Inverse eigenproblem for centrosymmetric and centroskew matrices and their approximation

Zheng-jian Bai [a,*], Raymond H. Chan [a,1]

[a] *Department of Mathematics, Chinese University of Hong Kong, Shatin, NT, Hong Kong, China*

Abstract

In this paper, we first give the solvability condition for the following inverse eigenproblem (IEP): given a set of vectors $\{\mathbf{x}_i\}_{i=1}^m$ in $\mathbb{C}^n$ and a set of complex numbers $\{\lambda_i\}_{i=1}^m$, find a centrosymmetric or centroskew matrix C in $\mathbb{R}^{n\times n}$ such that $\{\mathbf{x}_i\}_{i=1}^m$ and $\{\lambda_i\}_{i=1}^m$ are the eigenvectors and eigenvalues of C respectively. We then consider the best approximation problem for the IEPs that are solvable. More precisely, given an arbitrary matrix B in $\mathbb{R}^{n\times n}$, we find the matrix C which is the solution to the IEP and is closest to B in the Frobenius norm. We show that the best approximation is unique and derive an expression for it.

Key words: Eigenproblem; Centrosymmetric matrix; Centroskew matrix

1 Introduction

Let J_n be the n-by-n anti-identity matrix, i.e, J_n has 1 on the anti-diagonal and zeros elsewhere. An n-by-n matrix C is said to be *centrosymmetric* (or *persymmetric*) if $C = J_nCJ_n$, and it is called *centroskew* (or *skew-centrosymmetric*) if $C = -J_nCJ_n$. The centrosymmetric and centroskew matrices play an important role in many areas [7,16] such as signal processing [8,11], the numerical solution of differential equations [2], and Markov processes [17].

In this paper, we consider two problems related to centrosymmetric and centroskew matrices. Both problems are on numerical and approximate computing

* Corresponding author.
Email addresses: `zjbai@math.cuhk.edu.hk` (Zheng-jian Bai), `rchan@math.cuhk.edu.hk` (Raymond H. Chan).

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but here we solve them algebraically, based on some explicit expressions for the solutions of overdetermined linear systems of equations. The first problem is an inverse eigenproblem. There are many applications of structured inverse eigenproblems, see for instance the expository paper [5]. In particular, the inverse eigenproblem for Toeplitz matrices (a special case of centrosymmetric matrices) arises in trigonometric moment problem [10] and signal processing [9]. The inverse eigenproblem for centrosymmetric Jacobi matrices also comes from inverse Sturm-Liouville problem [19, p.70]. There are also different types of inverse eigenproblem, for instances multiplicative type and additive type [19, Chapter 4]. Here we consider the following type of inverse eigenproblem which appeared in the design of Hopfield neural networks [4,13].

Problem I. Given $X = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m]$ in $\mathbb{C}^{n\times m}$ and $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$ in $\mathbb{C}^{m\times m}$, find a centrosymmetric or centroskew matrix C in $\mathbb{R}^{n\times n}$ such that $CX = X\Lambda$.

The second problem we consider in this paper is the problem of best approximation:

Problem II. Let $\mathcal{L}^S$ be the solution set of Problem I. Given a matrix $B \in \mathbb{R}^{n\times n}$, find $C^* \in \mathcal{L}^S$ such that

$$\|B - C^*\| = \min_{C\in\mathcal{L}^S} \|B - C\|,$$

where $\|\cdot\|$ is the Frobenius norm.

The best approximation problem occurs frequently in experimental design, see for instance [14, p.123]. Here the matrix B may be a matrix obtained from experiments, but it may not satisfy the structural requirement (centrosymmetric or centroskew) and/or spectral requirement (having eigenpairs X and Λ). The best estimate C^* is the matrix that satisfies both requirements and is the best approximation of B in the Frobenius norm. In addition, because there are fast algorithms for solving various kinds of centrosymmetric and centroskew matrices [12], the best approximate C^* of B can also be used as a preconditioner in the preconditioned conjugate gradient method for solving linear systems with coefficient matrix B, see for instance [1].

