Energy Method for Multi-dimensional Balance Laws with Non-local Dissipation

Renjun Duan*

Johann Radon Institute for Computational and Applied Mathematics
Austrian Academy of Sciences
Altenbergerstrasse 69, A-4040 Linz, Austria
Email: renjun.duan@oeaw.ac.at

Klemens Fellner

DAMTP, Centre of Mathematical Sciences, University of Cambridge
Wilberforce Road, Cambridge CB3 0WA, UK
On leave from: Faculty of Mathematics, University of Vienna
Nordbergstr. 15, 1090 Wien, Austria
Email: K.Fellner@damtp.cam.ac.uk

Changjiang Zhu

Department of Mathematics, Huazhong Normal University
Wuhan 430079, P.R. China
Email: cjzhu@mail.ccnu.edu.cn

Abstract

In this paper, we are concerned with a class of multi-dimensional balance laws with a non-local dissipative source which arise as simplified models for the hydrodynamics of radiating gases. At first we introduce the energy method in the setting of smooth perturbations and study the stability of constants states. Precisely, we employ Fourier space analysis to quantify the energy dissipation rate and recover the optimal time-decay estimates for perturbed solutions via an interpolation inequality in Fourier space. As application, the developed energy method is used to prove stability of smooth planar waves in all dimensions $n \geq 2$, and also to show existence and stability of time-periodic solutions in the presence of the time-periodic source. Optimal rates of convergence of solutions towards the planar waves or time-periodic states are also shown provided initially L^1-perturbations.

Résumé

Dans ce papier, on s'intéresse à une classe de lois de balance multi-dimensionnelles avec une source non locale, qui résultent del modèles simplifiés pour l'hydrodynamique des gaz irradiants. En utilisant la méthode de l'énergie, la stabilité, le taux de convergence des solutions près des états constants, les ondes planes régulières et les états périodiques sont étudiés.

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*Corresponding author.

1 Introduction

In this paper, we are concerned with the hyperbolic-elliptic coupled system in several dimensions:

$$\partial_t u + \nabla \cdot [f(u) + q] = S, \tag{1.1}$$

$$-\nabla[\nabla \cdot q - u] + q = 0. \tag{1.2}$$

Here $u = u(t,x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $q = q(t,x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ are unknown, and $n \geq 1$ denotes the space dimension. $f(u) = (f_1(u), f_2(u), \cdots, f_n(u)) : \mathbb{R} \to \mathbb{R}^n$ and $S = S(t,x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are given functions. It is supposed that the flux function $f(u)$ is smooth in u. Additionally, in Section 3 dealing with planar rarefaction waves, we shall assume that the first component f_1 is strictly convex.

The system (1.1)-(1.2) can alternatively be written as a single equation. In fact, let P be a pseudo-differential operator defined by

$$\widehat{Pu}(\xi) = \frac{1}{1+|\xi|^2}\widehat{u}(\xi),$$

and then taking the divergence of (1.2) gives

$$\nabla \cdot q = -\Delta P u.$$

Thus, the system (1.1)-(1.2) reduces to

$$\partial_t u + \nabla \cdot f(u) = \Delta P u + S. \tag{1.3}$$

Note that Pu is equivalent to the non-local convolution operator

$$Pu = K * u,$$

where the kernel K is the Bessel potential given by

$$K(x) = \frac{1}{(4\pi)^{n/2}} \int_0^\infty s^{-\frac{n}{2}} e^{-s-\frac{|x|^2}{4s}} ds,$$

satisfying the following basic properties:

$$K(x) = K(|x|) \geq 0, \qquad \int_{\mathbb{R}^n} K(x)dx = 1, \qquad \Delta K * u = -u + K * u.$$

The main goal of this paper is to introduce the classical energy method to prove the stability and convergence rate of smooth solutions to the model system (1.1)-(1.2) or equivalently (1.3) near some existing equilibria such as constant states, one-dimensional smooth rarefaction wave and possible time-periodic states. For the proof of stability, the essential issue is to obtain the dissipation rate of the temporal energy in order to further control the nonlinear conservative term. In fact, the linear non-local dissipative term $\Delta P u$ in (1.3) can be written in terms of the inverse Fourier transformation as

$$\Delta P u = -\left(\frac{|\xi|^2}{1+|\xi|^2}\widehat{u}(\xi)\right)^{\vee}. \tag{1.4}$$

As pointed out in [34, 35], eq. (1.4) can be read in the way that $\Delta P u$ behaves qualitatively like the usual Navier-Stokes viscosity Δu at the low-ξ, while it may act as the damping force $-u$ at the high-ξ. From the series of work in [35, 19, 22, 27, 8], this characteristic of the linear operator $\Delta P u$ implies that (1.3) retains many of the properties exposed by the viscous conservation laws

$$\partial_t u + \nabla \cdot f(u) = \Delta u,$$

essentially since the high-ξ components to the solutions of (1.3) decay exponentially in time. However, it should be also noted that the derivative of solutions to (1.3) may blow up in finite time [19] due to the weaker smoothing property of $\Delta P u$. We remark on the other hand that the non-local term $\Delta P u$ was also derived by Rosenau in [34] as the corresponding extension of the Navier-Stokes equations via the regularisation of the Chapman-Enskog expansion from the Boltzmann equation, which is intended to obtain a bounded approximation of the linearised collision operator for both low and high frequencies.

Let us comment on related mathematical models. The model system (1.1)-(1.2) or its equivalent form (1.3) appears in radiative hydrodynamics [40] and in the context of self-gravitating fluids modeled by the Euler-Poisson system [44], where the nonlocal forcing in (1.3) reflects the global influence of heat sources or gravitation fields, respectively. In particular for radiative hydrodynamics, the fluid gains or loses energy and momentum by interacting with electromagnetic radiation through the emission, absorption and scattering of photons, see [30, 40]. An underlying hyperbolic-elliptic system describing the one-dimensional motion of the radiating fluid takes the form of

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \\ \partial_t[\rho(e + \frac{1}{2}u^2)] + \partial_x[\rho u(e + \frac{1}{2}u^2) + pu + q] = 0, \\ -\partial_x^2 q + 3a^2 q + 4a\kappa\partial_x\theta^4 = 0, \end{cases} \tag{1.5}$$

where the hydrodynamic functions $\rho \geq 0$, u, p, e and $\theta \geq 0$ denotes the mass density, velocity, pressure, internal energy and absolute temperature of the fluid, respectively, and q is the radiative heat flux, and $a > 0$, $\kappa > 0$ are the absorption coefficient and the Boltzmann constant, respectively, see [40]. A full mathematical model of radiating plasma flow is given by the compressible Euler equations as in (1.5) coupled with a linear Boltzmann equation for radiation density [40]. We also note that nonlocal models of the above type have also been derived in traffic simulation [37].

Note that the hyperbolic-elliptic coupled system (1.5) can be written in the following general form:

$$\begin{cases} \partial_t U + \partial_x(f(U) + \mathbb{M}_1 V) = 0, \\ -\partial_x^2 V + \mathbb{M}_2 V + \nu(U)\partial_x g(U) = 0, \end{cases} \tag{1.6}$$

where $U \in \mathbb{R}^m$ and $V \in \mathbb{R}^n$ are unknown functions of the single spatial variable $x \in \mathbb{R}$, and $f : \mathbb{R}^m \to \mathbb{R}^m$, $g : \mathbb{R}^m \to \mathbb{R}^n$, $\nu : \mathbb{R}^m \to \mathbb{R}$ are given smooth mappings, and $\mathbb{M}_1, \mathbb{M}_2$ are $m \times n$ and $n \times n$ matrices, see [14, 17, 18]. Furthermore, by taking the proper approximation of (1.5), Hamer in [13] derived a simplified model system in the form of

$$\begin{cases} \partial_t u + \partial_x(\frac{1}{2}u^2 + q) = 0, \\ -\partial_x^2 q + q + \partial_x u = 0, \end{cases} \tag{1.7}$$

which as far as we know, is the simplest model for the hyperbolic-elliptic coupled system appearing in the study of radiation hydrodynamics. We refer also to [7] for the existence theory and the large time analysis of non-linear variants of (the second equation of) (1.7) replacing $\partial_x u$ by a nonlinear gradient $\partial_x B(u)$. A typical choice is $B(u) = \sigma u^4$ with $\sigma > 0$, which is derived from Planck's law of black body radiation and corresponds to the nonlinearity appearing in the equation of the temperature in (1.5). A similar system with general nondecreasing $B(u)$ has been derived in [4, 5] as a simplified model for describing non-local energy transports in a radiation fluid. We refer to [33] for a investigation of this special model as the non-relativistic limit.

Concerning higher dimensions, [13] proposed the model system (1.1)-(1.2) as a simplification of the multi-dimensional version of (1.5); see [11] for the formal derivation from (1.5) in the case of several dimensions to (1.1)-(1.2).

There is extensive mathematical work on the macroscopic systems at the different levels mentioned above, and in what follows let us review some of results related to them. Firstly, the global well-posedness and large-time behaviour of solutions to the Cauchy problem of the one-dimensional model system (1.7) are well-established. In fact, (1.7) was first studied in [35], where the propagation of smoothness of solutions with small initial data, existence of travelling waves, existence of entropy solutions with BV initial data and zero relaxation limit were considered. The further investigations of the existence and uniqueness of entropy solutions were made by Ito [16] for BV data and by Lattanzio-Marcati [25] for $L^1 \cap L^\infty$ data, see also the recent work [36]. In addition, a series of work was done by Kawashima et al.

to study the stability of wave patterns such as travelling wave and rarefaction wave in [19, 20, 21, 22], see also Liu-Tadmor [29]. Recently, a sharp rate of solutions to the travelling wave was given in [31] through the weighted energy method, and also the optimal L^1 time-decay rate of solutions to the diffusion wave defined by the viscous Burgers' equation

$$\partial_t u + u\partial_x u = \partial_x^2 u$$

was given in [8] by means of entropy production method. For the system (1.5) or its generalised form (1.6), the large-time behaviour and singular limit of solutions for different types of initial data were considered in [17] and [18, 32], and later [14] exposed the pointwise estimate of solutions by using the Green's function method; see also the recent work [41]. [28] considered the existence and regularity of smooth travelling waves, and [26] proved the existence of admissible radiative shock wave. The system (1.1)-(1.2) or (1.3) has been also extensively studied in the past few years. [6] generalised the previous result [25] to the case of several dimensions. The stability and convergence rate of solutions near the planar rarefaction waves were obtained in [11, 12] for the case of the space dimension $2 \leq n \leq 5$. In this context we also mention more general results on the stability of viscous shock waves, see e.g. [45] and the reference therein. The local- and global-in-time well-posedness of solutions in L^1 was given by [2, 1].

Finally, we mention [42, 23] about the studies of stability of planar rarefaction waves in the context of the viscous conservation laws and [43, 33] for the BV estimates on solutions via a discrete difference scheme. For the general information about the hyperbolic conservation laws, refer to [3].

The results of this paper are organised as follows: In Section 2 we consider (1.3) without external source S. The main result Theorem 2.1 shows stability and optimal convergence rates of smooth solutions near a constant state. The proof employs an energy method in the setting of smooth perturbations based on the following Plancherel-identities

$$\int_{\mathbb{R}^n} \partial^\alpha \Delta P u \, \partial^\alpha u \, dx = -\int_{\mathbb{R}^n} \frac{|\xi|^2 \xi^{2\alpha}}{1+|\xi|^2} |\widehat{u}|^2 d\,\xi,$$

which (by taking a suitable linear combination over all α with $|\alpha| = k$ for any integer $k \geq 0$) sum up to yield the desired energy dissipation of u:

$$\int_{\mathbb{R}^n} \frac{|\xi|^{2+2k}}{1+|\xi|^2} |\widehat{u}|^2 d\,\xi.$$

It remains to control the nonlinear conservative term $\nabla \cdot f(u)$, for which we derive first a zero-order ($k = 0$) energy equality and, subsequently, high-order energy inequalities. While the zero-order equation is closed (see Lemma 2.1), the high-order estimates (see Lemma 2.2) require to assume an *a priori* smallness to control the derivatives of $\nabla \cdot f(u)$ in terms of the energy dissipation. We remark that the here presented way of estimating the conservative flux term can also be applied to related physical models.

Moreover in Section 2, we prove in Lemma 2.4 an interpolation inequality for the lowest-order term of the energy dissipation rate via a frequency decomposition argument. Since solutions to (1.3) are contractive in L^1, we conclude time-decay rate in L^2 of perturbed solutions in a typical way via time-weighted estimates on the energy (see the end of Section 2 for the proof of Theorem 2.1).

In Section 3 the energy method is subsequently applied to study stability and convergence rates of smooth planar waves. In contrast to the recent works [11, 12], we are here able to demonstrate a unified proof of stability of smooth planar waves for any spatial dimension $n \geq 2$. As firstly considered in [15], the main difficulty comes from the degeneration of planar wave profile in the transversal directions which can be overcome by the time-decay of the planar wave profile itself, see the proof of Theorem 3.1.

