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Optimal Time Decay of the Vlasov-Poisson-Boltzmann System in $\mathbb{R}^3$

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Abstract

The Vlasov-Poisson-Boltzmann System governs the time evolution of the distribution function for the dilute charged particles in the presence of a self-consistent electric potential force through the Poisson equation. In this paper, we are concerned with the rate of convergence of solutions to equilibrium for this system over $\mathbb{R}^3$. It is shown that the electric field which is indeed responsible for the lowest-order part in the energy space reduces the speed of convergence and hence the dispersion of this system over the full space is slower than that of the Boltzmann equation without forces, where the exact difference between both power indices in the algebraic rates of convergence is 1/4. For the proof, in the linearized case with a given non-homogeneous source, Fourier analysis is employed to obtain time-decay properties of the solution operator. In the nonlinear case, the combination of the linearized results and the nonlinear energy estimates with the help of the proper Lyapunov-type inequalities leads to the optimal time-decay rate of perturbed solutions under some conditions on initial data.

Contents

1 Introduction

The Vlasov-Poisson-Boltzmann (called VPB in the sequel for simplicity) system is a physical model describing the time evolution of dilute charged particles (e.g., electrons) under a given external magnetic field [21, 1]. The VPB system for one-species of particles in the whole space $\mathbb{R}^3$ reads

$$\partial_t f + \xi \cdot \nabla_x f + \nabla_x \Phi \cdot \nabla_\xi f = Q(f,f), \tag{1.1}$$

$$\Delta_x \Phi = \int_{\mathbb{R}^3} f d\xi - \bar{\rho}(x), \tag{1.2}$$

with initial data

$$f(0,x,\xi) = f_0(x,\xi). \tag{1.3}$$

Here, the unknown $f = f(t,x,\xi)$ is a non-negative function standing for the number density of gas particles which have position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and velocity $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ at time $t > 0$. Q is the bilinear collision operator for the hard-sphere model defined by

$$\begin{gathered}
Q(f,g) = \int_{\mathbb{R}^3 \times S^2} (f' g'_* - f g_*) |(\xi - \xi_*) \cdot \omega| d\omega d\xi_*, \\
f = f(t,x,\xi), \quad f' = f(t,x,\xi'), \quad g_* = g(t,x,\xi_*), \quad g'_* = g(t,x,\xi'_*), \\
\xi' = \xi - [(\xi - \xi_*) \cdot \omega]\omega, \quad \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \omega]\omega, \quad \omega \in S^2.
\end{gathered}$$

The potential function $\Phi = \Phi(t,x)$ generating the self-consistent electric field in (1.1) is coupled with $f(t,x,\xi)$ through the Poisson equation (1.2). $\bar{\rho}(x)$ denotes the stationary background density satisfying

$$\bar{\rho}(x) \to \rho_\infty \quad \text{as} \quad |x| \to \infty,$$

for a positive constant state $\rho_\infty > 0$.

The existence of stationary solutions to the system (1.1)-(1.2) and the nonlinear stability of solutions to the Cauchy problem (1.1)-(1.3) near the stationary state were obtained in [8], and the corresponding results have been recalled in the appendix. In this paper, we are concerned with the rate of convergence of solutions towards the stationary states. Since the background density does not produce any essential difficulty, for simplicity it is supposed throughout this paper that

$$\bar{\rho}(x) \equiv \rho_\infty = 1, \quad x \in \mathbb{R}^3. \tag{1.4}$$

In this case, the VPB system (1.1)-(1.2) has a stationary solution (f_*, Φ_*) with

$$f_* = \mathbf{M}, \quad \Phi_* = 0,$$

where

$$\mathbf{M} = \frac{1}{(2\pi)^{3/2}} \exp\left(-|\xi|^2/2\right)$$

is a normalized global Maxwellian in three dimensions which has zero bulk velocity and unit density and temperature. One of the main results of this paper is stated as follows. Notations and norms will be explained at the end of this section.

Theorem 1.1. *Let $N\geq 4$ and $w(\xi)=(1+|\xi|^2)^{1/2}$. Assume that $f_0\geq 0$ and*

$$\left\|\frac{f_0-\mathbf{M}}{\sqrt{\mathbf{M}}}\right\|_{H^N\cap L^2_w\cap Z_1} \tag{1.5}$$

is sufficiently small. Let $f\geq 0$ be the solution obtained in Proposition 5.2 to the Cauchy problem (1.1), (1.2) *and* (1.3) *under the assumption* (1.4). *Then, f enjoys the estimate with algebraic rate of convergence:*

$$\left\|\frac{f(t)-\mathbf{M}}{\sqrt{\mathbf{M}}}\right\|_{H^N}\leq C\left\|\frac{f_0-\mathbf{M}}{\sqrt{\mathbf{M}}}\right\|_{H^N\cap Z_1}(1+t)^{-\frac{1}{4}}. \tag{1.6}$$

Furthermore, under the following additional conditions on f_0, f also enjoys some estimates with extra rates of convergence:

Case 1. If

$$\int_{\mathbb{R}^3}(f_0(x,\xi)-\mathbf{M})d\xi\equiv 0 \tag{1.7}$$

holds for any $x\in\mathbb{R}^3$, then one has

$$\left\|\frac{f(t)-\mathbf{M}}{\sqrt{\mathbf{M}}}\right\|_{H^N}\leq C\left\|\frac{f_0-\mathbf{M}}{\sqrt{\mathbf{M}}}\right\|_{H^N\cap Z_1}(1+t)^{-\frac{3}{4}}. \tag{1.8}$$

Case 2. Fix any $0<\epsilon\leq 3/4$ and suppose that

$$\left\|\frac{f_0-\mathbf{M}}{\sqrt{\mathbf{M}}}\right\|_{H^N_w\cap Z_1} \tag{1.9}$$

is sufficiently small. Then one has

$$\begin{aligned}&\left\|\frac{f(t)-\sum_{j=0}^4\langle e_j,f(t)\rangle e_j\mathbf{M}}{\sqrt{\mathbf{M}}}\right\|_{H^N_w}\\&\qquad\qquad+\|\langle e_0,f(t)-\mathbf{M}\rangle\|_{H^N_x}+\sum_{j=1}^4\|\nabla_x\langle e_j,f(t)\rangle\|_{H^{N-1}_x}\\&\leq C\left\|\frac{f_0-\mathbf{M}}{\sqrt{\mathbf{M}}}\right\|_{H^N_w\cap Z_1}(1+t)^{-\frac{3}{4}+\epsilon},\end{aligned} \tag{1.10}$$

where $\{e_j\}_{j=0}^4$ is the orthonormal set in $L^2(\mathbb{R}^3;\mathbf{M}d\xi)$ defined by

$$e_0=1,\quad e_j=\xi_j\,(1\leq j\leq 3),\quad e_4=\frac{|\xi|^2-3}{\sqrt{6}}. \tag{1.11}$$

Remark 1.1. *The rates of convergence in* (1.6) *and* (1.8) *are optimal under the corresponding assumptions in the sense that they coincide with those rates given in Theorem 3.1 at the level of linearization. Similarly, the rate of convergence in* (1.10) *is almost optimal. To our knowledge, these results are the first ones in the study of rate of convergence for the nonlinear VPB system in* $\mathbb{R}^3$. *Of course, it is an interesting problem to improve the almost optimal rate in* (1.10) *to the optimal rate.*

Remark 1.2. *In [32] on the same issue, some slower algebraic rates of convergence of solutions in the space* $L^\infty_x(L^2_\xi)$ *were obtained on the basis of the pure energy estimates and time-decay of some ordinary differential equation. Here, under the assumptions of* (1.9), *from the Sobolev inequality,* (1.10) *shows that*

$$\left\|\frac{f(t)-\mathbf{M}}{\sqrt{\mathbf{M}}}\right\|_{L^\infty_x(H^{N-2}_{\xi,w})} \leq C\left\|\frac{f_0-\mathbf{M}}{\sqrt{\mathbf{M}}}\right\|_{H^N_w\cap Z_1}(1+t)^{-\frac{3}{4}+\epsilon},$$

where $L^\infty_x(H^{N-2}_{\xi,w}) = L^\infty(\mathbb{R}^3_x; H^{N-2}(\mathbb{R}^3_\xi; wd\xi))$. *The above inequality implies the almost optimal rate of convergence of solutions. The main reason why one can here obtain the optimal or almost optimal rates is that we make full use of the time-decay properties of solutions to the linearized system with nonhomogeneous sources. This will be carried out in Section 3, where we shall also state another main result Theorem 3.1 in this paper.*

Remark 1.3. *As shown in [29], for the Boltzmann equation without external forces, the power index in* (1.6) *takes the value* $3/4$. *Hence, the dispersion of the VPB system in* $\mathbb{R}^3$ *is slower than that of the Boltzmann equation. This is essentially caused by the self-induced potential force. Furthermore, if one decomposes* f *as the summation of three parts*

$$f = \langle e_0, f\rangle e_0\mathbf{M} + \sum_{j=1}^{4}\langle e_j, f\rangle e_j\mathbf{M} + \left\{f - \sum_{j=0}^{4}\langle e_j, f\rangle e_j\mathbf{M}\right\},$$

then by comparing (1.6) *and* (1.10), *one can find out that the effect of the self-induced potential force on rates of convergence only happens in the above second part which corresponds to the projections of* f *along the momentum components* e_j $(1\leq j\leq 3)$ *and the temperature component* e_4. *This phenomenon is consistent with that recently obtained by [17] for the Navier-Stokes-Poisson system in* $\mathbb{R}^3$.

The time rate of convergence to equilibrium is an important topic in the mathematical theory of the physical world. As pointed out in [30], there exist general structures in which the interaction between a conservative part and a degenerate dissipative part lead to convergence to equilibrium, where this property was called *hypocoercivity*. Theorem 1.1 indeed provides a concrete example of the hypocoercivity property for the nonlinear VPB system in the framework of perturbations. The key of the method to study hypocoercivity provided by this paper is to carefully capture the time-decay rates for the perturbed macroscopic system of equations with the hyperbolic-parabolic structure, which is in the same spirit of the Kawashima's work [16].

There has been extensive investigations on the rate of convergence for the nonlinear Boltzmann equation or related spatially non-homogeneous kinetic equations with relaxations. In what follows let us mention some of them. In the context of perturbed solutions, the first result was given by Ukai [28], where the spectral analysis was used to obtain the exponential rates for the Boltzmann equation with hard potentials on torus. The results in [28] were improved by Ukai-Yang [29] in order to consider existence of time-periodic states in the presence of time-periodic sources, which was later extended by Duan-Ukai-Yang-Zhao [10] to the case with time-periodic external forcing by using the energy-spectrum method. We also mention Glassey-Strauss [12] for the study of the essential spectra of the solution operator of the VPB system. Recently, Strain-Guo [27] developed a weighted energy method to get the exponential rate of convergence for the Boltzmann equation and Landau equation with soft potentials on the torus. Earlier but along the same line of research, Strain-Guo [26] developed a general theory of polynomial decay rates to any order in a unified framework and applied it to four kinetic equations, the Vlasov-Maxwell-Boltzmann System, the relativistic Landau-Maxwell System, the Boltzmann equation with cutoff soft-potentials and the Landau equation all on the torus.

Another powerful tool is entropy method which has general applications in the existence theory for nonlinear equations. By using this method as well as the elaborate analysis of functional inequalities, time-derivative estimates and interpolation, Desvillettes-Villani [2] obtained first the almost exponential rate of convergence of solutions to the Boltzmann equation on torus with soft potentials for large initial data under the additional regularity conditions that all the moments of f are uniformly bounded in time and f is bounded in all Sobolev spaces uniformly in time. See Villani [30] for extension and simplification of results in [2] still conditionally to smoothness bounds by further designing a new auxiliary functional. Notice that [26] provided a simple proof of [2] for the unconditional perturbative regime. Recently, by finding some proper Lyapunov functional defined over the Hilbert space, Mouhot-Neumann [23] obtained the exponential rates of convergence for some kinetic models with general structures in the case of torus; see also [30] for the similar study. An extension of [23] to models with additional confining potential forces was given by Dolbeault-Mouhot-Schmeiser [5].

Besides those methods mentioned above for the study of rates of convergence, the method of Green's functions was also founded by Liu-Yu [20] to expose the pointwise large-time behavior of solutions to the Boltzmann equation in the full space $\mathbb{R}^3$.

Here, we mention that if there is no collision in (1.1)-(1.2), that is to consider the Vlasov-Poisson system, then the so-called Landau damping comes out. This was recently studied by Mouhot-Villani [24] on torus, where it was shown that even though in the absence of kinetic relaxation, in the analytic regime the solution still converges weakly in some sense to certain large-time states determined by the initial data and the nonlinear system itself, for any interaction potential less singular than Coulomb. In the Coulomb case, they established Landau damping over exponentially long times.

Finally, we also mention some of results on the existence theory of the VPB system and related kinetic equations: global existence of renormalized solutions with large initial data [4, 3, 22], global existence of classical solutions near Maxwellians [19, 18, 31], [14, 13, 25] and [6, 8], and global existence of solutions near vacuum [15, 9].

The rest of this paper is organized as follows. In Section 2 we make some preparations to reformulate the Cauchy problem of the VPB system, make the macro-micro decomposition for both the solution and equations, obtain a system of equations describing the evolution of some macroscopic velocity moment functions for the later analysis in both the linear and nonlinear cases, and reduce Theorem 1.1 in an equivalent form to Proposition 2.1. In Section 3 and Section 4, we obtain the time-decay rates of perturbed solutions under some conditions on initial data in the linear and nonlinear cases, respectively. Here, Theorem 3.1 which is another main result of this paper is applied together with some energy estimates to prove Proposition 2.1. Some free energy functionals and Lyapunov-type inequalities play a key role in the proof of Theorem 3.1 and Proposition 2.1. Finally, we conclude this paper with an appendix in Section 5 by listing some results obtained in [8] about the existence of the stationary solution and its nonlinear stability for the VPB system (1.1)-(1.2).

Notations. Throughout this paper, C denotes some positive (generally large) constant and λ denotes some positive (generally small) constant, where both C and λ may take different values in different places. In addition, $A \sim B$ means $\lambda_1 A \leq B \leq \lambda_2 A$ for two generic constants $\lambda_1 > 0$ and $\lambda_2 > 0$. For an integrable function $g : \mathbb{R}^n \to \mathbb{R}$, its Fourier transform $\widehat{g} = \mathcal{F}g$ is defined by

$$\widehat{g}(k) = \mathcal{F}g(k) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot k} g(x)dx, \quad x \cdot k =: \sum_{j=1}^{n} x_j k_j,$$

for $k \in \mathbb{R}^n$, where $i = \sqrt{-1} \in \mathbb{C}$ is the imaginary unit. For two complex vectors $a, b \in \mathbb{C}^n$, $(a \mid b) = a \cdot \overline{b}$ denotes the dot product over the complex field, where $\overline{b}$ is the complex conjugate of b. For any integer $m \geq 0$, we use H^m, H^m_x, H^m_ξ to denote the usual Hilbert spaces $H^m(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, $H^m(\mathbb{R}^n_x)$, $H^m(\mathbb{R}^n_\xi)$, respectively, where L^2, L^2_x, L^2_ξ are used for the case when $m = 0$. For a Banach space X, $\|\cdot\|_X$ denotes the corresponding norm, while $\|\cdot\|$ always denotes the norm $\|\cdot\|_{L^2}$ or $\|\cdot\|_{L^2_x}$ for simplicity. We also use $\|\cdot\|_{H^m_w}$ for the norm of the weighted Hilbert space $H^m(\mathbb{R}^n_x \times \mathbb{R}^n_\xi; w(\xi)dxd\xi)$ and $\|\cdot\|_{L^2_w}$ for $L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi; w(\xi)dxd\xi)$. We use $\langle \cdot, \cdot \rangle$ to denote the inner product over the Hilbert space L^2_ξ, i.e.

$$\langle g, h \rangle = \int_{\mathbb{R}^n} g(\xi)h(\xi)d\xi, \quad g, h \in L^2_\xi.$$

For $q \geq 1$, we also define

$$Z_q = L^2_\xi(L^q_x) = L^2(\mathbb{R}^n_\xi; L^q(\mathbb{R}^n_x)), \quad \|g\|_{Z_q} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |g(x,\xi)|^q dx\right)^{2/q} d\xi\right)^{1/2}.$$

For the multiple indices $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$, as usual we denote

$$\partial^\alpha_x \partial^\beta_\xi = \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n} \partial^{\beta_1}_{\xi_1} \cdots \partial^{\beta_n}_{\xi_n}.$$

The length of α is $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For simplicity, we also use ∂_j to denote ∂_{x_j} for each $j = 1, \cdots, n$. Generally, except in Section 3, we consider the case of $n = 3$ dimensions.