Problems I and II have been solved for different classes of structured matrices, see for instance [18,20]. In this paper, we extend the results in [18,20] to the classes of centrosymmetric and centroskew matrices. We first give a solvability condition for Problem I and also the form of its general solution. Then in the case when Problem I is solvable, we show that Problem II has a unique solution and we give a formula for the minimizer C^*.

The paper is organized as follows: In §2 we first characterize the class of centrosymmetric matrices and give the solvability condition of Problem I over this class of matrices. In §3, we derive a formula for the best approximation of Problem II, give the algorithm for finding the minimizer, and study the stability of the problem. In §4 we give an example to illustrate the theory. In the last section, we extend the results in §§2–3 to centroskew matrices.

2 Solvability Condition for Problem I

We first characterize the set of all centrosymmetric matrices. For all positive integers k, let

$$K_{2k} = \frac{1}{\sqrt{2}} \begin{bmatrix} I_k & I_k \\ J_k & -J_k \end{bmatrix} \quad \text{and} \quad K_{2k+1} = \frac{1}{\sqrt{2}} \begin{bmatrix} I_k & \mathbf{0} & I_k \\ \mathbf{0} & \sqrt{2} & \mathbf{0} \\ J_k & \mathbf{0} & -J_k \end{bmatrix}.$$

Clearly K_n is orthogonal for all n. The matrix K_n plays an important role in analyzing the properties of centrosymmetric matrices, see for example [6]. In particular, we have the following splitting of centrosymmetric matrices into smaller submatrices using K_n.

Lemma 1 [6] *Let $\mathcal{C}_n$ be the set of all centrosymmetric matrices in $\mathbb{R}^{n\times n}$. We have*

$$\mathcal{C}_{2k} = \left\{ \begin{bmatrix} E & FJ_k \\ J_kF & J_kEJ_k \end{bmatrix} \;\middle|\; E, F \in \mathbb{R}^{k\times k} \right\},$$

$$\mathcal{C}_{2k+1} = \left\{ \begin{bmatrix} E & \mathbf{a} & FJ_k \\ \mathbf{b}^T & c & \mathbf{b}^TJ_k \\ J_kF & J_k\mathbf{a} & J_kEJ_k \end{bmatrix} \;\middle|\; E, F \in \mathbb{R}^{k\times k}, \mathbf{a}, \mathbf{b} \in \mathbb{R}^k, c \in \mathbb{R} \right\}.$$

Moreover, for all $n = 2k$ and $2k+1$, we have

$$\mathcal{C}_n = \left\{ K_n \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} K_n^T \;\middle|\; G_1 \in \mathbb{R}^{(n-k)\times(n-k)}, G_2 \in \mathbb{R}^{k\times k} \right\}. \qquad (1)$$

Before we come to Problem I, we first note that we can assume without loss of generality that X and Λ are real matrices. In fact, since $\mathcal{C}_n \subset \mathbb{R}^{n\times n}$, the complex eigenvectors and eigenvalues of any $C \in \mathcal{C}_n$ will appear in complex conjugate pairs. If $\alpha \pm \beta\sqrt{-1}$ and $\mathbf{x} \pm \sqrt{-1}\mathbf{y}$ are one of its eigenpair, then we have $C\mathbf{x} = \alpha\mathbf{x} - \beta\mathbf{y}$ and $C\mathbf{y} = \alpha\mathbf{y} + \beta\mathbf{x}$, i.e.

$$C[\mathbf{x}, \mathbf{y}] = [\mathbf{x}, \mathbf{y}] \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}.$$

Hence we can assume without loss of generality that $X \in \mathbb{R}^{n \times m}$ and

$$\Lambda = \mathrm{diag}(\Phi_1, \Phi_2, \dots, \Phi_l, \gamma_1, \dots, \gamma_{m-2l}) \in \mathbb{R}^{m \times m}, \tag{2}$$

where $\Phi_i = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix}$ with α_i, β_i and γ_i in $\mathbb{R}$.

Next, we investigate the solvability of Problem I. We need the following lemma where U^+ denotes the Moore-Penrose pseudo-inverse of U.

Lemma 2 **[15, Lemma 1.3]** *Let $U, V \in \mathbb{R}^{n \times m}$ be given. Then $YU = V$ is solvable if and only if $VU^+U = V$. In this case the general solution is*

$$Y = VU^+ + Z(I - UU^+),$$

where $Z \in \mathbb{R}^{n \times n}$ is arbitrary.

