Mean-Variance, Mean-VaR, and Mean-CVaR Models for Portfolio Selection with Background Risk

Xu Guo

School of Statistics, Beijing Normal University

Raymond H. Chan

Department of Mathematics, The Chinese University of Hong Kong

Wing-Keung Wong

Department of Finance, Fintech Center, and Big Data Research Center, Asia University

Department of Medical Research, China Medical University Hospital

Department of Economics and Finance, Hang Seng Management College

Department of Economics, Lingnan University

Lixing Zhu

School of Statistics, Beijing Normal University

Department of Mathematics, Hong Kong Baptist University

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Corresponding author: Wing-Keung Wong, Department of Finance, College of Management, Asia University, 500, Lioufeng Rd., Wufeng, Taichung, Taiwan. Email: wong@asia.edu.tw.

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Abstract: This paper extends Jiang, *et al.* (2010), Guo, *et al.* (2018), and others by investigating the impact of background risk on an investor's portfolio choice in the mean-VaR, mean-CVaR, and mean-variance framework, and analyzes the characterization of the mean-variance, mean-VaR, and mean-CVaR boundaries and efficient frontiers in the presence of background risk. We derive the conditions that the portfolios lie on the mean-variance, mean-VaR, and mean-CVaR boundaries with and without background risk. We show that the MV, VaR, and CVaR boundary depends on the covariance vector between the returns of the risky assets and that of the background asset and also the variance of the return of the background asset. We develop properties on MV, mean-VaR, and mean-CVaR efficient frontiers. In addition, we establish some new properties for the case with a risk-free security, extend our work to the non-normality situation, and examine the economic implication of the mean-VaR/CVaR model.

Keywords: Background risk; Portfolio selection; VaR; CVaR; mean-variance model

JEL Classification : C0, D81, G11

1 Introduction

Markowitz (1952) and others develop the mean-variance (MV) framework and MV boundary. Alexander and Baptista (2002) extend the theory to mean- value-at-risk (mean-VaR) model for portfolio selection while Alexander and Baptista (2004) further extend theory to include mean-conditional VaR (mean-CVaR) model for portfolio selection.

On the other hand, Bodie, *et al.* (1992) and many others have found that background risk is necessary in many empirical analysis. Jiang, *et al.* (2010) and others extend the theory by investigating the impact of background risk on an investor's portfolio choice in a mean-variance framework, and analyzes the properties of efficient portfolios as well as the investor's hedging behaviour in the presence of background risk. In addition, Guo, *et al.* (2018) survey existing results on the mean-variance approach and the expected-utility approach of risk preferences with multiple additive risks.

This paper extends the theory further by investigating the impact of background risk on an investor's portfolio choice in the mean-VaR, mean-CVaR, and mean-variance framework, and analyzes the characterization of the mean-variance, mean-VaR, and mean-CVaR boundaries and efficient frontiers in the presence of background risk. We devise some new results in this framework. For example, we derive the conditions that the portfolios lie on the mean-variance, mean-VaR, and mean-CVaR boundaries with and without background risk. We show that the mean-variance (VaR, CVaR) boundary depends on the covariance vector between the returns of the risky assets and that of the background asset and also the variance of the return of the background asset. We develop properties on MV, mean-VaR, and mean-CVaR efficient. In addition, we establish some new properties for the case with a risk-free security, extend our work to the non-normality situation, and examine the economic implication of the mean-VaR/CVaR model. We note that all the above is the contribution of our paper to the literature.

The remainder of the paper is organized as follows. Section 2 reviews related literature. Section 3 describes the model, and characterizes the mean-variance boundary with background risk, minimum VaR/CVaR portfolio, and the mean-VaR efficient frontiers. Section 4 addresses the case of risk-free security. Section 5 examines the case when distributions of the returns are non-normal. Section 6 concludes. All proofs are given in the appendix.

2 Literature Review

2.1 Portfolio Selection

Markowitz (1952) develops the mean-variance (MV) portfolio optimization theory which is the milestone of modern finance theory to obtain optimal portfolio. Bai, *et al.* (2009) prove that when the dimension of the data is large, the MV optimization procedure will result in a serious departure of the optimal return estimate for the theoretical Markowitz model. Thereafter, they propose a bootstrap-corrected estimator to correct the overestimation. Leung, *et al.* (2012) further extend the model to obtain closed form of the estimator.

2.2 Mean-Variance, Mean-VaR, Mean-CVaR Models

There are many studies on the mean-variance (MV) framework. For example, Markowitz (1952) establishes a MV rule (of risk averters) for the returns Y and Z of any two assets or portfolios with means μ_Y and μ_Z and standard deviations σ_y and σ_Z such that $\mu_Y \geq \mu_Z$ and $\sigma_y \leq \sigma_Z$, then Y is preferred to Z for risk averters. Meyer (1987) compares the distributions that differ only by location and scale parameters while analyzing the class of expected utility functions with convexity or concavity restrictions. Wong (2007) and Guo and Wong (2016) show that if Y dominates Z by the MV rule, then risk averters with $u^{(1)} > 0$ and $u^{(2)} < 0$ will attain higher expected utility by holding Y instead of holding Z when Y and Z follow a location-scale (LS) family or a linear combination of location-scale families where $u^{(i)}$ is the i^{th} derivative of utility function u. Wong and Ma (2008) extend the work on location-scale (LS) family with general n random seed sources and some general non-expected utility functions defined over the LS family. Bai, *et al.* (2012) show that the mean-variance-ratio (MVR) statistic for comparing the performance of prospects after eliminating the effect of the background risk produces a uniformly most powerful unbiased (UMPU) test. Broll, *et al.* (2010) develop properties of indifference curves and hedging decisions within the prospect theory.

Lajeri-Chaherli (2002) shows that proper risk aversion is equivalent to both quasi-concavity of a mean-variance utility function and DARA, while Lajeri-Chaherli (2004) shows that the mean-variance framework and expected utility specifications are fully compatible for the concept of standard risk aversion but not for the concepts of proper risk aversion and proper prudence. Eichner and Wagener (2003) derive the necessary and sufficient conditions for variance vulnerability, and provide connections between the mean-variance properties for risk vulnerability within the EU framework. In addition, Eichner (2008) shows that risk vulnerability is equivalent to the slope of the mean-variance indifference curve being decreasing in mean and increasing in variance.

Alexander and Baptista (2002) relate value at risk (VaR) to mean-variance analysis and examine the economic implications of using a mean-VaR model for portfolio selection. In addition, Alexander and Baptista (2004) analyze the portfolio selection implications arising from imposing a value-at-risk (VaR) constraint on the mean-variance model, and compare them with those arising from the imposition of a conditional value-at-risk (CVaR) constraint.

There are many applications of the mean-variance, mean-VaR, mean-CVaR models. For example, Broll, *et al.* (2006) analyze export production in the presence of exchange rate uncertainty under mean-variance preferences while Broll, *et al.* (2015) analyze a bank's risk taking in a two-moment decision framework. On the other hand, Alghalith, *et al.* (2017) analyze the impacts of joint energy and output prices uncertainties on the input demands in a MV framework.

2.3 Background Risk

There are many studies on background risk. For example, Lusk and Coble (2008) find that individuals are more risk-averse when facing an unfair or mean-preserving background risk. Bodie, *et al.* (1992), Heaton and Lucas (2000), and Viceira (2001) investigate the relation between labour income variations and investors' portfolio decisions and confirm the relevance of labour income risk to asset allocations. In addition, Cocco (2005) and Pelizzon and Weber (2009) analyze the impact of the housing investment on the composition of an investor's portfolio, and conclude that the investment in housing plays an important role in asset accumulation and in portfolio choice among financial assets.

Eichner and Wagener (2009) document the comparative statics with both an endogenous risk and a background risk for an agent with mean-variance preferences in a generic decision model, and confirm that the agent becomes less risk-averse in response to an increase in the expected value of the background risk or a decrease in its variability if the preferences exhibit DARA or variance vulnerability.