2 Macro-micro decomposition

In this section, we shall make some preparations for the later analysis in both the linear and nonlinear cases, and moreover reduce Theorem 1.1 to Proposition 2.1 in the equivalent form. Firstly, one can reformulate the Cauchy problem (1.1), (1.2) and (1.3) as follows. Set the perturbation $u = u(t, x, \xi)$ by

$$f = \mathbf{M} + \sqrt{\mathbf{M}}u. \tag{2.1}$$

Then u and Φ satisfy the perturbed system:

$$\partial_t u + \xi \cdot \nabla_x u + \nabla_x \Phi \cdot \nabla_\xi u - \frac{1}{2}\xi \cdot \nabla_x \Phi u - \nabla_x \Phi \cdot \xi \sqrt{\mathbf{M}} = \mathbf{L}u + \Gamma(u, u), \tag{2.2}$$

$$\Delta_x \Phi = \int_{\mathbb{R}^3} \sqrt{\mathbf{M}} u d\xi, \tag{2.3}$$

with given initial data

$$u(0, x, \xi) = u_0(x, \xi) \equiv \frac{f_0 - \mathbf{M}}{\sqrt{\mathbf{M}}}, \tag{2.4}$$

where $\mathbf{L}u$ and $\Gamma(u, u)$ are denoted by

$$\mathbf{L}u = \frac{1}{\sqrt{\mathbf{M}}}\left[Q(\mathbf{M}, \sqrt{\mathbf{M}}u) + Q(\sqrt{\mathbf{M}}u, \mathbf{M})\right], \tag{2.5}$$

$$\Gamma(u, u) = \frac{1}{\sqrt{\mathbf{M}}} Q(\sqrt{\mathbf{M}}u, \sqrt{\mathbf{M}}u). \tag{2.6}$$

It is well-known that for the linearized collision operator $\mathbf{L}$, one has

$$\begin{aligned}
(\mathbf{L}u)(\xi) &= -\nu(\xi)u(\xi) + (Ku)(\xi),\\
\nu(\xi) &= \int_{\mathbb{R}^3 \times S^2} |(\xi - \xi_*) \cdot \omega| \mathbf{M}_* \, d\omega d\xi_*,\\
(Ku)(\xi) &= \int_{\mathbb{R}^3 \times S^2} \left(-\sqrt{\mathbf{M}}u_* + \sqrt{\mathbf{M}'_*}u' + \sqrt{\mathbf{M}'}u'_*\right) |(\xi - \xi_*) \cdot \omega| \sqrt{\mathbf{M}_*} d\omega d\xi_*\\
&= \int_{\mathbb{R}^3} K(\xi, \xi_*) u(\xi_*) d\xi_*,
\end{aligned}$$

where $\nu(\xi)$ is called the collision frequency and K is a self-adjoint compact operator on $L^2(\mathbb{R}^3_\xi)$ with a real symmetric integral kernel $K(\xi, \xi_*)$. The null space of the operator $\mathbf{L}$ is the five dimensional space spanned by the collision invariants

$$\mathcal{N} = Ker\mathbf{L} = \operatorname{span}\left\{\sqrt{\mathbf{M}};\ \xi_i \sqrt{\mathbf{M}}, i = 1, 2, 3;\ |\xi|^2 \sqrt{\mathbf{M}}\right\}.$$

From the Boltzmann's H-theorem, the linearized collision operator $\mathbf{L}$ is non-positive and moreover, $-\mathbf{L}$ is locally coercive in the sense that there is a constant $\lambda > 0$ such that

$$-\int_{\mathbb{R}^3} u \mathbf{L} u \, d\xi \geq \lambda \int_{\mathbb{R}^3} \nu(\xi) \left(\{\mathbf{I} - \mathbf{P}\}u\right)^2 d\xi, \quad \forall\, u \in D(\mathbf{L}),$$

where for fixed (t,x), $\mathbf{P}$ denotes the projection operator from L^2_ξ to $\mathcal{N}$ and $D(\mathbf{L})$ is the domain of $\mathbf{L}$ given by

$$D(\mathbf{L}) = \left\{ u \in L^2_\xi \;\middle|\; \sqrt{\nu(\xi)}u \in L^2_\xi \right\}.$$

In addition, for the hard sphere model, ν satisfies

$$\nu(\xi) \sim (1+|\xi|^2)^{\frac{1}{2}} = w(\xi).$$

This property will be used throughout this paper.

Given any $u(t,x,\xi)$, we define the projection operator, $\mathbf{P}u$, as

$$\mathbf{P}u = \left\{ a^u(t,x) + \sum_{j=1}^{3} b_j^u(t,x)\xi_j + c^u(t,x)|\xi|^2 \right\} \sqrt{\mathbf{M}}. \tag{2.7}$$

Since $\mathbf{P}$ is a projector, it holds that

$$\int_{\mathbb{R}^3} (1,\xi,|\xi|^2)\sqrt{\mathbf{M}}\{\mathbf{I}-\mathbf{P}\}u d\xi = 0,$$

i.e. $\{\mathbf{I}-\mathbf{P}\}u$ is orthogonal to $\mathcal{N}$, which together with the form (2.7) of $\mathbf{P}$ imply

$$a^u = \frac{1}{2}\int_{\mathbb{R}^3} (5-|\xi|^2)\sqrt{\mathbf{M}} u d\xi, \tag{2.8}$$

$$b^u = \int_{\mathbb{R}^3} \xi\sqrt{\mathbf{M}} u d\xi, \tag{2.9}$$

$$c^u = \frac{1}{6}\int_{\mathbb{R}^3} (|\xi|^2-3)\sqrt{\mathbf{M}} u d\xi, \tag{2.10}$$

where $b^u = (b_1^u, b_2^u, b_3^u)$. Thus, $u(t,x,\xi)$ can be uniquely decomposed into

$$\begin{cases} u(t,x,\xi) = \mathbf{P}u \oplus \{\mathbf{I}-\mathbf{P}\}u, \\ \mathbf{P}u = \left\{ a^u(t,x) + b^u(t,x)\cdot\xi + c^u(t,x)|\xi|^2 \right\} \sqrt{\mathbf{M}} \in \mathcal{N}, \\ \{\mathbf{I}-\mathbf{P}\}u \in \mathcal{N}^\perp, \end{cases} \tag{2.11}$$

where $\mathbf{P}u$ is called the macroscopic component of $u(t,x,\xi)$ with coefficients (a^u, b^u, c^u), and $\{\mathbf{I}-\mathbf{P}\}u$ the microscopic component of $u(t,x,\xi)$. For later use, one can further decompose $\mathbf{P}u$ as

$$\begin{cases} \mathbf{P}u = \mathbf{P}_0 u \oplus \mathbf{P}_1 u, \\ \mathbf{P}_0 u = (a^u + 3c^u)\sqrt{\mathbf{M}}, \quad a^u + 3c^u = \displaystyle\int_{\mathbb{R}^3} \sqrt{\mathbf{M}} u d\xi, \\ \mathbf{P}_1 u = \{b^u\cdot\xi + c^u(|\xi|^2-3)\}\sqrt{\mathbf{M}}, \end{cases} \tag{2.12}$$

where $\mathbf{P}_0$ and $\mathbf{P}_1$ are the projectors corresponding to the hyperbolic and parabolic parts of the macroscopic (fluid) component, respectively.

In what follows, we shall apply the macro-micro decomposition (2.11) to the system (2.2)-(2.3) to deduce the macroscopic balance laws satisfied by (a^u, b^u, c^u). Firstly, by multiplying (1.1) by the collision invariants $1, \xi, |\xi|^2$, one can get the local conservation laws

$$\begin{aligned}\partial_t \int_{\mathbb{R}^3} f d\xi + \nabla_x \cdot \int_{\mathbb{R}^3} \xi f d\xi = 0,\\ \partial_t \int_{\mathbb{R}^3} \xi f d\xi + \nabla_x \cdot \int_{\mathbb{R}^3} \xi \otimes \xi f d\xi - \nabla_x \Phi \int_{\mathbb{R}^3} f d\xi = 0,\\ \partial_t \int_{\mathbb{R}^3} |\xi|^2 f d\xi + \nabla_x \cdot \int_{\mathbb{R}^3} |\xi|^2 \xi f d\xi - \nabla_x \Phi \cdot \int_{\mathbb{R}^3} \xi f d\xi = 0.\end{aligned}$$

Plugging $f = \mathbf{M} + \sqrt{\mathbf{M}}\mathbf{P}u + \sqrt{\mathbf{M}}\{\mathbf{I} - \mathbf{P}\}u$ into the above equations as well as into the Poisson equation (1.2) gives

$$\partial_t(a^u + 3c^u) + \nabla_x \cdot b^u = 0, \tag{2.13}$$

$$\partial_t b^u + \nabla_x(a^u + 5c^u) + \nabla_x \cdot \langle \xi \otimes \xi\sqrt{\mathbf{M}}, \{\mathbf{I} - \mathbf{P}\}u\rangle - \nabla_x\Phi = (a^u + 3c^u)\nabla_x\Phi, \tag{2.14}$$

$$\partial_t(3a^u + 15c^u) + 5\nabla_x \cdot b^u + \nabla_x \cdot \langle |\xi|^2\xi\sqrt{\mathbf{M}}, \{\mathbf{I} - \mathbf{P}\}u\rangle = b^u \cdot \nabla_x\Phi, \tag{2.15}$$

and

$$\Delta_x\Phi = a^u + 3c^u. \tag{2.16}$$

Here and hereafter we use the moment values of the normalized global Maxwellian $\mathbf{M}$:

$$\begin{aligned}&\langle 1, \mathbf{M}\rangle = 1,\\ &\langle |\xi_j|^2, \mathbf{M}\rangle = 1, \quad \langle |\xi|^2, \mathbf{M}\rangle = 3,\\ &\langle |\xi_j|^2|\xi_m|^2, \mathbf{M}\rangle = 1, \quad j \neq m,\\ &\langle |\xi_j|^4, \mathbf{M}\rangle = 3, \quad \langle |\xi|^2|\xi_j|^2, \mathbf{M}\rangle = 5, \quad \langle |\xi|^4, \mathbf{M}\rangle = 15,\\ &\langle |\xi|^4|\xi_j|^2, \mathbf{M}\rangle = 35, \quad \langle |\xi|^6, \mathbf{M}\rangle = 105.\end{aligned}$$

Notice that (2.13) and (2.15) implies

$$\partial_t c^u + \frac{1}{3}\nabla_x \cdot b^u + \frac{1}{6}\nabla_x \cdot \langle |\xi|^2\xi\sqrt{\mathbf{M}}, \{\mathbf{I} - \mathbf{P}\}u\rangle = 0,$$

which is more convenient to be used to replace the time derivative of c^u in the proof.

As in [7], we also have to consider the evolution of higher-order moments of $\{\mathbf{I} - \mathbf{P}\}u$:

$$\langle \xi \otimes \xi\sqrt{\mathbf{M}}, \{\mathbf{I} - \mathbf{P}\}u\rangle, \quad \langle |\xi|^2\xi\sqrt{\mathbf{M}}, \{\mathbf{I} - \mathbf{P}\}u\rangle,$$

which have appeared in (2.14) and (2.15), respectively. The equation (2.2) can be rewritten as

$$\partial_t\mathbf{P}u + \xi \cdot \nabla_x\mathbf{P}u - \nabla_x\Phi \cdot \xi\sqrt{\mathbf{M}} = -\partial_t\{\mathbf{I} - \mathbf{P}\}u + R + G, \tag{2.17}$$

where the linear term R and the nonlinear term G are denoted by

$$R = -\xi \cdot \nabla_x\{\mathbf{I} - \mathbf{P}\}u + \mathbf{L}\{\mathbf{I} - \mathbf{P}\}u, \tag{2.18}$$

$$G = \Gamma(u, u) - \nabla_x\Phi \cdot \nabla_\xi u + \left(\frac{1}{2}\xi \cdot \nabla_x\Phi\right) u. \tag{2.19}$$

One can use (2.11) to further write (2.17) as

$$\begin{aligned}
&\partial_t a^u\sqrt{\mathbf{M}}+\sum_j\{\partial_t b_j^u+\partial_j a^u-\partial_j\Phi\}\xi_j\sqrt{\mathbf{M}}+\sum_j\{\partial_t c^u+\partial_j b_j^u\}|\xi_j|^2\sqrt{\mathbf{M}}\\
&\qquad+\sum_{j<m}\{\partial_j b_m^u+\partial_m b_j^u\}\xi_j\xi_m\sqrt{\mathbf{M}}+\sum_j\partial_j c^u|\xi|^2\xi_j\sqrt{\mathbf{M}}\\
&=-\partial_t\{\mathbf{I}-\mathbf{P}\}u+R+G. \qquad (2.20)
\end{aligned}$$

Define the high-order moment functions $A=(A_{jm})_{3\times 3}$ and $B=(B_1,B_2,B_3)$ by

$$A_{jm}(u)=\langle(\xi_j\xi_m-1)\sqrt{\mathbf{M}},u\rangle,\quad B_j(u)=\langle(|\xi|^2-5)\xi_j\sqrt{\mathbf{M}},u\rangle. \qquad (2.21)$$

Applying $A_{jm}(\cdot)$ and $B_j(\cdot)$ to both sides of (2.20) and using the conservation law of mass (2.13), one has

$$\partial_t[A_{jj}(\{\mathbf{I}-\mathbf{P}\}u)+2c^u]+2\partial_j b_j^u=A_{jj}(R+G), \qquad (2.22)$$

$$\partial_t A_{jm}(\{\mathbf{I}-\mathbf{P}\}u)+\partial_j b_m^u+\partial_m b_j^u=A_{jm}(R+G),\quad j\neq m, \qquad (2.23)$$

$$\partial_t B_j(\{\mathbf{I}-\mathbf{P}\}u)+\partial_j c^u=B_j(R+G), \qquad (2.24)$$

where (2.23) also holds for $j>m$ since it is symmetric for (j,m) due to the symmetry of A_{jm}. The main observation which initially came from [13] and later [7] is that for fixed m, from (2.22) and (2.23), one can deduce

$$\begin{aligned}
&-\partial_t\left[\sum_j\partial_j A_{jm}(\{\mathbf{I}-\mathbf{P}\}u)+\frac{1}{2}\partial_m A_{mm}(\{\mathbf{I}-\mathbf{P}\}u)\right]-\Delta_x b_m-\partial_m\partial_m b_m\\
&\qquad=\frac{1}{2}\sum_{j\neq m}\partial_m A_{jj}(R+G)-\sum_j\partial_j A_{jm}(R+G). \qquad (2.25)
\end{aligned}$$

Remark 2.1. *It should be pointed out that the proof of* (5.11) *in Lemma 5.1 for the macroscopic dissipation is only based on similar moment equations in presence of external forcing corresponding to* (2.13)-(2.15), (2.16), (2.22)-(2.24) *and* (2.25). *The derivation of these moment equations was inspired by [13] and firstly given by [6] and [7] in the case of the Boltzmann equation. They play a key role in the choice of the solution space without time derivatives. Actually, the method of excluding time derivatives in [6, 7, 8] is much useful since it will also be applied in Section 3 of this paper to the study of the linearized system in order to obtain the time-decay property of solutions.*

Next, as mentioned at the beginning of this section, we claim that Theorem 1.1 is indeed implied by the following proposition in terms of perturbation u. Hence, the rest of this paper is devoted to the proof of this proposition.