In the remaining part of the paper, we will only give the theorems and the proofs for even n. The case where n is odd can be proved similarly. Thus we let $n = 2k$.

Theorem 1 *Given $X \in \mathbb{R}^{n \times m}$ and Λ as in (2), let*

$$K_n^T X = \begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{bmatrix}, \tag{3}$$

where $\tilde{X}_2 \in \mathbb{R}^{k \times m}$. Then there exists a matrix $C \in \mathcal{C}_n$ such that $CX = X\Lambda$ if and only if

$$\tilde{X}_1 \Lambda \tilde{X}_1^+ \tilde{X}_1 = \tilde{X}_1 \Lambda \quad \text{and} \quad \tilde{X}_2 \Lambda \tilde{X}_2^+ \tilde{X}_2 = \tilde{X}_2 \Lambda. \tag{4}$$

In this case, the general solution to $CX = X\Lambda$ is given by

$$C_s = C_0 + K_n \begin{bmatrix} Z_1(I_{n-k} - \tilde{X}_1 \tilde{X}_1^+) & 0 \\ 0 & Z_2(I_k - \tilde{X}_2 \tilde{X}_2^+) \end{bmatrix} K_n^T, \tag{5}$$

where $Z_1 \in \mathbb{R}^{(n-k)\times(n-k)}$ *and* $Z_2 \in \mathbb{R}^{k\times k}$ *are both arbitrary, and*

$$C_0 = K_n \begin{bmatrix} \tilde{X}_1\Lambda\tilde{X}_1^+ & 0 \\ 0 & \tilde{X}_2\Lambda\tilde{X}_2^+ \end{bmatrix} K_n^T. \tag{6}$$

Proof: From (1), $C \in \mathcal{C}_n$ is a solution to Problem I if and only if there exist $G_1 \in \mathbb{R}^{(n-k)\times(n-k)}$ and $G_2 \in \mathbb{R}^{k\times k}$ such that

$$C = K_n \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} K_n^T \tag{7}$$

and

$$\left(K_n \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} K_n^T\right) X = X\Lambda. \tag{8}$$

Using (3), (8) is equivalent to

$$G_1\tilde{X}_1 = \tilde{X}_1\Lambda \quad \text{and} \quad G_2\tilde{X}_2 = \tilde{X}_2\Lambda. \tag{9}$$

According to Lemma 2, equations (9) have solutions if and only if equations (4) hold. Moreover in this case, the general solution of (9) is given by

$$G_1 = \tilde{X}_1\Lambda\tilde{X}_1^+ + Z_1(I_{n-k} - \tilde{X}_1\tilde{X}_1^+), \tag{10}$$
$$G_2 = \tilde{X}_2\Lambda\tilde{X}_2^+ + Z_2(I_k - \tilde{X}_2\tilde{X}_2^+), \tag{11}$$

where $Z_1 \in \mathbb{R}^{(n-k)\times(n-k)}$ and $Z_2 \in \mathbb{R}^{k\times k}$ are both arbitrary. Putting (10) and (11) into (7), we get (5). $\square$

3 The Minimizer of Problem II

Let $\mathcal{C}_n^S$ be the solution set of Problem I over $\mathcal{C}_n$. In this section, we solve Problem II over $\mathcal{C}_n^S$ when $\mathcal{C}_n^S$ is nonempty.

Theorem 2 *Given* $X \in \mathbb{R}^{n\times m}$ *and* Λ *as in (2), let the solution set* $\mathcal{C}_n^S$ *of Problem I be nonempty. Then for any* $B \in \mathbb{R}^{n\times n}$*, the problem* $\min_{C\in\mathcal{C}_n^S} \|B - C\|$

has a unique solution C^* given by

$$C^* = C_0 + K_n \begin{bmatrix} \tilde{B}_{11}(I_{n-k} - \tilde{X}_1\tilde{X}_1^+) & 0 \\ 0 & \tilde{B}_{22}(I_k - \tilde{X}_2\tilde{X}_2^+) \end{bmatrix} K_n^T. \tag{12}$$

Here $\tilde{X}_1$, $\tilde{X}_2$, and C_0 are given in (3) and (6), and $\tilde{B}_{11}$ and $\tilde{B}_{22}$ are obtained by partitioning $K_n^T B K_n$ as

$$K_n^T B K_n = \begin{bmatrix} \tilde{B}_{11} & \tilde{B}_{12} \\ \tilde{B}_{21} & \tilde{B}_{22} \end{bmatrix}, \tag{13}$$

where $\tilde{B}_{22} \in \mathbb{R}^{k\times k}$.