Finally in Section 4, we consider (1.3) with a time-periodic source. In Theorem 4.1, we construct time-periodic solutions via an iterative scheme using (1.3) with linearised flux term, which passes to the limit provided that the time-periodic source term is small in sufficiently high Sobolev spaces. One of the main ideas, initially developed in [38] and later in [39, 10], is to write the equation with time-periodic coefficients as the mild form containing an infinite time integral.

We also remark that the time-decay estimate on the linearised solution operator plays a key role in obtaining the uniform bounds of the iterative solution sequence, see Lemma 4.1 and Lemma 4.2. In order for the time-decay rate to be integrable over all times we have to assume spatial dimension $n \geq 5$ in Theorem 4.1. The remaining dimensions $1 \leq n \leq 4$ are left open. The asymptotic stability of the time-periodic solution is discussed in Theorem 4.2. The proof of stability is analog to perturbations of constant states, whereas the convergence rate is obtained on the basis of the energy-spectrum method recently developed by [10, 9].

Notations. Through this paper, for integer $m \geq 0$, we use H^m denotes the Sobolev space with norm $\|\cdot\|_{H^m}$. As usual, we use $L^2 = H^0$ for $m = 0$ with norm $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ to denote the inner product in L^2. The notion $\widehat{u}(\xi)$ denotes the Fourier transformation of $u(x)$. We use

$$\nabla = (\partial_{x_1}, \cdots, \partial_{x_n}),$$

and for any integer $m \geq 0$, $\nabla^m u$ denotes all m-order derivatives of the function u. For the multiple index $\alpha = (\alpha_1, \cdots, \alpha_n)$, we denote

$$\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}, \quad \xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}.$$

The length of α is $|\alpha| = \alpha_1 + \cdots + \alpha_n$. $\beta \leq \alpha$ means $\beta_i \leq \alpha_i$ for all $1 \leq i \leq n$. In addition, C denotes a generic positive (generally large) constant and λ denotes a generic positive (generally small) constant.

2 L^2 energy method for smooth perturbations

In this section, we consider the Cauchy problem of the equation (1.3) in the absence of sources

$$\partial_t u + \nabla \cdot f(u) = \Delta P u, \quad t > 0, \quad x \in \mathbb{R}^n, \tag{2.1}$$

for smooth flux function $f(u)$ and with given initial data

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^n. \tag{2.2}$$

We moreover introduce the operator $\sqrt{P}$ by

$$\widehat{\sqrt{P}u}(\xi) = \frac{1}{\sqrt{1+|\xi|^2}} \widehat{u}(\xi)$$

in order to state the main result of this section:

Theorem 2.1. *Let $n \geq 1$ and $N \geq 2[n/2] + 2$. Then, there are constants $\epsilon_0 > 0$, $\lambda > 0$ and C such that for $\|u_0\|_{H^N} \leq \epsilon_0$ the Cauchy problem (2.1)-(2.2) admits a unique smooth solution $u(t,x)$ satisfying*

$$\begin{aligned} \|u(t)\|_{H^N}^2 &+ \lambda \int_0^t \|\nabla u(s)\|_{H^{N-1}}^2 ds \\ &+ \lambda \int_0^t (\|\nabla \sqrt{P} u(s)\|^2 + \|\nabla^{N+1} \sqrt{P} u(s)\|^2) ds \leq C \|u_0\|_{H^N}^2, \end{aligned} \tag{2.3}$$

for any $t \geq 0$. Moreover, if $\|u_0\|_{L^1}$ is bounded, then the obtained solution u enjoys the time-decay estimate

$$\|u(t)\| \leq C \|u_0\|_{L^2 \cap L^1} (1+t)^{-\frac{n}{4}}. \tag{2.4}$$

Proof. Theorem 2.1 will be proven by continuing a unique local solution using a uniform *a priori* estimate in the setting of small data. Recalling that the non-local term on right hand side of (1.3) can be written as $\Delta K * u = -u + K * u$ and K being the bounded, integrable Bessel potential, the local existence of a unique smooth solution on a time interval $[0,T]$ for $T > 0$ follows from a standard fix-point argument, see e.g. [24]. Moreover, due to the construction of the fix-point Banach space, we have

$$\sup_{0 \leq t \leq T} \|u(t)\|_{H^N} \leq \epsilon, \tag{2.5}$$

for the constant $0 < \epsilon \leq 1$ small enough, and $u \in C([0,T]; H^N)$ is the unique solution to the Cauchy problem (2.1)-(2.2) on $[0,T]$.

In what follows, we show that in the setting of small data the norm $\|u(t)\|_{H^N}$ is actually subject to an energy estimate and, thus, non-increasing in time, which permits to continue the solution $u(t)$ globally. Notice that by Sobolev embeddings, it holds that

$$\sup_{0\leq t\leq T}\|u(t)\|_{W^{m,\infty}}\leq C\epsilon, \tag{2.6}$$

for any integer $0\leq m\leq N-[n/2]-1$.

In the following two lemmas we are concerned with the energy estimates on solutions of zero-order and high-order, respectively.

Lemma 2.1. *Let u be the solution to the Cauchy problem (2.1)-(2.2) over $[0,T]$. Then it holds that*

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2+\int_{\mathbb{R}^n}\frac{|\xi|^2}{1+|\xi|^2}|\widehat{u}(t)|^2d\xi=0, \tag{2.7}$$

for any $0\leq t\leq T$.

Proof. The zero-order energy estimate gives

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2+\sum_{j=1}^n\langle f_j(u)_{x_j},u\rangle=\langle\Delta Pu,u\rangle.$$

It follows from the Plancherel theorem that

$$\langle\Delta Pu,u\rangle=\langle\widehat{\Delta Pu},\overline{\widehat{u}}\rangle=-\int_{\mathbb{R}^n}\frac{|\xi|^2}{1+|\xi|^2}|\widehat{u}|^2d\xi,$$

and also from the integration by part that

$$\langle f_j(u)_{x_j},u\rangle=-\langle f_j(u)-f_j(0),u_{x_j}\rangle=-\int_{\mathbb{R}^n}\left[\int_0^u(f_j(\eta)-f_j(0))\right]_{x_j}dx=0$$

for each $1\leq j\leq n$. Then Lemma 2.1 is proved. □

Lemma 2.2. *Let u be the solution to the Cauchy problem (2.1)-(2.2) over $[0,T]$ which satisfies (2.5). Then, for any $1\leq k\leq N$, it holds that*

$$\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|=k}C_\alpha^k\|\partial^\alpha u(t)\|^2+\int_{\mathbb{R}^n}\frac{|\xi|^{2+2k}}{1+|\xi|^2}|\widehat{u}(t)|^2d\xi\leq C\epsilon\sum_{1\leq m\leq N}\|\nabla^m u(t)\|^2, \tag{2.8}$$

for any $0\leq t\leq T$, where $C_\alpha^k=\frac{k!}{\alpha!}=\frac{k!}{\alpha_1!\alpha_2!\cdots\alpha_n!}$ for $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$.

Proof. Fix k with $1\leq k\leq N$. The high-order energy estimate gives

$$\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|=k}C_\alpha^k\|\partial^\alpha u(t)\|^2+\sum_{j=1}^n\sum_{|\alpha|=k}C_\alpha^k\langle\partial_{x_j}\partial^\alpha f_j(u),\partial^\alpha u\rangle=\sum_{|\alpha|=k}C_\alpha^k\langle\Delta\partial^\alpha Pu,\partial^\alpha u\rangle.$$

Similarly before, it follows from the Plancherel theorem that

$$\sum_{|\alpha|=k}C_\alpha^k\langle\Delta\partial^\alpha Pu,\partial^\alpha u\rangle=-\sum_{|\alpha|=k}C_\alpha^k\int_{\mathbb{R}^n}\frac{|\xi|^2}{1+|\xi|^2}|i\xi|^{2\alpha}|\widehat{u}|^2d\xi=-\int_{\mathbb{R}^n}\frac{|\xi|^{2+2k}}{1+|\xi|^2}|\widehat{u}|^2d\xi,$$

where we used the identity $(\xi_1^2+\xi_2^2+\cdots+\xi_n^2)^k=\sum_{|\alpha|=k}C_\alpha^k\xi_1^{2\alpha_1}\xi_2^{2\alpha_2}\cdots\xi_n^{2\alpha_n}$. Then, one has

$$\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|=k}C_\alpha^k\|\partial^\alpha u(t)\|^2+\int_{\mathbb{R}^n}\frac{|\xi|^{2+2k}}{1+|\xi|^2}|\widehat{u}|^2d\xi=\sum_{j=1}^n\sum_{|\alpha|=k}C_\alpha^kI_{j,\alpha}, \tag{2.9}$$

where

$$I_{j,\alpha} = \langle \partial_{x_j} \partial^\alpha f_j(u), -\partial^\alpha u \rangle.$$

Next, we estimate $I_{j,\alpha}$ for a fixed j with $1 \le j \le n$ and $|\alpha| = k$. From the Taylor expansion

$$f_j(u) = f_j(0) + f_j'(0)u + F_j(u)u^2$$

with

$$F_j(u) \equiv \int_0^1 (1-\theta) f_j''(\theta u) d\theta,$$

it holds that

$$I_{j,\alpha} = f_j'(0)\langle \partial_{x_j}\partial^\alpha u, -\partial^\alpha u\rangle + \langle \partial_{x_j}\partial^\alpha [F_j(u)u^2], -\partial^\alpha u\rangle,$$

where the first term vanishes from the integration by part. Let $\beta = \alpha + e_j$ with $e_j = (0, \cdots, 0, 1, 0, \cdots, 0)$, and then it further holds that

$$\begin{aligned} I_{j,\alpha} &= \langle \partial_{x_j}\partial^\alpha [F_j(u)u^2], -\partial^\alpha u\rangle \\ &= \sum_{\beta_1+\beta_2+\beta_3=\beta} C^\beta_{\beta_1,\beta_2,\beta_3} \langle \partial^{\beta_1} F_j(u) \partial^{\beta_2} u \partial^{\beta_3} u, -\partial^\alpha u\rangle, \end{aligned} \tag{2.10}$$

where $C^\beta_{\beta_1,\beta_2,\beta_3} = C^{\lambda_1}_{(\gamma_1^1,\gamma_2^1,\gamma_3^1)} C^{\lambda_2}_{(\gamma_1^2,\gamma_2^2,\gamma_3^2)} \cdots C^{\lambda_n}_{(\gamma_1^n,\gamma_2^n,\gamma_3^n)}$ with $\beta = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ and $\beta_i = (\gamma_i^1, \gamma_i^2, \cdots, \gamma_i^n)$, $i = 1, 2, 3$.

For the later use, let us give a complete proof for the estimates on $I_{j,\alpha}$ as follows.

Claim:

$$I_{j,\alpha} \le C\epsilon \sum_{1 \le m \le N} \|\nabla^m u\|^2. \tag{2.11}$$

Proof of Claim: We prove (2.11) by two cases.

Case 1: $\beta_1 = 0$.

Subcase 1.1: $\beta_2 = \beta$ or $\beta_3 = \beta$. In this case, without loss of generality, one can suppose $\beta_2 = 0$ and $\beta_3 = \beta$. Then one has

$$\begin{aligned} \langle \partial^{\beta_1} F_j(u) \partial^{\beta_2} u \partial^{\beta_3} u, -\partial^\alpha u\rangle &= \langle F_j(u) u \partial_{x_j} \partial^\alpha u, -\partial^\alpha u\rangle \\ &= \frac{1}{2}\langle F_j(u) \partial_{x_j} u, (\partial^\alpha u)^2\rangle + \frac{1}{2}\langle F_j'(u) u \partial_{x_j} u, (\partial^\alpha u)^2\rangle \\ &\le C\epsilon \|\partial^\alpha u\|^2, \end{aligned}$$

where (2.6) can be used since $N - [n/2] - 1 \ge [n/2] + 1 \ge 1$ by the assumption of Theorem 2.1.

Subcase 1.2: $\beta_2 < \beta$ and $\beta_3 < \beta$. In this case,

$$1 \le |\beta_2| \le |\alpha| = k \le N, \quad 1 \le |\beta_3| \le |\alpha| = k \le N.$$

Moreover, since

$$|\beta_2| + |\beta_3| = k + 1 \le N + 1,$$

and the symmetry of β_2 and β_3, one can suppose $|\beta_2| \le N/2$ for even N, and $|\beta_2| \le (N+1)/2$ for odd N. We only consider the case of even N since for odd N, $N \ge 2[n/2] + 2$ implies $N \ge 2[n/2] + 3$ and thus the estimates can be similarly made. Therefore, it holds that

$$\begin{aligned} \langle \partial^{\beta_1} F_j(u) \partial^{\beta_2} u \partial^{\beta_3} u, -\partial^\alpha u\rangle &= \langle F_j(u) \partial^{\beta_2} u \partial^{\beta_3} u, -\partial^\alpha u\rangle \\ &\le \|F_j(u)\|_{L^\infty} \|\partial^{\beta_2} u\|_{L^\infty} \|\partial^{\beta_3} u\| \cdot \|\partial^\alpha u\| \\ &\le C\epsilon \|\partial^\alpha u\|^2 + C\epsilon \sum_{1 \le m \le N} \|\nabla^m u\|^2, \end{aligned}$$

where (2.6) can be still used since $|\beta_2|\leq N/2\leq N-[n/2]-1$, and also $|\beta_2|\geq 1, |\beta_3|\geq 1$ were used.