Proposition 2.1. *Let* $N\geq 4$ *and* $w(\xi)=(1+|\xi|^2)^{1/2}$. *Assume that* $f_0\equiv\mathbf{M}+\sqrt{\mathbf{M}}u_0\geq 0$ *and* $\|u_0\|_{H^N\cap L^2_w\cap Z_1}$ *is sufficiently small. Let* $f\equiv\mathbf{M}+\sqrt{\mathbf{M}}u\geq 0$ *be the solution obtained in*

Proposition 5.2 to the Cauchy problem (2.2), (2.3) *and* (2.4) *under the assumption* (1.4). *Then, u enjoys the estimate with algebraic decay rate in time:*

$$\|u(t)\|_{H^N} \leq C\,\|u_0\|_{H^N\cap Z_1}(1+t)^{-\frac{1}{4}}. \tag{2.26}$$

Furthermore, under the following additional conditions on u_0, u also enjoys some estimates with extra decay rates in time:

Case 1. If

$$\mathbf{P}_0 u_0 \equiv 0 \tag{2.27}$$

holds for any $x\in\mathbb{R}^3$, then one has

$$\|u(t)\|_{H^N} \leq C\,\|u_0\|_{H^N\cap Z_1}(1+t)^{-\frac{3}{4}}. \tag{2.28}$$

Case 2. Suppose that $\|u_0\|_{H^N_w\cap Z_1}$ is further sufficiently small. Then, for any $0<\epsilon\leq 3/4$, there is $\eta>0$ depending only on ϵ such that whenever

$$\|u_0\|_{H^N\cap Z_1}\leq \eta,$$

one has

$$\|\{\mathbf{I}-\mathbf{P}_1\}u(t)\|_{H^N_w}+\|\nabla_x\mathbf{P}_1 u(t)\|_{L^2_\xi(H^{N-1}_x)} \leq C\,\|u_0\|_{H^N_w\cap Z_1}(1+t)^{-\frac{3}{4}+\epsilon}. \tag{2.29}$$

Proof of Theorem 1.1: By the definition (2.1) for perturbation u, (1.6) and (1.8) are equivalent with (2.26) and (2.28), respectively, and conditions (1.5) and (1.9) coincide with those in Proposition 2.1, and the condition (1.7) is equivalent with (2.27) since by (2.12),

$$\mathbf{P}_0 u_0=\int_{\mathbb{R}^3}\sqrt{\mathbf{M}}u_0 d\xi\sqrt{\mathbf{M}}=\int_{\mathbb{R}^3}(f_0-\mathbf{M})d\xi\sqrt{\mathbf{M}}$$

holds for any $(x,\xi)\in\mathbb{R}^3\times\mathbb{R}^3$. Finally, (1.10) is implied by (2.29). This can be seen from

$$\|\{\mathbf{I}-\mathbf{P}_1\}u(t)\|_{H^N_w}\sim\|\{\mathbf{I}-\mathbf{P}\}u(t)\|_{H^N_w}+\|\mathbf{P}_0 u(t)\|_{L^2_\xi(H^N_x)},$$

and further from

$$\mathbf{P}_0 u=\langle e_0, f-\mathbf{M}\rangle e_0\sqrt{\mathbf{M}},\quad \mathbf{P}_1 u=\sum_{j=1}^{4}\langle e_j, f\rangle e_j\sqrt{\mathbf{M}},$$

and

$$\begin{aligned}\{\mathbf{I}-\mathbf{P}\}u&=u-\sum_{j=0}^{4}\langle e_j,\sqrt{\mathbf{M}}u\rangle e_j\sqrt{\mathbf{M}}\\&=\frac{f-\mathbf{M}}{\sqrt{\mathbf{M}}}-\sum_{j=0}^{4}\langle e_j, f-\mathbf{M}\rangle e_j\sqrt{\mathbf{M}}\\&=\frac{f-\sum_{j=0}^{4}\langle e_j,f\rangle e_j\mathbf{M}}{\sqrt{\mathbf{M}}}.\end{aligned}$$

Here, the orthonormal set $\{e_j\}_{j=0}^4$ is defined by (1.11). Therefore, Theorem 1.1 is proved.

Remark 2.2. *Let us explain that conditions of Proposition 2.1 make sure those of Proposition 5.2 are satisfied. Actually, this follows from the fact that*

$$\mathcal{E}(u_0) \leq C\|u_0\|^2_{H^N \cap Z_1},$$

where $\mathcal{E}(u_0)$ *is defined in* (5.3). *It suffices to verify*

$$\|\nabla_x \Delta_x^{-1} \mathbf{P}_0 u_0\|^2 \leq C\|u_0\|^2_{L^2 \cap Z_1}. \tag{2.30}$$

In fact, it follows from the definition of $\mathbf{P}_0$ *and an interpolation inequality that*

$$\|\nabla_x \Delta_x^{-1} \mathbf{P}_0 u_0\|^2 \sim \|\nabla_x \Phi_0\|^2 \leq C\|\rho_0\|^{\frac{2}{3}}_{L^2_x} \|\rho_0\|^{\frac{4}{3}}_{L^1_x},$$

where Φ_0 *and* ρ_0 *are defined by*

$$\Phi_0 = -\frac{1}{4\pi|x|} * \rho_0, \quad \rho_0 = \int_{\mathbb{R}^3} \sqrt{\mathbf{M}} u_0 d\xi.$$

Furthermore, one has

$$\|\rho_0\|_{L^2_x} \leq \|u_0\|, \quad \|\rho_0\|_{L^1_x} \leq \|u_0\|_{Z_1}.$$

Then, (2.30) *follows.*

3 The linearized system

In this section, let us consider the Cauchy problem of the linearized system with a non-homogeneous source:

$$\begin{cases} \partial_t u + \xi \cdot \nabla_x u - \nabla_x \Phi \cdot \xi \sqrt{\mathbf{M}} = \mathbf{L} u + h, \\ \Delta_x \Phi = \displaystyle\int_{\mathbb{R}^n} \sqrt{\mathbf{M}} u d\xi, \quad t > 0, x \in \mathbb{R}^n, \xi \in \mathbb{R}^n, \\ u|_{t=0} = u_0, \quad x \in \mathbb{R}^n, \end{cases} \tag{3.1}$$

where $h = h(t, x, \xi)$ and $u_0 = u_0(x, \xi)$ are given, the spatial dimension $n \geq 1$ is supposed to be arbitrary in order to see how it enters into the time-decay rate at the level of linearization, and for simplicity we still use $\mathbf{M}$ to denote the normalized n-dimensional Maxwellian

$$\mathbf{M} = \frac{1}{(2\pi)^{n/2}} e^{-|\xi|^2/2}.$$

Formally, the solution to the Cauchy problem (3.1) can be written as the mild form

$$u(t) = e^{t\mathbf{B}} u_0 + \int_0^t e^{(t-s)\mathbf{B}} h(s) ds, \tag{3.2}$$

where $e^{t\mathbf{B}}$ denotes the solution operator to the Cauchy problem of the linearized equation without source corresponding to (3.1) for $h \equiv 0$. The goal of this section is to show that

$e^{t\mathbf{B}}$ has the proposed algebraic decay properties as time tends to infinity. The idea of proofs is to make energy estimates for pointwise time t and frequency variable k which corresponds to the spatial variable x. To the end, for $1 \leq q \leq 2$ and integer m, set the index $\sigma_{q,m}$ of the time-decay rate by

$$\sigma_{q,m} = \frac{n}{2}\left(\frac{1}{q} - \frac{1}{2}\right) + \frac{m}{2}.$$

As mentioned at the end of Remark 1.2, another main result of this paper is stated in the following

Theorem 3.1. *Let $1 \leq q \leq 2$ and $n \geq 1$. Let $\mathbf{P}$ and $\mathbf{P}_0$ be defined in (3.6).*

(i) For any α, α' with $\alpha' \leq \alpha$, and for any u_0 satisfying $\partial_x^\alpha u_0 \in L^2$ and $\partial_x^{\alpha'} u_0 \in Z_q$, one has

$$\|\partial_x^\alpha e^{t\mathbf{B}} u_0\| + \|\partial_x^\alpha \nabla_x \Delta_x^{-1} \mathbf{P}_0 e^{t\mathbf{B}} u_0\| \leq C(1+t)^{-\sigma_{q,m-1}} (\|\partial_x^{\alpha'} u_0\|_{Z_q} + \|\partial_x^\alpha u_0\|), \tag{3.3}$$

and

$$\begin{aligned} &\|\partial_x^\alpha e^{t\mathbf{B}} \{\mathbf{I} - \mathbf{P}_0\} u_0\| + \|\partial_x^\alpha \nabla_x \Delta_x^{-1} \mathbf{P}_0 e^{t\mathbf{B}} \{\mathbf{I} - \mathbf{P}_0\} u_0\| \\ &\qquad \leq C(1+t)^{-\sigma_{q,m}} (\|\partial_x^{\alpha'} \{\mathbf{I} - \mathbf{P}_0\} u_0\|_{Z_q} + \|\partial_x^\alpha \{\mathbf{I} - \mathbf{P}_0\} u_0\|), \end{aligned} \tag{3.4}$$

for $t \geq 0$ with $m = |\alpha - \alpha'|$, where C is a positive constant depending only on n, m, q.

(ii) Similarly, for any α, α' with $\alpha' \leq \alpha$, and for any h satisfying $\nu(\xi)^{-1/2} \partial_x^\alpha h(t) \in L^2$ and $\nu(\xi)^{-1/2} \partial_x^{\alpha'} h(t) \in Z_q$ for $t \geq 0$, one has

$$\begin{aligned} &\left\| \partial_x^\alpha \int_0^t e^{(t-s)\mathbf{B}} \{\mathbf{I} - \mathbf{P}\} h(s) ds \right\|^2 + \left\| \partial_x^\alpha \nabla_x \Delta_x^{-1} \mathbf{P}_0 \int_0^t e^{(t-s)\mathbf{B}} \{\mathbf{I} - \mathbf{P}\} h(s) ds \right\|^2 \\ &\leq C \int_0^t (1+t-s)^{-2\sigma_{q,m}} \\ &\qquad (\|\nu^{-1/2} \partial_x^{\alpha'} \{\mathbf{I} - \mathbf{P}\} h(s)\|_{Z_q}^2 + \|\nu^{-1/2} \partial_x^\alpha \{\mathbf{I} - \mathbf{P}\} h(s)\|^2) ds, \end{aligned} \tag{3.5}$$

for $t \geq 0$ with $m = |\alpha - \alpha'|$, where C is a positive constant depending only on n, m, q.

Remark 3.1. *In the case of the Boltzmann equation [28] or the Navier-Stokes-Poisson system [17], some similar algebraic time-decay estimates in Theorem 3.1 were obtained on the basis of the spectral analysis of the solution semigroup. However, so far there has not been known results applying the direct spectral analysis as in [28] to the study of the VPB system. Here, the proof of Theorem 3.1 that we shall show can provide a robust method to get the time-decay estimates in L^2 space for not only the VPB system but also some other kinetic models such as the Boltzmann equation and Landau equation, relativistic or non-relativistic, which is a future research goal in the general framework as in [30].*

Remark 3.2. *In the case when the spatial domain is a torus $\mathbb{T}^n$, Glassey-Strauss [11] and Mouhot-Neumann [23] obtained the exponential time-decay rate*

$$\left\| e^{t\mathbf{B}} \{\mathbf{I} - \mathbf{P}\} u_0 \right\|_X \leq C e^{-\lambda t} \|u_0\|_X,$$

where $X = L^2(\mathbb{T}^n \times \mathbb{R}^n)$ in [11] and $X = H^1(\mathbb{T}^n \times \mathbb{R}^n)$ in [23]. See also [30, 5, 7] for the other kinetic models on torus. Actually, the proof of Theorem 3.1 can be modified in a simple way so that the above inequality with exponential rates is recovered. We shall further explain this point at the end of this section; see Theorem 3.2.

To prove Theorem 3.1, let $u(t)$, formally defined by (3.2), be the solution to the linearized non-homogeneous Cauchy problem (3.1), where $u_0(x,\xi)$ and $h(t,x,\xi)$ with $\mathbf{P}h \equiv 0$ are given. We decompose u as

$$\begin{cases} u(t,x,\xi) = \mathbf{P}u \oplus \{\mathbf{I}-\mathbf{P}\}u, \\ \mathbf{P}u = \mathbf{P}_0 u \oplus \mathbf{P}_1 u \equiv \{a^u + b^u\cdot\xi + c^u|\xi|^2\}\sqrt{\mathbf{M}}, \\ \mathbf{P}_0 u = (a^u + nc^u)\sqrt{\mathbf{M}}, \\ \mathbf{P}_1 u = [b^u\cdot\xi + c^u(|\xi|^2 - n)]\sqrt{\mathbf{M}}, \end{cases} \tag{3.6}$$

where a^u, b^u, c^u are the macro moment functions of u given by

$$\begin{aligned} a^u &= \frac{1}{2}\int_{\mathbb{R}^3} [(n+2) - |\xi|^2]\sqrt{\mathbf{M}} u\, d\xi, \\ b^u &= \int_{\mathbb{R}^3} \xi\sqrt{\mathbf{M}} u\, d\xi, \\ c^u &= \frac{1}{2n}\int_{\mathbb{R}^3} (|\xi|^2 - n)\sqrt{\mathbf{M}} u\, d\xi, \end{aligned}$$

which are the n dimensional generalizations of (2.8), (2.9), (2.10) when $n=3$. Notice that although a^u, b^u and c^u also depend on the spatial dimension, we used the same notations as in Section 2 for simplicity. Then, from the same procedure as in Section 2, one has the macroscopic balance laws satisfied by a^u, b^u, c^u:

$$\partial_t(a^u + nc^u) + \nabla_x\cdot b^u = 0, \tag{3.7}$$

$$\partial_t b_j^u + \partial_j(a^u + nc^u) + 2\partial_j c^u + \sum_m \partial_m A_{jm}(\{\mathbf{I}-\mathbf{P}\}u) - \partial_j\Phi = 0, \tag{3.8}$$

$$\partial_t c^u + \frac{1}{n}\nabla_x\cdot b^u + \frac{1}{2n}\sum_j \partial_j B_j(\{\mathbf{I}-\mathbf{P}\}u) = 0, \tag{3.9}$$

$$\Delta_x\Phi = a^u + nc^u, \tag{3.10}$$

and

$$\partial_t[A_{jj}(\{\mathbf{I}-\mathbf{P}\}u) + 2c^u] + 2\partial_j b_j^u = A_{jj}(R+h), \tag{3.11}$$

$$\partial_t A_{jm}(\{\mathbf{I}-\mathbf{P}\}u) + \partial_j b_m^u + \partial_m b_j^u = A_{jm}(R+h),\ j\neq m, \tag{3.12}$$

$$\partial_t B_j(\{\mathbf{I}-\mathbf{P}\}u) + \partial_j c^u = B_j(R+h), \tag{3.13}$$

for $1\le j,m\le n$. Here, the velocity moment functions $A_{jm}(\cdot)$ and $B_j(\cdot)$ are given by

$$A_{jm}(u) = \langle(\xi_j\xi_m - 1)\sqrt{\mathbf{M}}, u\rangle,\quad B_j(u) = \langle[|\xi|^2 - (n+2)]\xi_j\sqrt{\mathbf{M}}, u\rangle,$$

where these moment functions correspond to (2.21) for $n = 3$, and we again used the same notations as in Section 2 for simplicity. R has the same form with (2.18), given by

$$R = -\xi \cdot \nabla_x \{\mathbf{I} - \mathbf{P}\}u + \mathbf{L}\{\mathbf{I} - \mathbf{P}\}u. \tag{3.14}$$

Notice that the source term h does not appear in the first $n+2$ equations (3.7)-(3.8) due to $\mathbf{P}h = 0$. Furthermore, similar to derive (3.15) from (2.22) and (2.23), it follows from (3.11) and (3.12) that

$$\begin{aligned} -\partial_t \left[\sum_j \partial_j A_{jm}(\{\mathbf{I} - \mathbf{P}\}u) + \frac{1}{2}\partial_m A_{mm}(\{\mathbf{I} - \mathbf{P}\}u) \right] - \Delta_x b_m^u - \partial_m \partial_m b_m^u \\ = \frac{1}{2}\sum_{j\neq m} \partial_m A_{jj}(R+h) - \sum_j \partial_j A_{jm}(R+h). \end{aligned} \tag{3.15}$$

Lemma 3.1. *There is a free energy functional $E_{free}(\widehat{u}(t,k))$ which is local in the time and frequency and takes the form of*

$$\begin{aligned} &E_{free}(\widehat{u}(t,k)) \\ &= \kappa_1 \sum_m \left(\sum_j \frac{ik_j}{1+|k|^2} A_{jm}(\{\mathbf{I} - \mathbf{P}\}\widehat{u}) + \frac{1}{2}\frac{ik_m}{1+|k|^2} A_{mm}(\{\mathbf{I} - \mathbf{P}\}\widehat{u}) \mid -b_m^{\widehat{u}} \right) \\ &\quad + \kappa_1 \sum_j \left(B_j(\{\mathbf{I} - \mathbf{P}\}\widehat{u}) \mid \frac{ik_j}{1+|k|^2} c^{\widehat{u}} \right) \\ &\quad + \sum_m \left(b_m^{\widehat{u}} \mid \frac{ik_m}{1+|k|^2}(a^{\widehat{u}} + nc^{\widehat{u}}) \right) \end{aligned} \tag{3.16}$$

for some constant $\kappa_1 > 0$, such that one has

$$\begin{aligned} \partial_t \mathrm{Re}\, E_{free}(\widehat{u}(t,k)) + \lambda \frac{|k|^2}{1+|k|^2}\left(|b^{\widehat{u}}|^2 + |c^{\widehat{u}}|^2\right) + \lambda |a^{\widehat{u}} + nc^{\widehat{u}}|^2 \\ \leq C\left(\|\{\mathbf{I} - \mathbf{P}\}\widehat{u}\|_{L^2_\xi}^2 + \|\nu^{-1/2}\{\mathbf{I} - \mathbf{P}\}\widehat{h}\|_{L^2_\xi}^2 \right) \end{aligned} \tag{3.17}$$

for any $t \geq 0$ and $k \in \mathbb{R}^n$.