Alghalith, *et al.* (2016) present two dynamic models of background risk. They present a stochastic factor model with an additive background risk and a dynamic model of simultaneous (correlated) multiplicative background risk and additive background risk. Alghalith, *et al.* (2017) analyze the impacts of joint energy and output prices uncertainties on the inputs demands in a mean-variance framework.

2.4 Portfolio Selection with Background Risk

Classical portfolio theory (Markowitz, 1952; Merton, 1969, 1971; Samuelson, 1969) do not include background risk because the market is assumed to be complete. This assumption infers that background assets can be spanned and priced by tradable financial assets.

Nonetheless, Campbell (2006) shows that standard portfolio theory fails to explain household investment decisions in practice. To circumvent the limitation of the classical portfolio theory, academics introduce background risk in the study of portfolio compositions. For example, it is found that there are strong cross-sectional correlations between health and both financial and non-financial assets, and that adverse health shocks discourage risky asset holdings, see, for example, Rosen and Wu (2004), Berkowitz and Qiu (2006), Edwards (2008), and Fan and Zhao (2009) for more discussion.

There are some studies examine the properties in mean-variance framework in the presence of the background risk. For example, Jiang, *et al.* (2010) investigate the impact of background risk on an investor's portfolio choice in a mean-variance framework, and analyzes the properties of efficient portfolios as well as the investor's hedging behaviour in the presence of background risk.

With multiple additive risks, the mean-variance approach and the expected-utility approach of risk preferences are compatible if all attainable distributions belong to the same location-scale family. Under this proviso, Guo, *et al.* (2018) survey existing results on the parallels of the two approaches with respect to risk attitudes, the changes thereof, and the comparative statics for simple, linear choice problems under risks. In mean-variance approach all effects can be couched in terms of the marginal rate of substitution between mean and variance. In addition, they apply the theory to study the behavior of banking firm and study risk taking behavior with background risk in the mean-variance model.

3 Mean-VaR/CVaR/variance Boundaries and Efficient Frontiers with Background Risk but without Risk-Free Security

The theory developed in this paper is to extend the mean-variance portfolio frontier model without background asset which has been well established and studied. The theory without background asset can be obtained by setting the background asset to be zero in the theory with background asset developed in this paper or other papers. Some results of the theory without background asset will be given in this section and the next section when we compare the results with and without background asset. Thus, readers can understand the theory without background asset by reading our paper. However, Readers may read Huang and Litzenberger (1988) and others for the discussion of the mean-variance portfolio frontier model without background asset.

3.1 Models and Notations

Following Das, *et al.* (2010), Baptista (2012), and others, we first assume that there is no risk-free asset and there are n different risky assets in the market with returns $r = (r_1, r_2, \cdots, r_n)^\tau$ where r_i is the return of asset i and the superscript τ represents the transpose operation. We let $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^\tau$ be a portfolio in which ω_i is the proportion (weight) of the portfolio invested in asset i with a positive (negative) weight represents a long (short) position and $\sum_{i=1}^n \omega_i = 1$. Thus, the return of the portfolio is $r_p = \omega^\tau r$. We denote r_b to be the return of the background asset. Then, the mean and variance of total return r_ω putting weight ω on financial assets with background risk

$$r_\omega = \omega^\tau r + r_b \tag{3.1}$$

are given by

$$\begin{aligned} E(r_\omega) &= \omega^\tau E(r) + E(r_b) \,, \\ \sigma^2(r_\omega) &= \omega^\tau V \omega + 2\omega^\tau \mathrm{Cov}(r, r_b) + \mathrm{Var}(r_b) \,, \end{aligned} \tag{3.2}$$

where V is a non-singular matrix representing the covariance matrix of financial asset returns, $\mathrm{cov}(r, r_b)$ is an n-column covariance vector in which its i^{th} item represents the covariance between the i^{th} financial asset return and the background asset return, and $\mathrm{Var}(r_b)$ is the variance of the return of the background asset.

Hull and White (1998) suggest that normality is a common assumption when computing a portfolio's VaR. In addition, Duffie and Pan (1997) demonstrate that normality assumption is is a good approximation to daily VaR of the S&P 500. We follow their suggestion to assume that security rates of return and the return of the background asset have a multivariate normal distribution. We will discuss the extension of our results to non-normality in Section 5.

We note that Baptista (2012) assumes that there are M mental accounting with $M \geq 2$ and there is background risk in each mental accounting. In addition, he assumes that investor faces possibly different background risks in different accounts. Different from Baptista (2012), in this paper we assume that all background risks are combined in one unobserved background asset with return r_b. The background asset could be real estate and/or other assets exposed in background risk so that the return of background asset r_b could be from the return of real estate and/or other assets exposed in background risk. Nonetheless, we follow Baptista (2012) to let the weight of the background asset to be one.[1] We also note that some researchers assume that background risk is an independent, zero-mean risk, and then analyze its impacts on an agent's risk activity. Nonetheless, Lusk and Coble (2008) point out that the independent background risk has no impact on the investor's portfolio choice. Thus, it is more reasonable to assume background asset and financial assets are dependent, see, for example, Baptista (2008) and Jiang et al. (2010). In this paper, we adopt this assumption. Baptista (2008) assumes the non-tradeable income (background asset) with a zero expected value. In this paper, we relax this assumption by allowing the background risk to have non-zero expected value.

Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) are two of the most popular risk management tools to be employed in the financial industry in rencent years. Baumol (1963) first introduces the concept of VaR when examining a model referred as

[1] One could assume that an individual who could invest ω_f of her wealth in financial assets and invest $\omega_b = 1 - \omega_f$ in background asset. However, one could make some simple adjustment such that $\omega_f = 1 = \sum_{i=1}^{n} \omega_i$ and $\omega_b = 1$ as shown in Equation (3.2).

the expected gain-confidence limit criterion. Jorion (2000), Linsmeier and Pearson (2000), and others document that VaR is a widely used measure by corporate treasurers, dealers, fund managers, financial institutions, and regulators. We state the definition of VaR as follows:

Definition 3.1 *The VaR at the* $100t\%$ *confidence level of a risky portfolio for a specific time period is the rate of return* $V[t, r_\omega]$ *such that the probability of that portfolio having a rate of return of* $-V[t, r_\omega]$ *or less is* $1-t$*. In other words, the VaR of the portfolio* ω*'s return at the* $100t\%$ *confidence level is*

$$V[t, r_\omega] = -F_\omega^{-1}(1-t) \ , \tag{3.3}$$

where $F_\omega(\cdot)$ *is the cumulative distribution function of* r_ω*.*

We let $\Phi(\cdot)$ be the standard normal cumulative distribution function and $\phi(\cdot)$ the standard normal density function. In addition, for any confidence level $t \in (1/2, 1)$, we let $\Phi(-z_t) = (1-t)$. Thus, we have

$$\int_{-\infty}^{-z_t} \phi(x)dx = 1 - t \ .$$

Using the definition of VaR in (3.3), we can have

$$V[t, r_\omega] = z_t \sigma(r_\omega) - E(r_\omega).$$

Nonetheless, some researchers (Artzner, *et al.,* 1999; Basak and Shapiro, 2001) have pointed out the shortcomings of VaR as a measure of risk. In this connection, some researchers have recommended to use CVaR instead. We follow their suggestion to define CVaR as follows:

Definition 3.2 *A portfolio's CVaR is the loss one expects to suffer at that confidence level by holding it over the investment period, given that the loss is equal to or larger than its VaR. Formally, the CVaR of the portfolio* ω*'s return at the* $100t\%$ *confidence level is*

$$L[t, r_\omega] = -E\{r_\omega | r_\omega \le -V[t, r_\omega]\} \ . \tag{3.4}$$

In this paper, we first follow Hull and White (1998), Alexander and Baptista (2004) to assume that (r, r_b) have a multivariate normal distribution. From the definition of CVaR in (3.4), we get

$$L[t, r_\omega] = k_t \sigma(r_\omega) - E(r_\omega) \ ,$$

where

$$k_t = \frac{-\int_{-\infty}^{-z_t} x\phi(x)dx}{1-t} \ . \tag{3.5}$$

Clearly, we have $k_t > z_t$, consequently, we also have $L[t, r_\omega] > V[t, r_\omega]$.