Case 2: $|\beta_1|\geq 1$. In this case, $|\beta_2|\leq k$ and $|\beta_3|\leq k$. One can write

$$\begin{aligned}\langle\partial^{\beta_1}F_j(u)\partial^{\beta_2}u\partial^{\beta_3}u,-\partial^\alpha u\rangle &= \langle[\partial^{\beta_1}F_j(u)-F_j'(u)\partial^{\beta_1}u]\partial^{\beta_2}u\partial^{\beta_3}u,-\partial^\alpha u\rangle\\ &\quad+\langle F_j'(u)\partial^{\beta_1}u\partial^{\beta_2}u\partial^{\beta_3}u,-\partial^\alpha u\rangle. \end{aligned}\tag{2.12}$$

For the second term on the r.h.s. of (2.12), one can estimate it as in the Case 1 by considering $\beta_1=\beta=\alpha+e_j$ and $\beta_1<\beta$, and hence it holds that

$$\langle F_j'(u)\partial^{\beta_1}u\partial^{\beta_2}u\partial^{\beta_3}u,-\partial^\alpha u\rangle\leq C\epsilon\|\partial^\alpha u\|^2+C\epsilon\sum_{1\leq m\leq N}\|\nabla^m u\|^2.$$

On the other hand, for the first term, one can suppose $|\beta_1|\geq 2$ since it vanishes for $|\beta_1|=1$. Notice that

$$\partial^{\beta_1}F_j(u)-F_j'(u)\partial^{\beta_1}u=\sum_{m=2}^{|\beta_1|}F_j^{(m)}(u)\sum_{\substack{\gamma_1+\cdots+\gamma_m=\beta_1\\ \gamma_\ell\geq 1,1\leq\ell\leq m}}C_{\gamma_1,\cdots,\gamma_m}\prod_{\ell=1}^m\partial^{\gamma_\ell}u,$$

which by (2.6) gives

$$|\partial^{\beta_1}F_j(u)-F_j'(u)\partial^{\beta_1}u|\leq C\epsilon\sum_{1\leq m\leq|\beta_1|-1}|\nabla^m u|.$$

Then, one has

$$\begin{aligned}\langle[\partial^{\beta_1}F_j(u)-F_j'(u)\partial^{\beta_1}u]\partial^{\beta_2}u\partial^{\beta_3}u,-\partial^\alpha u\rangle &\leq C\epsilon\sum_{1\leq m\leq|\beta_1|-1}\int_{\mathbb{R}^n}|\nabla^m u|\cdot|\partial^{\beta_2}u|\cdot|\partial^{\beta_3}u|\cdot|\partial^\alpha u|dx\\ &\leq C\epsilon\|\partial^\alpha u\|^2+C\epsilon\sum_{1\leq m\leq N}\|\nabla^m u\|^2,\end{aligned}$$

where (2.6) can be used again since $|\beta_1|+|\beta_2|+|\beta_3|\leq N+1$ and $|\beta_1|\geq 2$ imply that at most one of the following inequality holds

$$|\beta_1|-1\geq N-\left[\frac{n}{2}\right]-1,\quad |\beta_2|\geq N-\left[\frac{n}{2}\right]-1,\quad |\beta_3|\geq N-\left[\frac{n}{2}\right]-1.$$

Therefore, combining Case 1 and Case 2, (2.11) follows.

It follows from the above estimates that

$$\sum_{j=1}^n\sum_{|\alpha|=k}C_\alpha^k I_{j,\alpha}\leq C\epsilon\sum_{1\leq m\leq N}\|\nabla^m u\|^2$$

which together with (2.9) gives (2.8). This completes the proof of Lemma 2.2. □

To obtain the time-decay rate of solutions, we need the following two lemmas about the estimates of solutions in L^1 and the interpolation inequality for the low-order.

Lemma 2.3. *Let u be the solution to the Cauchy problem* (2.1)-(2.2) *over* $[0,T]$. *If $\|u_0\|_{L^1}$ is bounded, then it holds that*

$$\|u(t)\|_{L^1}\leq\|u_0\|_{L^1}\tag{2.13}$$

for any $0\leq t\leq T$.

Proof. Let φ be the standard mollifier and $\varphi_\delta(x)=\frac{1}{\delta}\varphi(\frac{x}{\delta})$ for $\delta>0$. Furthermore, let sgn be the sign function over $\mathbb{R}$ and its mollified function $\mathrm{sgn}_\delta=\varphi_\delta*\mathrm{sgn}$ for $\delta>0$. Multiplying (2.1) by $\mathrm{sgn}_\delta(u)$ and taking integration over $\mathbb{R}^n$, one has

$$\frac{d}{dt}\int_{\mathbb{R}^n}\int_0^u\mathrm{sgn}_\delta(\eta)d\eta dx+\int_{\mathbb{R}^n}\mathrm{sgn}_\delta(u)\nabla\cdot f(u)dx+\int_{\mathbb{R}^n}\mathrm{sgn}_\delta'(u)\nabla Pu\cdot\nabla udx=0.\tag{2.14}$$

Notice that

$$
\begin{aligned}
\mathrm{sgn}_\delta(u)\nabla\cdot f(u) &= \nabla\cdot[\mathrm{sgn}_\delta(u)(f(u)-f(0))]\\
&\quad -\nabla\cdot\left[\int_0^u \mathrm{sgn}'_\delta(\eta)(f(\eta)-f(0))d\eta\right].
\end{aligned}
$$

Thus the second term on the l.h.s. of (2.14) vanishes, and hence one has

$$
\frac{d}{dt}\int_{\mathbb{R}^n}\int_0^u \mathrm{sgn}_\delta(\eta)d\eta dx + \int_{\mathbb{R}^n}\mathrm{sgn}'_\delta(u)\nabla Pu\cdot\nabla u dx = 0. \tag{2.15}
$$

Now, (2.13) follows from (2.15) by taking time integration over $[0,t]$ for any $0\le t\le T$ and passing to the limit $\delta\to 0$, see [15] for the details. This completes the proof of Lemma 2.3. $\square$

Lemma 2.4 (interpolation inequality)**.** *Let $n\ge 1$. It holds that*

$$
\int_{\mathbb{R}^n}\frac{1}{1+|\xi|^2}|\widehat{u}|^2 d\xi \le C\left(\int_{\mathbb{R}^n}\frac{|\xi|^2}{1+|\xi|^2}|\widehat{u}|^2 d\xi\right)^{\frac{n}{n+2}}\|\widehat{u}\|_{L^\infty}^{\frac{4}{n+2}} \tag{2.16}
$$

for some constant C.

Proof. For $R>0$ to be chosen we split the integral $\int_{\mathbb{R}^n}=\int_{|\xi|\ge R}+\int_{|\xi|\le R}$ and estimate

$$
\begin{aligned}
\int_{\mathbb{R}^n}\frac{1}{1+|\xi|^2}|\widehat{u}|^2 d\xi &\le \frac{1}{R^2}\int_{|\xi|\ge R}\frac{|\xi|^2}{1+|\xi|^2}|\widehat{u}|^2 d\xi + \|\widehat{u}\|_{L^\infty}^2\int_{|\xi|\le R}\frac{1}{1+|\xi|^2}d\xi\\
&\le \frac{A}{R^2}+CBR^n,
\end{aligned}
$$

where

$$
A=\int_{\mathbb{R}^n}\frac{|\xi|^2}{1+|\xi|^2}|\widehat{u}|^2 d\xi,\quad B=\|\widehat{u}\|_{L^\infty}^2. \tag{2.17}
$$

Hence, (2.16) follows by choosing R such that $A/B=CR^{n+2}$. $\square$

<u>Continuation of the Proof of Theorem 2.1</u>: As mentioned before, for the global existence of solutions, we only have to obtain the uniform *a priori* estimates under the assumption (2.5). In fact, notice that

$$
\begin{aligned}
\frac{2|\xi|^2}{1+|\xi|^2}+\sum_{k=1}^N\frac{2|\xi|^{2+2k}}{1+|\xi|^2} &= \frac{|\xi|^2}{1+|\xi|^2}+\sum_{k=1}^N\left(\frac{|\xi|^{2+2(k-1)}}{1+|\xi|^2}+\frac{|\xi|^{2+2k}}{1+|\xi|^2}\right)+\frac{|\xi|^{2+2N}}{1+|\xi|^2}\\
&= \frac{|\xi|^2}{1+|\xi|^2}+\sum_{k=1}^N|\xi|^{2k}+\frac{|\xi|^{2+2N}}{1+|\xi|^2}.
\end{aligned}
$$

Thus, adding up (2.7) and (2.8) over $1\le k\le N$ gives

$$
\begin{aligned}
\frac{d}{dt}\sum_{k\le N}\sum_{|\alpha|=k}C_\alpha^k\|\partial^\alpha u(t)\|^2+\sum_{k=1}^N\int_{\mathbb{R}^n}|\xi|^{2k}|\widehat{u}|^2 d\xi+\int_{\mathbb{R}^n}\left(\frac{|\xi|^2}{1+|\xi|^2}+\frac{|\xi|^{2+2N}}{1+|\xi|^2}\right)|\widehat{u}|^2 d\xi&\\
\le C\epsilon\sum_{1\le m\le N}\|\nabla^m u(t)\|^2.&
\end{aligned}
$$

By noticing

$$
\sum_{1\le m\le N}\|\nabla^m u(t)\|^2\le C\sum_{1\le m\le N}\int_{\mathbb{R}^n}|\xi|^{2m}|\widehat{u}|^2 d\xi,
$$

and taking $\epsilon > 0$ small such that $C\epsilon \le 1/2$, then it follows that

$$\frac{d}{dt}\sum_{k\le N}\sum_{|\alpha|=k} C_\alpha^k\|\partial^\alpha u(t)\|^2 + \frac{1}{2}\sum_{k=1}^N \int_{\mathbb{R}^n}|\xi|^{2k}|\widehat{u}|^2 d\xi + \int_{\mathbb{R}^n}\left(\frac{|\xi|^2}{1+|\xi|^2}+\frac{|\xi|^{2+2N}}{1+|\xi|^2}\right)|\widehat{u}|^2 d\xi \le 0.$$

One can take the further time integration and use the following equivalent relations

$$\sum_{k\le N}\sum_{|\alpha|=k} C_\alpha^k\|\partial^\alpha u(t)\|^2 \sim \|u(t)\|_{H^N}^2,$$

and

$$|\xi|^{2m} \sim \sum_{|\alpha|=m}(i\xi)^{2\alpha},$$

for any integer m. Therefore, (2.3) follows for any $0 \le t \le T$, which gives the uniform *a priori* estimates and hence the global existence is proven.

To obtain the decay rates, let us suppose that $\|u_0\|_{L^1}$ is bounded in the sequel. Take $a \ge 0$ to be determined later and then it follows from (2.7) that

$$(1+t)^a\|u(t)\|^2 + 2\int_0^t\int_{\mathbb{R}^n}(1+s)^a\frac{|\xi|^2}{1+|\xi|^2}|\widehat{u}|^2 d\xi ds \le \|u_0\|^2 + a\int_0^t\int_{\mathbb{R}^n}(1+s)^{a-1}|\widehat{u}|^2 d\xi ds. \tag{2.18}$$

To estimate the r.h.s. integral term of (2.18), one can write it as

$$I = I_1 + I_2. \tag{2.19}$$

with

$$\begin{aligned} I_1 &= a\int_0^t\int_{\mathbb{R}^n}(1+s)^{a-1}\frac{|\xi|^2}{1+|\xi|^2}|\widehat{u}|^2 d\xi ds,\\ I_2 &= a\int_0^t\int_{\mathbb{R}^n}(1+s)^{a-1}\frac{1}{1+|\xi|^2}|\widehat{u}|^2 d\xi ds. \end{aligned}$$

