A Note on the Besov Space $B_2^{\frac{1}{2}}$ †

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Abstract. We consider complex-valued functions f defined on the unit circle $\boldsymbol{T}$ that are continuous for all $t \in \boldsymbol{T}$ except at a point t_0 where the left- and right-hand limits of f both exist. Using matrix methods, we show that if f is in the Besov class $B_2^{\frac{1}{2}}(\boldsymbol{T})$, then f is continuous at t_0. In particular, we prove that if the left- and right-hand limits of f are not equal at t_0, then $\sum_{k=-\infty}^{\infty} |k||a_k[f]|^2 = \infty$, where $a_k[f]$ are the Fourier coefficients of f.

Key Words. Besov class, Toeplitz matrix, Circulant matrix, Hilbert matrix.

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1. Introduction.

Let $\boldsymbol{T}$ be the unit circle in the complex plane. For $1 \leq p < \infty$, let L^p be the Banach space of all complex-valued Lebesgue measurable functions f on $\boldsymbol{T}$ for which the L^p norm

$$\|f\|_p \equiv \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(e^{i\theta}) \right|^p d\theta \right\}^{\frac{1}{p}}$$

is finite. For $\phi \in \boldsymbol{R}$, the set of real numbers, we define the operator δ_ϕ as

$$(\delta_\phi f)(e^{i\theta}) \equiv f\left(e^{i(\theta+\phi)}\right) - f\left(e^{i\theta}\right), \quad \forall \theta \in \boldsymbol{R}.$$

Then for all natural number n, we let

$$\delta_\phi^n \equiv \delta_\phi \delta_\phi^{n-1}.$$

For $\alpha > 0$ and $1 \leq p < \infty$, the Besov class B_p^α is defined as

$$B_p^\alpha = \left\{ f \in L^p : \int_{-\pi}^{\pi} |\phi|^{-1-\alpha p} \|\delta_\phi^n f\|_p^p d\phi < \infty \right\}$$

where n is any integer such that $n > \alpha$.

A well-known theorem about the class B_p^α states that if $1 < p < \infty$ and $\alpha > 1/p$, then all functions in B_p^α are continuous functions, see Böttcher and Silbermann [1, p.44]. In this paper, we will use matrix methods to discuss the case when $p = 2$ and $\alpha = 1/2$. Our main result is the following

Theorem 1. *If $f \in B_2^{\frac{1}{2}}$ is continuous at every point $t \in \boldsymbol{T} \backslash \{-1\}$ and both*

$$f(-1+0) \equiv \lim_{\theta \to 0^+} f\left(e^{i(\pi-\theta)}\right)$$

and

$$f(-1-0) \equiv \lim_{\theta \to 0^+} f\big(e^{i(-\pi+\theta)}\big)$$

exist, then $f(-1+0) = f(-1-0)$.

As an immediate corollary, we also prove

Theorem 2. *Let* f *be any arbitrary complex-valued function defined on* $\boldsymbol{T}$. *If* f *is continuous at every point* $t \in \boldsymbol{T}\backslash\{-1\}$ *and both* $f(-1+0)$ *and* $f(-1-0)$ *exist but* $f(-1+0) \neq f(-1-0)$, *then*

$$\sum_{k=-\infty}^{\infty} |k||a_k[f]|^2 = \infty,$$

where $a_k[f]$ *are the Fourier coefficients of* f.

Before carrying out our proof, we need several definitions and lemmas.

2. Definitions and Lemmas.

Given $f \in L^1$, we define its Fourier coefficients $a_k[f]$ by

$$a_k[f] = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-ik\theta} d\theta \ , \quad k = 0, \pm 1, \pm 2, \cdots \ .$$

Let $A_n[f]$ denote the n-by-n Toeplitz matrix with the (j,ℓ)th entry given by $a_{j-\ell}[f]$. If f is real-valued, then $a_{-k}[f] = \overline{a}_k[f]$ and hence $A_n[f]$ is a Hermitian matrix. Let $C_n[f]$ be the n-by-n circulant matrix in which the (j,ℓ)th entry is given by $c_{j-\ell}[f]$ where

$$c_k[f] = \begin{cases} \dfrac{(n-k)a_k[f] + k a_{k-n}[f]}{n} & 0 \le k < n, \\[2ex] c_{n+k}[f] & 0 < -k < n \ . \end{cases}$$

Clearly, $C_n[f]$ will be a Hermitian matrix if f is real-valued.

A sequence of matrices $\{M_n\}_{n=1,2,\cdots}$ is said to have clustered spectra if for any $\epsilon > 0$, there exists an $N > 0$ such that for all $n \geq 1$, at most N eigenvalues of M_n have absolute values exceeding ϵ. As examples, we consider the following Lemmas.