3.2 Mean-Variance, Mean-VaR, Mean-CVaR Boundaries with Background Risk

Now we are ready to discuss the theory of mean-variance, mean-VaR, and mean-CVaR boundaries with background risk. We first provide definitions of return-risk boundaries with background risk when variance, VaR, and CVaR are used as measures of risk. For any $\bar{E} \in R$, we let $W(\bar{E}) = \{\omega \in W : E[r_\omega] = \bar{E}\}$ be the set of portfolios with expected return equal to $\bar{E}$ in which W is the set of portfolios. The definitions of mean-variance, mean-VaR, and mean-CVaR boundaries with background risk can then be defined as follows:

Definition 3.3 *A portfolio $\bar{\omega} \in W(\bar{E})$ is on the mean-variance boundary with background risk if and only if for $\bar{E} \in R$, $\bar{\omega}$ is the solution of solving* $\min_{\omega \in W(\bar{E})} \sigma_\omega^2$ *where σ_ω^2 is defined in (3.2).*

Definition 3.4 *A portfolio $\bar{\omega} \in W(\bar{E})$ is on to the mean-VaR boundary with background risk if and only if for $\bar{E} \in R$, $\bar{\omega}$ is the solution of solving* $\min_{\omega \in W(\bar{E})} V[t, r_\omega]$ *where $V[t, r_\omega]$ is defined in (3.3).*

Definition 3.5 *A portfolio $\bar{\omega} \in W(\bar{E})$ is on the mean-CVaR boundary with background risk if and only if for $\bar{E} \in R$, $\bar{\omega}$ is the solution of solving* $\min_{\omega \in W(\bar{E})} L[t, r_\omega]$ *where $L[t, r_\omega]$ is defined in (3.4).*

From the above definitions, one could infer that a portfolio is on the mean-VaR or mean-CVaR boundary with background risk if and only if it is on the mean-variance boundary with background risk since z_t and $k_t > 0$ where z_t and k_t are defined in (3.5).

For the mean-variance boundary with background risk, we develop the following proposition:

Proposition 3.1 *Portfolio ω is on the mean-variance boundary with background risk if and only if*

$$\frac{\sigma_\omega^2}{a} - \frac{(E(r_\omega) - E(r_b) - (A - EC + FA/C)^2}{Da/C} = 1 \ , \tag{3.6}$$

where $A = I^\tau V^{-1}E(r)$, $B = E(r)^\tau V^{-1}E(r)$, $C = I^\tau V^{-1}I$, $D = BC - A^2$, $E = \mathrm{Cov}(r, r_b)^\tau V^{-1}E(r)$, $F = \mathrm{Cov}(r, r_b)^\tau V^{-1}I$, $a = (1+F)^2/C - \mathrm{Cov}(r, r_b)^\tau V^{-1}\mathrm{Cov}(r, r_b) + \mathrm{Var}(r_b)$.

When the return of the background risk r_b is independent with the return of the financial assets, we can have $\mathrm{Cov}(r, r_b) = 0$ and, as a result, we have $E = F = 0$. So, we obtain the following corollary as a special case:

Corollary 3.1 *When r_b is independent of r, portfolio ω is on the mean-variance boundary with background risk if and only if*

$$\frac{\sigma_\omega^2}{a} - \frac{(E(r_\omega) - E(r_b) - A/C)^2}{Da/C} = 1 \ , \tag{3.7}$$

where $A = I^\tau V^{-1}E(r)$, $B = E(r)^\tau V^{-1}E(r)$, $C = I^\tau V^{-1}I$, $D = BC - A^2$, $a = 1/C + \mathrm{Var}(r_b)$.

From Proposition 3.1, we observe that any portfolio ω that satisfies Equation (3.6) will belong to both mean-VaR and mean-CVaR boundaries with background risk. Merton (1972) proves that portfolio ω is on the mean-variance boundary without background risk if and only if

$$\frac{\sigma_\omega^2}{1/C} - \frac{(E(r_\omega) - A/C)^2}{D/C^2} = 1.$$

Consequently, the variance of the minimum variance portfolio without background risk is $1/C$, while that with background risk is a. In addition, whether the variance of the minimum variance portfolio with background risk is larger than that of the minimum variance portfolio without background risk depends on the covariance vector $\mathrm{Cov}(r, r_b)$ and the covariance matrix V.

3.3 Mean-Variance, Mean-VaR, and Mean-CVaR Efficient Frontiers with Background Risk

We turn to develop properties for the mean-variance, mean-VaR, and mean-CVaR efficient frontiers with background risk. To do so, we first provide the notions of efficiency associated with the mean-variance, mean-VaR, and mean-CVaR boundaries as shown in the following definitions:

Definition 3.6 *A portfolio $\omega \in W$ is on the mean-variance efficient frontier with background risk if and only if there is no portfolio $\nu \in W$ such that $E(r_\nu) \geq E(r_\omega)$ and $\sigma(r_\nu) \leq \sigma(r_\omega)$ with at least one of the inequalities holds strictly where r_u and $\sigma(r_u)$ are defined in (3.3) with $u = \nu$ or ω.*

Definition 3.7 *A portfolio $\omega \in W$ is on the mean-VaR efficient frontier with background risk if and only if there is no portfolio $\nu \in W$ such that $E(r_\nu) \geq E(r_\omega)$ and $V[t, r_\nu] \leq V[t, r_\omega]$, with at least one of the inequalities holds strictly where r_u is defined in (3.2) and $V[t, r_u]$ is defined in (3.3) with $u = \nu$ or ω.*

Definition 3.8 *A portfolio $\omega \in W$ is on the mean-CVaR efficient frontier with background risk if and only if there is no portfolio $\nu \in W$ such that $E(r_\nu) \geq E(r_\omega)$ and $L[t, r_\nu] \leq L[t, r_\omega]$ with at least one of the inequalities holds strictly where r_u is defined in (3.2) and $V[t, r_u]$ is defined in (3.4) with $u = \nu$ or ω.*

3.3.1 Characterizations of the Minimum-VaR and Minimum-CVaR Portfolios

We begin by developing the minimum-VaR portfolio with background risk as stated in the following proposition:

Proposition 3.2

1. *If the minimum-VaR-portfolio exists, then it is both mean-variance and mean-CVaR efficient, and*

2. *if the minimum-CVaR portfolio exists, then it is mean-variance efficient.*

The proof of Proposition 3.2 is given in the appendix.

Assuming that both global minimum-VaR and global minimum-variance portfolios exist, we let $\omega_{V(t)} \in W$ denote the global minimum-VaR portfolio and $\omega_\sigma \in W$ denote the global minimum-variance portfolio at the $100t\%$ confidence level. We first establish the following proposition to describe the condition for the existence of the minimum-VaR portfolio:

Proposition 3.3

1. *The minimum-VaR portfolio exists if and only if* $z_t > \sqrt{D/C}$.

2. *Furthermore, if* $z_t > \sqrt{D/C}$, *then*

$$E(r_{\omega_{V(t)}}) = E(r_b) + \frac{A - EC + FA}{C} + \sqrt{\frac{D}{C}\Big(\frac{aCz_t^2}{Cz_t^2 - D} - a\Big)}.$$

From Proposition 3.3, we know that if $z_t \leq \sqrt{D/C}$, then there is no solution for the globally minimizing-VaR problem. Thus, academics and practitioners need to be careful when they select the confidence level for the globally minimizing-VaR problem.