For I_1, we claim that

$$I_1 \le \eta\int_0^t\int_{\mathbb{R}^n}(1+s)^a\frac{|\xi|^2}{1+|\xi|^2}|\widehat{u}|^2 d\xi ds + C_{a,\eta}\|u_0\|^2, \tag{2.20}$$

for any $\eta > 0$, where $C_{a,\eta} \ge 1$. In fact, if $0 \le a \le 1$, it holds that

$$I_1 \le \int_0^t\int_{\mathbb{R}^n}\frac{|\xi|^2}{1+|\xi|^2}|\widehat{u}|^2 d\xi ds \le \frac{\|u_0\|^2}{2}$$

by using (2.18) for the case of $a = 0$. On the other hand, if $a > 1$, it follows from the Young inequality $(a-1)/a + 1/a = 1$ that

$$I_1 \le \eta\int_0^t\int_{\mathbb{R}^n}(1+s)^a\frac{|\xi|^2}{1+|\xi|^2}|\widehat{u}|^2 d\xi ds + C_{a,\eta}\int_0^t\int_{\mathbb{R}^n}\frac{|\xi|^2}{1+|\xi|^2}|\widehat{u}|^2 d\xi ds,$$

for any $\eta > 0$. Thus (2.20) follows again by using (2.18) for $a = 0$. For I_2, it holds that

$$\begin{aligned} I_2 &\le C\int_0^t (1+s)^{a-1}A^{\frac{n}{n+2}}B^{\frac{2}{n+2}}ds\\ &\le \eta\int_0^t(1+s)^a A ds + C_{a,\eta}\int_0^t(1+s)^{a-\frac{n+2}{2}}B ds\\ &\le \eta\int_0^t(1+s)^a A ds + C_{a,\eta}\|u_0\|_{L^1}^2\int_0^t(1+s)^{a-\frac{n}{2}-1}ds\\ &\le \eta\int_0^t\int_{\mathbb{R}^n}(1+s)^a\frac{|\xi|^2}{1+|\xi|^2}|\widehat{u}|^2 d\xi ds + C_{a,\eta}\|u_0\|_{L^1}^2(1+t)^{a-\frac{n}{2}}, \end{aligned} \tag{2.21}$$

where we used the same notation (2.17) for A and B, and also we used Lemma 2.4, Lemma 2.3 and the Young inequality $n/(n+2)+2/(n+2)=1$, and $a>n/2$ was chosen. Therefore, putting (2.20) and (2.21) into (2.19) together with (2.18), one has

$$(1+t)^a\|u(t)\|^2+\lambda\int_0^t\int_{\mathbb{R}^n}(1+s)^a\frac{|\xi|^2}{1+|\xi|^2}|\widehat{u}|^2d\xi ds\leq C\|u_0\|_{L^2}^2+C\|u_0\|_{L^1}^2(1+t)^{a-\frac{n}{2}}$$

for $a>n/2$ and any $t\geq 0$. Then (2.4) follows. This completes the proof of Theorem 2.1. □

3 Stability of smooth planar rarefaction waves in dimensions $n\geq 2$

In this section, we still consider the Cauchy problem of the equation (1.3) without sources:

$$\partial_t u+\nabla\cdot f(u)=\Delta Pu,\quad t>0,\quad x\in\mathbb{R}^n. \tag{3.1}$$

As we are interested in planar rarefaction waves, we shall additionally suppose the condition that the first component of the flux $f_1(\cdot)$ is uniformly convex over $\mathbb{R}$, i.e., there is $\kappa>0$ such that

$$f_1''(u)\geq 2\kappa>0 \tag{3.2}$$

for any $u\in\mathbb{R}$, and that the solution is subject to initial data

$$u(0,x)=u_0(x),\quad x\in\mathbb{R}^n, \tag{3.3}$$

where u_0 may have the different end states at infinity along the x_1-direction:

$$u_0(x)\to u_\pm\ \text{ as } x_1\to\pm\infty.$$

Throughout this section, we are interested in the case when $u_-<u_+$. In this case, it is well-known (see e.g. [22, 11]) that there exists the 1-D smoothed rarefaction wave $\phi(t,x_1)$ which is determined by

$$\begin{gathered}\partial_t\phi_t+\partial_{x_1}f_1(\phi)=(1-\partial_{x_1}^2)^{-1}\partial_{x_1}^2\phi,\\ \phi(t,x_1)\to u_\pm\ \text{ as }\ x_1\to\pm\infty,\\ \phi_{x_1}(t,x_1)\geq 0,\quad u_-<u_+,\end{gathered}$$

Moreover, $\phi(t,x_1)$ enjoys the following time-decay estimates.

Proposition 3.1 (see [22, 11])**.** *Let $1\leq p\leq\infty$ and integer $k\geq 1$. Then it holds that*

$$\|\partial_{x_1}^k\phi(t)\|_{L^p}\leq C|u_+-u_-|(1+t)^{-\frac{1}{2}(k-\frac{1}{p})}\log(2+t),$$

for any $t\geq 0$.

The goal of this section is to prove the stability of the above 1-D smoothed rarefaction wave and further obtain the rate of convergence of solutions to it by using the energy method developed in Section 2. For this purpose, let us set the perturbation v by

$$u(t,x)=v(t,x)+\phi(t,x_1).$$

The Cauchy problem (3.1)-(3.3) is reformulated as

$$v_t+\nabla\cdot[f(\phi+v)-f(\phi)]=\Delta Pv,\quad t>0,\quad x\in\mathbb{R}^n, \tag{3.4}$$

$$v(0,x)=v_0(x),\quad x\in\mathbb{R}^n. \tag{3.5}$$

The main result of this section improves the previous stability results for dimensions up to $n=5$ [15, 11, 12] by providing a unified proof for all dimensions $n\geq 2$.

Theorem 3.1. *Let $n \geq 2$, $N \geq 2[n/2]+2$, and $\delta = |u_+ - u_-|$ with $u_- < u_+$. Assume that the first component of the flux f_1 is strictly convex as in (3.2). Then, there are constants $\epsilon_0 > 0$, $\lambda > 0$ and C such that if $\|v_0\|_{H^N} + \delta \leq \epsilon_0$ then the Cauchy problem (3.4)-(3.5) admits a unique smooth solution $v(t,x)$ satisfying*

$$\|v(t)\|_{H^N}^2 + \lambda \sum_{k\leq N} \int_0^t \int_{\mathbb{R}^n} \phi_{x_1}(t,x_1)|\partial^k v(s,x)|^2 dx ds + \lambda \int_0^t \|\nabla v(s)\|_{H^{N-1}}^2 ds$$
$$+ \lambda \int_0^t (\|\nabla \sqrt{P} v(s)\|^2 + \|\nabla^{N+1}\sqrt{P}v(s)\|^2) ds \leq C(\|v_0\|_{H^N}^2 + \delta^2), \quad (3.6)$$

for any $t \geq 0$. Moreover, if $\|v_0\|_{L^1}$ is bounded, then the obtained solution v enjoys the time-decay estimate

$$\|v(t)\| \leq C\|v_0\|_{L^2\cap L^1}(1+t)^{-\frac{n}{4}}. \quad (3.7)$$

Proof. Similar to Section 2, the existence of local smooth solutions follows by standard arguments, and we are left to give a proof of (3.6) and (3.7) for solutions satisfying

$$\sup_{0\leq t\leq T} \|v(t)\|_{H^N} \leq \epsilon, \quad |u_+ - u_-| \leq \delta,$$

for constants $0 < \epsilon \leq 1$ and $0 < \delta \leq 1$ small enough, where $v(t)$ is the solution to the Cauchy problem (3.4)-(3.5) over $[0,T]$ for $T > 0$.

Step 1. The first step of the energy method is to obtain the zero-order estimate:

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2 + \kappa \int_{\mathbb{R}^n} \phi_{x_1}(t,x_1)|v(t,x)|^2 dx + \int_{\mathbb{R}^n} \frac{|\xi|^2}{1+|\xi|^2}|\widehat{v}(t)|^2 d\xi \leq 0. \quad (3.8)$$

In fact, as before, the energy integration gives

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2 + \langle \nabla\cdot[f(\phi+v) - f(\phi)], v\rangle = \langle \Delta P v, v\rangle.$$

We only have to consider the term related to f. One has the following identity

$$\nabla\cdot[f(\phi+v)-f(\phi)]v = \nabla\cdot\left\{[f(\phi+v)-f(\phi)]v - \int_0^v [f(\phi+\eta)-f(\phi)]d\eta\right\}$$
$$+ \int_0^v [f_1'(\phi+\eta) - f_1'(\phi)]d\eta \phi_{x_1}(t,x_1),$$

where the uniform convexity of $f_1(\cdot)$ implies

$$\int_0^v [f_1'(\phi+\eta)-f_1'(\phi)]d\eta \geq \kappa |v|^2.$$

Thus the nonnegativity of $\phi_{x_1}(t,x_1)$ gives

$$\langle \nabla\cdot[f(\phi+v)-f(\phi)], v\rangle \geq \kappa \int_{\mathbb{R}^n} \phi_{x_1}(t,x_1)|v|^2 dx.$$

Hence (3.8) is proved.

Step 2. The second step of the energy method is to obtain the estimates for the high-order:

$$\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|=k} C_\alpha^k \|\partial^\alpha v(t)\|^2 + \kappa \sum_{|\alpha|=k} C_\alpha^k \int_{\mathbb{R}^n} \phi_{x_1}(t,x_1)|\partial^\alpha v(t)|^2 dx + \int_{\mathbb{R}^n} \frac{|\xi|^{2+2k}}{1+|\xi|^2}|\widehat{v}(t)|^2 d\xi$$
$$\leq C(\epsilon+\delta)\sum_{1\leq m\leq k}\|\nabla^m v(t)\|^2 + C\sum_{m<k}\int_{\mathbb{R}^n}\phi_{x_1}(t,x_1)|\nabla^m v(t)|^2 dx + \frac{C\delta^2}{(1+t)^\sigma}, \quad (3.9)$$

for $1 \le k \le N$ for some constant $\sigma > 1$. In fact, similarly as before, one has

$$\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|=k} C_\alpha^k \|\partial^\alpha v(t)\|^2 + \int_{\mathbb{R}^n} \frac{|\xi|^{2+2k}}{1+|\xi|^2}|\hat{v}(t)|^2 d\xi = \sum_{j=1}^n \sum_{|\alpha|=k} C_\alpha^k I_{j,\alpha}, \tag{3.10}$$

where

$$I_{j,\alpha} = -\langle \partial_{x_j}\partial^\alpha [f_j(\phi+v) - f_j(\phi)], \partial^\alpha v\rangle. \tag{3.11}$$

Putting the identity

$$f_j(\phi+v) - f_j(\phi) = f_j'(\phi)v + \int_0^1 (1-\theta) f_j''(\phi+\theta v) d\theta v^2$$

into (3.11), one has

$$I_{j,\alpha} = I_{j,\alpha}^1 + I_{j,\alpha}^2, \tag{3.12}$$

with

$$\begin{aligned} I_{j,\alpha}^1 &= -\langle \partial_{x_j}\partial^\alpha [f_j'(\phi)v], \partial^\alpha v\rangle, \\ I_{j,\alpha}^2 &= -\left\langle \partial_{x_j}\partial^\alpha \left(\int_0^1 (1-\theta) f_j''(\phi+\theta v) d\theta v^2 \right), \partial^\alpha v \right\rangle. \end{aligned}$$

Next, we give the estimates on $I_{j,\alpha}^1$ and $I_{j,\alpha}^2$ similarly as before.

Estimate on $I_{j,\alpha}^1$. We claim that

$$I_{j,\alpha}^1 \le -\kappa\delta_{1j}\int_{\mathbb{R}^n} \phi_{x_1}|\partial^\alpha v|^2 dx + C\sum_{m<k}\int_{\mathbb{R}^n} \phi_{x_1}|\nabla^m v|^2 dx + C\delta \sum_{1\le m\le k}\int_{\mathbb{R}^n} |\nabla^m v|^2 dx, \tag{3.13}$$

where δ_{ij} denotes the Kronecker Delta symbol.

In fact, from the integration by part, $I_{j,\alpha}^1$ can be further written as

$$\begin{aligned} I_{j,\alpha}^1 &= \langle \partial^\alpha [f_j'(\phi)v], \partial_{x_j}\partial^\alpha v\rangle \\ &= \langle f_j'(\phi)\partial^\alpha v, \partial_{x_j}\partial^\alpha v\rangle + \sum_{\beta<\alpha}\langle \partial^{\alpha-\beta} f_j'(\phi)\partial^\beta v, \partial_{x_j}\partial^\alpha v\rangle \\ &= I_{j,\alpha}^{1,1} + I_{j,\alpha}^{1,2}. \end{aligned} \tag{3.14}$$

Here, from the uniform convexity of $f_1(\cdot)$ and nonnegativity of ϕ_{x_1}, one has

$$\begin{aligned} I_{j,\alpha}^{1,1} &= -\frac{1}{2}\langle \partial_{x_j} f_j'(\phi), (\partial^\alpha v)^2\rangle = -\frac{1}{2}\delta_{1j}\langle f_j''(\phi)\phi_{x_1}, (\partial^\alpha v)^2\rangle \\ &\le -\kappa\delta_{1j}\int_{\mathbb{R}^n} \phi_{x_1}|\partial^\alpha v|^2 dx. \end{aligned} \tag{3.15}$$

On the other hand, for $I_{j,\alpha}^{1,2}$, it holds that

$$I_{j,\alpha}^{1,2} = -\delta_{1j}\sum_{\beta<\alpha}\langle \partial^{\alpha-\beta} f_j''(\phi)\phi_{x_1}\partial^\beta v, \partial^\alpha v\rangle - \sum_{\beta<\alpha}\langle \partial^{\alpha-\beta} f_j'(\phi)\partial^\beta\partial_{x_j} v, \partial^\alpha v\rangle,$$

where one further has

$$\begin{aligned} -\delta_{1j}\sum_{\beta<\alpha}\langle \partial^{\alpha-\beta} f_j''(\phi)\phi_{x_1}\partial^\beta v, \partial^\alpha v\rangle &\le C\sum_{\beta<\alpha}\int_{\mathbb{R}^n} \phi_{x_1}(|\partial^\beta v|^2 + |\partial^\alpha v|^2) dx \\ &\le C\sum_{m<k}\int_{\mathbb{R}^n} \phi_{x_1}|\nabla^m v|^2 dx + C\delta\int_{\mathbb{R}^n} |\nabla^k v|^2 dx, \end{aligned}$$

and

$$-\sum_{\beta<\alpha}\langle\partial^{\alpha-\beta}f_j'(\phi)\partial^\beta\partial_{x_j}v,\partial^\alpha v\rangle\leq C\sum_{\beta<\alpha}\int_{\mathbb{R}^n}|\partial^{\alpha-\beta}f_j'(\phi)|(|\partial^\beta\partial_{x_j}v|^2+|\partial^\alpha v|^2)dx$$
$$\leq C\delta\sum_{1\leq m\leq k}\int_{\mathbb{R}^n}|\nabla^m v|^2dx.$$

Here we used

$$|\phi_{x_1}|\leq C\delta,\quad |\partial^{\alpha-\beta}f_j'(\phi)|\leq C\delta, \tag{3.16}$$

since $\beta<\alpha$ implies that the l.h.s. of (3.16) takes zero or contains the derivatives of $\phi(t,x_1)$.