Lemma 1. *Let $\{M_n\}_{n=1,2,\cdots}$ be a sequence of Hermitian matrices. If $\sup_n \|M_n\|_F < \infty$ where $\|\cdot\|_F$ denotes the Frobenius norm, then $\{M_n\}$ has clustered spectra.*

Proof. Since the square of the Frobenius norm of a Hermitian matrix is equal to the sum of the square of its eigenvalues, it follows that for any given $\epsilon > 0$, M_n has at most $\sup_n \|M_n\|_F^2/\epsilon^2$ eigenvalues with absolute values greater than ϵ. $\square$

Lemma 2. *Let f be a real-valued continuous function on $\boldsymbol{T}$. Then the sequence of matrices*

$$\triangle_n[f] \equiv A_n[f] - C_n[f], \quad n = 0, 1, 2, \cdots$$

has clustered spectra.

Proof. See Chan and Yeung [2, Theorem 1]. $\square$

Lemma 3. *If f is a real-valued function in $B_2^{1/2}$, then $\{\triangle_n[f]\}$ has clustered spectra.*

Proof. We first note that the space $B_2^{1/2}$ admits a very simple description, namely

$$f \in B_2^{1/2} \Longleftrightarrow \sum_{k=-\infty}^{\infty} (|k|+1)|a_k[f]|^2 < \infty \ , \tag{1}$$

see for instance, Böttcher and Silbermann [1, p.44]. Since the first row of the Hermitian Toeplitz matrix $\triangle_n[f] = A_n[f] - C_n[f]$ is given by

$$\Big(0, \frac{1}{n}\big(a_{-1}[f] - a_{n-1}[f]\big), \frac{2}{n}\big(a_{-2}[f] - a_{n-2}[f]\big), \cdots, \frac{n-1}{n}\big(a_{-n+1}[f] - a_1[f]\big)\Big),$$

we have

$$
\begin{aligned}
\|\triangle_n[f]\|_F^2 &= 2\sum_{k=1}^{n-1} \frac{(n-k)k^2}{n^2}\big|a_{-k}[f]-a_{n-k}[f]\big|^2 \\
&\le 4\sum_{k=1}^{n-1} \frac{(n-k)k^2}{n^2}\left(|a_{-k}[f]|^2+|a_{n-k}[f]|^2\right) \\
&= 4\sum_{k=1}^{n-1} \left\{\frac{(n-k)k^2}{n^2}|a_{-k}[f]|^2+\frac{(n-k)^2k}{n^2}|a_k[f]|^2\right\} \\
&= 4\sum_{k=1}^{n-1} \frac{n-k}{n}\cdot k|a_k[f]|^2 \\
&\le 4\sum_{k=1}^{n-1} k|a_k[f]|^2 \\
&\le 2\sum_{k=-\infty}^{\infty} (|k|+1)|a_k[f]|^2 < \infty.
\end{aligned}
$$

By Lemma 1, $\{\triangle_n[f]\}$ has clustered spectra. $\square$

Lemma 4. *Let H_n be the n-by-n Hilbert matrix, i.e.*

$$
H_n = \begin{bmatrix}
1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\
\frac{1}{2} & \frac{1}{3} & & \cdot & \vdots \\
\frac{1}{3} & & \cdot & & \vdots \\
\vdots & \cdot & & & \vdots \\
\frac{1}{n} & \cdots & \cdots & \cdots & \frac{1}{2n-1}
\end{bmatrix}.
$$

Then for any $\epsilon > 0$, the number of eigenvalues of H_n which exceed $\epsilon > 0$ is asymptotically equal to

$$
\frac{2}{\pi}\log n \ \operatorname{sech}^{-1}\frac{\epsilon}{\pi}\ .
$$

In other words, $\{H_n\}$ does not have clustered spectra.

Proof. See Widom [3, p.31]. $\square$

3. Proofs of Theorems.

Proof of Theorem 1: It is enough to prove the theorem for real-valued functions. Thus let f be a real-valued function in $B_2^{1/2}$. Assume that f is continuous at every point $t \in \boldsymbol{T}\backslash\{-1\}$ with both $f(-1+0) = \lim_{\theta\to 0^+} f(e^{i(\pi-\theta)})$ and $f(-1-0) = \lim_{\theta\to 0^+} f(e^{i(-\pi+\theta)})$ exist, but $f(-1+0) \neq f(-1-0)$.

Define $g(e^{i\theta}) = \theta$ for all $-\pi < \theta \leq \pi$ and let

$$\beta = \frac{f(-1+0) - f(-1-0)}{2\pi} \neq 0.$$

Then $f - \beta g$ is a continuous function on $\boldsymbol{T}$. By Lemmas 3 and 2, both $\{\triangle_n[f]\}$ and $\{\triangle_n[f-\beta g]\}$ have clustered spectra. Since $g = \frac{1}{\beta}\big(f - (f - \beta g)\big)$,

$$\triangle_n[g] = \frac{1}{\beta}\triangle_n[f] - \frac{1}{\beta}\triangle_n[f-\beta g]$$

and hence $\{\triangle_n[g]\}$ has clustered spectra by Cauchy's interlace theorem, see for instance Wilkinson [4, p.101].