Assuming that the minimum-CVaR portfolio exists, we let $\omega_{L(t)} \in W$ be the $100t\%$ confidence level minimum-CVaR portfolio. Then, one could easily obtain the result for the minimum-CVaR portfolio similar to Proposition 3.3 replacing z_t and $\omega_{v(t)}$ by k_t and $\omega_{L(t)}$, respectively. Since $k_t > z_t$, the minimum-CVaR portfolio exists if the minimum-VaR portfolio exists.

Applying Proposition 3.3, we establish the following corollary:

Corollary 3.2

1. *If the minimum-VaR portfolio exists, then* $E[r_{\omega_{V(t)}}] > E[r_{\omega_{L(t)}}]$*, and*

2. *if the minimum-CVaR portfolio exists, then* $E[r_{\omega_{L(t)}}] > E[r_{\omega_\sigma}]$*.*

The above result infers that for the solutions to the problems of the VaR minimization, the CVaR minimization and the variance minimization are all distinct from each other. It also shows that the minimum-VaR portfolio lies above the minimum-CVaR portfolio which, in turn, lies above the minimum-variance portfolio on the mean-variance efficient frontier. From Corollary 3.2, we obtain the follow corollary:

Corollary 3.3 *At any confidence level* $t < 1$*,*

1. *the minimum-variance portfolio is mean-VaR inefficient,*

2. *the minimum-variance portfolio is mean-CVaR inefficient, and*

3. *the minimum-CVaR portfolio is mean-VaR inefficient.*

From Corollary 3.2, one could obtain Corollary 3.3. For example, consider the minimum-CVaR portfolio $\omega_{L(t)}$, we know from Corollary 3.2 that $E[r_{\omega_{V(t)}}] > E[r_{\omega_{L(t)}}]$. Moreover, we can have $V[r_{\omega_{V(t)}}] \leq V[r_{\omega_{L(t)}}]$. Thus, $\omega_{L(t)}$ is mean-VaR inefficient.

3.3.2 Characterization of Mean-VaR and Mean-CVaR Efficiency

We turn to discuss the characterization of mean-VaR and mean-CVaR efficiency. We first develop the characterization of mean-VaR efficiency with background risk in the following proposition:

Proposition 3.4

1. *If* $z_t > \sqrt{D/C}$*, then a portfolio* ω *is mean-VaR efficient if and only if it is on the mean-VaR boundary and* $E[r_\omega] \geq E[r_{\omega_{v(t)}}]$*, and*

2. *if* $z_t \leq \sqrt{D/C}$*, then there is no mean-VaR efficient portfolio.*

We note that one could obtain the result for the mean-CVaR efficient frontier from Proposition 3.4 when z_t and $\omega_{v(t)}$ are replaced by k_t and $\omega_{L(t)}$, respectively. We state this result in the following corollary:

Corollary 3.4

1. *If* $k_t > \sqrt{D/C}$*, then a portfolio* ω *is mean-CVaR efficient if and only if it is on the mean-CVaR boundary and* $E[r_\omega] \geq E[r_{\omega_{L(t)}}]$*, and*

2. *if* $k_t \leq \sqrt{D/C}$*, then there is no mean-CVaR efficient portfolio.*

From Propositions 3.3 and 3.4, one could easily obtain the following corollary to compare the mean-variance, mean-VaR, and mean-CVaR efficient frontiers with background risk:

Corollary 3.5

1. *If* $k_t \leq \sqrt{D/C}$*, then both mean-VaR and mean-CVaR efficient frontiers are empty;*

2. *if* $z_t \leq \sqrt{D/C} < k_t$*, then the mean-VaR efficient frontier is empty but the mean-CVaR efficient frontier is a nonempty proper subset of the mean-variance efficient frontier; and*

3. *if* $z_t > \sqrt{D/C}$*, then a portfolio is on the mean-VaR efficient frontier if and only if it is on the mean-CVaR efficient frontier and* $E[r_\omega] \geq E[r_{\omega_{v(t)}}]$*; that is, the mean-VaR efficient frontier is a nonempty proper subset of the mean-CVaR efficient frontier.*

4 Adding a Risk-free Security

Our previous analysis in Section 3 for the mean-variance/VaR/CVaR boundary assumes that there is no risk-free security in the economy. Now, we turn to develop the theory by assuming that there is a risk-free security with rate of return $r_f \geq 0$ under the situations that agents can lend but cannot borrow and can lend and borrow. We first discuss the theory in which the agents can lend but cannot borrow in next subsection and discuss theory that agents can lend and borrow in the subsection thereafter.

4.1 Adding a Risk-free Lending but No Borrowing

We now develop the theory by assuming that there is a risk-free security with rate of return $r_f \geq 0$ under the situations that agents can lend but cannot borrow. Supposing there are n risky securities, we let $W_f = \{(\omega, \omega_f) \in R^n \times R : \sum_{j=1}^n \omega_j + \omega_f = 1\}$. In this situation, different from those expressed in (3.2), the expectation and variance of total return

$$r_\omega = \omega_f r_f + \omega^\tau r + r_b \tag{4.1}$$

become

$$\begin{aligned} E(r_\omega) &= \omega_f r_f + \omega^\tau E(r) + E(r_b) \ , \\ \sigma^2(r_\omega) &= \omega^\tau V \omega + 2\omega^\tau \mathrm{Cov}(r, r_b) + \mathrm{Var}(r_b) \ , \end{aligned} \tag{4.2}$$

respectively. We denote w_1 be the tangency portfolio associated with the risk-free lending rate, and assume that it lies above the minimum-variance portfolio in the absence of the risk-free security. By doing this, we develop the following proposition to give the characterization of the mean-variance boundary with both background risk and risk-free lending but no borrowing security:

Proposition 4.1 *Portfolio ω is on the mean-variance boundary with both background* *risk and risk-free security if and only if*

$$\frac{\sigma_\omega^2}{a} - \frac{(E(r_\omega) - E(r_b) - (A - EC + FA/C)^2}{Da/C} = 1 \quad \text{when} \quad E(r_\omega) > E(r_{\omega_1}) \ , \quad \text{and}$$
$$\frac{\sigma_\omega^2}{a^*} - \frac{(E(r_\omega) - E(r_b) - (r_f + r_f F - E))^2}{Ha^*} = 1 \quad \text{when} \quad E(r_\omega) < E(r_{\omega_1}) \ , \qquad (4.3)$$

where $H = B - 2r_f A + r_f^2 C$ *and* $a^* = -\mathrm{Cov}(r, r_b)^\tau V^{-1}\mathrm{Cov}(r, r_b) + \mathrm{Var}(r_b) > 0.$

Merton (1972) shows that when a risk-free security exists in the economy, a portfolio $\omega \in W_f$ is on the mean-variance boundary without background risk if and only if

$$\sigma_\omega^2 = \frac{(E(r_\omega) - r_f)^2}{H}.$$

Form Proposition 4.1, one could conclude that the presence of background risk changes the shape of the mean-variance boundary when there is a risk-free asset. Furthermore, since $a > a^*$, from Proposition 4.1, we obtain the following corollary:

Corollary 4.1 *When there is an additive background risk, the variance of the minimum-variance portfolio without risk-free security is larger than that with risk-free security.*

In other words, when background risk exists, adding a risk-free security result in reducing the risk of the minimum-variance portfolio.

From Proposition 4.1, we also note that even when we put all asset into the risk-free lending, that is, $\omega_f = 1$, because of the existence of the background risk, we still have a nonzero variance. This can be seen clearly from Figure 4.1.

