On the Full Dissipative Property of the Vlasov-Poisson-Boltzmann System

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Abstract

In this paper, we present a new approach of studying the full dissipative property of the Vlasov-Poisson-Boltzmann system over the whole space. The key part of this approach is to design the interactive functional to capture the dissipation of the system along the degenerate components. The developed approach is generally applicable to other relevant models arising from plasma physics both at the kinetic and fluid levels.

1 Introduction

In this paper we present a recent work [7] for the study of the optimal large-time behavior on the Vlasov-Poisson-Boltzmann system (VPB). The approach that we have developed in [7] is a combination of the time-decay property of the linearized nonhomogeneous system in terms of the Fourier analysis and the bootstrap to the nonlinear system. The key part of this approach is to design some interactive functionals to capture the dissipation of the system along the degenerate components. Moreover, the approach used in [7] has been applied to some models with the electromagnetic field in the context of plasma physics both at the kinetic and fluid levels, such as the Vlasov-Maxwell-Boltzmann system [8] and the Euler-Maxwell system with relaxation [3]. A new feature in the presence of the coupled Maxwell system which was found in [8] and [3] is that the full system admits the regularity-loss property. From the analysis of the Green's function in the case of the Euler-Maxwell system, this kind of regularity-loss property corresponds to the fact that the real part of eigenvalues along the electromagnetic component tends to zero as the frequency goes to infinity. It is believed that the ideas and methods that are established in [7, 8, 3] can also be applicable to many other relevant physical models.

In what follows we first formulate the problem considered in [7] and then give a review of the related references; the statement of the main results in [7] is left to the next subsection. The VPB system is a physical model describing the time evolution of dilute charged particles (e.g., electrons) under a given external magnetic field [20, 1]. For one-species of particles, it reads

$$\partial_t f + \xi \cdot \nabla_x f + \nabla_x \Phi \cdot \nabla_\xi f = Q(f, f), \tag{1.1}$$

$$\Delta_x \Phi = \int_{\mathbb{R}^3} f d\xi - \bar{\rho}(x), \tag{1.2}$$

with initial data

$$f(0, x, \xi) = f_0(x, \xi). \tag{1.3}$$

Here, the unknown $f = f(t, x, \xi)$ is a non-negative function standing for the number density of gas particles which have position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and velocity $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ at time $t > 0$. Q is the bilinear collision operator for the hard-sphere model defined by

$$Q(f, g) = \int_{\mathbb{R}^3 \times S^2} (f' g'_* - f g_*) |(\xi - \xi_*) \cdot \omega| d\omega d\xi_*,$$
$$f = f(t, x, \xi), \quad f' = f(t, x, \xi'), \quad g_* = g(t, x, \xi_*), \quad g'_* = g(t, x, \xi'_*),$$
$$\xi' = \xi - [(\xi - \xi_*) \cdot \omega]\omega, \quad \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \omega]\omega, \quad \omega \in S^2.$$

The potential function $\Phi = \Phi(t, x)$ generating the self-consistent electric field in (1.1) is coupled with $f(t, x, \xi)$ through the Poisson equation (1.2). $\bar{\rho}(x)$ denotes the stationary background density satisfying $\bar{\rho}(x) \to \rho_\infty$ as $|x| \to \infty$ for a positive constant state $\rho_\infty > 0$.

The existence of stationary solutions to the system (1.1)-(1.2) and the nonlinear stability of solutions to the Cauchy problem (1.1)-(1.3) near the stationary state were obtained in [9] in terms of the energy method [18, 17, 14, 13, 28, 29, 10]. Notice that when $\bar{\rho}$ is a positive constant, the global existence was also studied in [14, 28, 29] in the energy space with time derivatives, but [9] used a solution space without initial layer. After [9], we obtained in [7] the optimal rate of convergence of solutions towards the stationary states. To consider this issue in a simple way, it is here supposed that

$$\bar{\rho}(x) \equiv \rho_\infty = 1, \quad x \in \mathbb{R}^3. \tag{1.4}$$

In this case, the VPB system (1.1)-(1.2) has a stationary solution (f_*, Φ_*) with

$$f_* = \mathbf{M} = (2\pi)^{-3/2} \exp\left(-|\xi|^2/2\right), \quad \Phi_* = 0,$$

where $\mathbf{M}$ is a normalized global Maxwellian.