Proof: When $\mathcal{C}_n^S$ is nonempty, it is easy to verify from (5) that $\mathcal{C}_n^S$ is a closed convex set. Since $\mathbb{R}^{n\times n}$ is a uniformly convex Banach space under the Frobenius norm, there exists a unique solution for Problem II [3, p. 22]. Moreover, because the Frobenius norm is unitary invariant, Problem II is equivalent to

$$\min_{C\in\mathcal{C}_n^S} \|K_n^T BK - K_n^T CK\|^2. \tag{14}$$

By (5), we have

$$\|K_n^T BK - K_n^T CK\|^2 = \left\| \begin{bmatrix} \tilde{B}_{11} - \tilde{X}_1\Lambda\tilde{X}_1^+ & \tilde{B}_{12} \\ \tilde{B}_{21} & \tilde{B}_{22} - \tilde{X}_2\Lambda\tilde{X}_2^+ \end{bmatrix} - \begin{bmatrix} Z_1P & 0 \\ 0 & Z_2Q \end{bmatrix} \right\|^2,$$

where

$$P = I_{n-k} - \tilde{X}_1\tilde{X}_1^+ \quad \text{and} \quad Q = I_k - \tilde{X}_2\tilde{X}_2^+. \tag{15}$$

Thus (14) is equivalent to

$$\min_{Z_1\in\mathbb{R}^{(n-k)\times(n-k)}} \|\tilde{B}_{11} - \tilde{X}_1\Lambda\tilde{X}_1^+ - Z_1P\|^2 + \min_{Z_2\in\mathbb{R}^{k\times k}} \|\tilde{B}_{22} - \tilde{X}_2\Lambda\tilde{X}_2^+ - Z_2Q\|^2.$$

Clearly, the solution is given by Z_1 and Z_2 such that

$$Z_1P = \tilde{B}_{11} - \tilde{X}_1\Lambda\tilde{X}_1^+ \quad \text{and} \quad Z_2Q = \tilde{B}_{22} - \tilde{X}_2\Lambda\tilde{X}_2^+.$$

Notice that by (15), P and Q are projection matrices, i.e. $P^2 = P$ and $Q^2 = Q$. Therefore $Z_1P = (\tilde{B}_{11} - \tilde{X}_1\Lambda\tilde{X}_1^+)P$ and $Z_2Q = (\tilde{B}_{22} - \tilde{X}_2\Lambda\tilde{X}_2^+)Q$. Notice further that because $\tilde{X}_1^+\tilde{X}_1\tilde{X}_1^+ = \tilde{X}_1^+$, we have

$$\begin{aligned}(\tilde{B}_{11} - \tilde{X}_1\Lambda\tilde{X}_1^+)P &= \tilde{B}_{11} - \tilde{B}_{11}\tilde{X}_1\tilde{X}_1^+ - \tilde{X}_1\Lambda\tilde{X}_1^+ + \tilde{X}_1\Lambda\tilde{X}_1^+\tilde{X}_1\tilde{X}_1^+ \\ &= \tilde{B}_{11} - \tilde{B}_{11}\tilde{X}_1\tilde{X}_1^+ = \tilde{B}_{11}P.\end{aligned}$$

Similarly, $Z_2Q = (\tilde{B}_{22} - \tilde{X}_2\Lambda\tilde{X}_2^+)Q = \tilde{B}_{22}Q$. Hence the unique solution for Problem II is given by (12). $\square$

Based on Theorem 2, we give the following algorithm for solving Problem II for $n = 2k$.