Below we give a characterization of the tangency portfolio ω_1. At this point, the shape of the curve

$$\frac{\sigma_\omega^2}{a} - \frac{(E(r_\omega) - E(r_b) - (A - EC + FA/C)^2}{Da/C} = 1 \ ,$$

and that of the curve

$$\frac{\sigma_\omega^2}{a^*} - \frac{(E(r_\omega) - E(r_b) - (r_f + r_f F - E))^2}{Ha^*} = 1$$

is the same. In other words, we have

$$\sqrt{H}\frac{\sigma}{\sqrt{\sigma^2 - a^*}} = \sqrt{\frac{D}{C}}\frac{\sigma}{\sqrt{\sigma^2 - a}}.$$

This follows that at the tangency portfolio ω_1, the variance should be $\frac{HCa - Da^*}{HC - D}$.

Thereafter, we develop the following proposition to state the conditions for the existence of the minimum-VaR portfolio when there is a risk-free security:

Proposition 4.2

1. *The minimum-VaR portfolio exists if and only if* $z_t > \sqrt{H}$.

2. *Furthermore, if* $z_t > \sqrt{H}$, *then*

$$E(r_{\omega_{V(t)}}) = E(r_b) + (r_f + r_f F - E) + \sqrt{H\Big(\frac{a^* z_t^2}{z_t^2 - H} - a^*\Big)}.$$

This result can be obtained by using a similar argument as used in Proposition 4.2. In addition, from applying Proposition 4.2, we establish have the following two corollaries:

Corollary 4.2

1. *If the minimum-VaR portfolio exists, then* $E[r_{\omega_{V(t)}}] > E[r_{\omega_{L(t)}}]$, *and*

2. *if the minimum-CVaR portfolio exists, then* $E[r_{\omega_{L(t)}}] > E[r_{\omega_\sigma}]$.

Corollary 4.3 *At any confidence level* $t < 1$,

1. *the minimum-variance portfolio is mean-VaR inefficient,*

2. *the minimum-variance portfolio is mean-CVaR inefficient, and*

3. *the minimum-CVaR portfolio is mean-VaR inefficient.*

In addition, we develop the following to characterize the mean-VaR efficiency when there is a risk-free security:

Proposition 4.3

1. *If $z_t > \sqrt{H}$, then a portfolio ω is mean-VaR efficient if and only if it is on the mean-VaR boundary and $E[r_\omega] \geq E[r_{\omega_{v(t)}}]$, and*

2. *if $z_t \leq \sqrt{H}$, then the mean-VaR efficient portfolio does not exist.*

We note that a result similar to Proposition 4.3 holds for the mean-CVaR efficient frontier when z_t and $\omega_{v(t)}$ are replaced by k_t and $\omega_{L(t)}$, respectively. We state it in the following corollary:

Corollary 4.4

1. *If $k_t > \sqrt{H}$, then a portfolio ω is mean-CVaR efficient if and only if it is on the mean-CVaR boundary and $E[r_\omega] \geq E[r_{\omega_{L(t)}}]$, and*

2. *if $k_t \leq \sqrt{H}$, then the mean-CVaR efficient portfolio does not exist.*

In addition, similar to Corollary 3.5, we establish the following corollary to compare the mean-variance, mean-VaR, mean-CVaR efficient frontier with both background risk and risk-free security:

Corollary 4.5

1. *If $k_t \leq \sqrt{H}$, then both mean-VaR and mean-CVaR efficient frontiers are empty.*

2. *If $z_t \leq \sqrt{H} < k_t$, then the mean-VaR efficient frontier is empty but the mean-CVaR efficient frontier is a nonempty proper subset of the mean-variance efficient frontier;*

3. *if $z_t > \sqrt{H}$, then a portfolio is on the mean-VaR efficient frontier if and only if it is on the mean-CVaR efficient frontier and $E[r_\omega] \geq E[r_{\omega_{v(t)}}]$; that is, the mean-VaR efficient frontier is a nonempty proper subset of the mean-CVaR efficient frontier.*

4.2 Allowing for Both Risk-Free Lending and Borrowing

In this section, we suppose that both risk-free lending and borrowing are allowed and that the borrowing rate r_{fb} is higher than the risk-free lending rate r_{fl}. The set of portfolios with well-defined expected rates of return is then given by letting $W_f = \{(\omega, \omega_{fl}, \omega_{fb}) \in$

$R^n \times R_+ \times R_- : \sum_{j=1}^n \omega_j + \omega_{fl} + \omega_{fb} = 1\}$, where ω_{fl} and ω_{fb} are the proportions of wealth lend and borrowed at r_{fl} and r_{fb}, respectively, and R_+ and R_- denote the sets of non-negative and non-positive real numbers, respectively.

We let ω_2 be the tangency portfolio associated with the risk-free borrowing rate and assume that it lies above the minimum-variance portfolio in the absence of the risk-free security. By doing this, we develop the following proposition to give the characterization of the mean-variance boundaries with both background risk and risk-free lending and borrowing security:

Proposition 4.4 *Portfolio ω is on the mean-variance boundaries with both background risk and risk-free security if and only if*

$$\begin{aligned}
&\frac{\sigma_\omega^2}{a^*} - \frac{(E(r_\omega) - E(r_b) - (r_{fb} + r_{fb}F - E))^2}{H_2 a^*} = 1 \quad \text{when} \quad E(r_\omega) > E(r_{\omega_2}) \ , \\
&\frac{\sigma_\omega^2}{a} - \frac{(E(r_\omega) - E(r_b) - (A - EC + FA/C)^2}{Da/C} = 1 \quad \text{when} \quad E(r_{\omega_2}) > E(r_\omega) > E(r_{\omega_1}) \ , \quad \text{and} \\
&\frac{\sigma_\omega^2}{a^*} - \frac{(E(r_\omega) - E(r_b) - (r_{fl} + r_{fl}F - E))^2}{H_1 a^*} = 1 \quad \text{when} \quad E(r_\omega) < E(r_{\omega_1}) \ ,
\end{aligned}$$

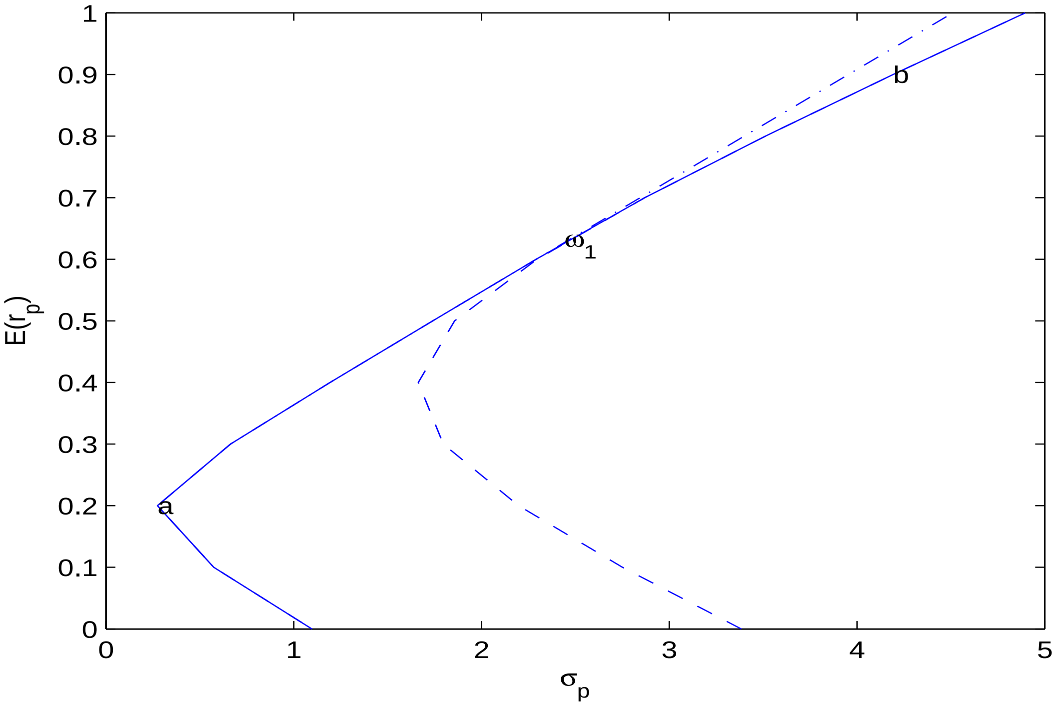
where $H_1 = B - 2r_{fl}A + r_{fl}^2 C$ and $H_2 = B - 2r_{fb}A + r_{fb}^2 C$.

