). Then, for any $\varepsilon>0$, we can find a constant r_0 depending only on ε but independent of k, such that

$$\forall\ r\geq r_0,\quad \int_0^1\int_{B_r}\int_{\mathbb{R}^n}\nu(\xi)\left|z_k(s,x,\xi)\right|^2\,d\xi\,dx\,ds>1-\varepsilon.$$

This, together with (3.3) and the relation

$$\begin{aligned}\int_0^1\int_{B_r}\int_{\mathbb{R}^n} Kz_k(s,x,\xi)z_k(t,x,\xi)\,d\xi\,dx\,ds &= \int_0^1\int_{B_r}\int_{\mathbb{R}^n}\nu(\xi)\left|z_k(s,x,\xi)\right|^2\,d\xi\,dx\,ds\\ &\quad -\int_0^1\int_{B_r}\int_{\mathbb{R}^n} Lz_k(s,x,\xi)z_k(t,x,\xi)\,d\xi\,dx\,ds,\end{aligned}$$

implies

$$\forall\ r\geq r_0,\quad \int_0^1\int_{B_r}\int_{\mathbb{R}^n} Kz_k(s,x,\xi)z_k(t,x,\xi)\,d\xi\,dx\,ds\geq 1-\varepsilon-\frac{1}{k}.$$

Note that the number r_0 is independent of k. Then letting $k\to+\infty$ we have, by virtue of (3.7),

$$\forall\ r\geq r_0,\quad \int_0^1\int_{B_r}\int_{\mathbb{R}^n} Kz(s,x,\xi)z(t,x,\xi)\,d\xi\,dx\,ds\geq 1-\varepsilon. \tag{3.8}$$

Moreover the identity

$$\int_0^1\langle Kz_k(s),z_k(s)\rangle\,ds = \int_0^1\left\|z_k(s)\right\|_\nu^2 ds-\int_0^1\langle Lz_k(s),z_k(s)\rangle\,ds$$

as well as (3.2) and the positivity of L give

$$\int_0^1 \int_{B_r} \int_{\mathbb{R}^n} Kz_k(s,x,\xi) z_k(t,x,\xi)\, d\xi\, dx\, ds \le \int_0^1 \langle Kz_k(s), z_k(s)\rangle\, ds \le 1,$$

and thus

$$\forall\ r > 0, \quad \int_0^1 \int_{B_r} \int_{\mathbb{R}^n} Kz(s,x,\xi) z(t,x,\xi)\, d\xi\, dx\, ds \le 1 \tag{3.9}$$

due to (3.7). This along with (3.8) shows that

$$\lim_{r\to+\infty} \int_0^1 \int_{B_r} \int_{\mathbb{R}^n} Kz(s,x,\xi) z(t,x,\xi)\, d\xi\, dx\, ds = 1.$$

As a result, using the Lebesgue convergence theorem due to (3.9), we obtain the desired estimate (3.6).

Step 4. We show $z \not\equiv 0$ in $L^2\left((0,1); L^2_\nu\right)$ and

$$z(t,x,\xi) = \{a(t,x) + b(t,x)\cdot\xi + c(t,x)|\xi|^2\}\mathcal{M}^{1/2}$$

with $e^{-\phi(x)/2}a, e^{-\phi(x)/2}b, e^{-\phi(x)/2}c \in L^2((0,1)\times\mathbb{R}^3_x)$. In fact, using the positivity of L and the fact that

$$\int_0^1 \langle Lz(s), z(s)\rangle\, ds = \int_0^1 \left\|z(s)\right\|_\nu^2 ds - \int_0^1 \langle Kz(s), z(s)\rangle\, ds \le 1 - \int_0^1 \langle Kz(s), z(s)\rangle\, ds = 0$$

due to (3.6), one has

$$\int_0^1 \langle Lz(s), z(s)\rangle\, ds = 0, \quad \int_0^1 \left\|z(s)\right\|_\nu^2 ds = \int_0^1 \langle Kz(s), z(s)\rangle\, ds = 1.$$

Therefore, $z \not\equiv 0$, and

$$\{\mathbf{I} - \mathbf{P}\}z(t,x,\xi) = 0$$

for almost every $(t,x,\xi) \in (0,1)\times\mathbb{R}^{2n}$, by the coercivity estimate

$$\int_0^1 \langle Lz(s), z(s)\rangle\, ds \ge C\int_0^1 \|\{\mathbf{I}-\mathbf{P}\}z\|_\nu^2\, ds.$$