ALGORITHM I

(a) Compute $\tilde{X}_1$ and $\tilde{X}_2$ by (3) and then compute $\tilde{X}_1^+$ and $\tilde{X}_2^+$.
(b) If $\tilde{X}_1\Lambda\tilde{X}_1^+\tilde{X}_1 = \tilde{X}_1\Lambda$ and $\tilde{X}_2\Lambda\tilde{X}_2^+\tilde{X}_2 = \tilde{X}_2\Lambda$, then the solution set $\mathcal{C}_n^S$ to Problem I is nonempty and we continue. Otherwise we stop.
(c) Partition $K_n^T B K_n$ as in (13) to get $\tilde{B}_{11}$ and $\tilde{B}_{22}$.
(d) Compute

$$
\begin{aligned}
W_1 &= \tilde{X}_1\Lambda\tilde{X}_1^+ + \tilde{B}_{11} - \tilde{B}_{11}\tilde{X}_1\tilde{X}_1^+,\\
W_2 &= X_2\Lambda X_2^+ + \tilde{B}_{22} - \tilde{B}_{22}\tilde{X}_2\tilde{X}_2^+.
\end{aligned}
$$

(e) Then $C^* = K_n \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} K_n^T$.

Next we consider the computational complexity of our algorithm. For Step (a), since K_n has only 2 nonzero entries per row, it requires $O(nm)$ operations to compute $\tilde{X}_1$ and $\tilde{X}_2$. Then using singular value decomposition to compute $\tilde{X}_1^+$ and $\tilde{X}_2^+$ requires $O(n^2m+m^3)$ operations. Step (b) obviously requires $O(n^2m)$ operations. For Step (c), because of the sparsity of K_n, the operations required is $O(n^2)$ only. For Step (d), if we compute $\tilde{B}_{ii}\tilde{X}_i\tilde{X}_i^+$ as $[(\tilde{B}_{ii}\tilde{X}_i)\tilde{X}_i^+]$, then the cost will only be of $O(n^2m)$ operations. Finally, because of the sparsity of K_n again, Step (e) requires $O(n^2)$ operations. Thus the total complexity of the algorithm is $O(n^2m + m^3)$. We remark that in practice, $m \ll n$.

Before we end this section, we give a stability analysis for Problem II, that is, we study how the solution of Problem II is affected by a small perturbation of B. We have the following result.

Corollary 1 *Given $B^{(i)} \in \mathbb{R}^{n\times n}, i = 1,2$. Let $C^{*(i)} = \arg\min_{C\in\mathcal{C}_n^S}\|B^{(i)} - C\|$ for $i = 1,2$. Then there exists a constant α independent of $B^{(i)}, i = 1,2$, such that*

$$
\|C^{*(2)} - C^{*(1)}\| \le \alpha\|B^{(2)} - B^{(1)}\|. \tag{16}
$$

Proof: By Theorem 2, $C^{*(i)}$ is given by

$$C^{*(i)} = C_0 + K_n \begin{bmatrix} \tilde{B}_{11}^{(i)} P & 0 \\ 0 & \tilde{B}_{22}^{(i)} Q \end{bmatrix} K_n^T, \quad i = 1, 2,$$

where $\tilde{B}_{22}^{(i)}$ are the blocks of $K_n^T B^{(i)} K_n$ as defined in (13), and P and Q are given in (15). Thus we have

$$\begin{aligned} \|C^{*(2)} - C^{*(1)}\| &= \left\| K_n \begin{bmatrix} \left(\tilde{B}_{11}^{(2)} - \tilde{B}_{11}^{(1)}\right) P & 0 \\ 0 & \left(\tilde{B}_{22}^{(2)} - \tilde{B}_{22}^{(1)}\right) Q \end{bmatrix} K_n^T \right\| \\ &\leq \left\| \begin{bmatrix} \tilde{B}_{11}^{(2)} - \tilde{B}_{11}^{(1)} & 0 \\ 0 & \tilde{B}_{22}^{(2)} - \tilde{B}_{22}^{(1)} \end{bmatrix} \right\| \left\| \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \right\| \\ &\leq \left\| K_n^T \left(B^{(2)} - B^{(1)} \right) K_n \right\| \left\| \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \right\| \leq \alpha \left\| B^{(2)} - B^{(1)} \right\|, \end{aligned}$$