Similar to the analysis on ω_1, we can also conclude that at the tangency portfolio ω_2, the variance should be $\frac{H_2 Ca - Da^*}{H_2 C - D}$. We give an illustration of the mean-variance boundary in this case in Figure 4.2.

5 Extension to Non-normality Case

Now, we assume that the rates of returns for the risky securities and the background asset in the economy are not multivariate normal distributed. Suppose that for any portfolio in the mean-VaR boundary, (i) we include a position with many stocks, and (ii) it is sufficiently diversified so that we can use the central limit theorem in our analysis (Duffie and Pan, 1997). Applying the central limit theorem, the rate of return of any portfolio

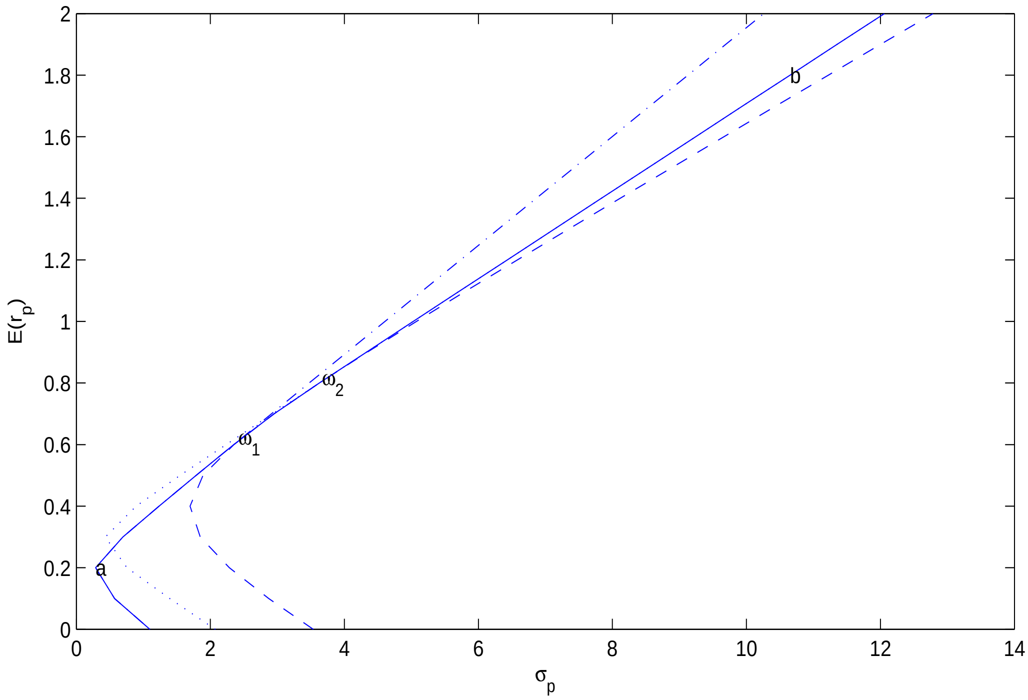
Figure 4.1: The mean-variance boundary when risk-free lending is allowed but no borrowing.



(b)

Note: The mean-variance boundary is the solid line that touches a, continues until it passes through the tangency portfolio ω_1, and then continues on the curved segment reaches b.

Figure 4.2: The mean-variance boundary when both risk-free lending and borrowing are allowed.



Note: The mean-variance frontier is the solid line that reaches a, continues until it touches the tangency portfolio ω_1, then continues on the curved segment until it passes through the other tangency portfolio ω_2, and continues on the curved segment touches b.

is approximately normal distributed in the mean-VaR boundary. As a result, the theory developed in Section 4 under the normality assumption is still valid. At least the theory is a good approximation when the distributions of stocks are skewness or fat tailed or both.

In addition, we now study a generalization of the theory developed in Section 4 to include the situation in which the distribution is not normal. We remark that the theory developed in Section 4 can be extended to include the situation in which the rates of returns are multivariate elliptical distributed rates. Suppose that the rates of return of n risky securities and that of the background asset (r, r_b) are jointly multivariate elliptical distributed, denoted by $E_{n+1}(\mu^*, \Sigma^*, g)$, where g is the density generator (Landsman and Valdez, 2003). Hence, $r_\omega = \omega^\tau r + r_b \sim E_1(E(r_\omega), \sigma^2(r_\omega), g)$. Here, $E(r_\omega)$ and $\sigma^2(r_\omega)$ are the same as those used in Section 4.

Let $F_Z(\cdot)$ be the standard elliptical CDF and $f_Z(\cdot)$ the standard elliptical PDF. Assuming that $F(-z_t^*) = (1-t)$ and applying VaR , we have

$$V[t, r_\omega] = z_t^* \sigma(r_\omega) - E(r_\omega)$$

for any confidence level $t \in (1/2, 1)$.

$$\begin{aligned} L[t, r_\omega] &= -E(r_\omega | r_\omega < -V[t, r_\omega]) \\ &= -\frac{\int_{-\infty}^{-V[t,r_\omega]} r_\omega f_r(r_\omega) dr_\omega}{1-t} \\ &= -\frac{\int_{-\infty}^{-z_t^*} (\sigma(r_\omega) z + E(r_\omega) f_Z(z) dz}{1-t} \\ &= k_t^* \sigma(r_\omega) - E(r_\omega). \end{aligned}$$

where

$$k_t^* = \frac{-\int_{-\infty}^{-z_t^*} z f_Z(z) dz}{1-t}$$

Therefore, the theory developed in Section 4 is still valid when the rates of return have a multivariate elliptical distribution, and z_t^* and k_t^* are used instead of z_t and k_t, respectively.

6 Conclusion

This paper investigates the impact of background risk on an investor's portfolio choice in the mean-VaR, mean-CVaR, and mean-variance framework, and analyzes the characterization of the mean-variance, mean-VaR, and mean-CVaR boundaries and efficient frontiers in the presence of background risk. We first derive the conditions that the portfolios lie on the mean-variance, mean-VaR, and mean-CVaR boundaries with and without background risk so that one could know what is the difference between the conditions in which the portfolios lie on the mean-variance, mean-VaR, and mean-CVaR boundaries with and without background risk. We show that the mean-variance (VaR, CVaR) boundary depends on the covariance vector between the returns of the risky assets and that of the background asset and also the variance of the return of the background asset. Thus, even when the return of the background risk is independent with that of the financial assets, the mean-variance (VaR, CVaR) boundary with background risk is still different from that without background risk. We also show that if the minimum-VaR-portfolio exists, then it is both mean-variance and mean-CVaR efficient, and if the minimum-CVaR portfolio exists, then it is mean-variance efficient. The existence condition of the minimum-VaR (CVaR) is also derived. It is found that this condition is the same for portfolio choice problem with or without background asset. In addition, we find that the minimum-variance portfolio is both mean-VaR and mean-CVaR inefficient, while the minimum-CVaR portfolio is mean-VaR inefficient.

Thereafter, we consider the case with a risk-free security. We show that adding a risk-free security results will reduce the risk of the minimum variance portfolio. The characterizations of the mean-variance, mean-VaR, and mean-CVaR boundaries and mean-variance, mean-VaR, and mean-CVaR efficient frontiers in this situation are also investigated and derived. Finally, we extend our work to the non-normality situation and examine the economic implication of the mean-VaR/CVaR model. In addition, we find that the main results developed in this paper are still valid when the rates of returns are multivariate elliptical distributed.