The time rate of convergence to equilibrium is an important topic in the mathematical theory of the physical world. As pointed out in [26], there exist general structures in which the interaction between a conservative part and a degenerate dissipative part lead to convergence to equilibrium, where this property was called *hypocoercivity*. Our study in [7] indeed provides a concrete example of the hypocoercivity property for the nonlinear VPB system in the framework of perturbations. The key of the method to study hypocoercivity provided in [7] is to carefully capture the dissipative property for the perturbed macroscopic system of equations with the hyperbolic-parabolic structure, which is in the same spirit of the Kawashima's work [15].

There has been extensive investigations on the rate of convergence for the nonlinear Boltzmann equation or related spatially non-homogeneous kinetic equations with relaxations. We shall mention some of them; interested readers can refer to [7, 8], [22] and more references therein. In the context of perturbed solutions, the first result was given by Ukai [25], where the spectral analysis is used to obtain the exponential rates for the Boltzmann equation with hard potentials on torus. When there is an external force, a new approach of the energy-spectrum combination was developed by Duan-Ukai-Yang-Zhao [10]. The compensation function method of the Boltzmann equation was found by Kawashima [16]. This method was later applied by Glassey-Strauss [11] to the time decay of the linearized VPB system on torus, and the recent applications was made by Yang-Yu [27]. Here, we also mention Glassey-Strauss [12] for the study of the essential spectra of the solution operator of the VPB system. Moreover, Strain-Guo [24] developed a weighted energy method to get the exponential rate of convergence for the Boltzmann equation and Landau equation with soft potentials on the torus; the earlier work [23] was done by the same authors.

Another powerful tool is entropy method which has general applications in the existence theory for nonlinear equations. By using this method, Desvillettes-Villani [2] obtained first the almost exponential rate of convergence of solutions to the Boltzmann equation on torus with soft potentials for large initial data under the additional regularity conditions; see Villani [26] for extension and simplification of results. Recently, by finding some proper Lyapunov functional defined over the Hilbert space, Mouhot-Neumann [21] obtained the exponential rates of convergence for some kinetic models with general structures in the case of torus; see also [26] for the systematic study of this topic.

Besides those methods mentioned above for the study of rates of convergence, the method of Green's functions was also founded by Liu-Yu [19] to expose the pointwise large-time behavior of solutions to the Boltzmann equation in the full space $\mathbb{R}^3$.

2 Main results

As mentioned before, our work [7] is mainly concerned with the study of the large-time behavior for the Cauchy problem (1.1), (1.2) and (1.3) of the VPB system, especially the rates of convergence of solutions trending towards the global Maxwellian $\mathbf{M}$. The main result obtained in [7] is stated as follows.

Theorem 2.1. *Let $N \geq 4$ and $w(\xi) = (1+|\xi|^2)^{1/2}$. Assume that $f_0 \geq 0$ and $\|(f_0-\mathbf{M})\mathbf{M}^{-1/2}\|_{H^N\cap L^2_w\cap Z_1}$ is sufficiently small. Let $f \geq 0$ be the solution to the Cauchy problem* (1.1), (1.2) *and* (1.3) *under the assumption* (1.4). *Then, f enjoys the estimate with algebraic rate of convergence:*

$$\|(f(t)-\mathbf{M})\mathbf{M}^{-1/2}\|_{H^N} \leq C\|(f_0-\mathbf{M})\mathbf{M}^{-1/2}\|_{H^N\cap Z_1}(1+t)^{-\frac{1}{4}}. \quad (2.1)$$

Furthermore, under the following additional conditions on f_0, f also enjoys some estimates with extra rates of convergence:

Case 1. If $\int_{\mathbb{R}^3}(f_0(x,\xi)-\mathbf{M})d\xi \equiv 0$ holds for any $x\in\mathbb{R}^3$, then one has

$$\|(f(t)-\mathbf{M})\mathbf{M}^{-1/2}\|_{H^N} \leq C\|(f_0-\mathbf{M})\mathbf{M}^{-1/2}\|_{H^N\cap Z_1}(1+t)^{-\frac{3}{4}}. \quad (2.2)$$