Therefore, for $I^{1,2}_{j,\alpha}$, it holds that

$$I^{1,2}_{j,\alpha}\leq C\sum_{m<k}\int_{\mathbb{R}^n}\phi_{x_1}|\nabla^m v|^2dx+C\delta\sum_{1\leq m\leq k}\int_{\mathbb{R}^n}|\nabla^m v|^2dx. \tag{3.17}$$

Then, (3.13) follows from (3.15) and (3.17) together with (3.14).

Estimate on $I^2_{j,\alpha}$. This estimates contain the key argument of the proof to control the transversal directions of the x_1-directed rarefraction wave $\phi(t,x_1)$. Similar to (2.10), $I^2_{j,\alpha}$ can be written as

$$I^2_{j,\alpha}=\sum_{\beta_1+\beta_2+\beta_3=\beta}C^\beta_{\beta_1,\beta_2,\beta_3}I_{\beta_1,\beta_2,\beta_3},$$

with

$$I_{\beta_1,\beta_2,\beta_3}=\left\langle\partial^{\beta_1}\left(\int_0^1(1-\theta)f_j''(\phi+\theta v)d\theta\right)\partial^{\beta_2}v\partial^{\beta_3}v,-\partial^\alpha v\right\rangle,$$

where $\beta=\alpha+e_j$ with $|\alpha|=k$ and $1\leq k\leq N$. We claim that

$$I^2_{j,\alpha}\leq C(\epsilon+\delta)\sum_{1\leq m\leq N}\int_{\mathbb{R}^n}|\nabla^m v(t)|^2dx+\frac{C\delta^2}{(1+t)^\sigma}, \tag{3.18}$$

for some constant $2>\sigma>1$. In fact, when $\beta_1=0$, $I_{\beta_1,\beta_2,\beta_3}$ is bounded similarly to before by the first term of the r.h.s. of (3.18). The main trouble lies in the case of $|\beta_1|\geq 1$ since ϕ depends only on x_1 and thus it is not integrable over $\mathbb{R}^n$ with $n\geq 2$. However, these bad terms have good time-decay estimates. Next, we consider the case of $|\beta_1|\geq 1$. If $|\beta_2|\geq 1$ or $|\beta_3|\geq 1$ which leads to $|\beta_1|\leq N$, then $I_{\beta_1,\beta_2,\beta_3}$ is still bounded by the first term of the r.h.s. of (3.18) from the Sobolev inequality. Therefore, we are left the case $\beta_1=\alpha+e_j$ and $\beta_2=\beta_3=0$, for which one can compute

$$\begin{aligned}I_{\beta_1,\beta_2,\beta_3}&=\left\langle\partial_{x_j}\partial^\alpha\left(\int_0^1(1-\theta)f_j''(\phi+\theta v)d\theta\right)v^2,-\partial^\alpha v\right\rangle\\&=\delta_{1j}\left\langle\partial^\alpha\left(\int_0^1(1-\theta)f_j^{(3)}(\phi+\theta v)d\theta\,\phi_{x_1}(t,x_1)\right)v^2,-\partial^\alpha v\right\rangle\\&\quad+\left\langle\partial^\alpha\left(\int_0^1(1-\theta)f_j^{(3)}(\phi+\theta v)\theta d\theta\,\partial_{x_j}v\right)v^2,-\partial^\alpha v\right\rangle.\end{aligned} \tag{3.19}$$

For the second term on the r.h.s. of (3.19), it can be rewritten as follows:

$$
\begin{aligned}
&\left\langle \partial^\alpha \left(\int_0^1 (1-\theta) f_j^{(3)}(\phi+\theta v)\theta d\theta\, \partial_{x_j} v \right) v^2, -\partial^\alpha v \right\rangle \\
&= \left\langle \left(\int_0^1 (1-\theta) f_j^{(3)}(\phi+\theta v)\theta d\theta \right) 2(\partial_{x_j}\partial^\alpha v) v^2, -\partial^\alpha v \right\rangle \\
&\quad + \sum_{\gamma<\alpha} C_\gamma^\alpha \left\langle \partial^{\alpha-\gamma} \left(\int_0^1 (1-\theta) f_j^{(3)}(\phi+\theta v)\theta d\theta \right) (\partial_{x_j}\partial^\gamma v) v^2, -\partial^\alpha v \right\rangle \\
&= \frac{1}{2} \left\langle \partial_{x_j} \left(\int_0^1 (1-\theta) f_j^{(3)}(\phi+\theta v)\theta d\theta\, v^2 \right), (\partial^\alpha v)^2 \right\rangle \\
&\quad + \sum_{\gamma<\alpha} C_\gamma^\alpha \left\langle \partial^{\alpha-\gamma} \left(\int_0^1 (1-\theta) f_j^{(3)}(\phi+\theta v)\theta d\theta \right) (\partial_{x_j}\partial^\gamma v) v^2, -\partial^\alpha v \right\rangle,
\end{aligned}
$$

which is again bounded by the first term of the r.h.s. of (3.18) from the Sobolev inequality since $1 \le |\alpha| = k \le N$. For the first term on the r.h.s. of (3.19), if ∂^α contains transversal x_ℓ-derivative for some $2 \le \ell \le n$, it is also bounded by the first term of the r.h.s. of (3.18) similarly, and hence we only have to consider the case of $\partial^\alpha = \partial_{x_1}^{|\alpha|}$ to obtain the estimates on

$$
\delta_{1j} \left\langle \partial_{x_1}^k \left(\int_0^1 (1-\theta) f_j^{(3)}(\phi+\theta v) d\theta\, \phi_{x_1}(t,x_1) \right) v^2, -\partial^\alpha v \right\rangle .
$$

The above term can be written as

$$
\begin{aligned}
&\delta_{1j} \left\langle \partial_{x_1}^{k-1} \left\{ \int_0^1 (1-\theta) f_j^{(4)}(\phi+\theta v) d\theta\, [\phi_{x_1}(t,x_1)]^2 \right\} v^2, -\partial^\alpha v \right\rangle \\
&+\delta_{1j} \left\langle \partial_{x_1}^{k-1} \left\{ \int_0^1 (1-\theta) f_j^{(3)}(\phi+\theta v) d\theta\, \phi_{x_1x_1}(t,x_1) \right\} v^2, -\partial^\alpha v \right\rangle \\
&+\delta_{1j} \left\langle \partial_{x_1}^{k-1} \left\{ \int_0^1 (1-\theta) f_j^{(4)}(\phi+\theta v)\theta d\theta\, \phi_{x_1}(t,x_1)\partial_{x_1} v \right\} v^2, -\partial^\alpha v \right\rangle,
\end{aligned}
$$

where the third term is bounded by the first term of the r.h.s. of (3.18), and the sum of the first and second terms is bounded by

$$
\begin{aligned}
&C \int_{\mathbb{R}^n} |\partial^\alpha v|^2 dx \\
&+C \int_{\mathbb{R}^n} \left| \partial_{x_1}^{k-1} \left\{ \int_0^1 (1-\theta) f_j^{(4)}(\phi+\theta v) d\theta [\phi_{x_1}(t,x_1)]^2 \right\} \right|^2 |v|^2 dx \\
&+C \int_{\mathbb{R}^n} \left| \partial_{x_1}^{k-1} \left\{ \int_0^1 (1-\theta) f_j^{(3)}(\phi+\theta v) d\theta \phi_{x_1x_1}(t,x_1) \right\} \right|^2 |v|^2 dx \\
&\le C\epsilon \int_{\mathbb{R}^n} |\partial^\alpha v|^2 dx + \frac{C\epsilon\delta^2}{(1+t)^\sigma} \int_{\mathbb{R}^n} |v|^2 dx,
\end{aligned}
$$

by using Proposition 3.1, for $2 > \sigma > 1$. Thus, (3.18) holds for the case of $\partial^\alpha = \partial_{x_1}^{|\alpha|}$. Collecting the estimates for all cases, (3.18) is proved.

Therefore, from (3.12), putting the estimates (3.13) and (3.18) into (3.10) yields (3.9).

Step 3. The linear combination of the energy estimates (3.8) for the zero-order and (3.9) for the high-order gives the uniform energy inequality (3.6) after taking time integrations. In fact, one can first take a suitable linear combination of (3.9) for $1 \le k \le N$ so that the second terms on the r.h.s. of (3.9) are dominated by the second terms on the l.h.s. for all orders $m < k$, except for $m = 0$ and the r.h.s. takes the form

$$
C(\epsilon+\delta) \sum_{1\le m\le N} \|\nabla^m v(t)\|^2 + C \int_{\mathbb{R}^n} \phi_{x_1}(t,x_1)|v(t)|^2 dx + \frac{C\delta^2}{(1+t)^\sigma},
$$

for $2>\sigma>1$. The obtained energy inequality can further be made the proper linear combination with (3.8) so that one has

$$\frac{d}{dt}\mathcal{E}(v(t))+\lambda\mathcal{D}(v(t))\leq C(\epsilon+\delta)\sum_{1\leq m\leq N}\|\nabla^m v(t)\|^2+\frac{C\delta^2}{(1+t)^\sigma}, \tag{3.20}$$

for $2>\sigma>1$, where $\mathcal{E}(v(t))$ and $\mathcal{D}(v(t))$ are the equivalent energy functional and dissipation rate, respectively, which as in the proof Theorem 2.1 take

$$\begin{aligned}\mathcal{E}(v(t)) &= \|v(t)\|_{H^N}^2,\\ \mathcal{D}(v(t)) &= \sum_{k\leq N}\int_{\mathbb{R}^n}\phi_{x_1}(t,x_1)|\partial^k v(t,x)|^2dx\\ &\quad+\|\nabla\sqrt{P}v(t)\|^2+\|\nabla v(t)\|_{H^{N-1}}^2+\|\nabla^{N+1}\sqrt{P}v(t)\|^2.\end{aligned}$$

Thus, the first term on the r.h.s. of (3.20) is absorbed by the dissipation $\mathcal{D}(v(t))$ since ϵ,δ are small enough. Then, the time integration of (3.20) gives (3.6) since $2>\sigma>1$. This completes the proof of the uniform *a priori* estimates and hence the global existence.

Step 4. To the end, let $\|v_0\|_{L^1}$ be bounded. Notice that (3.4) can be rewritten as

$$\partial_t v+\nabla\cdot[f(\phi+v)-f(\phi)-\nabla Pv]=0,$$

which is in the form of the conservation. Thus the same argument as in Lemma 2.3 leads to

$$\|v(t)\|_{L^1}\leq\|v_0\|_{L^1},$$

for any $t\geq 0$. Hence, (3.7) follows from the zero-order estimate (3.8) by using the completely same proof as for (2.4). This completes the proof of Theorem 3.1. □

4 Time-periodic solutions

In this section, we consider the time-periodic problem of

$$\partial_t u+\nabla\cdot f(u)=\Delta Pu+S,\quad t\in\mathbb{R},\quad x\in\mathbb{R}^n, \tag{4.1}$$

where $S=S(t,x)$ is time-periodic in time with period $T\geq 0$. We will prove the existence and stability of the time-periodic solutions under some conditions. Without loss of generality, we suppose

$$f_i(0)=f_i'(0)=0,\quad 1\leq i\leq n. \tag{4.2}$$

Otherwise, one can take change of variables

$$\tilde{t}=t,\quad \tilde{x}_i=x_i-f_i'(0)t,$$

and denote $\tilde{f}(\cdot)$ by

$$\tilde{f}(u)=f(u)-f(0)-f'(0)u,$$

so that the form of (4.1) remains unchanged but (4.2) still holds for $\tilde{f}$. In what follows, for simplicity we use $C_{per}([0,T];H^N)$ to denote the Banach space of functions which are time-periodic with period T and have weak derivatives up to N-order, and we also define the triple norm $|\!|\!|\cdot|\!|\!|_{H^N_{per}}$ by

$$|\!|\!|u|\!|\!|_{H^N_{per}}=\sup_{0\leq t\leq T}\|u(t)\|_{H^N}$$

for any $u\in C_{per}([0,T];H^N)$.

4.1 Existence of time-periodic solutions

The main result of this subsection is stated as follows.