Proof. We shall make estimates on $b^{\widehat{u}}, c^{\widehat{u}}$ and $a^{\widehat{u}} + nc^{\widehat{u}}$ individually and then take the proper linear combination to deduce the desired free energy inequality (3.17). Firstly, notice that

$$\mathcal{F}a^u = a^{\mathcal{F}u}, \quad \mathcal{F}b^u = b^{\mathcal{F}u}, \quad \mathcal{F}c^u = c^{\mathcal{F}u},$$

and likewise for the high-order moment functions $A_{jm}(\cdot)$ and $B_j(\cdot)$.

Estimate on $b^{\widehat{u}}$. We claim that for any $0<\delta_1<1$, it holds that

$$\begin{aligned}
&\partial_t \mathrm{Re}\sum_m \left(\sum_j ik_j A_{jm}(\{\mathbf{I}-\mathbf{P}\}\widehat{u}) + \frac{1}{2} ik_m A_{mm}(\{\mathbf{I}-\mathbf{P}\}\widehat{u}) \mid -b_m^{\widehat{u}}\right) + (1-\delta_1)|k|^2|b^{\widehat{u}}|^2\\
&\leq \delta_1(1+|k|^2)|a^{\widehat{u}}+nc^{\widehat{u}}|^2 + \delta_1|k|^2|c^{\widehat{u}}|^2\\
&\quad + \frac{C}{\delta_1}(1+|k|^2)\left(\|\{\mathbf{I}-\mathbf{P}\}\widehat{u}\|_{L^2_\xi}^2 + \|\nu^{-1/2}\{\mathbf{I}-\mathbf{P}\}\widehat{h}\|_{L^2_\xi}^2\right).
\end{aligned} \tag{3.18}$$

In fact, the Fourier transform of (3.15) gives

$$\begin{aligned}
-\partial_t\left[\sum_j ik_j A_{jm}(\{\mathbf{I}-\mathbf{P}\}\widehat{u}) + \frac{1}{2} ik_m A_{mm}(\{\mathbf{I}-\mathbf{P}\}\widehat{u})\right] + |k|^2 b_m^{\widehat{u}} + k_m^2 b_m^{\widehat{u}}\\
= \frac{1}{2}\sum_{j\neq m} ik_m A_{jj}(\widehat{R}+\widehat{h}) - \sum_j ik_j A_{jm}(\widehat{R}+\widehat{h}).
\end{aligned}$$

Taking further the complex inner product with $b_m^{\widehat{u}}$ gives

$$\begin{aligned}
&\partial_t\left(\sum_j ik_j A_{jm}(\{\mathbf{I}-\mathbf{P}\}\widehat{u}) + \frac{1}{2} ik_m A_{mm}(\{\mathbf{I}-\mathbf{P}\}\widehat{u}) \mid -b_m^{\widehat{u}}\right) + (|k|^2+k_m^2)|b_m^{\widehat{u}}|^2\\
&= \left(\frac{1}{2}\sum_{j\neq m} ik_m A_{jj}(\widehat{R}+\widehat{h}) - \sum_j ik_j A_{jm}(\widehat{R}+\widehat{h}) \mid b_m^{\widehat{u}}\right)\\
&\quad + \left(\sum_j ik_j A_{jm}(\{\mathbf{I}-\mathbf{P}\}\widehat{u}) + \frac{1}{2} ik_m A_{mm}(\{\mathbf{I}-\mathbf{P}\}\widehat{u}) \mid -\partial_t b_m^{\widehat{u}}\right) = I_1 + I_2.
\end{aligned} \tag{3.19}$$

I_1 is bounded by

$$I_1 \leq \delta_1|k|^2|b_m^{\widehat{u}}|^2 + \frac{C}{\delta_1}\sum_{jm}(|A_{jm}(\widehat{R})|^2 + |A_{jm}(\widehat{h})|^2).$$

For I_2, one can use the Fourier transforms of (3.8) and (3.10):

$$\partial_t b_j^{\widehat{u}} + ik_j[a^{\widehat{u}} + (n+2)c^{\widehat{u}}] + \sum_m ik_m A_{jm}(\{\mathbf{I}-\mathbf{P}\}\widehat{u}) - ik_j\widehat{\Phi} = 0, \tag{3.20}$$

$$-|k|^2\widehat{\Phi} = a^{\widehat{u}} + nc^{\widehat{u}} \tag{3.21}$$

to estimate it as

$$I_2 \leq \delta_1(1+|k|^2)|a^{\widehat{u}}+nc^{\widehat{u}}|^2 + \delta_1|k|^2|c^{\widehat{u}}|^2 + \frac{C}{\delta_1}(1+|k|^2)\sum_{jm}|A_{jm}(\{\mathbf{I}-\mathbf{P}\}\widehat{u})|^2.$$

On the other hand, notice from (3.14) that

$$\widehat{R} = -\xi\cdot k\{\mathbf{I}-\mathbf{P}\}\widehat{u} + \mathbf{L}\{\mathbf{I}-\mathbf{P}\}\widehat{u}$$

which implies

$$|A_{jm}(\widehat{R})|^2 \le C(1+|k|^2)\|\{\mathbf{I}-\mathbf{P}\}\widehat{u}\|_{L^2_\xi}^2.$$

Similarly it holds that

$$|A_{jm}(\widehat{h})|^2 \le C\|\nu^{-1/2}\{\mathbf{I}-\mathbf{P}\}\widehat{h}\|_{L^2_\xi}^2, \quad |A_{jm}(\{\mathbf{I}-\mathbf{P}\}\widehat{u})|^2 \le C\|\{\mathbf{I}-\mathbf{P}\}\widehat{u}\|_{L^2_\xi}^2.$$

Thus, (3.18) follows from taking the real part of (3.19) and plugging the estimates of I_1, I_2 into it.

Estimate on $c^{\widehat{u}}$. We claim that for any $0<\delta_2<1$, it holds that

$$\begin{aligned}
&\partial_t \mathrm{Re}\sum_j \Big(B_j(\{\mathbf{I}-\mathbf{P}\}\widehat{u}) \mid ik_j c^{\widehat{u}}\Big) + (1-\delta_2)|k|^2|c^{\widehat{u}}|^2\\
&\le \delta_2|k|^2|b^{\widehat{u}}|^2 + \frac{C}{\delta_2}(1+|k|^2)\|\{\mathbf{I}-\mathbf{P}\}\widehat{u}\|_{L^2_\xi}^2 + \frac{C}{\delta_2}\|\nu^{-1/2}\{\mathbf{I}-\mathbf{P}\}\widehat{h}\|_{L^2_\xi}^2. \qquad (3.22)
\end{aligned}$$

In fact, similarly as before, from the Fourier transform of (3.13)

$$\partial_t B_j(\{\mathbf{I}-\mathbf{P}\}\widehat{u}) + ik_j c^{\widehat{u}} = B_j(\widehat{R}+\widehat{h}),$$

one can get

$$\begin{aligned}
&\partial_t \Big(B_j(\{\mathbf{I}-\mathbf{P}\}\widehat{u}) \mid ik_j c^{\widehat{u}}\Big) + |k_j|^2|c^{\widehat{u}}|^2\\
&= \Big(B_j(\widehat{R}+\widehat{h}) \mid ik_j c^{\widehat{u}}\Big) + \Big(B_j(\{\mathbf{I}-\mathbf{P}\}\widehat{u}) \mid ik_j \partial_t c^{\widehat{u}}\Big) = I_3 + I_4. \qquad (3.23)
\end{aligned}$$

I_3 is bounded by

$$I_3 \le \delta_2|k_j|^2|c^{\widehat{u}}|^2 + \frac{C}{\delta_2}\sum_j(|B_j(\widehat{R})|^2 + |B_j(\widehat{h})|^2),$$

and from the Fourier transform of (3.9)

$$\partial_t c^{\widehat{u}} + \frac{1}{n} ik\cdot b^{\widehat{u}} + \frac{1}{2n}\sum_j ik_j B_j(\{\mathbf{I}-\mathbf{P}\}\widehat{u}) = 0,$$

I_4 is bounded by

$$I_4 \le \frac{\delta_2}{n}|k|^2|b^{\widehat{u}}|^2 + \frac{C}{\delta_2}|k|^2\sum_j |B_j(\{\mathbf{I}-\mathbf{P}\}\widehat{u})|^2.$$

Notice that similar to A_{jm}, it holds that

$$\begin{aligned}
&|B_j(\widehat{R})|^2 \le C(1+|k|^2)\|\{\mathbf{I}-\mathbf{P}\}\widehat{u}\|_{L^2_\xi}^2,\\
&|B_j(\widehat{h})|^2 \le C\|\nu^{-1/2}\{\mathbf{I}-\mathbf{P}\}\widehat{h}\|_{L^2_\xi}^2, \quad |B_j(\{\mathbf{I}-\mathbf{P}\}\widehat{u})|^2 \le C\|\{\mathbf{I}-\mathbf{P}\}\widehat{u}\|_{L^2_\xi}^2.
\end{aligned}$$

Then, (3.22) follows from (3.23) by taking summation over $1 \le i \le n$, taking the real part and then applying the estimates of I_3 and I_4.

Estimate on $a^{\widehat{u}} + nc^{\widehat{u}}$. We claim that for any $0 < \delta_3 < 1$, it holds that

$$\partial_t \mathrm{Re} \sum_m \left(b_m^{\widehat{u}} \mid ik_m(a^{\widehat{u}} + nc^{\widehat{u}}) \right) + (1-\delta_3)(1+|k|^2)|a^{\widehat{u}} + nc^{\widehat{u}}|^2$$
$$\le |k|^2 |b^{\widehat{u}}|^2 + \frac{C}{\delta_3}|k|^2|c^{\widehat{u}}|^2 + \frac{C}{\delta_3}|k|^2 \|\{\mathbf{I}-\mathbf{P}\}\widehat{u}\|^2_{L^2_\xi}. \tag{3.24}$$

In fact, by taking the complex inner product with $ik_j(a^{\widehat{u}} + nc^{\widehat{u}})$ and then taking summation over $1 \le j \le n$, it follows from (3.20) that

$$\partial_t \sum_j (b_j^{\widehat{u}} \mid ik_j(a^{\widehat{u}} + nc^{\widehat{u}})) + |k|^2 |a^{\widehat{u}} + nc^{\widehat{u}}|^2 + \sum_j (-ik_j \widehat{\Phi} \mid ik_j(a^{\widehat{u}} + nc^{\widehat{u}}))$$
$$= (-2ik_j c^{\widehat{u}} \mid ik_j(a^{\widehat{u}} + nc^{\widehat{u}})) + \sum_{jm} (-ik_j A_{jm}(\{\mathbf{I}-\mathbf{P}\}\widehat{u}) \mid ik_j(a^{\widehat{u}} + nc^{\widehat{u}}))$$
$$+ \sum_j (b^{\widehat{u}} \mid ik_j \partial_t (a^{\widehat{u}} + nc^{\widehat{u}})). \tag{3.25}$$

Using (3.21), one has

$$\sum_j (-ik_j \widehat{\Phi} \mid ik_j(a^{\widehat{u}} + nc^{\widehat{u}})) = \sum_j (k_j^2 \frac{a^{\widehat{u}} + nc^{\widehat{u}}}{|k|^2} \mid a^{\widehat{u}} + nc^{\widehat{u}}) = |a^{\widehat{u}} + nc^{\widehat{u}}|^2.$$

The first two terms on the r.h.s. of (3.25) are bounded by

$$\delta_3 |k|^2 |a^{\widehat{u}} + nc^{\widehat{u}}|^2 + \frac{C}{\delta_3}|k|^2|c^{\widehat{u}}|^2 + \frac{C}{\delta_3}|k|^2 \|\{\mathbf{I}-\mathbf{P}\}\widehat{u}\|^2_{L^2_\xi},$$

while for the third term, it holds that

$$\sum_j (b_j^{\widehat{u}} \mid ik_j \partial_t (a^{\widehat{u}} + nc^{\widehat{u}})) = \sum_j (b_j^{\widehat{u}} \mid ik_j(-ik \cdot b^{\widehat{u}})) = |k \cdot b^{\widehat{u}}|^2 \le |k|^2 |b^{\widehat{u}}|^2,$$

where we used the Fourier transform of (3.7):

$$\partial_t (a^{\widehat{u}} + nc^{\widehat{u}}) + ik \cdot b^{\widehat{u}} = 0. \tag{3.26}$$

Then, putting the above estimates into (3.25) and taking the real part yields (3.24).

Therefore, (3.17) follows from the proper linear combination of (3.18), (3.22) and (3.24) by taking $0 < \delta_1, \delta_2, \delta_3 < 1$ small enough and also $\kappa_0 > 0$ large enough. This completes the proof of Lemma 3.1. □

Lemma 3.2. *It holds that*

$$\partial_t \left(\|\widehat{u}\|_{L^2_\xi} + \frac{|a^{\widehat{u}} + nc^{\widehat{u}}|^2}{|k|^2} \right) + \lambda \|\nu^{1/2}\{\mathbf{I}-\mathbf{P}\}\widehat{u}\|^2_{L^2_\xi} \le C \|\nu^{-1/2}\{\mathbf{I}-\mathbf{P}\}\widehat{h}\|^2_{L^2_\xi} \tag{3.27}$$

for any $t \ge 0$ and $k \in \mathbb{R}^n$.