Case 2. Fix any $0<\epsilon\leq 3/4$ and suppose that $\|(f_0-\mathbf{M})\mathbf{M}^{-1/2}\|_{H^N_w\cap Z_1}$ is sufficiently small. Then one has

$$\begin{aligned}\|f(t)\mathbf{M}^{-1/2}-\sum_{j=0}^{4}\langle e_j,f(t)\rangle e_j\mathbf{M}^{1/2}\|_{H^N_w}+\sum_{j=1}^{4}\|\nabla_x\langle e_j,f(t)\rangle\|_{H^{N-1}_x}\\ +\|\langle e_0,f(t)-\mathbf{M}\rangle\|_{H^N_x}\\ \leq C\|(f_0-\mathbf{M})\mathbf{M}^{-1/2}\|_{H^N_w\cap Z_1}(1+t)^{-\frac{3}{4}+\epsilon}, \quad (2.3)\end{aligned}$$

where $\{e_j\}_{j=0}^4$ is the orthonormal set in $L^2(\mathbb{R}^3;\mathbf{M}d\xi)$ defined by

$$e_0=1,\quad e_j=\xi_j\,(1\leq j\leq 3),\quad e_4=\frac{|\xi|^2-3}{\sqrt{6}}.$$

Some remarks are given as follows. From the proof in [7], the condition in Case 1 in Theorem 2.1 can be weakened as

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3}(f_0(x,\xi)-\mathbf{M})dxd\xi=0.$$

Hence, the rate in (2.2) for the VPB system is the same as one for the Boltzmann equation provided that the zero-mass perturbation is postulated. On the other hand, from (2.1), for the general initial perturbation, the time-decay of the VPB system over $\mathbb{R}^3$ is slower than that of the Boltzmann equation. This is essentially caused by the self-induced potential force. Furthermore, if one decomposes f as the summation of three parts

$$f = \langle e_0, f\rangle e_0\mathbf{M} + \sum_{j=1}^{4}\langle e_j, f\rangle e_j\mathbf{M} + \{f - \sum_{j=0}^{4}\langle e_j, f\rangle e_j\mathbf{M}\},$$

then by comparing (2.1) and (2.3), one can find out that the effect of the self-induced potential force on rates of convergence only occurs in the above second part which corresponds to the projections of f along the momentum and temperature components e_j, $1 \leq j \leq 4$.

3 Dissipative property of the linearized system

In this section we emphasize one of the key parts in the proof of Theorem 2.1. It is connected with the Fourier analysis of the following Cauchy problem of the linearized system with a nonhomogeneous source:

$$\begin{cases} \partial_t u + \xi\cdot\nabla_x u - \nabla_x\Phi\cdot\xi\sqrt{\mathbf{M}} = \mathbf{L}u + h, \\ \Delta_x\Phi = \displaystyle\int_{\mathbb{R}^n}\sqrt{\mathbf{M}}u d\xi, \quad t>0, x\in\mathbb{R}^n, \xi\in\mathbb{R}^n, \\ u|_{t=0} = u_0, \quad x\in\mathbb{R}^n, \end{cases} \tag{3.1}$$

where $h = h(t,x,\xi)$ and $u_0 = u_0(x,\xi)$ are given, the spatial dimension $n \geq 2$ is arbitrary in order to determine how it enters into the time-decay rate at the level of linearization, and for simplicity we still use $\mathbf{M}$ to denote the normalized n-dimensional Maxwellian $\mathbf{M} = (2\pi)^{-n/2}e^{-|\xi|^2/2}$. Formally, the solution to the Cauchy problem (3.1) can be written as the mild form

$$u(t) = e^{t\mathbf{B}}u_0 + \int_0^t e^{(t-s)\mathbf{B}}h(s)ds,$$

where $e^{t\mathbf{B}}$ denotes the solution operator to the Cauchy problem of the linearized equation without source corresponding to (3.1) for $h \equiv 0$. A vital goal is to show that $e^{t\mathbf{B}}$ has the proposed algebraic decay properties as time tends to infinity. The idea for that purpose is to make energy estimates for pointwise time t and frequency variable k which corresponds to the spatial variable x. We decompose u as