Theorem 4.1. *Let $T \geq 0$, $n \geq 5$ and $N \geq 2[n/2]+2$ be given, and let (4.2) hold. Suppose that $S = S(t,x)$ is time-periodic with period T. Then, there are $\delta > 0$, $\epsilon > 0$ such that if*

$$\sup_{0\leq t\leq T} \|S(t)\|_{H^{N+1}\cap L^1} \leq \delta, \tag{4.3}$$

then the equation (4.1) admits a unique solution $u_*(t,x) \in C_{per}([0,T];H^N)$ with

$$\|u_*\|_{H^N_{per}} \leq \epsilon. \tag{4.4}$$

From now on we devote ourselves to the proof of Theorem 4.1. We first consider the Cauchy problem on the linearised equation:

$$\partial_t u - \Delta P u = 0, \quad t > 0, \quad x \in \mathbb{R}^n, \tag{4.5}$$

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^n. \tag{4.6}$$

Let us formally denote the solution to the Cauchy problem (4.5)-(4.6) by

$$u(t) = e^{\Delta P t}u_0.$$

One has the following lemma about the time-decay estimates on the linear solution semigroup $e^{\Delta P t}$.

Lemma 4.1. *Let $k \geq 0$ and $n \geq 1$ be integers and $1 \leq p \leq 2$. Then, it holds that*

$$\|\nabla^k e^{\Delta P t}u_0\| \leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}(\|u_0\|_{L^p} + \|\nabla^k u_0\|), \tag{4.7}$$

for any $t \geq 0$.

Proof. In fact, let us denote $u(t) = e^{\Delta P t}u_0$, and then it follows from (4.5)-(4.6) that

$$\widehat{u}(t) = e^{-\frac{|\xi|^2}{1+|\xi|^2}t}\widehat{u}_0.$$

Thus for $k \geq 0$, it holds that

$$\|\nabla^k u(t)\|^2 \leq C\int_{\mathbb{R}^n} |\xi|^{2k}|\widehat{u}(t)|^2 d\xi \leq C\int_{\mathbb{R}^n} |\xi|^{2k} e^{-\frac{2|\xi|^2}{1+|\xi|^2}t}|\widehat{u}_0|^2 d\xi.$$

From the usual proof by dividing the integration domain by $\{|\xi| \leq 1\} \cup \{|\xi| \geq 1\}$ and using the Hölder and Young inequalities, one has

$$\int_{\mathbb{R}^n} |\xi|^{2k} e^{-\frac{2|\xi|^2}{1+|\xi|^2}t}|\widehat{u}_0|^2 d\xi \leq C(1+t)^{-\frac{n}{p}+\frac{n-2k}{2}}\|u_0\|_{L^p}^2 + Ce^{-\nu t}\|\nabla^k u_0\|^2,$$

for some constant $\nu > 0$ and for any $t \geq 0$ and $1 \leq p \leq 2$. Then (4.7) holds from the above estimate. This completes the proof of Lemma 4.1. □

Next, we extend Lemma 4.1 to the case of the linear equation with variable coefficients for later use. Precisely, we will use the energy-spectrum method to obtain the time-decay estimates on the linear solution operator $\mathbb{A}_v(t,s)$, $-\infty < s \leq t < \infty$ for fixed v which is small in some Sobolev space. Here, $\mathbb{A}_v(t,s)$ is defined in the way that for any $u_0 \in H^\ell$, $\mathbb{A}_v(t,s)u_0$ is the solution to the Cauchy problem

$$\partial_t u + \sum_{j=1}^n f_j'(v)u_{x_j} = \Delta P u, \quad t > s, \quad x \in \mathbb{R}^n, \tag{4.8}$$

$$u|_{t=s} = u_0, \quad x \in \mathbb{R}^n, \tag{4.9}$$

where $v \in C_b(\mathbb{R}_t; H_x^\ell)$ is given, and $\ell \geq 0$ is an integer. It should be pointed out that if $v = 0$, the linear solution operator $\mathbb{A}_v(t,s)$ reduces to $e^{\Delta P(t-s)}$ due to the assumption (4.2). One has the following

Lemma 4.2. *Let $n \geq 3$ and $\ell \geq 2[n/2]+2$. There is $\epsilon_\ell > 0$ small enough such that if*

$$\|v\|_{L_t^\infty(H_x^\ell)} \leq \epsilon_\ell, \tag{4.10}$$

then it holds that

$$\|\mathbb{A}_v(t,s)u_0\|_{H^\ell} \leq C_*(\epsilon_\ell)(1+t-s)^{-\frac{n}{4}}\|u_0\|_{H^\ell\cap L^1}, \tag{4.11}$$

for any $-\infty < s \leq t < \infty$ and any $u_0 \in H^\ell \cap L^1$, where $C_*(\cdot)$ is a non-negative and non-increasing function in the argument.

Proof. Fix v with (4.10) holding for small $\epsilon_\ell > 0$. Without loss of generality one can suppose $s = 0$. Let $u(t,x) = \mathbb{A}_v(t,0)u_0$ be the solution to the Cauchy problem (4.8)-(4.9) for brevity. Similarly as before, from (4.8)-(4.9), for each $1 \leq k \leq \ell$, one has the high-order energy estimate

$$\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|=k} C_\alpha^k\|\partial^\alpha u(t)\|^2 + \int_{\mathbb{R}^n}\frac{|\xi|^{2+2k}}{1+|\xi|^2}|\widehat{u}(t)|^2 d\xi = \sum_{|\alpha|=k}\sum_{j=1}^n \langle -\partial^\alpha[f_j'(v)u_{x_j}], \partial^\alpha u\rangle,$$

where the general term on the r.h.s. can be rewritten as

$$\langle -\partial^\alpha[f_j'(v)u_{x_j}], \partial^\alpha u\rangle = \langle -f_j'(v)\partial^\alpha u_{x_j}, \partial^\alpha u\rangle + \sum_{\beta<\alpha} C_\beta^\alpha\langle -\partial^{\alpha-\beta}f_j'(v)\partial^\beta u_{x_j}, \partial^\alpha u\rangle. \tag{4.12}$$

For the first term on the r.h.s. of (4.12), it follows from integration by part and (4.10) that

$$\langle -f_j'(v)\partial^\alpha u_{x_j}, \partial^\alpha u\rangle = \frac{1}{2}\langle f_j''(v)v_{x_j}, (\partial^\alpha u)^2\rangle \leq C\|v_{x_j}\|_{L^\infty}\|\partial^\alpha u\|^2 \leq C\epsilon_\ell\|\partial^\alpha u\|^2.$$

For the general one of the second term on the r.h.s. of (4.12), one can use the Sobolev inequality and (4.10) to estimate

$$\begin{aligned}
|\langle -\partial^{\alpha-\beta}f_j'(v)\partial^\beta u_{x_j}, \partial^\alpha u\rangle| &\leq \|\partial^{\alpha-\beta}f_j(v)\|_{L^2}\|\partial^\beta u_{x_j}\|_{L^\infty}\|\partial^\alpha u\|_{L^2}\\
&\leq C\|\nabla v\|_{H^{\ell-1}}\|\nabla u\|_{H^{\ell-1}}\|\partial^\alpha u\|\\
&\leq C\epsilon_\ell\|\nabla u\|_{H^{\ell-1}}^2
\end{aligned}$$

for the case when $|\beta| \leq \ell-[n/2]-2$ which implies $|\beta|+1+[n/2]+1 \leq \ell$, and

$$\begin{aligned}
|\langle -\partial^{\alpha-\beta}f_j'(v)\partial^\beta u_{x_j}, \partial^\alpha u\rangle| &\leq \|\partial^{\alpha-\beta}f_j(v)\|_{L^\infty}\|\partial^\beta u_{x_j}\|_{L^2}\|\partial^\alpha u\|_{L^2}\\
&\leq C\|\nabla v\|_{H^{\ell-1}}\|\partial^\beta u_{x_j}\|_{L^2}\|\partial^\alpha u\|_{L^2}\\
&\leq C\epsilon_\ell\|\nabla u\|_{H^{\ell-1}}^2
\end{aligned}$$

for the case when $|\beta| \geq \ell-[n/2]-1$ which together with $\beta < \alpha$ gives $1 \leq |\alpha-\beta| \leq [n/2]+1$. Then, from the above estimates, one has

$$\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|=k} C_\alpha^k\|\partial^\alpha u(t)\|^2 + \int_{\mathbb{R}^n}\frac{|\xi|^{2+2k}}{1+|\xi|^2}|\widehat{u}(t)|^2 d\xi \leq C\epsilon_\ell\sum_{1\leq m\leq\ell}\|\nabla^m u(t)\|^2 \tag{4.13}$$

for $t \geq 0$.

Next, we combine the above energy inequalities with the spectral analysis result Lemma 4.1 to obtain the time-decay of u. In fact, by taking the proper linear combination over $1 \leq k \leq \ell$ and using the smallness of ϵ_ℓ, it follows from (4.13) that there exists the high-order energy functional $\mathcal{E}_{high}(u(t))$ which is equivalent with $\|\nabla u(t)\|_{H^{\ell-1}}$ such that one has

$$\frac{d}{dt}\mathcal{E}_{high}(u(t)) + \lambda\mathcal{E}_{high}(u(t)) \leq C\|\nabla u(t)\|^2 \tag{4.14}$$

for any $t \geq 0$. Since u can be written as

$$u(t) = e^{\Delta Pt}u_0 + \int_0^t e^{\Delta P(t-s)}\left\{-\sum_{j=1}^n f_j'(v)u_{x_j}\right\}ds, \tag{4.15}$$

it follows from Lemma 4.1 that

$$\begin{aligned}\|\nabla u(t)\| &\leq C\|u_0\|_{H^1\cap L^1}(1+t)^{-\frac{n}{4}-\frac{1}{2}}+\sum_{j=1}^{n} C\int_0^t (1+t-s)^{-\frac{n}{4}-\frac{1}{2}}\|f_j'(v)u_{x_j}\|_{H^1\cap L^1}ds\\ &\leq C\|u_0\|_{H^1\cap L^1}(1+t)^{-\frac{n}{4}-\frac{1}{2}}+C\epsilon_\ell\int_0^t (1+t-s)^{-\frac{n}{4}-\frac{1}{2}}\|\nabla u\|_{H^1}ds. \end{aligned}\tag{4.16}$$

Define

$$M(t)=\sup_{0\leq s\leq t}(1+s)^{\frac{n}{2}+1}\mathcal{E}_{high}(u(s)).$$

Since $n\geq 3$ implies $n/4+1/2>1$, it follows from (4.16) that

$$\|\nabla u(t)\|\leq C(1+t)^{-\frac{n}{4}-\frac{1}{2}}\left(\|u_0\|_{H^1\cap L^1}+\epsilon_\ell\sqrt{M(t)}\right).$$

Plugging the above inequality into (4.14) and using the Gronwall inequality, one has

$$\begin{aligned}\mathcal{E}_{high}(u(t))&\leq e^{-\lambda t}\mathcal{E}_{high}(u_0)+C\int_0^t e^{-\lambda(t-s)}\|\nabla u(s)\|^2ds\\ &\leq C(1+t)^{-\frac{n}{2}-1}\left(\|u_0\|^2_{H^\ell\cap L^1}+\epsilon_\ell^2M(t)\right),\end{aligned}$$

which implies that

$$M(t)\leq C\left(\|u_0\|^2_{H^\ell\cap L^1}+\epsilon_\ell^2M(t)\right).$$

Since $\epsilon_\ell>0$ is small, it follows that

$$M(t)\leq C\|u_0\|^2_{H^\ell\cap L^1},$$

that is,

$$\|\nabla u(t)\|_{H^{\ell-1}}\leq C\mathcal{E}_{high}(u(t))\leq C\|u_0\|_{H^\ell\cap L^1}(1+t)^{-\frac{n}{4}-\frac{1}{2}}\tag{4.17}$$

for any $t\geq 0$. To obtain the time-decay of zero-order, one again uses (4.15) to get

$$\|u(t)\|\leq C\|u_0\|_{L^2\cap L^1}(1+t)^{-\frac{n}{4}}+C\epsilon_\ell\int_0^t (1+t-s)^{-\frac{n}{4}}\|\nabla u(s)\|ds,$$

which further from (4.17) gives

$$\|u(t)\|\leq C\|u_0\|_{H^\ell\cap L^1}(1+t)^{-\frac{n}{4}}\tag{4.18}$$

due to $n\geq 3$. Therefore, (4.11) follows from (4.17) together with (4.18). This completes the proof of Lemma 4.2. □

Proof of Theorem 4.1: Suppose all conditions in Theorem 4.1 hold. Let us define the function sequence $\{u^m\}_{m=0}^\infty$ by iteration as follows:

$$\partial_t u^{m+1}+\sum_{j=1}^{n} f_j'(u^m)u^{m+1}_{x_j}=\Delta P u^{m+1}+S,\quad t\in\mathbb{R},\quad x\in\mathbb{R}^n,$$

where $u^0\equiv 0$ and $S=S(t,x)\in C_{per}([0,T];H^{N+1})$. We claim that if (4.3) holds for some small constant $\delta>0$, then for each $m\geq 1$, u^m is well-defined and satisfies