Proof. The Fourier transform of $(3.1)_1$ and $(3.1)_2$ gives

$$\partial_t\widehat{u}+i\xi\cdot k\widehat{u}+i\frac{k}{|k|^2}(a^{\widehat{u}}+nc^{\widehat{u}})\cdot\xi\sqrt{\mathbf{M}}=\mathbf{L}\widehat{u}+\widehat{h}.$$

Further taking the complex inner product with $\widehat{u}$ and taking the real part yield

$$\begin{aligned}\frac{1}{2}\partial_t\|\widehat{u}\|_{L^2_\xi}^2&+\mathrm{Re}\int_{\mathbb{R}^n}\left(i\frac{k}{|k|^2}(a^{\widehat{u}}+nc^{\widehat{u}})\cdot\xi\sqrt{\mathbf{M}}\mid\widehat{u}\right)d\xi\\&=\mathrm{Re}\int_{\mathbb{R}^n}(\mathbf{L}\widehat{u}\mid\widehat{u})d\xi+\mathrm{Re}\int_{\mathbb{R}^n}(\widehat{h}\mid\widehat{u})d\xi.\end{aligned}\tag{3.28}$$

For the second term on the l.h.s. of (3.28), from (3.26), one has

$$\begin{aligned}\mathrm{Re}\int_{\mathbb{R}^n}\left(i\frac{k}{|k|^2}(a^{\widehat{u}}+nc^{\widehat{u}})\cdot\xi\sqrt{\mathbf{M}}\mid\widehat{u}\right)d\xi&=\mathrm{Re}\int_{\mathbb{R}^n}\left(\frac{1}{|k|^2}(a^{\widehat{u}}+nc^{\widehat{u}})\mid -ik\cdot b^{\widehat{u}}\right)d\xi\\&=\mathrm{Re}\int_{\mathbb{R}^n}\left(\frac{1}{|k|^2}(a^{\widehat{u}}+nc^{\widehat{u}})\mid\partial_t(a^{\widehat{u}}+nc^{\widehat{u}})\right)d\xi\\&=\frac{1}{2|k|^2}\partial_t|a^{\widehat{u}}+nc^{\widehat{u}}|^2.\end{aligned}$$

For two terms on the r.h.s. of (3.28), one has

$$\mathrm{Re}\int_{\mathbb{R}^n}(\mathbf{L}\widehat{u}\mid\widehat{u})d\xi\leq-\lambda\|\nu^{1/2}\{\mathbf{I}-\mathbf{P}\}\widehat{u}\|_{L^2_\xi}^2$$

and

$$\begin{aligned}\mathrm{Re}\int_{\mathbb{R}^n}(\widehat{h}\mid\widehat{u})d\xi&=\mathrm{Re}\int_{\mathbb{R}^n}(\{\mathbf{I}-\mathbf{P}\}\widehat{h}\mid\{\mathbf{I}-\mathbf{P}\}\widehat{u})d\xi\\&\leq\frac{\lambda}{2}\|\nu^{1/2}\{\mathbf{I}-\mathbf{P}\}\widehat{u}\|_{L^2_\xi}^2+C\|\nu^{-1/2}\{\mathbf{I}-\mathbf{P}\}\widehat{h}\|_{L^2_\xi}^2,\end{aligned}$$

where $\mathbf{P}h\equiv 0$ was used. Plugging the above estimates into (3.28) gives (3.27). This completes the proof of Lemma 3.2. □

Proof of Theorem 3.1: Let $\kappa_2>0$ be a small constant to be determined later. Define

$$E(\widehat{u}(t,k))=\|\widehat{u}(t,k)\|_{L^2_\xi}^2+\frac{1}{|k|^2}|a^{\widehat{u}}+nc^{\widehat{u}}|^2+\kappa_2\mathrm{Re}\,E_{free}(\widehat{u}(t,k))$$

for $t\geq 0$ and $k\in\mathbb{R}^n$, where $E_{free}(\widehat{u}(t,k))$ is given by (3.16). Notice from (3.16) that

$$\begin{aligned}|E_{free}(\widehat{u}(t,k))|&\leq C(|b^{\widehat{u}}|^2+|c^{\widehat{u}}|^2+|a^{\widehat{u}}+nc^{\widehat{u}}|^2)\\&\quad+\sum_{ij}(|A_{ij}(\{\mathbf{I}-\mathbf{P}\}\widehat{u})|^2+|B_i(\{\mathbf{I}-\mathbf{P}\}\widehat{u})|^2)\\&\leq C(\|\mathbf{P}\widehat{u}\|_{L^2_\xi}^2+\|\{\mathbf{I}-\mathbf{P}\}\widehat{u}\|_{L^2_\xi}^2)\\&\leq C\|\widehat{u}(t,k)\|_{L^2_\xi}^2.\end{aligned}$$

Therefore, one can choose $\kappa_2 > 0$ small enough such that

$$E(\widehat{u}(t,k)) \sim \|\widehat{u}(t,k)\|_{L^2_\xi}^2 + \frac{1}{|k|^2}|a^{\widehat{u}} + nc^{\widehat{u}}|^2 \tag{3.29}$$

holds. By further letting $\kappa_2 > 0$ be small enough, the linear combination of (3.27) and (3.17) implies

$$\begin{aligned}\partial_t E(\widehat{u}(t,k)) + \lambda\|\nu^{1/2}\{\mathbf{I}-\mathbf{P}\}\widehat{u}\|_{L^2_\xi}^2 + \lambda\frac{|k|^2}{1+|k|^2}\left(|b^{\widehat{u}}|^2 + |c^{\widehat{u}}|^2\right)&\\ +\lambda|a^{\widehat{u}} + nc^{\widehat{u}}|^2 \leq C\|\nu^{-1/2}\{\mathbf{I}-\mathbf{P}\}\widehat{h}\|_{L^2_\xi}^2.&\end{aligned}$$

Notice that

$$\begin{aligned}&\|\nu^{1/2}\{\mathbf{I}-\mathbf{P}\}\widehat{u}\|_{L^2_\xi}^2 + \frac{|k|^2}{1+|k|^2}\left(|b^{\widehat{u}}|^2 + |c^{\widehat{u}}|^2\right) + |a^{\widehat{u}} + nc^{\widehat{u}}|^2\\ &\geq \lambda\frac{|k|^2}{1+|k|^2}\left(\|\{\mathbf{I}-\mathbf{P}\}\widehat{u}\|_{L^2_\xi}^2 + |a^{\widehat{u}} + nc^{\widehat{u}}|^2 + |b^{\widehat{u}}|^2 + |c^{\widehat{u}}|^2 + \frac{1}{|k|^2}|a^{\widehat{u}} + nc^{\widehat{u}}|^2\right)\\ &\geq \lambda\frac{|k|^2}{1+|k|^2}\left(\|\{\mathbf{I}-\mathbf{P}\}\widehat{u}\|_{L^2_\xi}^2 + \|\mathbf{P}\widehat{u}\|_{L^2_\xi}^2 + \frac{1}{|k|^2}|a^{\widehat{u}} + nc^{\widehat{u}}|^2\right)\\ &\geq \lambda\frac{|k|^2}{1+|k|^2}\left(\|\widehat{u}\|_{L^2_\xi}^2 + \frac{1}{|k|^2}|a^{\widehat{u}} + nc^{\widehat{u}}|^2\right) \geq \lambda\frac{|k|^2}{1+|k|^2}E(\widehat{u}(t,k)).\end{aligned}$$

Then, it follows that

$$\partial_t E(\widehat{u}(t,k)) + \frac{\lambda|k|^2}{1+|k|^2}E(\widehat{u}(t,k)) \leq C\|\nu^{-1/2}\{\mathbf{I}-\mathbf{P}\}\widehat{h}\|_{L^2_\xi}^2, \tag{3.30}$$

which by the Gronwall inequality, implies

$$\begin{aligned}E(\widehat{u}(t,k)) \leq E(\widehat{u}(0,k))e^{-\frac{\lambda|k|^2}{1+|k|^2}t}&\\ +C\int_0^t e^{-\frac{\lambda|k|^2}{1+|k|^2}(t-s)}\|\nu^{-1/2}\{\mathbf{I}-\mathbf{P}\}\widehat{h}(s,k)\|_{L^2_\xi}^2 ds&\end{aligned} \tag{3.31}$$

for any $t \geq 0$ and $k \in \mathbb{R}^n$.

Now, to prove (3.3) and (3.4), let $h = 0$ so that $u(t) = e^{t\mathbf{B}}u_0$ is the solution to the Cauchy problem (3.1) and hence satisfies the estimate (3.31) with $h = 0$. Write $k^\alpha = k_1^{\alpha_1}k_2^{\alpha_2}\cdots k_n^{\alpha_n}$. By noticing

$$\begin{aligned}&\|\partial_x^\alpha e^{t\mathbf{B}}u_0\|^2 + \|\partial_x^\alpha \nabla_x \Delta_x^{-1}\mathbf{P}_0 e^{t\mathbf{B}}u_0\|^2\\ &= \int_{\mathbb{R}^n_k} |k^{2\alpha}| \cdot \|\widehat{u}(t,k)\|_{L^2_\xi}^2 dk + \int_{\mathbb{R}^n_k} |k^{2\alpha}| \cdot \frac{1}{|k|^2}|a^{\widehat{u}} + nc^{\widehat{u}}|^2 dk\\ &\leq C\int_{\mathbb{R}^n_k} |k^{2\alpha}|\,|E(\widehat{u}(t,k))|\,dk,\end{aligned} \tag{3.32}$$

then, from (3.31) with $h = 0$ and (3.29), one has

$$
\begin{aligned}
&\|\partial_x^\alpha e^{t\mathbf{B}}u_0\|^2 + \|\partial_x^\alpha \nabla_x \Delta_x^{-1}\mathbf{P}_0 e^{t\mathbf{B}}u_0\|^2 \\
&\leq C\int_{\mathbb{R}^n_k} |k^{2\alpha}| e^{-\frac{\lambda|k|^2}{1+|k|^2}t} \|\widehat{u_0}(k)\|^2_{L^2_\xi} dk + C\int_{\mathbb{R}^n_k} \frac{|k^{2\alpha}|}{|k|^2} e^{-\frac{\lambda|k|^2}{1+|k|^2}t} \|\widehat{\mathbf{P}_0 u_0}(k)\|^2_{L^2_\xi} dk. \quad (3.33)
\end{aligned}
$$

As in [16], one can further estimate the first term on the r.h.s. of (3.33) by

$$
\begin{aligned}
&\int_{\mathbb{R}^n_k} |k^{2\alpha}| e^{-\frac{\lambda|k|^2}{1+|k|^2}t} \|\widehat{u_0}(k)\|^2_{L^2_\xi} dk \\
&\leq \int_{|k|\leq 1} |k^{2(\alpha-\alpha')}| e^{-\frac{\lambda|k|^2}{1+|k|^2}t} |k^{2\alpha'}| \cdot \|\widehat{u_0}(k)\|^2_{L^2_\xi} dk + \int_{|k|\geq 1} e^{-\frac{\lambda}{2}t} |k^{2\alpha}| \cdot \|\widehat{u_0}(k)\|^2_{L^2_\xi} dk \\
&\leq C(1+t)^{-\frac{n}{q}+\frac{n-2|\alpha-\alpha'|}{2}} \|\partial_x^{\alpha'} u_0\|^2_{Z_q} + Ce^{-\frac{\lambda}{2}t}\|\partial_x^\alpha u_0\|^2, \quad (3.34)
\end{aligned}
$$

where the Hölder and Hausdorff-Young inequalities were used, and similarly for the second term on the r.h.s. of (3.33), it holds that

$$
\begin{aligned}
&\int_{\mathbb{R}^n_k} \frac{|k^{2\alpha}|}{|k|^2} e^{-\frac{\lambda|k|^2}{1+|k|^2}t} \|\widehat{\mathbf{P}_0 u_0}(k)\|^2_{L^2_\xi} dk \leq \int_{\mathbb{R}^n_k} \frac{|k^{2\alpha}|}{|k|^2} e^{-\frac{\lambda|k|^2}{1+|k|^2}t} \|\widehat{u_0}(k)\|^2_{L^2_\xi} dk \\
&\leq C(1+t)^{-\frac{n}{q}+\frac{n-2(|\alpha-\alpha'|-1)}{2}} \|\partial_x^{\alpha'} u_0\|^2_{Z_q} + Ce^{-\frac{\lambda}{2}t}\|\partial_x^\alpha u_0\|^2. \quad (3.35)
\end{aligned}
$$

Thus, (3.3) follows from (3.33) together with (3.34) and (3.35). Moreover, to prove (3.4), notice $\mathbf{P}_0\{\mathbf{I}-\mathbf{P}_0\}u_0 = 0$ and hence (3.4) similarly follows only from (3.33) and (3.34) since the second term on the r.h.s. of (3.33) vanishes.

Finally, to prove (3.5), let $u_0 = 0$ so that

$$
u(t) = \int_0^t e^{(t-s)\mathbf{B}}\{\mathbf{I}-\mathbf{P}\}h(s)ds
$$

is the solution to the Cauchy problem (3.1) and hence satisfies the estimate (3.31) with $u_0 = 0$. Then, similar to (3.32), one has

$$
\begin{aligned}
&\left\|\partial_x^\alpha \int_0^t e^{(t-s)\mathbf{B}}\{\mathbf{I}-\mathbf{P}\}h(s)ds\right\|^2 + \left\|\partial_x^\alpha \nabla_x\Delta_x^{-1}\mathbf{P}_0 \int_0^t e^{(t-s)\mathbf{B}}\{\mathbf{I}-\mathbf{P}\}h(s)ds\right\|^2 \\
&\leq C\int_0^t \int_{\mathbb{R}^n_k} |k^{2\alpha}| e^{-\frac{\lambda|k|^2}{1+|k|^2}(t-s)} \|\nu^{-1/2}\{\mathbf{I}-\mathbf{P}\}\widehat{h}(s,k)\|^2_{L^2_\xi} dkds.
\end{aligned}
$$

Therefore, (3.5) follows in the same way as in (3.34). This completes the proof of Theorem 3.1.

We conclude this section by extending Theorem 3.1 to the exponential time-decay rate in the case of $\mathbb{T}^n$ as mentioned in Remark 3.2.

Theorem 3.2. *Suppose*

$$\int_{\mathbb{T}^n} \mathbf{P}u_0 dx = 0 \tag{3.36}$$

for any $\xi \in \mathbb{R}^n$. *Let* $e^{t\mathbf{B}}u_0$ *be the solution to the Cauchy problem*

$$\begin{cases} \partial_t u + \xi \cdot \nabla_x u - \nabla_x \Phi \cdot \xi\sqrt{\mathbf{M}} = \mathbf{L}u, \\ \Delta_x \Phi = \int_{\mathbb{T}^n} \sqrt{\mathbf{M}} u d\xi, \ \int_{\mathbb{T}^n} \Phi dx = 0, \quad t > 0, x \in \mathbb{T}^n, \xi \in \mathbb{R}^n, \\ u|_{t=0} = u_0, \quad x \in \mathbb{T}^n. \end{cases} \tag{3.37}$$

Then, there are constants $C > 0$, $\lambda > 0$ *such that*

$$\|e^{t\mathbf{B}}u_0\|_{L^2(\mathbb{T}^n \times \mathbb{R}^n)} \leq Ce^{-\lambda t}\|u_0\|_{L^2(\mathbb{T}^n \times \mathbb{R}^n)}, \tag{3.38}$$

for any $t \geq 0$.

Proof. In fact, on the one hand, (3.17) and (3.27) with $h = 0$ in Lemma 3.1 and Lemma 3.2 still hold for any $t \geq 0$ and $k \in \mathbb{Z}^n$ for the solution $u(t) = e^{t\mathbf{B}}u_0$ to the Cauchy problem (3.37) in the torus case. On the other hand, from (3.7)-(3.9), one has the total conservation laws

$$\int_{\mathbb{T}^n} (a^u, b^u, c^u) dx \bigg|_{t>0} = \int_{\mathbb{T}^n} (a^{u_0}, b^{u_0}, c^{u_0}) dx,$$

which together with the assumption (3.36) imply

$$\int_{\mathbb{T}^n} (a^u, b^u, c^u) dx \bigg|_{t \geq 0} \equiv 0.$$

Thus, it follows that

$$(a^{\widehat{u}}, b^{\widehat{u}}, c^{\widehat{u}}) \Big|_{t \geq 0, k=0} \equiv 0,$$

which yields

$$\frac{2|k|^2}{1+|k|^2}(|b^{\widehat{u}}|^2 + |c^{\widehat{u}}|^2 + |a^{\widehat{u}} + nc^{\widehat{u}}|^2) \geq |b^{\widehat{u}}|^2 + |c^{\widehat{u}}|^2 + |a^{\widehat{u}} + nc^{\widehat{u}}|^2,$$

for any $t \geq 0$ and $k \in \mathbb{Z}^n$. Therefore, similar to get (3.30), it holds that

$$\partial_t E(\widehat{u}(t,k)) + \lambda E(\widehat{u}(t,k)) \leq 0,$$

where $E(\widehat{u}(t,k))$ is still given by (3.29). Then,

$$E(\widehat{u}(t,k)) \leq e^{-\lambda t} E(\widehat{u}(0,k)). \tag{3.39}$$

holds for any $t \geq 0$ and $k \in \mathbb{Z}^n$. Notice that in the torus case, one has

$$\begin{aligned} \int_{\mathbb{Z}^n} E(\widehat{u}(t,k)) dk &\sim \int_{\mathbb{Z}^n} \|\widehat{u}(t,k)\|_{L^2_\xi}^2 dk + \int_{\mathbb{Z}^n} \frac{1}{|k|^2} |a^{\widehat{u}} + nc^{\widehat{u}}|^2 dk \\ &\sim \int_{\mathbb{Z}^n} \|\widehat{u}(t,k)\|_{L^2_\xi}^2 dk + \int_{\mathbb{Z}^n} |a^{\widehat{u}} + nc^{\widehat{u}}|^2 dk \\ &\sim \int_{\mathbb{Z}^n} \|\widehat{u}(t,k)\|_{L^2_\xi}^2 dk = \|u(t)\|_{L^2(\mathbb{T}^n \times \mathbb{R}^n)}^2. \end{aligned}$$

Therefore, the integration of (3.39) over $k \in \mathbb{Z}^n$ gives (3.38). This completes the proof of Theorem 3.2. $\square$

4 The nonlinear system

4.1 Energy estimates

From now on, we devote ourselves to the proof of Proposition 2.1. For that, assume that all conditions of Proposition 2.1 hold, particularly

$$\|u_0\|_{H^N \cap L^2_w \cap Z_1}$$

is supposed to be sufficiently small throughout this section. Let u be the solution to the Cauchy problem on the nonlinear VPB system (2.2), (2.3) and (2.4) obtained by Proposition 5.2. Here, notice from Remark 2.2 that the solution u indeed exists under the assumptions of Proposition 2.1. Therefore, from (5.5),

$$\mathcal{E}(u(t)) + \lambda \int_0^t \mathcal{D}(u(s))ds \le \mathcal{E}(u_0) \tag{4.1}$$

holds for any $t \ge 0$, where $\mathcal{E}(u(t))$ and $\mathcal{D}(u(t))$ are defined by (5.3) and (5.4), respectively. Thus, one can suppose that the energy functional $\mathcal{E}(u(t))$ is small enough uniformly in time. We also remark that those uniform *a priori* estimates given in Lemma 5.1 will be used later on in the proof.