Thus $z = \mathbf{P}z = \{\tilde a(t,x) + \tilde b(t,x)\cdot\xi + \tilde c(t,x)|\xi|^2\}M^{1/2} = \{a(t,x) + b(t,x)\cdot\xi + c(t,x)|\xi|^2\}\mathcal{M}^{1/2}$ with $a = \tilde a\, e^{\phi(x)/2}$, $b = \tilde b\, e^{\phi(x)/2}$, etc. Note that $\tilde a, \tilde b$ and $\tilde c$ can be solved as linear combination of

$$\int z(t,x,\xi) M^{1/2}\, d\xi, \ \int z(t,x,\xi)\xi M^{1/2}\, d\xi, \ \int z(t,x,\xi)\, |\xi|^2\, M^{1/2}\, d\xi,$$

and thus $\tilde a, \tilde b, \tilde c \in L^2((0,1)\times\mathbb{R}^3_x)$ due to the fact that $z \in L^2\left((0,1); L^2_\nu\right)$.

Step 6. We prove that z satisfies in the sense of distribution the transport equation

$$\partial_t z + \xi\cdot\nabla_x z - \nabla_x\phi\cdot\nabla_\xi z = 0.$$

For this purpose, for any $\psi \in C_0^\infty\left((0,1)\times\mathbb{R}^n\times\mathbb{R}^n\right)$, we have

$$\begin{aligned} -\iiint z_k \partial_t\psi\, d\xi\, dx\, dt - \iiint z_k\xi\cdot\nabla_x\psi\, d\xi\, dx\, dt + \iiint z_k \nabla_x\phi\cdot\nabla_\xi\psi\, d\xi\, dx\, dt \\ + \iiint \psi\nu(\xi) z_k\, d\xi\, dx\, dt - \iiint \psi K z_k\, d\xi\, dx\, dt \ = 0, \end{aligned}$$

since z_k is a solution to (1.11). On the other hand, from the weak convergence (3.4) and strong convergence (3.5), it follows that, as $k \to +\infty$,

$$\begin{aligned} &-\iiint z_k \partial_t\psi\, d\xi\, dx\, dt - \iiint z_k\xi\cdot\nabla_x\psi\, d\xi\, dx\, dt + \iiint z_k \nabla_x\phi\cdot\nabla_\xi\psi\, d\xi\, dx\, dt \\ &\quad + \iiint \psi\nu(\xi) z_k\, d\xi\, dx\, dt \to -\iiint z\partial_t\psi\, d\xi\, dx\, dt - \iiint z\xi\cdot\nabla_x\psi\, d\xi\, dx\, dt \\ &\qquad + \iiint z\nabla_x\phi\cdot\nabla_\xi\psi\, d\xi\, dx\, dt + \iiint \psi\nu(\xi) z\, d\xi\, dx\, dt, \end{aligned}$$

and

$$\iiint \psi K z_k \, d\xi \, dx \, dt \to \iiint \psi K z \, d\xi \, dx \, dt.$$

Combining the above relations, we conclude, for any $\psi \in C_0^\infty((0,1) \times \mathbb{R}^n \times \mathbb{R}^n)$,

$$-\iiint z\partial_t\psi \, d\xi \, dx \, dt - \iiint z\xi \cdot \nabla_x \psi \, d\xi \, dx \, dt + \iiint z\nabla_x\phi \cdot \nabla_\xi \psi \, d\xi \, dx \, dt \\ + \iiint \psi\nu(\xi) z \, d\xi \, dx \, dt - \iiint \psi K z \, d\xi \, dx \, dt = 0.$$

Moreover note $z \in \ker L = \ker(\nu - K)$ in view of the conclusion of the previous step. Then the above equation becomes

$$\iiint z(t,x,\xi) \left(-\partial_t - \xi \cdot \nabla_x + \nabla_x\phi \cdot \nabla_\xi\right) \psi(t,x,\xi) \, d\xi \, dx \, dt = 0,$$

that is, z satisfies in the sense of distribution the equation $(\partial_t + \xi \cdot \nabla_x - \nabla_x\phi \cdot \nabla_\xi) \, z = 0$. This, together with the conclusion that z satisfies the integrated conservation laws (2.3), (2.4) and (2.5) shown in Step 1, implies $z(t,x,\xi) \equiv 0$ by Lemma 2.1, which is a contradiction with $z \not\equiv 0$. Then the proof of Proposition 3.1 is complete. □

Proof of Theorem 1.1. Let $0 < \lambda < 1$ be a small number to be determined later, and let f be a solution to (1.11) with initial data $F_0 = \mathcal{M} + \mathcal{M}^{1/2} f_0$ satisfying the conservation laws (1.8), (1.9) and (1.10). Firstly we have the equation for $e^{\lambda t} f$:

$$(\partial_t + \xi \cdot \nabla_x - \nabla_x\phi \cdot \nabla_\xi + L)\,(e^{\lambda t} f) = \lambda e^{\lambda t} f.$$