where $\alpha = \|P\| + \|Q\|$. Thus (16) holds. $\square$

4 Demonstration by an Example

Let us first compute the input matrices X and Λ for which Problem I has a solution. We start by choosing a random matrix $\hat{C}$ in $\mathcal{C}_n$:

$$\hat{C} = \begin{bmatrix} 0.1749 & 0.0325 & -0.2046 & 0.0932 & 0.0315 \\ 0.0133 & -0.0794 & -0.0644 & 0.1165 & -0.0527 \\ 0.1741 & 0.0487 & 0.1049 & 0.0487 & 0.1741 \\ -0.0527 & 0.1165 & -0.0644 & -0.0794 & 0.0133 \\ 0.0315 & 0.0932 & -0.2046 & 0.0325 & 0.1749 \end{bmatrix} \in \mathbb{C}_5.$$

Then we compute its eigenpairs. The eigenvalues of $\hat{C}$ are $0.1590 \pm 0.2841\sqrt{-1}$, -0.1836, 0.1312, and 0.0304. Let $\mathbf{x}_1 \pm \sqrt{-1}\mathbf{x}_2$, $\mathbf{x}_3, \mathbf{x}_4$, and $\mathbf{x}_5$ be the corresponding eigenvectors. Then we take

$$X = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5] = \begin{bmatrix} 0.4815 & 0.2256 & -0.2455 & -0.7071 & -0.1313 \\ 0.0118 & 0.1700 & 0.7071 & -0.1427 & -0.7071 \\ 0.4322 & -0.5120 & 0.2235 & 0 & 0 \\ 0.0118 & 0.1700 & 0.7071 & 0.1427 & 0.7071 \\ 0.4815 & 0.2256 & -0.2455 & 0.7071 & 0.1313 \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} 0.1590 & 0.2841 & 0 & 0 & 0 \\ -0.2841 & 0.1590 & 0 & 0 & 0 \\ 0 & 0 & 0.0304 & 0 & 0 \\ 0 & 0 & 0 & 0.1312 & 0 \\ 0 & 0 & 0 & 0 & -0.1836 \end{bmatrix}.$$

Given this X and Λ, clearly we have a solution to Problem I, namely $\hat{C}$. Thus $\mathcal{C}_5^S$ is nonempty. Next we perturb $\hat{C}$ by a random matrix to obtain a matrix $B(\epsilon) \notin \mathcal{C}_5$:

$$B(\epsilon) = \hat{C} + \epsilon \cdot \begin{bmatrix} 1.4886 & -0.9173 & 1.2688 & -0.1869 & -1.0830 \\ 1.2705 & -1.1061 & -0.7836 & 1.0132 & 1.0354 \\ -1.8561 & 0.8106 & 0.2133 & 0.2484 & 1.5854 \\ 2.1343 & 0.6985 & 0.7879 & 0.0596 & 0.9157 \\ 1.4358 & -0.4016 & 0.8967 & 1.3766 & -0.5565 \end{bmatrix}.$$

Then we apply our algorithm in §3 to obtain $C^*(\epsilon)$ corresponding to $B(\epsilon)$. In Figure 1, we plot the following two quantities for ϵ between 10^{-10} to 10^{10}: $\log_{10} \|B(\epsilon) - C^*(\epsilon)\|$ (marked by "$*$") and $\log_{10} \|\hat{C} - C^*(\epsilon)\|$ (marked by "$+$"). We can see that as ϵ goes to zero, $C^*(\epsilon)$ approaches $B(\epsilon)$ as expected. Also when $\epsilon \leq 10^{-1}$, $C^*(\epsilon) = \hat{C}$ up to the machine precision (we use MATLAB which has machine precision around 10^{-16}).