The Fourier coefficients $a_k[g]$ of g are given by

$$a_k[g] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta e^{-ik\theta} d\theta = \begin{cases} 0 & k = 0, \\ \dfrac{(-1)^k}{k} i & k = \pm 1, \pm 2, \cdots . \end{cases}$$

Therefore, for all $m > 0$, the first row of the $2m$-by-$2m$ Hermitian Toeplitz matrix $\triangle_{2m}[g]$ is given by

$$\begin{aligned} &\left(0, \frac{1}{2m}\big(a_{-1}[g] - a_{2m-1}[g]\big), \frac{2}{2m}\big(a_{-2}[g] - a_{2m-2}[g]\big), \cdots, \frac{2m-1}{2m}\big(a_{-2m+1}[g] - a_1[g]\big)\right) \\ &= \left(0, \frac{1}{2m-1} i, \frac{-1}{2m-2} i, \cdots, \frac{(-1)^{k+1}}{2m-k} i, \cdots, i\right) . \end{aligned}$$

Let P_m and Q_m denote the m-by-m diagonal matrices with $(-1)^{j+1}i$ and $(-1)^{m+j+1}$ as their (j,j)th entries respectively and let $\triangle_{2m}[g]$ be partitioned as

$$\triangle_{2m}[g] = \begin{bmatrix} W_m & U_m \\ U_m^* & W_m \end{bmatrix}$$

where W_m and U_m are m-by-m Toeplitz matrices. Then

$$\begin{aligned} \begin{bmatrix} P_m & 0 \\ 0 & Q_m \end{bmatrix} \triangle_{2m}[g] \begin{bmatrix} P_m^* & 0 \\ 0 & Q_m^* \end{bmatrix} &= \begin{bmatrix} P_m W_m P_m^* & P_m U_m Q_m^* \\ Q_m U_m^* P_m^* & Q_m W_m Q_m^* \end{bmatrix} \\ &= \begin{bmatrix} P_m W_m P_m^* & H_m J_m \\ J_m H_m & Q_m W_m Q_m^* \end{bmatrix} \end{aligned}$$

where

$$J_m = \begin{bmatrix} 0 & & & 1 \\ & & 1 & \\ & \cdot & & \\ \cdot & & & \\ 1 & & & 0 \end{bmatrix},$$

is the m-by-m anti-identity matrix and H_m is the m-by-m Hilbert matrix. Let

$$X_{2m} = \begin{bmatrix} P_m W_m P_m^* & 0 \\ 0 & Q_m W_m Q_m^* \end{bmatrix}$$

and

$$Y_{2m} = \begin{bmatrix} 0 & H_m J_m \\ J_m H_m & 0 \end{bmatrix}$$

Then we have

$$\begin{bmatrix} P_m & 0 \\ 0 & Q_m \end{bmatrix} \triangle_{2m}[g] \begin{bmatrix} P_m^* & 0 \\ 0 & Q_m^* \end{bmatrix} = X_{2m} + Y_{2m}. \qquad (2)$$

Since

$$\begin{aligned} \|X_{2m}\|_F^2 &= \|P_m W_m P_m^*\|_F^2 + \|Q_m W_m Q_m^*\|_F^2 \\ &= 2\|W_m\|_F^2 = 4\sum_{k=1}^{m-1} \frac{m-k}{(2m-k)^2} \\ &\le 4\int_0^1 \frac{1-t}{(2-t)^2}dt = 4\log 2 - 2 \ , \end{aligned}$$

$\{X_{2m}\}$ has clustered spectra by Lemma 1. Recall that $\{\triangle_{2m}[g]\}$ also has clustered spectra, therefore from (2) and Cauchy's interlace theorem, $\{Y_{2m}\}$ has clustered spectra.

Let

$$R_{2m} = \frac{1}{\sqrt{2}} \begin{bmatrix} I_m & I_m \\ J_m & -J_m \end{bmatrix}$$

where I_m is the $m \times m$ identity matrix. Clearly, $R_{2m}^* R_{2m} = I_{2m}$. Hence $\{R_{2m}^* Y_{2m} R_{2m}\}$ has clustered spectra. However,

$$R_{2m}^* Y_{2m} R_{2m} = \frac{1}{2} \begin{bmatrix} H_m & 0 \\ 0 & -H_m \end{bmatrix} .$$

This implies that $\{H_m\}$ has clustered spectra, a contradiction to Lemma 4. $\square$

Proof of Theorem 2: Just use (1) and Theorem 1. $\square$

We finally note that since estimates of the form (1) only hold for Besov space B_p^α where $p = 2$ and $\alpha = 1/2$, the matrix method used here will not work for larger Besov spaces.

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