Extension of our paper includes adding background risk to other risk measures, for example, Niu, *et al.* (2017). In this paper, the background asset is additive. One could consider to extend the approach to include multiplicative background asset. In addition, many mean-variance portfolio selection models without background risk have been developed. For example, Zhou and Li (2000) develop a continuous-time mean-variance portfolio selection model while Zhou and Yin (2003) further develop a continuous-time version of the Markowitz mean-variance portfolio selection model with regime switching. Li and Ng (2000) extend the single-period mean-variance portfolio selection model to multi-period and Li, *et al.* (2010) develop the mean-variance-skewness model for portfolio selection with fuzzy returns. The above models could be extended to include background risk.

Appendix.

Proof of Proposition 3.1: We assume that investors are mean-variance optimizers who will solve the following optimization problem for their investment decision making:

$$\min \frac{1}{2} Var(r_\omega) \quad \text{s.t.} \quad \omega^\tau E(r) = \mu \ \text{ and } \ \omega^\tau I = 1 \tag{A.1}$$

where I is a vector of ones.

Using Lagrange multipliers to solve the two constraints in Problem (A.1), the first order optimality condition is

$$V\omega + \text{Cov}(r, r_b) - \lambda_1 E(r) - \lambda_2 I = 0$$

Consequently, we can have

$$\omega = V^{-1}(\lambda_1 E(r) + \lambda_2 I - \text{Cov}(r, r_b)) \tag{A.2}$$

Plugging Equation (A.2) into the constraints in Equation (A.1) obtains the following Lagrange multipliers

$$\lambda_1 = \frac{C\mu - A + (EC - FA)}{D} \quad \text{and} \quad \lambda_2 = \frac{B - \mu A + (FB - AE)}{D} \ ,$$

where $A = I^\tau V^{-1} E(r)$, $B = E(r)^\tau V^{-1} E(r)$, $C = I^\tau V^{-1} I$, $D = BC - A^2$, $E = \text{Cov}(r, r_b)^\tau V^{-1} E(r)$, and $F = \text{Cov}(r, r_b)^\tau V^{-1} I$.

Further, for the portfolio q on the mean-variance boundary, we can have:

$$\begin{aligned}
\sigma_\omega^2 &= \omega^\tau V\omega + 2\omega^\tau \text{Cov}(r, r_b) + \text{Var}(r_b) \\
&= \omega^\tau(\lambda_1 E(r) + \lambda_2 I - \text{Cov}(r, r_b)) + 2\omega^\tau \text{Cov}(r, r_b) + \text{Var}(r_b) \\
&= \lambda_1 \mu + \lambda_2 + \omega^\tau \text{Cov}(r, r_b) + \text{Var}(r_b) \\
&= \lambda_1(\mu + E) + \lambda_2(1 + F) - \text{Cov}(r, r_b)^\tau V^{-1} \text{Cov}(r, r_b) + \text{Var}(r_b) \\
&= \frac{C\mu - A + (EC - FA)}{D}(\mu + E) + \frac{B - \mu A + (FB - AE)}{D}(1 + F) \\
&\quad - \text{Cov}(r, r_b)^\tau V^{-1} \text{Cov}(r, r_b) + \text{Var}(r_b) \\
&= \frac{C\mu^2 - 2(A - EC + FA)\mu - 2AE + CE^2 - 2AEF + B(1 + F)^2}{D} \\
&\quad - \text{Cov}(r, r_b)^\tau V^{-1} \text{Cov}(r, r_b) + \text{Var}(r_b) \\
&= \frac{C}{D}(\mu - \frac{A - EC + FA}{C})^2 - \frac{C^2E^2 - 2CEA(1 + F) + A^2(1 + F)^2}{CD} \\
&\quad + \frac{-2AE + CE^2 - 2AEF + B(1 + F)^2}{D} - \text{Cov}(r, r_b)^\tau V^{-1} \text{Cov}(r, r_b) + \text{Var}(r_b) \\
&= \frac{C}{D}(\mu - \frac{A - EC + FA}{C})^2 + a \ ,
\end{aligned}$$

where $a = \frac{(1+F)^2}{C} - \text{Cov}(r, r_b)^\tau V^{-1} \text{Cov}(r, r_b) + \text{Var}(r_b) > 0$.

Consequently, we can have that portfolio ω is on the mean-variance boundary with background risk if and only if

$$\frac{\sigma_\omega^2}{a} - \frac{(E(r_\omega) - E(r_b) - (A - EC + FA)/C)^2}{Da/C} = 1 \ ,$$

and thus, Proposition 3.1 follows. □

Proof of Proposition 3.2:

Over here, we only prove that the minimum-VaR portfolio is mean-CVaR efficient. The other two assertions of Proposition 3.2 can be obtained similarly. Assuming ω is the minimum-VaR portfolio but is not mean-CVaR efficient. From Definition 3.8, we notice that there is a portfolio ν such that $E[r_\nu] \geq E[r_\omega]$ and $L[t, r_\nu] \leq L[t, r_\omega]$ with at least one of the inequalities holds strictly. As a result, we have

$$\begin{aligned} V(t, r_\nu) &= z_t\sigma(r_\nu) - E(r_\nu) = \frac{z_t}{k_t}\big[k_t\sigma(r_\nu) - E(r_\nu)\big] - (1 - \frac{z_t}{k_t})E(r_\nu) \\ &= \frac{z_t}{k_t}L(t, r_\nu) - (1 - \frac{z_t}{k_t})E(r_\nu) \\ &< V(t, r_\omega). \end{aligned}$$

This contradicts the assumption. Hence, the minimum-VaR portfolio is mean-CVaR efficient if it exists. Thus, the assertion of Proposition 3.2 holds. □

Proof of Proposition 3.3 : We first prove that the condition $z_t > \sqrt{D/C}$ is a necessary and sufficient condition that the minimum-VaR portfolio exists at the $100t\%$ confidence level. For any mean-variance efficient portfolio ω, applying Equation (3.6), we get

$$E(r_\omega) = E(r_b) + \frac{A - EC + FA}{C} + \sqrt{\frac{D}{C}(\sigma_\omega^2 - a)}. \tag{A.3}$$

Now, we have to solve

$$\min_{\omega \in W} V[t; r_\omega] \ .$$

Applying Equation (3.6), we also know that $\sigma(r_{\omega_\sigma}) = \sqrt{a}$. Hence, in order to obtain the VaR of the minimum-VaR portfolio, we use Proposition 3.2 and both equations (3.6) and (A.3) to solve

$$\min_{\sigma \in [\sqrt{a}, \infty]} z_t\sigma - \left(E(r_b) + \frac{A - EC + FA}{C} + \sqrt{\frac{D}{C}(\sigma^2 - a)}\right) \ . \tag{A.4}$$

We note that

$$\frac{\partial\left[z_t\sigma - \left(E(r_b) + \frac{A-EC+FA}{C} + \sqrt{\frac{D}{C}(\sigma^2 - a)}\right)\right]}{\partial\sigma} = z_t - \frac{\sigma\sqrt{D/C}}{\sqrt{\sigma^2 - a}}. \tag{A.5}$$

Clearly, $\bar{\sigma} = \sqrt{a}$ cannot be used to solve the minimization problem (A.4) because

$$\lim_{\sigma \to \sqrt{a}} z_t - \frac{\sigma\sqrt{D/C}}{\sqrt{\sigma^2 - a}} = -\infty.$$

Employing Equation (A.5), we obtain the following necessary condition for σ_{ω_V}:

$$z_t = \frac{\sigma_{\omega_V}\sqrt{D/C}}{\sqrt{\sigma^2_{\omega_V} - a}}$$

to solve the minimization problem (A.4).

From the above equation, we can have

$$\sigma^2_{\omega_V} = \frac{a z_t^2}{z_t^2 - D/C}.$$

Thus, $z_t > \sqrt{D/C}$ is found to be the necessary condition that the minimum-VaR portfolio exists at the $100t\%$ confidence level.