(i) $u^m\in C_{per}([0,T];H^{N+1})$;
(ii) there is $\epsilon_{N+1}>0$ such that for all m,

$$\|u^m\|_{H^{N+1}_{per}}\leq\epsilon_{N+1}.\tag{4.19}$$

We prove this claim by induction. In fact, suppose that (i) and (ii) hold for $m\geq 0$. As in [38], u^{m+1} can be written as the mild form

$$u^{m+1}(t)=\int_{-\infty}^t \mathbb{A}_{u^m}(t,s)\,S(s)\,ds.$$

Then, the periodicity of u^{m+1} is proved by the computations

$$\begin{aligned} u^{m+1}(t+T) &= \int_{-\infty}^{t+T} \mathbb{A}_{u^m}(t+T,s)\, S(s)\, ds \\ &= \int_{-\infty}^{t} \mathbb{A}_{u^m}(t+T,\tau+T)\, S(\tau+T)\, d\tau \\ &= \int_{-\infty}^{t} \mathbb{A}_{u^m}(t,\tau)\, S(\tau)\, d\tau = u^{m+1}(t), \end{aligned}$$

where the second equality follows from taking change of variable $s=\tau+T$, and the third equality holds since S is T-periodic and we used

$$\mathbb{A}_{u^m}(t+T,\tau+T) = \mathbb{A}_{u^m}(t,\tau)$$

due to the fact that u^m is also T-periodic. On the other hand, from Lemma 4.2 and the assumption $n\geq 5$, one has

$$\begin{aligned} \|u^{m+1}(t)\|_{H^{N+1}} &\leq \int_{-\infty}^{t} C_*(\epsilon_{N+1})(1+t-s)^{-\frac{n}{4}}\|S(s)\|_{H^{N+1}\cap L^1}ds \\ &\leq C_*(\epsilon_{N+1}) \sup_{0\leq t\leq T}\|S(t)\|_{H^{N+1}\cap L^1}\int_{-\infty}^{t}(1+t-s)^{-\frac{n}{4}}ds \\ &\leq CC_*(\epsilon_{N+1}) \sup_{0\leq t\leq T}\|S(t)\|_{H^{N+1}\cap L^1}, \end{aligned}$$

where $C_*(\cdot)$ is defined in (4.11) and $\epsilon_{N+1}>0$ is sufficiently small such that Lemma 4.2 can be applied. Thus, from (4.3), one can take $\delta>0$ small such that

$$CC_*(\epsilon_{N+1})\delta \leq \epsilon_{N+1}.$$

This leads to the uniform bound of u^{m+1} as follows

$$|\!|\!|u^{m+1}|\!|\!|_{H^{N+1}_{per}} = \sup_{0\leq t\leq T}\|u^{m+1}(t)\|_{H^{N+1}} \leq \epsilon_{N+1}.$$

Therefore, (i) and (ii) also hold for $m+1$. By induction, (i) and (ii) hold for all $m\geq 1$.

Next, let us define the infinite summation

$$u^0 + \sum_{m=0}^{\infty}(u^{m+1}-u^m) =: u^0 + \lim_{k\to\infty}\sum_{m=0}^{k}(u^{m+1}-u^m). \tag{4.20}$$

We further claim that the above infinite summation is absolutely convergent in $C_{per}([0,T];H^N)$. Actually, the difference $w=u^{m+1}-u^m$ satisfies the equation

$$\partial_t w + \sum_{j=1}^{n} f_j'(u^m)w_{x_j} = \Delta P w - \sum_{j=1}^{n}[f_j'(u^m)-f_j'(u^{m-1})]u^m_{x_j},$$

for any $t\in\mathbb{R}$ and $x\in\mathbb{R}^n$. Thus w also satisfies the following mild form

$$w(t) = \int_{-\infty}^{t}\mathbb{A}_{u^m}(t,s)\left\{-\sum_{j=1}^{n}[f_j'(u^m)-f_j'(u^{m-1})]u^m_{x_j}\right\}ds.$$

Similarly as before, it follows from Lemma 4.2 that

$$\begin{aligned} \|w(t)\|_{H^N} &\leq \sum_{j=1}^{n}\int_{-\infty}^{t} C_*(\epsilon_N)(1+t-s)^{-\frac{n}{4}}\|[f_j'(u^m)-f_j'(u^{m-1})]u^m_{x_j}\|_{H^N\cap L^1}ds \\ &\leq CC_*(\epsilon_{N+1})\int_{-\infty}^{t}(1+t-s)^{-\frac{n}{4}}\|u^m-u^{m-1}\|_{H^N}\|u^m\|_{H^{N+1}}ds \\ &\leq CC_*(\epsilon_{N+1})|\!|\!|u^m-u^{m-1}|\!|\!|_{H^N_{per}}|\!|\!|u^m|\!|\!|_{H^{N+1}_{per}}, \end{aligned}$$

which from (4.19) implies

$$\|w\|_{H^N_{per}} \leq CC_*(\epsilon_{N+1})\epsilon_{N+1}\|u^m - u^{m-1}\|_{H^N_{per}}.$$

Since δ can be arbitrarily small and so is ϵ_{N+1}, there is $\mu < 1$ such that

$$\|u^{m+1} - u^m\|_{H^N_{per}} = \|w\|_{H^N_{per}} \leq \mu\|u^m - u^{m-1}\|_{H^N_{per}}. \tag{4.21}$$

Therefore, the infinite summation given by (4.20) is indeed absolutely convergent in the Banach space $C_{per}([0,T];H^N)$. Let us denote the corresponding limit by $u_*(t,x)$. Then,

$$u_* \in C_{per}([0,T];H^N)$$

holds with

$$\|u_*\|_{H^N_{per}} \leq \liminf_{m\to\infty}\|u^m\|_{H^N_{per}} \leq \liminf_{m\to\infty}\|u^m\|_{H^{N+1}_{per}} \leq \epsilon_{N+1}.$$

Furthermore, by passing to the limit, u_* satisfies the equation

$$\partial_t u + \sum_{j=1}^n f_j'(u)u_{x_j} = \Delta P u + S, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n.$$

Hence, the equation (4.1) admits a solution u_* in $C_{per}([0,T];H^N)$ satisfying (4.4). For uniqueness, suppose that $\widetilde{u}_*$ is any other solution in $C_{per}([0,T];H^N)$ to (4.1) with (4.4) holding instead for $\widetilde{u}_*$. Then, the completely same proof as for (4.21) yields

$$\|u_* - \widetilde{u}_*\|_{H^N_{per}} \leq \mu\|u_* - \widetilde{u}_*\|_{H^N_{per}}$$

for some $\mu < 1$. Thus the uniqueness follows. This completes the proof of Theorem 4.1.

Let us conclude this subsection with a remark to point out how to extend Theorem 4.1 to the case of the usual viscous conservation laws with a conservative time-periodic source. The corresponding result turns out to hold even when the spatial dimension $n \geq 3$. To be precise, let us consider

$$\partial_t u + \nabla\cdot f(u) = \Delta u + \nabla\cdot G, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \tag{4.22}$$

where f satisfies (4.2) and G is T-periodic. It can be shown that if $n \geq 3$, $N \geq 2[n/2]+2$ are supposed, and

$$\int_0^T \|G(s)\|^2_{H^{N+1}\cap L^1} ds$$

is small, then (4.22) admits a unique time-periodic solution $u^{per}(t,x)$ satisfying

$$\|u^{per}\|^2 =: \sup_{0\leq t\leq T}\|u^{per}(t)\|^2_{H^N} + \int_0^T \|\nabla u^{per}(s)\|^2_{H^N} ds < C. \tag{4.23}$$

For the proof, as in [38, 10], the idea is to verify the nonlinear mapping

$$\Psi(u) =: \int_{-\infty}^t e^{\Delta(t-s)}\nabla\cdot[-f(u)+G]ds \tag{4.24}$$

satisfies the fixed point theorem in terms of the triple norm $\|\cdot\|$ defined in (4.23). The case of $3 \leq n \leq 4$ can be dealt with because compared with the case of the weak dissipation term $\Delta P u$, the viscosity term Δu produces one more derivative which is time-space integrable, and also because the source term in (4.24) is in divergence form.

4.2 Stability of time-periodic solutions

In this subsection, we are concerned with the stability of the time-periodic solution $u_*(t,x)$ obtained in Theorem 4.1. For this, let us consider the Cauchy problem

$$\partial_t u + \nabla \cdot f(u) = \Delta P u + S, \quad t > t_0, \quad x \in \mathbb{R}^n, \tag{4.25}$$

with given initial data

$$u(t_0, x) = u_0(x), \quad x \in \mathbb{R}^n, \tag{4.26}$$

for $t_0 \in \mathbb{R}$. The goal of this subsection is to prove that whenever $u_0(x)$ is a small smooth perturbation of $u_*(t_0,x)$, there exists a unique smooth solution $u(t,x)$ to the Cauchy problem (4.25)-(4.26) which is close to $u_*(t,x)$ for all the later time $t > t_0$ and moreover, under the additional condition that the initial perturbation is bounded in L^1, the solution $u(t,x)$ tends to u_* with an algebraic decaying rate when time tends to infinity.

Without loss of generality, we suppose $t_0 = 0$ in what follows. Set the perturbation $v = v(t,x)$ by

$$u(t,x) = u_*(t,x) + v(t,x),$$

and then the Cauchy problem (4.25)-(4.26) is reformulated as

$$\partial_t v + \nabla \cdot [f(u_* + v) - f(u_*)] = \Delta P v, \quad t > 0, \quad x \in \mathbb{R}^n, \tag{4.27}$$

$$v(0,x) = v_0(x) \equiv u_0(x) - u_*(0,x), \quad x \in \mathbb{R}^n. \tag{4.28}$$

The main result of this subsection is stated as follows.

Theorem 4.2. *Let $n \geq 3$ and $N \geq 2[n/2] + 2$ and let (4.2) hold. Suppose that $u_*(t,x)$ is a time-periodic solution satisfying*

$$\sup_{t \in \mathbb{R}} \|u_*(t)\|_{H^{N_0}} \leq \delta \tag{4.29}$$

for a small $\delta > 0$ and for $N_0 \geq N$. There are constants $\epsilon_0 > 0$, $\lambda > 0$ and C such that if $\|v_0\|_{H^N} \leq \epsilon_0$ then the Cauchy problem (4.27)-(4.28) admits a unique smooth solution $v(t,x)$ satisfying

$$\begin{aligned} \|v(t)\|_{H^N}^2 &+ \lambda \int_0^t \|\nabla v(s)\|_{H^{N-1}}^2 ds \\ &+ \lambda \int_0^t (\|\nabla \sqrt{P} v(s)\|^2 + \|\nabla^{N+1} \sqrt{P} v(s)\|^2) ds \leq C \|v_0\|_{H^N}^2, \end{aligned} \tag{4.30}$$

for any $t \geq 0$. Moreover, if $\|v_0\|_{L^1}$ is bounded, then the obtained solution v enjoys the time-decay estimate

$$\|v(t)\|_{H^N} \leq C \|v_0\|_{H^N \cap L^1} (1+t)^{-\frac{n}{4}}, \tag{4.31}$$

for any $t \geq 0$.

Proof. The proof can be done similarly as before, and hence we only give the proof of the uniform estimate (4.30) and the time-decay rate (4.31). To the end, suppose that

$$\sup_{t \geq 0} \|v(t)\|_{H^N} \leq \epsilon,$$

for small $\epsilon > 0$, and (4.29) holds for small $\delta > 0$. We divide the proof by two steps.

Step 1. The zero-order energy estimate gives

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|^2 + \int_{\mathbb{R}^n} \frac{|\xi|^2}{1+|\xi|^2} |\widehat{v}(t)|^2 d\xi \leq C(\epsilon + \delta) \|\nabla v(t)\|^2. \tag{4.32}$$

In fact, the zero-order energy integration shows that

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|^2 + \int_{\mathbb{R}^n} \frac{|\xi|^2}{1+|\xi|^2} |\widehat{v}(t)|^2 d\xi = \langle \nabla \cdot [f(u_* + v) - f(u_*)], -v \rangle.$$

By using the identity

$$f_j(u_*+v)-f(u_*)=f_j'(u_*)v+\int_0^1(1-\theta)f_j''(u_*+\theta v)d\theta v^2, \tag{4.33}$$

and also $f_j'(0)=0$ for each $1\leq j\leq n$, one has

$$\begin{aligned}\langle\nabla\cdot[f(u_*+v)-f(u_*)],-v\rangle &\leq C\int_{\mathbb{R}^n}(|u_*|+|v|)|v|\cdot|\nabla v|dx\\ &\leq C\|(u_*,v)(t)\|_{L^n}\|v(t)\|_{L^{\frac{2n}{n-2}}}\|\nabla v(t)\|\\ &\leq C(\delta+\epsilon)\|\nabla v(t)\|^2,\end{aligned}$$

where the Young inequality $1/n+(n-2)/(2n)+1/2=1$ and the Sobolev inequality were used. Then (4.32) is proved.