5 Extension to the Set of Centroskew Matrices

In this section, we extend our results in §§2–3 to centroskew matrices, i.e. matrices S such that $S = -J_n S J_n$. The results and the proofs are similar to

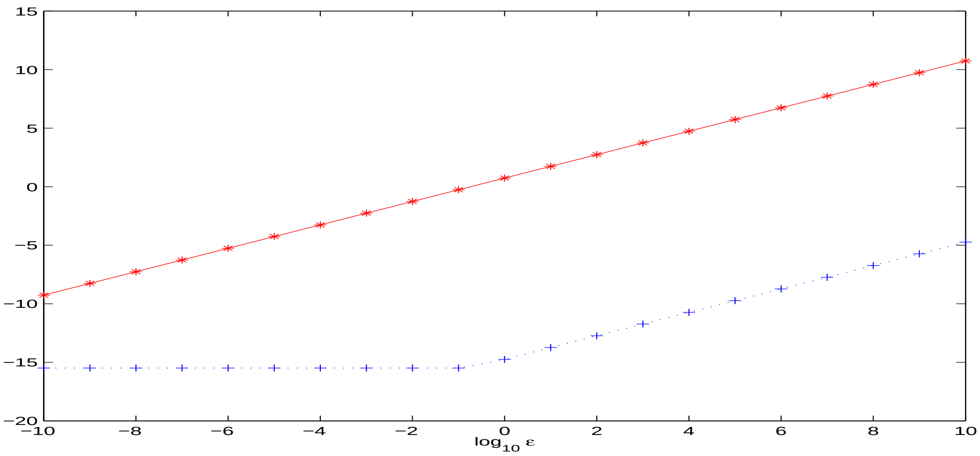

Fig. 1. $\log_{10} \|B(\epsilon) - C^*(\epsilon)\|$ ("$*$") and $\log_{10} \|\hat{C} - C^*(\epsilon)\|$ ("$+$") versus $\log_{10} \epsilon$.

the centrosymmetric case, and we only list the results for the case when n is even and omit the proofs. Let $n = 2k$. Considering Problem I for $\mathcal{S}_n$, we have the following theorem.

Theorem 3 *Given $X \in \mathbb{R}^{n\times m}$ and Λ as in (2), let $\tilde{X}_1$ and $\tilde{X}_2$ be as defined in (3). Then there exists $S \in \mathcal{S}_n$ such that $SX = X\Lambda$ if and only if*

$$\tilde{X}_1\Lambda\tilde{X}_2^+\tilde{X}_2 = \tilde{X}_1\Lambda \quad \text{and} \quad \tilde{X}_2\Lambda\tilde{X}_1^+\tilde{X}_1 = \tilde{X}_2\Lambda.$$

In this case, the general solution to $SX = X\Lambda$ is given by

$$S_s = S_0 + K_n \begin{bmatrix} 0 & Z_1(I_k - \tilde{X}_2\tilde{X}_2^+) \\ Z_2(I_k - \tilde{X}_1\tilde{X}_1^+) & 0 \end{bmatrix} K_n^T,$$

where $Z_1 \in \mathbb{R}^{k\times k}$ and $Z_2 \in \mathbb{R}^{k\times k}$ are both arbitrary, and

$$S_0 = K_n \begin{bmatrix} 0 & \tilde{X}_1\Lambda\tilde{X}_2^+ \\ \tilde{X}_2\Lambda\tilde{X}_1^+ & 0 \end{bmatrix} K_n^T. \tag{17}$$

For Problem II over the solution set $\mathcal{S}_n^S$ of Problem I for $\mathcal{S}_n$, we have the following result.

Theorem 4 *Given $X \in \mathbb{R}^{n\times m}$ and Λ as in (2), let the solution set $\mathcal{S}_n^S$ of Problem I be nonempty. Then for any $B \in \mathbb{R}^{n\times n}$, the problem $\min\limits_{S\in\mathcal{S}_n^S} \|B - S\|$ has a unique solution S^* given by*

$$S^* = S_0 + K_n \begin{bmatrix} 0 & \tilde{B}_{12}(I_k - \tilde{X}_2\tilde{X}_2^+) \\ \tilde{B}_{21}(I_{n-k} - \tilde{X}_1\tilde{X}_1^+) & 0 \end{bmatrix} K_n^T.$$

Here $\tilde{X}_1$, $\tilde{X}_2$, $\tilde{B}_{12}$, $\tilde{B}_{21}$, and S_0 are given in (3), (13), and (17). Moreover S^* is a continuous function of B.

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