In this subsection, for some preparations, we are concerned with energy estimates on the microscopic part $\{\mathbf{I}-\mathbf{P}\}u$ to obtain some Lyapunov-type inequalities. Recall from [8] that $\{\mathbf{I}-\mathbf{P}\}u$ satisfies

$$\begin{aligned}
&\partial_t\{\mathbf{I}-\mathbf{P}\}u + \xi\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}u + \nabla_x\Phi\cdot\nabla_\xi\{\mathbf{I}-\mathbf{P}\}u\\
=\ &\mathbf{L}\{\mathbf{I}-\mathbf{P}\}u + \Gamma(u,u) + \frac{1}{2}\xi\cdot\nabla_x\Phi\{\mathbf{I}-\mathbf{P}\}u\\
&-\{\mathbf{I}-\mathbf{P}\}(\xi\cdot\nabla_x\mathbf{P}u + \nabla_x\Phi\cdot\nabla_\xi\mathbf{P}u - \frac{1}{2}\xi\cdot\nabla_x\Phi\mathbf{P}u)\\
&-\mathbf{P}(\xi\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}u + \nabla_x\Phi\cdot\nabla_\xi\{\mathbf{I}-\mathbf{P}\}u - \frac{1}{2}\xi\cdot\nabla_x\Phi\{\mathbf{I}-\mathbf{P}\}u),
\end{aligned} \tag{4.2}$$

and also the following lemma was proved in [29, 10].

Lemma 4.1. *It holds that*

$$\|\nu^{-\gamma}\Gamma(u,v)\|_{L^2_\xi} \le C(\|\nu^{1-\gamma}u\|_{L^2_\xi}\|v\|_{L^2_\xi} + \|u\|_{L^2_\xi}\|\nu^{1-\gamma}v\|_{L^2_\xi}),$$

for $0 \le \gamma \le 1$, *and*

$$\begin{aligned}
\|Ku\|_{H^N} &\le C\|u\|_{H^N},\\
\|\Gamma(u,v)\|_{H^N} &\le C(\|u\|_{H^N_\nu}\|v\|_{H^N} + \|u\|_{H^N}\|v\|_{H^N_\nu}).
\end{aligned}$$

Firstly, corresponding to (5.6), it is straightforward to prove

Lemma 4.2. *It holds that*

$$\frac{d}{dt}\|\{\mathbf{I}-\mathbf{P}\}u\|^2+\lambda\|w^{1/2}\{\mathbf{I}-\mathbf{P}\}u\|^2\leq C\|\nabla_x\mathbf{P}u\|^2+C\mathcal{E}(u(t))\mathcal{D}(u(t)),\tag{4.3}$$

for any $t\geq 0$.

Proof. The direct zero-order energy estimate on (4.2) gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\{\mathbf{I}-\mathbf{P}\}u\|^2+\lambda\|\nu^{1/2}\{\mathbf{I}-\mathbf{P}\}u\|^2\\&\leq\int_{\mathbb{R}^3}\langle\Gamma(u,u),\{\mathbf{I}-\mathbf{P}\}u\rangle dx+\int_{\mathbb{R}^3}\langle\frac{1}{2}\xi\cdot\nabla_x\Phi,(\{\mathbf{I}-\mathbf{P}\}u)^2\rangle dx\\&\quad-\int_{\mathbb{R}^3}\langle\{\mathbf{I}-\mathbf{P}\}(\xi\cdot\nabla_x\mathbf{P}u+\nabla_x\Phi\cdot\nabla_\xi\mathbf{P}u-\frac{1}{2}\xi\cdot\nabla_x\Phi\mathbf{P}u),\{\mathbf{I}-\mathbf{P}\}u\rangle dx,\end{aligned}$$

which further implies

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\{\mathbf{I}-\mathbf{P}\}u\|^2+\lambda\|\nu^{1/2}\{\mathbf{I}-\mathbf{P}\}u\|^2\\&\leq C\sqrt{\mathcal{E}(u(t))\mathcal{D}(u(t))}\|\nu^{1/2}\{\mathbf{I}-\mathbf{P}\}u\|+C\sqrt{\mathcal{E}(u(t))}\|\nu^{1/2}\{\mathbf{I}-\mathbf{P}\}u\|^2\\&\quad+C\|\nabla_x\mathbf{P}u\|\|\nu^{1/2}\{\mathbf{I}-\mathbf{P}\}u\|,\end{aligned}$$

where Lemma 4.1 and Sobolev inequality were used. Then, (4.3) follows from the Cauchy's inequality, smallness of $\mathcal{E}(u(t))$ and the equivalence $w(\xi)\sim\nu(\xi)$. This completes the proof of Lemma 4.2. $\square$

Next, we consider the weighted energy estimates on $\{\mathbf{I}-\mathbf{P}\}u$, whose aim is to obtain the weighted high-order Lyapunov inequalities later.

Lemma 4.3. *It holds that*

$$\frac{d}{dt}\|w^{1/2}\{\mathbf{I}-\mathbf{P}\}u\|^2+\lambda\|w\{\mathbf{I}-\mathbf{P}\}u\|^2\leq C[1+\mathcal{E}(u(t))]\mathcal{D}(u(t)).\tag{4.4}$$

Furthermore

$$\begin{aligned}&\frac{d}{dt}\sum_{1\leq|\alpha|\leq N}\|w^{1/2}\partial_x^\alpha u\|^2+\lambda\sum_{1\leq|\alpha|\leq N}\|w\partial_x^\alpha u\|^2\\&\leq C\mathcal{D}(u(t))+C\mathcal{E}(u(t))\sum_{|\alpha|+|\beta|\leq N}\|w\partial_x^\alpha\partial_\xi^\beta\{\mathbf{I}-\mathbf{P}\}u\|^2,\end{aligned}\tag{4.5}$$

and

$$\begin{aligned}&\frac{d}{dt}\sum_{\substack{|\beta|\geq1\\|\alpha|+|\beta|\leq N}}\|w^{1/2}\partial_x^\alpha\partial_\xi^\beta\{\mathbf{I}-\mathbf{P}\}u\|^2+\lambda\sum_{\substack{|\beta|\geq1\\|\alpha|+|\beta|\leq N}}\|w\partial_x^\alpha\partial_\xi^\beta\{\mathbf{I}-\mathbf{P}\}u\|^2\\&\leq C\mathcal{D}(u(t))+C\mathcal{E}(u(t))\sum_{|\alpha|+|\beta|\leq N}\|w\partial_x^\alpha\partial_\xi^\beta\{\mathbf{I}-\mathbf{P}\}u\|^2,\end{aligned}\tag{4.6}$$

for any $t\geq 0$.

Proof. The proof of (4.4) is similar to that of (4.3) and hence is omitted. To prove (4.5), let $1 \le |\alpha| \le N$ and then it follows from (2.2) that

$$\begin{aligned} &\partial_t(\partial_x^\alpha u) + \xi \cdot \nabla_x(\partial_x^\alpha u) + \nabla_x\Phi \cdot \nabla_\xi(\partial_x^\alpha u) + \nu \partial_x^\alpha u \\ &= K\partial_x^\alpha u + \partial_x^\alpha \nabla_x \Phi \cdot \xi\sqrt{\mathbf{M}} + \partial_x^\alpha \Gamma(u,u) - [\partial_x^\alpha, \nabla_x\Phi \cdot \nabla_\xi]u, \end{aligned} \tag{4.7}$$

where the last term on the r.h.s. denotes the commutator. Then, (4.5) follows from multiplying the above equation by $w\partial_x^\alpha u$, taking integration over $\mathbb{R}^3 \times \mathbb{R}^3$ and then using integration by parts, Cauchy's inequality, Lemma 4.1 and Sobolev inequality.

Finally, we prove (4.6). Fix α, β with $|\alpha| + |\beta| \le N$ and $|\beta| \ge 1$. For simplicity, write $z = \partial_x^\alpha \partial_\xi^\beta \{\mathbf{I} - \mathbf{P}\}u$. Then, after applying $\partial_x^\alpha \partial_\xi^\beta$ to (4.2), z satisfies

$$\partial_t z + \xi \cdot \nabla_x z + \nabla_x \Phi \cdot \nabla_\xi z + \nu z = I, \tag{4.8}$$

where I is denoted by

$$I = I_1 + I_2 + I_3$$

with

$$\begin{aligned} I_1 &= \partial_\xi^\beta K \partial_x^\alpha \{\mathbf{I} - \mathbf{P}\}u + \partial_x^\alpha \partial_\xi^\beta \Gamma(u,u) + \frac{1}{2}\partial_x^\alpha \partial_\xi^\beta(\xi \cdot \nabla_x \Phi \{\mathbf{I} - \mathbf{P}\}u), \\ I_2 &= -\partial_x^\alpha \partial_\xi^\beta \{\mathbf{I} - \mathbf{P}\}(\xi \cdot \nabla_x \mathbf{P}u + \nabla_x\Phi \cdot \nabla_\xi \mathbf{P}u - \frac{1}{2}\xi \cdot \nabla_x \Phi \mathbf{P}u) \\ &\quad -\partial_x^\alpha \partial_\xi^\beta \mathbf{P}(\xi \cdot \nabla_x \{\mathbf{I} - \mathbf{P}\}u + \nabla_x \Phi \cdot \nabla_\xi \{\mathbf{I} - \mathbf{P}\}u - \frac{1}{2}\xi \cdot \nabla_x \Phi \{\mathbf{I} - \mathbf{P}\}u), \\ I_3 &= -[\partial_\xi^\beta, \xi \cdot \nabla_x]\partial_x^\alpha \{\mathbf{I} - \mathbf{P}\}u - [\partial_\xi^\beta, \nu(\xi)]\partial_x^\alpha \{\mathbf{I} - \mathbf{P}\}u \\ &\quad -[\partial_x^\alpha, \nabla_x \Phi \cdot \nabla_\xi]\partial_\xi^\beta \{\mathbf{I} - \mathbf{P}\}u. \end{aligned}$$

Here, notice that I_2 contains the macroscopic projection $\mathbf{P}$ which can absorb both the weight and derivative of velocity variable, and I_3 contains all the commutators in which the total order of differentiation of $\{\mathbf{I} - \mathbf{P}\}u$ is no more than N. Then, similarly for (4.7), the direct energy estimate of (4.8) yields (4.6). This also completes the proof of Lemma 4.3.

□

4.2 Time decay of energy

In this subsection, we shall prove (2.26) and (2.28) in Proposition 2.1. For this, define two temporal functions by

$$\mathcal{E}_\infty^{(m)}(t) = \sup_{0 \le s \le t} (1+s)^{\frac{1}{2}+m}\mathcal{E}(u(s)), \quad m = 0, 1, \tag{4.9}$$

where $\mathcal{E}(u(t))$ is defined by (5.3). For simplicity, also define two constants depending only on initial data by

$$\epsilon_0 = \|u_0\|_{H^N}^2 + \|u_0\|_{Z_1}^2, \quad \epsilon_{0,\nu} = \|u_0\|_{H^N}^2 + \|u_0\|_{Z_1}^2 + \|\nu^{1/2}u_0\|^2. \tag{4.10}$$

Proof of (2.26) **and** (2.28) **in Proposition 2.1:** We begin with the proof of (2.26). Firstly recall that from (5.5), one has the energy inequality

$$\frac{d}{dt}\mathcal{E}(u(t)) + \lambda\mathcal{D}(u(t)) \leq 0, \tag{4.11}$$

where $\mathcal{D}(u(t))$ is defined by (5.4). By comparing (5.3) with (5.4), it holds that

$$\mathcal{D}(u) + \|\mathbf{P}_1 u\|^2 + \|\nabla_x \Delta_x^{-1}\mathbf{P}_0 u\|^2 \geq \lambda\mathcal{E}(u).$$

Then, it follows from (4.11) that

$$\frac{d}{dt}\mathcal{E}(u(t)) + \lambda\mathcal{E}(u(t)) \leq C\|\mathbf{P}_1 u(t)\|^2 + C\|\nabla_x\Delta_x^{-1}\mathbf{P}_0 u(t)\|^2. \tag{4.12}$$

Also recall that the solution u to the Cauchy problem (2.2)-(2.4) of the nonlinear VPB system can be written as the mild form

$$u(t) = e^{\mathbf{B}t}u_0 + \int_0^t e^{\mathbf{B}(t-s)}G(s)ds, \tag{4.13}$$

where the source term G given by (2.19) is rewritten as

$$G = G_1 + G_2$$

with

$$G_1 = \Gamma(u,u), \quad G_2 = -\nabla_x\Phi\cdot\nabla_\xi u + \frac{1}{2}\xi\cdot\nabla_x\Phi u. \tag{4.14}$$

By checking $\mathbf{P}G_1 \equiv 0$ and $\mathbf{P}_0 G_2 \equiv 0$, one can decompose G as

$$G = \{\mathbf{I}-\mathbf{P}\}G_1 + \{\mathbf{I}-\mathbf{P}\}G_2 + \mathbf{P}_1 G_2. \tag{4.15}$$

By applying Theorem 3.1 with the spatial dimension $n = 3$ to (4.13) and using the decomposition (4.15), one has

$$\begin{aligned}
&\|\mathbf{P}_1 u(t)\|^2 + \|\nabla_x\Delta_x^{-1}\mathbf{P}_0 u(t)\|^2\\
&\leq C(1+t)^{-\frac{1}{2}}(\|u_0\|_{Z_1}^2 + \|u_0\|^2)\\
&\quad + \sum_{j=1}^2 C\int_0^t (1+t-s)^{-\frac{3}{2}}(\|\nu^{-1/2}\{\mathbf{I}-\mathbf{P}\}G_j(s)\|_{Z_1}^2 + \|\nu^{-1/2}\{\mathbf{I}-\mathbf{P}\}G_j(s)\|^2)ds\\
&\quad + C\left[\int_0^t (1+t-s)^{-\frac{3}{4}}(\|\mathbf{P}_1 G_2(s)\|_{Z_1} + \|\mathbf{P}_1 G_2(s)\|)ds\right]^2,
\end{aligned} \tag{4.16}$$

where the first term on the r.h.s. follows from (3.3), the second term from (3.5) and the third term from (3.4). One can further compute those terms of G_1, G_2 in (4.16) by

$$\begin{aligned}
&\|\nu^{-1/2}\{\mathbf{I}-\mathbf{P}\}G_1\|_{Z_1}^2 + \|\nu^{-1/2}\{\mathbf{I}-\mathbf{P}\}G_1\|^2\\
&= \|\nu^{-1/2}\Gamma(u,u)\|_{Z_1}^2 + \|\nu^{-1/2}\Gamma(u,u)\|^2\\
&\leq C\|\nu^{1/2}u\|^2\|u\|^2 + C\|\nu^{1/2}u\|^2 \sup_x \|u\|_{L^2_\xi}^2\\
&\leq C\|\nu^{1/2}u_0\|^2\mathcal{E}(u)
\end{aligned}$$

from Lemma 4.1, and

$$\begin{aligned}&\|\nu^{-1/2}\{\mathbf{I}-\mathbf{P}\}G_2\|_{Z_1}^2+\|\nu^{-1/2}\{\mathbf{I}-\mathbf{P}\}G_2\|^2\\&\leq C\|\nabla_x\Phi\|_{L_x^2\cap L_x^\infty}^2(\|\nabla_\xi u\|^2+\|\nu^{1/2}u\|^2)\\&\leq C(\mathcal{E}(u_0)+\|\nu^{1/2}u_0\|^2)\mathcal{E}(u),\end{aligned}$$

and

$$\|\mathbf{P}_1G_2\|_{Z_1}+\|\mathbf{P}_1G_2\|\leq C\|\nabla_x\Phi\|_{L_x^2\cap L_x^\infty}\|u\|\leq C\mathcal{E}(u).$$