The high-order energy estimates give

$$\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|=k}C_\alpha^k\|\partial^\alpha v(t)\|^2+\int_{\mathbb{R}^n}\frac{|\xi|^{2+2k}}{1+|\xi|^2}|\widehat{v}(t)|^2d\xi\leq C(\epsilon+\delta)\sum_{1\leq m\leq N}\|\nabla^m v(t)\|^2, \tag{4.34}$$

for each $1\leq k\leq N$. In fact, as before, it follows from (4.27) that

$$\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|=k}C_\alpha^k\|\partial^\alpha v(t)\|^2+\int_{\mathbb{R}^n}\frac{|\xi|^{2+2k}}{1+|\xi|^2}|\widehat{v}|^2d\xi=\sum_{j=1}^n\sum_{|\alpha|=k}C_\alpha^k(I_{j,\alpha}^1+I_{j,\alpha}^2), \tag{4.35}$$

where $I_{j,\alpha}^1, I_{j,\alpha}^2$ take the form of

$$\begin{aligned}I_{j,\alpha}^1&=\langle\partial_{x_j}\partial^\alpha[f_j'(u_*)v],-\partial^\alpha v\rangle,\\ I_{j,\alpha}^2&=\left\langle\partial_{x_j}\partial^\alpha\left(\int_0^1(1-\theta)f_j''(u_*+\theta v)d\theta v^2\right),-\partial^\alpha v\right\rangle.\end{aligned}$$

By using (4.29) and the same proof as before, one has

$$\begin{aligned}I_{j,\alpha}^1&\leq C\delta\sum_{1\leq m\leq N}\|\nabla^m v(t)\|^2,\\ I_{j,\alpha}^2&\leq C\epsilon\sum_{1\leq m\leq N}\|\nabla^m v(t)\|^2,\end{aligned}$$

which yield (4.35).

Therefore, (4.30) follows from taking the proper linear combination of (4.32) and (4.34), further taking time integrations over $[0,t]$ and then using the smallness of ϵ and δ.

Step 2. In this step, we prove the time-decay estimate (4.31). To the end, we suppose that $\|v_0\|_{L^1}$ is bounded. (4.27) can be written as the mild form:

$$v(t)=e^{\Delta Pt}v_0+\int_0^t e^{\Delta P(t-s)}\nabla\cdot[f(u_*)-f(u_*+v)]ds.$$

Applying Lemma 4.1 about the time-decay estimates on the linear solution operator $e^{\Delta Pt}$, one has

$$\begin{aligned}\|v(t)\|&\leq C(1+t)^{-\frac{n}{4}}\|v_0\|_{L^2\cap L^1}\\ &\quad+C\int_0^t(1+t-s)^{-\frac{n}{4}-\frac{1}{2}}(\|Q(s)\|_{L^1}+\|\nabla Q(s)\|)ds,\end{aligned} \tag{4.36}$$

where Q is denoted by

$$Q=f(u_*)-f(u_*+v).$$

It follows from (4.33) that

$$\|Q(s)\|_{L^1} + \|\nabla Q(s)\| \le C(\epsilon+\delta)\|\nabla v(s)\|. \tag{4.37}$$

On the other hand, from Step 1, one has

$$\frac{d}{dt}\mathcal{E}(v(t)) + \lambda\mathcal{E}(v(t)) \le C\|v(t)\|^2, \tag{4.38}$$

where $\mathcal{E}(v(t))$ is an equivalent energy with $\|v(t)\|_{H^N}$. Now, we claim that (4.31) follows from (4.36), (4.37) and (4.38). In fact, in [10], let us define

$$\mathcal{E}_\infty(t) = \sup_{0\le s\le t}(1+s)^{\frac{n}{2}}\mathcal{E}(v(s)). \tag{4.39}$$

Since $n \ge 3$ implies $n/4+1/2>1$, it follows from (4.36) and (4.37) that

$$\|v(t)\| \le C(1+t)^{-\frac{n}{4}}(\|v_0\|_{L^2\cap L^1} + (\epsilon+\delta)\sqrt{\mathcal{E}_\infty(t)}). \tag{4.40}$$

Then using the Gronwall inequality for (4.38) and combing it with (4.40), one has

$$\begin{aligned}\mathcal{E}(v(t)) &\le e^{-\lambda t}\mathcal{E}(v_0) + C\int_0^t e^{-\lambda(t-s)}\|v(s)\|^2 ds\\ &\le C(1+t)^{-\frac{n}{2}}(\|v_0\|^2_{L^2\cap L^1} + \mathcal{E}(v_0) + (\epsilon^2+\delta^2)\mathcal{E}_\infty(t)),\end{aligned}$$

for any $t \ge 0$, which further gives

$$\mathcal{E}_\infty(t) \le C(\|v_0\|^2_{L^2\cap L^1} + \mathcal{E}(v_0) + (\epsilon^2+\delta^2)\mathcal{E}_\infty(t)).$$

Since ϵ and δ are small, it follows that

$$\mathcal{E}_\infty(t) \le C(\|v_0\|^2_{L^2\cap L^1} + \mathcal{E}(v_0)),$$

which implies (4.31) from the definition (4.39). This completes the proof of Theorem 4.2. □

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References

[1] R.M. Colombo and G. Guerra, Hyperbolic balance laws with a dissipative non local source, *Comm. Pure Appl. Anal.* **7** (2008), 1077-1090.

[2] R.M. Colombo and G. Guerra, Hyperbolic balance laws with a non local source, *Comm. Partial Differential Equations* **32** (2007), 1917-1939.

[3] C.M. Dafermos, *Hyperbolic conservation laws in continuum physics*, Grundlehren der Mathematischen Wissenschaften **325**, Springer-Verlag, Berlin, 2005. xx+626 pp.

[4] A. Dedner and C. Rohde. FV-schemes for a scalar model problem of radiation magnetohydrodynamics. in: R. Herbin, D. Kröner (Eds.), Finite Volumes for Complex Applications III, Hermes Science Publications, 2002, pp. 179-186.

[5] A. Dedner and C. Rohde. Numerical approximation of entropy solutions for hyperbolic integral-differnetial equations. *Numer. Math.* **97** (3) (2004), 441-471.

[6] M. Di Francesco, Initial value problem and relaxation limits of the Hamer model for radiating gases in several space variables, *NoDEA Nonl. Differential Equations Appl.* **13** (2007), 531-562.

[7] M. Di Francesco, K. Fellner, H. Liu, A non-local conservation law with nonlinear "radiation" inhomogeneity, *Journal of Hyperbolic Differential Equations* **5** (1) (2008) pp. 1-23.

[8] M. Di Francesco and C. Lattanzio, Optimal L^1 rate of decay to diffusion waves for the Hamer model of radiating gases, *Appl. Math. Lett.* **19** (2006), 1046-1052.

[9] R.-J. Duan, S. Ukai and T. Yang, A combination of energy method and spectral analysis for studies on systems for gas motions, *Frontiers of Mathematics in China* **4** (2) (2009), 253-282.

[10] R.-J. Duan, S. Ukai, T. Yang and H.-J. Zhao, Optimal decay estimates on the linearized Boltzmann equation with time dependent force and their applications, *Comm. Math. Phys.* **277** (2008), 189-236.

[11] W.-L. Gao and C.-J. Zhu, Asymptotic decay toward the planar rarefaction waves for a model system of the radiating gas in two dimensions, *Math. Models Methods Appl. Sci.* **18** (2008), 511-541.

[12] W.-L. Gao, L.-Z. Ruan and C.-J. Zhu, Decay rates to the planar rarefaction waves for a model system of the radiating gas in n-dimensions, *J. Differential Equations* **244** (2008), 2614-2640.

[13] K. Hamer, Nonlinear effects on the propagation of sound waves in a radiating gas, *Quart. J. Mech. Appl. Math.* **24** (1971), 155-168.

[14] T. Iguchi and S. Kawashima, On space-time decay properties of solutions to hyperbolic-elliptic coupled systems, *Hiroshima Math. J.* **32** (2002), 229-308.

[15] K. Ito, Asymptotic decay toward the planar rarefaction waves of solutions for viscous conservation laws in several space dimentions, *Math. Models Methods Appl. Sci.* **6** (1996), 315-338.

[16] K. Ito, BV-solutions of a hyperbolic-elliptic system for a radiation gas, Preprint 386, Hokkaido University, 1997.

[17] S. Kawashima, Y. Nikkuni and S. Nishibata, Large-time behavior of solutions to hyperbolic-elliptic coupled systems, *Arch. Rational Mech. Anal.* **170** (2003), 297-329.

[18] S. Kawashima and S. Nishibata, A singular limit for hyperbolic-elliptic coupled systems in radiation hydrodynamics, *Indiana Univ. Math. J.* **50** (2001), 567-589.

[19] S. Kawashima and S. Nishibata, Weak solutions with a shock to a model system of the radiating gas, *Sci. Bull. Josai. Univ.* special issue **5** (1998), 119-130.

[20] S. Kawashima and S. Nishibata, Cauchy problem for a model system of the radiating gas: weak solutions with a jump and classical solutions, *Math. Models Methods Appl. Sci.* **9** (1999), 69-91.

[21] S. Kawashima and S. Nishibata, Shock waves for a model system of the radiating gas, *SIAM J. Math. Anal.* **30** (1999), 95-117.

[22] S. Kawashima and Y. Tanaka, Stability of rarefaction waves for a model system of a radiating gas, *Kyushu J. Math.* **58** (2004), 211-250.

[23] S. Kawashima, S. Nishibata and M. Nishikawa, L^p energy method for multi-dimensional viscous conservation laws and application to the stability of planar waves, *J. Hyperbolic Differential Equations* **1** (2004), 581-603.

[24] S. Kružkov, First order quasi-linear equations with several independent variables, *Mat. Sb.* **8**(123) (1970), 228-255.

[25] C. Lattanzio and P. Marcati, Global well-posedness and relaxation limits of a model for radiating gas, *J. Differential Equations* **190** (2003), 439-465.

[26] C. Lattanzio, C. Mascia and D. Serre, Shock waves for radiative hyperbolic-elliptic systems, *Indiana Univ. Math. J.* **56** (2007), 2601-2640.

[27] P. Laurençot, Asymptotic self-similarity for a simplified model for radiating gases, *Asymptot. Anal.* **42** (2005), 251-262.

[28] C.J. Lin, J.F. Coulombel and T. Goudon, Shock profiles for non-equilibrium radiating gases, *Phys. D* **218** (2006), 83-94.

[29] H. Liu and E. Tadmor, Critical thresholds in a convolution model for nonlinear conservation laws, *SIAM J. Math. Anal.* **33** (2001), 930-945.

[30] D. Mihalas and B. Weibel-Mihalas, *Foundations of Radiation Hydrodynamics*, Oxford Univ. Press, New York, 1984.

[31] M. Nishikawa and S. Nishibata, Convergence rates toward the travelling waves for a model system of the radiating gas, *Math. Methods Appl. Sci.* **30** (2007), 649-663.

[32] S. Nishibata, Asymptotic behavior of solutions to a model system of a radiating gas with discontinuous initial data, *Math. Models Methods Appl. Sci.* **10** (2000), 1209-1231.

[33] C. Rohde and W.-A. Yong, The nonrelativistic limit in radiation hydrodynamics, I., weak entropy solutions for a model problem, *J. Differential Equations* **234** (2007), 91-109.

[34] P. Rosenau, Extending hydrodynamics via the regularization of the Chapman-Enskog expansion, *Phys. Rev. A* **40** (3) (1989), 7193-7196.

[35] S. Schochet and E. Tadmor, The regularized Chapman-Enskog expansion for scalar conservation laws, *Arch. Rational Mech. Anal.* **119** (1992), 95-107.

[36] D. Serre, L^1-stability of constants in a model for radiating gases, *Comm. Math. Sci.* **1** (2003), 197-205.

[37] M. Treiber, A. Hennecke, and D. Helbing. Derivation, properties, and simulation of a gas-kinetic-based, non-local traffical model. *Phys. Rev. E*, 59:239-253, 1999.

[38] S. Ukai, Time-periodic solutions of the Boltzmann equation, *Discrete Contin. Dyn. Syst.* **14** (2006), 579-596.

[39] S. Ukai and T. Yang, The Boltzmann equation in the space $L^2 \cap L^\infty_\beta$: global and time-periodic solutions, *Anal. Appl. (Singap.)* **4** (2006), 263-310.

[40] W.G. Vincenti and C.H. Kruger, *Introduction to physical gas dynamics*, Wiley, New York, 1965.

[41] W.K. Wang and W.J. Wang, The pointwise estimates of solutions for a model system of the radiating gas in multi-dimensions, *Nonlinear Anal., TMA.* **71** (2009), 1180-1195.

[42] Z.P. Xin, Asymptotic stability of planar rarefaction waves for viscous conservation laws in several dimensions, *Trans. Amer. Math. Soc.* **319** (1990), 805-820.

[43] T. Yang and H.J. Zhao, BV estimates on Lax-Friedrichs' scheme for a model of radiating gas, *Appl. Anal.* **83** (2004), 533-539.

[44] Y.B. Zel'dovich and C.H. Rajzer. *Physics of shock waves and high temperature hydrodynamic phenomena.* Academic Press, 1966.

[45] K. Zumbrun, D. Serre, Viscous and inviscid stability of multidimensional planar shock fronts, *Indiana Univ. Math. J.* **48** (1999), 937-992.