We turn to show that $z_t > \sqrt{D/C}$ is the sufficient condition that The minimum-VaR portfolio exists at the $100t\%$ confidence level by proving that we are minimizing a convex function. Employing Equation (A.5), we get

$$\begin{aligned} & \frac{\partial^2 \left[z_t\sigma - \left(E(r_b) + \frac{A-EC+FA}{C} + \sqrt{\frac{D}{C}(\sigma^2 - a)} \right) \right]}{\partial \sigma^2} \\ = & \frac{\partial [z_t - \sigma\sqrt{D/C}/\sqrt{\sigma^2 - a}]}{\partial \sigma} \\ = & a\sqrt{\frac{D}{C}} \frac{1}{(\sigma^2 - a)^{3/2}} > 0. \end{aligned} \tag{A.6}$$

This completes the first part of our proof.

Further, from Equation (A.3), we can have, when $z_t > \sqrt{D/C}$,

$$E(r_{\omega_{V(t)}}) = E(r_b) + \frac{A - EC + FA}{C} + \sqrt{\frac{D}{C}(\frac{aCz_t^2}{Cz_t^2 - D} - a)}.$$

□

Proof of Corollary 3.2: For $m_{L(t)}$, we can have

$$E(r_{\omega_{L(t)}}) = E(r_b) + \frac{A - EC + FA}{C} + \sqrt{\frac{D}{C}(\frac{aCk_t^2}{Ck_t^2 - D} - a)}.$$

Note the following equation follows:

$$\frac{\partial E(r_{\omega_{V(t)}})}{\partial z_t} = -\sqrt{Da}\frac{D + Cz_t^2}{Cz_t^2 - D} < 0.$$

Since $k_t > z_t$, Corollary 3.2 follows. $\square$

Proof of Proposition 3.4: We first prove (i). To do so, we need to show prove that for every mean-VaR boundary portfolio ω, we have $\partial^2 V(t, r_\omega)/\partial E(r_\omega)^2 > 0$. Suppose that ω is a mean-VaR boundary portfolio being chosen arbitrarily. Then, employing Equation (4.3), we get

$$\sigma_\omega = \sqrt{a + \frac{(E_{r_\omega} - E(r_b) - (A - EC + FA)/C)^2}{D/C}}$$

As a result, we can have

$$\begin{aligned}
\frac{\partial V(t, r_\omega)}{\partial E(r_\omega)} &= \frac{\partial [z_t \sqrt{a + \frac{(E_{r_\omega} - E(r_b) - (A - EC + FA)/C)^2}{D/C}} - E(r_\omega)]}{\partial E(r_\omega)} \\
&= \frac{z_t C(E_{r_\omega} - E(r_b) - (A - EC + FA)/C)}{D\sqrt{a + \frac{(E_{r_\omega} - E(r_b) - (A - EC + FA)/C)^2}{D/C}}} - 1. \qquad \text{(A.7)}
\end{aligned}$$

Consequently, we can further have

$$\begin{aligned}
\frac{\partial^2 V(t, r_\omega)}{\partial E(r_\omega)^2} &= \frac{\partial [z_t \sqrt{a + \frac{(E_{r_\omega} - E(r_b) - (A - EC + FA)/C)^2}{D/C}} - E(r_\omega)]}{\partial E(r_\omega)} \\
&= \frac{z_t C a}{D\Big(a + \frac{(E_{r_\omega} - E(r_b) - (A - EC + FA)/C)^2}{D/C}\Big)^{3/2}} > 0.
\end{aligned}$$

Thus, the assertion of the first part of Proposition 3.4 holds.

Now, we turn to prove (ii). When $z_t \leq \sqrt{D/C}$, employing Equation (A.7), we find that for any mean-VaR boundary portfolio, we have $\frac{\partial V(t, r_\omega)}{\partial E(r_\omega)} < 0$ ω. Thus, there exists a portfolio $\nu \in W$ that has both higher expected return and smaller VaR for any portfolio $\omega \in W$. Hence, the assertion of the second part of Proposition 3.4 holds. $\square$

Proof of Proposition 4.1: We assume that the investor is a mean-variance optimizer who solves the following optimization problem for the decision in his/her investment:

$$\min \frac{1}{2} Var(r_\omega)$$

$$\text{s.t.}\omega^\tau E(r) = \mu - \omega_f r_f \tag{A.8}$$
$$\omega^\tau I = 1 - \omega_f.$$

where $r_\omega = \omega_f r_f + \omega^\tau r + r_b$, I is a vector of ones. .

From constraint Equation (A.8), we can have:

$$\mu = \omega_f r_f + \omega^\tau E(r) = (1 - \omega^\tau I) r_f + \omega^\tau E(r) = \omega^\tau (E(r) - r_f I) + r_f \tag{A.9}$$

The problem (A.8) can be handled by using Lagrange multipliers method

$$L = \frac{1}{2}(\omega^\tau V \omega + 2\omega^\tau \mathrm{Cov}(r, r_b)) + \lambda_1(1 - \omega_f - \omega^\tau I) + \lambda_2(\mu - \omega^\tau(E(r) - r_f I) - r_f).$$

Then, we obtain the following first-order optimality condition:

$$V\omega + \mathrm{Cov}(r, r_b) - \lambda_2(E(r) - r_f I) = 0$$
$$\omega_f = 1 - \omega^\tau I.$$

Consequently, we can have

$$\omega = V^{-1}(\lambda_2(E(r) - r_f I) - \mathrm{Cov}(r, r_b)) \tag{A.10}$$

Plugging Equation (A.10) into the constraints in Equation (A.9) obtains the following Lagrange multipliers

$$\mu - r_f = \omega^\tau (E(r) - r_f I) = \lambda_2(B - 2r_f A + r_f^2 C) - E + r_f F$$

here $A = I^\tau V^{-1} E(r), B = E(r)^\tau V^{-1} E(r), C = I^\tau V^{-1} I, D = BC - A^2, E = \mathrm{Cov}(r, r_b)^\tau V^{-1} E(r), F = \mathrm{Cov}(r, r_b)^\tau V^{-1} I$. It follows that

$$\lambda_2 = \frac{\mu - r_f + E - r_f F}{B - 2r_f A + r_f^2 C}.$$

Further, for the portfolio ω on the mean-variance boundary, we can have:

$$\begin{aligned}
\sigma_\omega^2 &= \omega^\tau V\omega + 2\omega^\tau \mathrm{Cov}(r, r_b) + \mathrm{Var}(r_b) \\
&= \omega^\tau(\lambda_2(E(r) - r_f I) - \mathrm{Cov}(r, r_b)) + 2\omega^\tau \mathrm{Cov}(r, r_b) + \mathrm{Var}(r_b) \\
&= \lambda_2(\mu - r_f) + \omega^\tau \mathrm{Cov}(r, r_b) + \mathrm{Var}(r_b) \\
&= \lambda_2(\mu - r_f) + \lambda_2 E - \lambda_2 r_f F - \mathrm{Cov}(r, r_b)^\tau V^{-1} \mathrm{Cov}(r, r_b) + \mathrm{Var}(r_b) \\
&= \frac{(\mu - r_f + E - r_f F)^2}{H} + a^*.
\end{aligned}$$

here $H = B - 2r_f A + r_f^2 C, a^* = -\mathrm{Cov}(r, r_b)^\tau V^{-1}\mathrm{Cov}(r, r_b) + \mathrm{Var}(r_b) > 0$.

Consequently, we can have that portfolio ω is on the mean-variance boundary with background risk if and only if

$$\frac{\sigma_\omega^2}{a^*} - \frac{(E(r_\omega) - E(r_b) - (r_f + r_f F - E))^2}{Ha^*} = 1 \ ,$$

and thus, Proposition 4.1 follows. □

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