Here and hereafter, one can use the uniform bounds of $\mathcal{E}(u(t))$ and $\|\nu^{1/2}u(t)\|$ due to

$$\begin{aligned}\sup_{t\geq 0}\mathcal{E}(u(t))\leq\mathcal{E}(u_0)\leq C\epsilon_0,\\ \sup_{t\geq 0}\|\nu^{1/2}u(t)\|^2\leq C\|\nu^{1/2}u_0\|^2+\mathcal{E}(u_0)\leq C\epsilon_{0,\nu},\end{aligned}$$

where the time integration of (4.11) and (4.4) was used. Plugging the above inequalities into (4.16), it follows that

$$\begin{aligned}&\|\mathbf{P}_1u(t)\|^2+\|\nabla_x\Delta_x^{-1}\mathbf{P}_0u(t)\|^2\\&\leq C(1+t)^{-\frac{1}{2}}\epsilon_0+C\epsilon_{0,\nu}\int_0^t(1+t-s)^{-\frac{3}{2}}\mathcal{E}(u(s))ds\\&\quad+C\int_0^t\left[(1+t-s)^{-\frac{3}{4}}\mathcal{E}(u(s))ds\right]^2.\end{aligned}$$

By the definition of $\mathcal{E}_\infty^{(0)}(t)$ in (4.9), it further follows that

$$\|\mathbf{P}_1u(t)\|^2+\|\nabla_x\Delta_x^{-1}\mathbf{P}_0u(t)\|^2\leq C(1+t)^{-\frac{1}{2}}\left\{\epsilon_0+\epsilon_{0,\nu}\mathcal{E}_\infty^{(0)}(t)+[\mathcal{E}_\infty^{(0)}(t)]^2\right\},\tag{4.17}$$

where we used

$$\begin{aligned}\int_0^t(1+t-s)^{-\frac{3}{2}}(1+s)^{-\frac{1}{2}}ds\leq C(1+t)^{-\frac{1}{2}},\\ \int_0^t(1+t-s)^{-\frac{3}{4}}(1+s)^{-\frac{1}{2}}ds\leq C(1+t)^{-\frac{1}{4}}.\end{aligned}$$

Due to the Gronwall inequality, (4.12) together with (4.17) yields

$$\begin{aligned}\mathcal{E}(u(t))&\leq\mathcal{E}(u_0)e^{-\lambda t}+C\int_0^te^{-\lambda(t-s)}\{\|\mathbf{P}_1u(s)\|^2+\|\nabla_x\Delta_x^{-1}\mathbf{P}_0u(s)\|^2\}ds\\&\leq C(1+t)^{-\frac{1}{2}}\left\{\epsilon_0+\epsilon_{0,\nu}\mathcal{E}_\infty^{(0)}(t)+[\mathcal{E}_\infty^{(0)}(t)]^2\right\},\end{aligned}$$

which implies

$$\mathcal{E}_\infty^{(0)}(t)\leq C\left\{\epsilon_0+\epsilon_{0,\nu}\mathcal{E}_\infty^{(0)}(t)+[\mathcal{E}_\infty^{(0)}(t)]^2\right\},$$

for any $t \geq 0$. Therefore, as long as both ϵ_0 and $\epsilon_{0,\nu}$ are small enough, one has

$$\sup_{t\geq 0} \mathcal{E}_\infty^{(0)}(t) \leq C\epsilon_0, \tag{4.18}$$

which proves (2.26) by the definition of $\mathcal{E}_\infty^{(0)}(t)$ in (4.9).

Next, one can modify the above proof of (2.26) to obtain (2.28) under the assumption (2.27). In fact, under the additional condition (2.27), (4.16) can be refined as

$$\begin{aligned}
&\|\mathbf{P}_1 u(t)\|^2 + \|\nabla_x \Delta_x^{-1} \mathbf{P}_0 u(t)\|^2 \\
&\leq C(1+t)^{-\frac{3}{2}}(\|u_0\|_{Z_1}^2 + \|u_0\|^2) \\
&\quad + \sum_{j=1}^{2} C \int_0^t (1+t-s)^{-\frac{3}{2}} (\|\nu^{-1/2}\{\mathbf{I}-\mathbf{P}\}G_j(s)\|_{Z_1}^2 + \|\nu^{-1/2}\{\mathbf{I}-\mathbf{P}\}G_j(s)\|^2) ds \\
&\quad + C\left[\int_0^t (1+t-s)^{-\frac{3}{4}} (\|\mathbf{P}_1 G_2(s)\|_{Z_1} + \|\mathbf{P}_1 G_2(s)\|) ds\right]^2,
\end{aligned} \tag{4.19}$$

where instead of using (3.3) to estimate the first term on the r.h.s. of (4.16), we used (3.4) since u_0 does not contain the hyperbolic component, i.e., $u_0 = \{\mathbf{I}-\mathbf{P}\}u_0$. Corresponding to obtain (4.17), from the definition of $\mathcal{E}_\infty^{(1)}(t)$ in (4.9), (4.19) implies

$$\|\mathbf{P}_1 u(t)\|^2 + \|\nabla_x \Delta_x^{-1} \mathbf{P}_0 u(t)\|^2 \leq C(1+t)^{-\frac{3}{2}} \left\{\epsilon_0 + \epsilon_{0,\nu} \mathcal{E}_\infty^{(1)}(t) + [\mathcal{E}_\infty^{(1)}(t)]^2\right\}. \tag{4.20}$$

Then, from the completely same proof as for (4.18), one can obtain

$$\sup_{t\geq 0} \mathcal{E}_\infty^{(1)}(t) \leq C\epsilon_0,$$

under the smallness condition of ϵ_0 and $\epsilon_{0,\nu}$. This proves (2.28) by the definition of $\mathcal{E}_\infty^{(1)}(t)$ in (4.9).

4.3 Time decay of high-order energy

At this time, in order to complete the proof of Proposition 2.1, it suffices to prove (2.29) in Proposition 2.1, which gives the time-decay rates for the high-order energy. First of all, let us obtain some Lyapunov-type inequalities of the high-order energy on the basis of Lemma 4.2, Lemma 4.3 and Lemma 5.1 in the following two lemmas.

Lemma 4.4. *There is a high-order energy functional $\mathcal{E}^h(u(t))$ defined by*

$$\mathcal{E}^h(u(t)) \sim \|\{\mathbf{I}-\mathbf{P}_1\}u(t)\|_{H^N}^2 + \|\nabla_x \mathbf{P}_1 u(t)\|_{L_\xi^2(H_x^{N-1})}^2, \tag{4.21}$$

such that

$$\frac{d}{dt}\mathcal{E}^h(u(t)) + \lambda \mathcal{D}(u(t)) \leq C\|\nabla_x \mathbf{P} u(t)\|^2 \tag{4.22}$$

holds for any $t \geq 0$, where $\mathcal{D}(u(t))$ is defined in (5.4).

Proof. Define

$$\mathcal{E}^h(u(t)) = \|\{\mathbf{I}-\mathbf{P}\}u(t)\|^2 + \sum_{1\leq|\alpha|\leq N}(\|\partial_x^\alpha u(t)\|^2 + \|\partial_x^\alpha\nabla_x\Phi(t)\|^2) \\ +\kappa_3\mathcal{E}_{x,\xi}(u(t)) + \kappa_4\mathcal{E}_{free}(u(t)), \tag{4.23}$$

where $\mathcal{E}_{x,\xi}(u(t))$ and $\mathcal{E}_{free}(u(t))$ are given by (5.9) and (5.10), respectively, and constants κ_3 and κ_4 to be determined later satisfies

$$0<\kappa_3\ll\kappa_4\ll 1. \tag{4.24}$$

Notice from the definition (5.10) of $\mathcal{E}_{free}(u(t))$ that

$$\begin{aligned}|\mathcal{E}_{free}(u(t))| &\leq C\sum_{|\alpha|\leq N-1}(\|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\|^2 + \|\partial_x^\alpha\mathbf{P}_0u\|^2 + \|\partial_x^\alpha\nabla_x b^u\|^2)\\ &\leq C\|\{\mathbf{I}-\mathbf{P}\}u(t)\|^2 + C\sum_{1\leq|\alpha|\leq N}(\|\partial_x^\alpha u(t)\|^2 + \|\partial_x^\alpha\nabla_x\Phi(t)\|^2),\end{aligned}$$

which together with (4.23) and (5.9) imply

$$\begin{aligned}\mathcal{E}^h(u(t)) &\sim \|\{\mathbf{I}-\mathbf{P}\}u(t)\|^2 + \sum_{1\leq|\alpha|\leq N}(\|\partial_x^\alpha u(t)\|^2 + \|\partial_x^\alpha\nabla_x\Phi(t)\|^2) + \kappa_3\mathcal{E}_{x,\xi}(u(t))\\ &\sim \|\{\mathbf{I}-\mathbf{P}\}u(t)\|_{H^N}^2 + \|\mathbf{P}_0u(t)\|^2_{L^2_\xi(H_x^{N-1})} + \|\nabla_x\mathbf{P}_1u(t)\|^2_{L^2_\xi(H_x^{N-1})}\\ &\sim \|\{\mathbf{I}-\mathbf{P}_1\}u(t)\|_{H^N}^2 + \|\nabla_x\mathbf{P}_1u(t)\|^2_{L^2_\xi(H_x^{N-1})},\end{aligned}$$

by taking $\kappa_4>0$ small enough and also letting $\kappa_3>0$. Thus, (4.21) holds true. Moreover, under the condition (4.24), the linear combination of (4.3), (5.7), (5.8) and (5.11) in the special case when $\delta_\phi=0$ from the assumption (1.4) yield

$$\frac{d}{dt}\mathcal{E}^h(u(t)) + \lambda\mathcal{D}(u(t)) \leq C\|\nabla_x\mathbf{P}u(t)\|^2 + C(\mathcal{E}(u(t)) + \sqrt{\mathcal{E}(u(t))})\mathcal{D}(u(t)),$$

where $\mathcal{D}(u(t))$ is given by (5.4). Therefore, (4.22) follows from the above inequality and smallness of $\mathcal{E}(u(t))$ as mentioned at the beginning of Subsection 4.1. This completes the proof of Lemma 4.4. □

Lemma 4.5. *There is a weighted high-order energy functional $\mathcal{E}_w^h(u(t))$ and a corresponding dissipation rate $\mathcal{D}_w(u(t))$, which are defined by*

$$\mathcal{E}_w^h(u(t)) \sim \|\{\mathbf{I}-\mathbf{P}_1\}u(t)\|^2_{H_w^N} + \|\nabla_x\mathbf{P}_1u(t)\|^2_{L^2_\xi(H_x^{N-1})}, \tag{4.25}$$

$$\mathcal{D}_w(u(t)) \sim \|\{\mathbf{I}-\mathbf{P}_1\}u(t)\|^2_{H^N_{w^2}} + \|\nabla_x\mathbf{P}_1u(t)\|^2_{L^2_\xi(H_x^{N-1})}, \tag{4.26}$$

such that

$$\frac{d}{dt}\mathcal{E}_w^h(u(t)) + \lambda\mathcal{D}_w(u(t)) \leq C\|\nabla_x\mathbf{P}u(t)\|^2 \tag{4.27}$$

holds for any $t\geq 0$.

Proof. Define

$$\mathcal{E}_w^h(u(t)) = \|w^{1/2}\{\mathbf{I}-\mathbf{P}\}u(t)\|^2 + \sum_{1\leq|\alpha|\leq N} \|w^{1/2}\partial_x^\alpha u(t)\|^2 + \sum_{\substack{|\beta|\geq 1\\ |\alpha|+|\beta|\leq N}} \|w^{1/2}\partial_x^\alpha\partial_\xi^\beta\{\mathbf{I}-\mathbf{P}\}u\|^2 + \kappa_5\mathcal{E}^h(u(t)),$$

where $\mathcal{E}^h(u(t))$ is given by (4.21) and $\kappa_5 > 0$ is a constant to be determined later. By using (4.21), it is straightforward to check

$$\begin{aligned}\mathcal{E}_w^h(u(t)) &\sim \|\{\mathbf{I}-\mathbf{P}\}u(t)\|^2_{H^N_{w^2}} + \|\nabla_x\mathbf{P}u(t)\|^2_{L^2_\xi(H^{N-1}_x)} + \kappa_5\mathcal{E}^h(u(t))\\ &\sim \|\{\mathbf{I}-\mathbf{P}\}u(t)\|^2_{H^N_{w^2}} + \|\nabla_x\mathbf{P}u(t)\|^2_{L^2_\xi(H^{N-1}_x)} + \|\mathbf{P}_0u(t)\|^2_{H^N}\\ &\sim \|\{\mathbf{I}-\mathbf{P}_1\}u(t)\|^2_{H^N_{w^2}} + \|\nabla_x\mathbf{P}_1u(t)\|^2_{L^2_\xi(H^{N-1}_x)},\end{aligned}$$

which implies (4.25). The rest is to verify (4.27). Actually, by taking $\kappa_5 > 0$ large enough, the linear combination of (4.4), (4.5), (4.6) and (4.22) yields

$$\frac{d}{dt}\mathcal{E}_w^h(u(t)) + \lambda\mathcal{D}_w(u(t)) \leq C\|\nabla_x\mathbf{P}u(t)\|^2 + C\mathcal{E}(u(t))\|\{\mathbf{I}-\mathbf{P}\}u(t)\|^2_{H^N_{w^2}}, \tag{4.28}$$

where $\mathcal{D}_w(u(t))$ takes the form

$$\mathcal{D}_w(u(t)) = \|w\{\mathbf{I}-\mathbf{P}\}u(t)\|^2 + \sum_{1\leq|\alpha|\leq N}\|w\partial_x^\alpha u(t)\|^2 + \sum_{\substack{|\beta|\geq 1\\ |\alpha|+|\beta|\leq N}}\|w\partial_x^\alpha\partial_\xi^\beta\{\mathbf{I}-\mathbf{P}\}u(t)\|^2 + \mathcal{D}(u(t)). \tag{4.29}$$

From the definition (5.4) of $\mathcal{D}(u(t))$, (4.29) implies that (4.26) holds true. Since for the second term on the r.h.s. of (4.28) it holds that

$$\mathcal{E}(u(t))\|\{\mathbf{I}-\mathbf{P}\}u(t)\|^2_{H^N_{w^2}} \leq C\mathcal{E}(u(t))\mathcal{D}_w(u(t)),$$

then (4.27) follows from (4.28) and smallness of $\mathcal{E}(u(t))$ again due to (4.1) and smallness of $\mathcal{E}(u_0)$. This completes the proof of Lemma 4.5. □

Now, we are in a position to prove (2.29). For this, as before, define a temporal function by

$$\mathcal{E}_{w,\infty}^h(t) = \sup_{0\leq s\leq t}(1+s)^{2(\frac{3}{4}-\epsilon)}\mathcal{E}_w^h(u(s)), \tag{4.30}$$

where $0 < \epsilon \leq 3/4$ is an arbitrary constant, and $\mathcal{E}_w^h(u)$ is given by (4.25). For simplicity, also denote a constant ϵ_1 depending only on initial data by

$$\epsilon_1 = \|u_0\|^2_{H^N_w} + \|u_0\|^2_{Z_1}. \tag{4.31}$$

Notice $\mathcal{E}_w^h(u_0) \leq C\epsilon_1$ which will be used later.