Then for any $N \in \mathbb{Z}_+$, one has

$$\int_0^N \left\langle (\partial_s + \xi \cdot \nabla_x - \nabla_x\phi \cdot \nabla_\xi + L)\,(e^{\lambda s} f),\ e^{\lambda s} f \right\rangle ds = \lambda \int_0^N e^{2\lambda s} \big\|f(s)\big\|^2 \, ds.$$

This implies

$$e^{\lambda N}\big\|f(N)\big\|^2 + 2\int_0^N e^{2\lambda s} \left\langle Lf(s),\ f(s)\right\rangle \, ds = \big\|f(0)\big\|^2 + 2\lambda \int_0^N e^{2\lambda s}\big\|f(s)\big\|^2 \, ds,$$

since $\xi \cdot \nabla_x - \nabla_x\phi \cdot \nabla_\xi$ is skew-adjoint in $L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$. We may rewrite the above equation as

$$e^{\lambda N}\big\|f(N)\big\|^2 + 2\sum_{k=0}^{N-1} \int_0^1 e^{2\lambda(s+k)} \left\langle Lf_k(s),\ f_k(s)\right\rangle \, ds \\ = \big\|f(0)\big\|^2 + 2\lambda \sum_{k=0}^{N-1} \int_0^1 e^{2\lambda(s+k)} \big\|f_k(s)\big\|^2 \, ds, \quad (3.10)$$

where $f_k(s,x,\xi) \overset{\text{def}}{=} f(s+k,x,\xi)$, $k = 0, \cdots, N-1$. Note that f_k satisfies the linear Boltzmann equation in the interval $[0,1]$. Then we use Proposition 3.1 to get

$$2\sum_{k=0}^{N-1} \int_0^1 e^{2\lambda(s+k)} \left\langle Lf_k(s),\ f_k(s)\right\rangle \, ds \geq 2\sum_{k=0}^{N-1} e^{2\lambda k} \int_0^1 \left\langle Lf_k(s),\ f_k(s)\right\rangle \, ds \\ \geq 2C \sum_{k=0}^{N-1} e^{2\lambda k} \int_0^1 \big\|f_k(s)\big\|^2 \, ds,$$

where we have used the fact that $\nu_0 \big\| \cdot \big\| \leq \big\| \cdot \big\|_\nu$ since $\nu = \nu(\xi)$ has a positive lower bound ν_0. On the other hand,

$$2\lambda \sum_{k=0}^{N-1} \int_0^1 e^{2\lambda(s+k)} \big\|f_k(s)\big\|^2 \, ds \leq 2e^{2\lambda}\lambda \sum_{k=0}^{N-1} e^{2\lambda k} \int_0^1 \big\|f_k(s)\big\|^2 \, ds.$$

As a result, we can choose $\lambda > 0$ sufficiently small such that

$$2\sum_{k=0}^{N-1}\int_0^1 e^{2\lambda(s+k)}\left\langle Lf_k(s),\ f_k(s)\right\rangle ds - 2\lambda\sum_{k=0}^{N-1}\int_0^1 e^{2\lambda(s+k)}\big\|f_k(s)\big\|^2 ds$$
$$\geq 2\left(C - e^{2\lambda}\lambda\right)\sum_{k=0}^{N-1} e^{2\lambda k}\int_0^1 \big\|f_k(s)\big\|^2 ds \geq 0.$$

This together with (3.10) imply

$$e^{\lambda N}\big\|f(N)\big\|^2 \leq \big\|f(0)\big\|^2. \tag{3.11}$$

Now, for any $t \geq 0$, we can find an integer N_t such that $0 \leq N_t \leq t \leq N_t + 1$. Using the relation

$$\int_{N_t}^t \left\langle (\partial_s + \xi\cdot\nabla_x - \nabla_x\phi\cdot\nabla_\xi + L)\, f,\ f\right\rangle ds = 0,$$

we get the L^2 energy estimate for the solution f on the interval $[N_t, t]$:

$$\big\|f(t)\big\|^2 + 2\int_{N_t}^t \left\langle Lf(s),\ f(s)\right\rangle ds = \big\|f(N_t)\big\|^2.$$

This, along with the positivity of L, yields

$$e^{\lambda t}\big\|f(t)\big\|^2 \leq e^{\lambda t}\big\|f(N_t)\big\|^2 = e^{\lambda(t-N_t)}e^{\lambda N_t}\big\|f(N_t)\big\|^2 \leq e\big\|f(0)\big\|^2,$$

where the last inequality follows from (3.11) and the fact that $e^{\lambda(t-N_t)} \leq e^{\lambda} \leq e$ since $\lambda < 1$. The desired estimate (1.13) follows by choosing $\sigma = \lambda/2$ and $C = \sqrt{e}$. This completes the proof of Theorem 1.1. □

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