$$\begin{cases} u(t,x,\xi) = \mathbf{P}u \oplus \{\mathbf{I}-\mathbf{P}\}u, \\ \mathbf{P}u = \{a^u + b^u\cdot\xi + c^u|\xi|^2\}\sqrt{\mathbf{M}}, \end{cases}$$

where $\mathbf{P}$ is the usual macroscopic projector, and a^u, b^u, c^u are the macro moment functions of u. Under the assumption of $\mathbf{P}h = 0$, one can obtain the macroscopic balance laws satisfied by a^u, b^u, c^u:

$$\begin{cases} \partial_t(a^u + nc^u) + \nabla_x \cdot b^u = 0, \\ \partial_t b_j^u + \partial_j(a^u + nc^u) + 2\partial_j c^u \\ \qquad\qquad + \sum_m \partial_m A_{jm}(\{\mathbf{I}-\mathbf{P}\}u) - \partial_j \Phi = 0, \\ \partial_t c^u + \frac{1}{n}\nabla_x \cdot b^u + \frac{1}{2n}\sum_j \partial_j B_j(\{\mathbf{I}-\mathbf{P}\}u) = 0, \\ \Delta_x \Phi = a^u + nc^u, \end{cases} \tag{3.2}$$

and also the high-order moment equations:

$$\begin{cases} \partial_t[A_{jj}(\{\mathbf{I}-\mathbf{P}\}u) + 2c^u] + 2\partial_j b_j^u = A_{jj}(R+h), \\ \partial_t A_{jm}(\{\mathbf{I}-\mathbf{P}\}u) + \partial_j b_m^u + \partial_m b_j^u = A_{jm}(R+h), \ j \neq m, \\ \partial_t B_j(\{\mathbf{I}-\mathbf{P}\}u) + \partial_j c^u = B_j(R+h), \end{cases} \tag{3.3}$$

for $1 \leq j, m \leq n$. Here, the velocity moment functions $A_{jm}(\cdot)$ and $B_j(\cdot)$ are given by

$$A_{jm}(u) = \langle (\xi_j\xi_m - 1)\sqrt{\mathbf{M}}, u\rangle, \ B_j(u) = \frac{\langle [|\xi|^2 - (n+2)]\xi_j\sqrt{\mathbf{M}}, u\rangle}{2(n+2)},$$

where R is given by $R = -\xi \cdot \nabla_x\{\mathbf{I}-\mathbf{P}\}u + \mathbf{L}\{\mathbf{I}-\mathbf{P}\}u$. From the above system (3.2)-(3.3), we showed in [7] the following

Theorem 3.1. *There exists an interactive time-frequency functional $\mathcal{E}_{\rm int}(\widehat{u}(t,k))$ whose modulus is bounded by the naturally existing Lyapunov functional of* (3.1)*, that is,*

$$|\mathcal{E}_{\rm int}(\widehat{u}(t,k))| \leq C\left(\|\widehat{u}(t,k)\|_{L^2_\xi}^2 + \frac{1}{|k|^2}|a^{\widehat{u}} + nc^{\widehat{u}}|^2\right),$$

such that

$$\begin{aligned} &\partial_t \Re\, \mathcal{E}_{\rm int}(\widehat{u}(t,k)) + \lambda\frac{|k|^2}{1+|k|^2}\left(|b^{\widehat{u}}|^2 + |c^{\widehat{u}}|^2\right) + \lambda|a^{\widehat{u}} + nc^{\widehat{u}}|^2 \\ &\qquad \leq C\left(\|\{\mathbf{I}-\mathbf{P}\}\widehat{u}\|_{L^2_\xi}^2 + \|\nu^{-1/2}\{\mathbf{I}-\mathbf{P}\}\widehat{h}\|_{L^2_\xi}^2\right) \end{aligned}$$

holds true for any $t \geq 0$ and $k \in \mathbb{R}^n$.

The proof of Theorem 3.1 is based on some observations from the previous work [5, 6, 4] together with the idea of carefully choosing a proper linear combination of the decomposed equations; this strategy was noted in [26]. Hence, the interactive time-frequency functional $\mathcal{E}_{\rm int}(\widehat{u}(t,k))$ indeed captures the dissipative property of all the degenerate components in the solution. The rest procedure to derive the time-decay of e^{tB} is based on energy estimates; refer to [7] for more details.

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