Proof of (2.29) **in Proposition 2.1:** We begin with the Lyapunov-type inequality (4.27) for the weighted high-order energy functional $\mathcal{E}_w^h(u(t))$. Since

$$\mathcal{E}_w^h(u(t)) \leq C\mathcal{D}_w(u(t))$$

holds by definitions (4.25)-(4.26) of $\mathcal{E}_w^h(u(t))$ and $\mathcal{D}_w(u(t))$, (4.27) implies

$$\frac{d}{dt}\mathcal{E}_w^h(u(t)) + \lambda\mathcal{E}_w^h(u(t)) \leq C\|\nabla_x \mathbf{P}u(t)\|^2.$$

From the Gronwall inequality, it follows that

$$\mathcal{E}_w^h(u(t)) \leq \mathcal{E}_w^h(u_0)e^{-\lambda t} + C\int_0^t e^{-\lambda(t-s)}\|\nabla_x \mathbf{P}u(s)\|^2 ds. \tag{4.32}$$

Notice that (4.32) together with (2.26) imply that $\mathcal{E}_w^h(u(t))$ has at least the same time-decay rate with the total energy $\mathcal{E}(u(t))$, that is,

$$\sup_{t\geq 0}(1+t)^{\frac{1}{2}}\mathcal{E}_w^h(u(t)) \leq C\epsilon_1, \tag{4.33}$$

if conditions of (2.26) hold and $\|u_0\|_{H_w^N}$ is bounded. In what follows, we shall improve (4.33) up to the almost optimal rate in the sense that for any $0 < \epsilon \leq 3/4$, there is $\eta > 0$ depending only on ϵ such that whenever $\epsilon_0 \leq \eta^2$ with ϵ_0 defined in (4.10), one has

$$\sup_{t\geq 0}(1+t)^{2(\frac{3}{4}-\epsilon)}\mathcal{E}_w^h(u(t)) \leq C\epsilon_1, \tag{4.34}$$

which implies (2.29). In fact, one can obtain a formal time-decay estimate on the first-order energy $\|\nabla_x \mathbf{P}u(t)\|^2$ in terms of $\mathcal{E}_{w,\infty}^h(t)$ in the same way as in (4.17) or (4.20). Firstly, similar to get (4.16), by applying Theorem 3.1 to (4.13) with the help of the decomposition (4.15) for the source term G, it follows that

$$\begin{aligned}\|\nabla_x \mathbf{P}u(t)\|^2 &\leq C(1+t)^{-\frac{3}{2}}(\|u_0\|_{Z_1}^2 + \|\nabla_x u_0\|^2)\\ &+\sum_{j=1}^2 C\int_0^t (1+t-s)^{-\frac{3}{2}}(\|\nu^{-1/2}\nabla_x\{\mathbf{I}-\mathbf{P}\}G_j(s)\|_{Z_1}^2 + \|\nu^{-1/2}\nabla_x\{\mathbf{I}-\mathbf{P}\}G_j(s)\|^2)ds\\ &+C\left[\int_0^t (1+t-s)^{-\frac{3}{4}}(\|\nabla_x \mathbf{P}_1 G_2(s)\|_{Z_1} + \|\nabla_x \mathbf{P}_1 G_2(s)\|)ds\right]^2,\end{aligned} \tag{4.35}$$

where G_1 and G_2 are defined in (4.14). Next, it is straightforward to check

$$\begin{aligned}&\|\nu^{-1/2}\nabla_x\{\mathbf{I}-\mathbf{P}\}G_j(t)\|_{Z_1}^2 + \|\nu^{-1/2}\nabla_x\{\mathbf{I}-\mathbf{P}\}G_j(t)\|^2\\ &\qquad\leq C\mathcal{E}_w^h(u(t))[\mathcal{E}_w^h(u(t)) + \mathcal{E}(u(t))]\end{aligned}$$

for $j = 1, 2$, and

$$\|\nabla_x \mathbf{P}_1 G_2(t)\|_{Z_1} + \|\nabla_x \mathbf{P}_1 G_2(t)\| \leq C[\mathcal{E}_w^h(u(t))\mathcal{E}(u(t))]^{\frac{1}{2}}.$$

Plugging the above inequalities into (4.35) gives

$$\begin{aligned}\|\nabla_x \mathbf{P} u(t)\|^2 \leq\ & C(1+t)^{-\frac{3}{2}}(\|u_0\|_{Z_1}^2+\|\nabla_x u_0\|^2)\\ &+C\int_0^t (1+t-s)^{-\frac{3}{2}}\mathcal{E}_w^h(u(s))[\mathcal{E}_w^h(u(s))+\mathcal{E}(u(s))]ds\\ &+C\left[\int_0^t (1+t-s)^{-\frac{3}{4}}[\mathcal{E}_w^h(u(s))\mathcal{E}(u(s))]^{\frac{1}{2}}ds\right]^2. \end{aligned}\tag{4.36}$$

Notice that from the time integrations of (4.11) and (4.22) as well as the definition (4.31) of ϵ_1, it holds that

$$\sup_{t\geq 0}[\mathcal{E}_w^h(u(t))+\mathcal{E}(u(t))]\leq C\epsilon_1,$$

and also recall from (4.18), (4.9) and (4.10) that

$$\mathcal{E}(u(t))\leq C\epsilon_0(1+t)^{-\frac{1}{2}},$$

where ϵ_0 is defined in (4.10). Then, it follows from (4.36) that

$$\begin{aligned}\|\nabla_x \mathbf{P} u(t)\|^2 \leq\ & C(1+t)^{-\frac{3}{2}}(\|u_0\|_{Z_1}^2+\|\nabla_x u_0\|^2)\\ &+C\epsilon_1\int_0^t (1+t-s)^{-\frac{3}{2}}\mathcal{E}_w^h(u(s))ds\\ &+C\epsilon_0\left[\int_0^t (1+t-s)^{-\frac{3}{4}}(1+s)^{-\frac{1}{4}}[\mathcal{E}_w^h(u(s))]^{\frac{1}{2}}ds\right]^2, \end{aligned}$$

which further from the definition (4.30) of $\mathcal{E}_{w,\infty}^h(t)$ implies

$$\begin{aligned}\|\nabla_x \mathbf{P} u(t)\|^2 \leq\ & C(1+t)^{-\frac{3}{2}}(\|u_0\|_{Z_1}^2+\|\nabla_x u_0\|^2)\\ &+C\epsilon_1\mathcal{E}_{w,\infty}^h(t)\int_0^t (1+t-s)^{-\frac{3}{2}}(1+s)^{-2(\frac{3}{4}-\epsilon)}ds\\ &+C\epsilon_0\mathcal{E}_{w,\infty}^h(t)\left[\int_0^t (1+t-s)^{-\frac{3}{4}}(1+s)^{-\frac{1}{4}-(\frac{3}{4}-\epsilon)}ds\right]^2. \end{aligned}\tag{4.37}$$

Notice that for $0<\epsilon\leq 3/4$,

$$\begin{aligned}&\int_0^t (1+t-s)^{-\frac{3}{2}}(1+s)^{-2(\frac{3}{4}-\epsilon)}ds\leq C(1+t)^{-2(\frac{3}{4}-\epsilon)},\\ &\int_0^t (1+t-s)^{-\frac{3}{4}}(1+s)^{-\frac{1}{4}-(\frac{3}{4}-\epsilon)}ds\leq C_\epsilon(1+t)^{-(\frac{3}{4}-\epsilon)},\end{aligned}$$

where $C>0$ in the first inequality can be taken uniformly in $0<\epsilon\leq 3/4$, while $C_\epsilon>0$ in the second inequality has to tend to infinity as ϵ goes to zero. Then, it follows from (4.37) that

$$\|\nabla_x \mathbf{P} u(t)\|^2\leq (1+t)^{-(\frac{3}{2}-2\epsilon)}[C\epsilon_0+(C\epsilon_1+C_\epsilon\epsilon_0)\mathcal{E}_{w,\infty}^h(t)].$$

Combined with the above estimate, (4.32) gives

$$\mathcal{E}_w^h(u(t)) \leq (1+t)^{-(\frac{3}{2}-2\epsilon)}[C\epsilon_1 + (C\epsilon_1 + C_\epsilon\epsilon_0)\mathcal{E}_{w,\infty}^h(t)],$$

for any $t \geq 0$, where $\max\{\mathcal{E}_w^h(u_0), \epsilon_0\} \leq C\epsilon_1$ was used. Thus, it follows that

$$\mathcal{E}_{w,\infty}^h(t) \leq C\epsilon_1 + (C\epsilon_1 + C_\epsilon\epsilon_0)\mathcal{E}_{w,\infty}^h(t),$$

that is,

$$\mathcal{E}_{w,\infty}^h(t) \leq C\epsilon_1 + C_\epsilon\epsilon_0\mathcal{E}_{w,\infty}^h(t),$$

since ϵ_1 is small enough. Therefore, given any $0 < \epsilon \leq 3/4$, one can choose $\eta = 1/\sqrt{2C_\epsilon}$ so that whenever $\epsilon_0 \leq \eta^2$, it holds that

$$\mathcal{E}_{w,\infty}^h(t) \leq C\epsilon_1,$$

for any $t \geq 0$, which proves (4.34) and hence (2.29) by the definition (4.30) of $\mathcal{E}_{w,\infty}^h(t)$. This also completes the proof of Proposition 2.1.

5 Appendix

In this appendix, we shall state the existence of the stationary solution and its nonlinear stability for the VPB system (1.1)-(1.2). For that, let us define the weighted norm $\|\cdot\|_{W_\theta^{m,\infty}}$ by

$$\|g\|_{W_\theta^{m,\infty}} = \sup_{x\in\mathbb{R}^3}(1+|x|)^\theta \sum_{|\alpha|\leq m} |\partial_x^\alpha g(x)|,$$

for suitable $g = g(x)$ and for an integer $m \geq 0$ and $\theta \geq 0$. [8] proved

Proposition 5.1 (existence of stationary solutions)**.** *Let the integer $m \geq 0$ and $\theta \geq 0$. Suppose that $\|\bar{\rho} - 1\|_{W_\theta^{m,\infty}}$ is small enough. Then the following elliptic equation with exponential nonlinearity:*

$$\Delta_x\phi = e^\phi - \bar{\rho}(x),$$

admits a unique solution $\phi = \phi(x)$ satisfying

$$\|\phi\|_{W_\theta^{m,\infty}} \leq C\|\bar{\rho} - 1\|_{W_\theta^{m,\infty}},$$

for some constant C.

From Proposition 5.1 above, it is straightforward to check that the VPB system (1.1)-(1.2) has a stationary solution (f_*, Φ_*) given by $f_* = e^\phi\mathbf{M}$, $\Phi_* = \phi$. To state the stability of the stationary state (f_*, Φ_*), set the perturbation $u = u(t, x, \xi)$ by

$$f = e^\phi\mathbf{M} + \sqrt{\mathbf{M}}u.$$

Then u satisfies the perturbed system:

$$\begin{cases} \partial_t u + \xi \cdot \nabla_x u + \nabla_x(\Phi + \phi) \cdot \nabla_\xi u - \dfrac{1}{2}\xi \cdot \nabla_x(\Phi+\phi)u - \xi \cdot \nabla_x \Phi e^{\phi}\sqrt{\mathbf{M}} \\ \qquad\qquad\qquad = e^{\phi}\mathbf{L}u + \Gamma(u,u), \\ \Phi = -\dfrac{1}{4\pi|x|} * \displaystyle\int_{\mathbb{R}^3} \sqrt{\mathbf{M}} u d\xi, \quad (t,x,\xi) \in (0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^3, \end{cases} \tag{5.1}$$

with given initial data

$$u(0,x,\xi) = u_0(x,\xi) \equiv \frac{f_0 - e^{\phi}\mathbf{M}}{\sqrt{\mathbf{M}}}, \quad (x,\xi) \in \mathbb{R}^3 \times \mathbb{R}^3. \tag{5.2}$$

Here $\mathbf{L}u$ and $\Gamma(u,u)$ are defined by in the same forms as in (2.5) and (2.6), respectively. [8] also proved

Proposition 5.2 (stability of stationary solutions)**.** *Let $N \geq 4$. Suppose that $\|\bar{\rho} - 1\|_{W_2^{N+1,\infty}}$ is small enough. Then, there are the equivalent energy functional $\mathcal{E}(\cdot)$ and the corresponding energy dissipation rate $\mathcal{D}(\cdot)$ defined by*

$$\mathcal{E}(u(t)) \sim \|u\|_{H^N}^2 + \|\nabla_x \Delta_x^{-1} \mathbf{P}_0 u\|^2, \tag{5.3}$$

$$\mathcal{D}(u(t)) \sim \|\{\mathbf{I} - \mathbf{P}_1\}u\|_{H_\nu^N}^2 + \|\nabla_x \mathbf{P}_1 u\|_{L_\xi^2(H_x^{N-1})}^2, \tag{5.4}$$

such that the following holds. If $f_0 = e^{\phi}\mathbf{M} + \sqrt{\mathbf{M}}u_0 \geq 0$ and $\mathcal{E}(u_0)$ is sufficiently small, then the Cauchy problem (5.1)-(5.2) *of the VPB system admits a unique global solution $u(t,x,\xi)$ satisfying $f(t,x,\xi) \equiv e^{\phi}\mathbf{M} + \sqrt{\mathbf{M}}u(t,x,\xi) \geq 0$, and*

$$\frac{d}{dt}\mathcal{E}(u(t)) + \lambda \mathcal{D}(u(t)) \leq 0. \tag{5.5}$$

The following lemma was obtained by [8] in the proof of uniform *a priori* estimates for the global-in-time stability of the stationary solution.

Lemma 5.1 (*a priori* estimates)**.** *Let all conditions of Proposition 5.2 hold and let u be the corresponding solution, and (a^u, b^u, c^u) be defined in* (2.8)-(2.10)*. Denote $\delta_\phi = \|\phi\|_{W_2^{N+1,\infty}}$. Then, the following uniform a priori estimates hold for any $t \geq 0$:*

(i) zero-order:

$$\begin{aligned} &\frac{d}{dt}\left(\|u\|^2 + \|\nabla_x \Phi\|^2 - 2\int_{\mathbb{R}^3} e^{-\phi}|b^u|^2 c^u \, dx\right) + \lambda \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \nu(\xi)|\{\mathbf{I} - \mathbf{P}\}u|^2 dx d\xi \\ &\leq C(\delta_\phi + \sqrt{\mathcal{E}(u(t))})\mathcal{D}(u(t)); \end{aligned} \tag{5.6}$$

(ii) spatial derivatives:

$$\begin{aligned} &\frac{d}{dt}\sum_{1 \leq |\alpha| \leq N} \left(\|\partial_x^\alpha u\|^2 + \|\partial_x^\alpha \nabla_x \Phi\|^2\right) + \lambda \sum_{1 \leq |\alpha| \leq N} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \nu(\xi)|\partial_x^\alpha \{\mathbf{I} - \mathbf{P}\}u|^2 dx d\xi \\ &\leq C(\delta_\phi + \sqrt{\mathcal{E}(u(t))})\mathcal{D}(u(t)); \end{aligned} \tag{5.7}$$

(iii) mixed spatial-velocity derivatives:

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{E}_{x,\xi}(u(t))+\lambda\sum_{\substack{|\beta|\geq 1\\ |\alpha|+|\beta|\leq N}}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\nu(\xi)|\partial_x^\alpha\partial_\xi^\beta\{\mathbf{I}-\mathbf{P}\}u|^2dxd\xi\\
&\leq C(\delta_\phi+\sqrt{\mathcal{E}(u(t))})\mathcal{D}(u(t))\\
&\quad+C\sum_{|\alpha|\leq N-1}\|\partial_x^\alpha\nabla_x(a^u,b^u,c^u)\|^2+C\sum_{|\alpha|\leq N}\|\nu^{1/2}\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\|^2,
\end{aligned}
\tag{5.8}
$$

where

$$
\mathcal{E}_{x,\xi}(u(t))=\sum_{k=1}^N C_{N,k}\sum_{\substack{|\beta|=k\\ |\alpha|+|\beta|\leq N}}\|\partial_x^\alpha\partial_\xi^\beta\{\mathbf{I}-\mathbf{P}\}u\|^2,
\tag{5.9}
$$

for some proper positive constants $C_{N,k}$;

(iv) macroscopic dissipation: There is a temporal free energy functional $\mathcal{E}_{free}(u(t))$ in the form of

$$
\begin{aligned}
\mathcal{E}_{free}(u(t))=\kappa_0&\sum_{|\alpha|\leq N-1}\sum_{ij}\langle A_{ij}(\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u),\partial_x^\alpha(\partial_i b_j^u+\partial_j b_i^u)\rangle\\
&+\kappa_0\sum_{|\alpha|\leq N-1}\sum_{j}\langle A_{jj}(\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u),\partial_x^\alpha\partial_j b_j^u\rangle\\
&+\kappa_0\sum_{|\alpha|\leq N-1}\sum_{i}\langle B_{i}(\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u),\partial_x^\alpha\partial_i c^u\rangle\\
&-\sum_{|\alpha|\leq N-1}\langle\partial_x^\alpha(a^u+3c^u),\partial_x^\alpha\nabla_x\cdot b^u\rangle
\end{aligned}
\tag{5.10}
$$

for some constant $\kappa_0>0$, such that one has

$$
\frac{d}{dt}\mathcal{E}_{free}(u(t))+\lambda\mathcal{D}_{mac}(u(t))\leq C\sum_{|\alpha|\leq N}\|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\|^2+\mathcal{E}(u(t))\mathcal{D}(u(t)),
\tag{5.11}
$$

where $\mathcal{D}_{mac}(u(t))$ is the macroscopic dissipation rate given by

$$
\mathcal{D}_{mac}(u(t))=\sum_{|\alpha|\leq N-1}\|\partial_x^\alpha\nabla_x(a^u+3c^u,b^u,c^u)\|^2+\|a^u+3c^u\|^2.
$$

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