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MATH1510 Calculus for Engineers
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Appendix 2: Continuity and Differentiability

2.1 Continuity

Theorem 1: Sequential criterion for continuity

For any function $f(x)$ and $a \in D_f$,

$$f(x) \text{ is continuous at } a \iff \forall (a_n)_{n\in\mathbb{Z}^+} \text{ such that } \lim_{n\to\infty} a_n = a, \quad \lim_{n\to\infty} f(a_n) = f(a)$$

Proof

$\Longrightarrow$

Given $(a_n)_{n\in\mathbb{Z}^+}$ such that $\lim_{n\to\infty} a_n = a$, for all $\epsilon > 0$,

$$f \text{ continuous at } a \implies \exists \delta > 0, \forall x \text{ such that } 0 < |x-a| < \delta, \quad |f(x) - f(a)| < \epsilon$$

In fact,

$$a \in D_f \implies \forall x \text{ such that } |x-a| < \delta, \quad |f(x) - f(a)| < \epsilon$$

Since $\lim_{n\to\infty} a_n = a$, we have

$$\exists N \in \mathbb{Z}^+, \forall n > N, \quad |a_n - a| < \delta \implies |f(a_n) - f(a)| < \epsilon$$

Hence, $\lim_{n\to\infty} f(a_n) = f(a)$.

$\Longleftarrow$

From the given condition,

$$\forall (a_n)_{n\in\mathbb{Z}^+} \text{ such that } a_n \neq a \text{ and } \lim_{n\to\infty} a_n = a, \quad \lim_{n\to\infty} f(a_n) = f(a)$$

By the sequential criterion of limits, we conclude that

$$\lim_{x\to a} f(x) = f(a),$$

which means f is continuous at a. $\square$

Proposition 1

(a) If f, g are continuous at a, then

$$f \pm g, \ f \cdot g \text{ and } \frac{f}{g} \text{ (if } g(a) \neq 0\text{) are all continuous at } a$$

(b) If f is continuous at a and g is continuous at $f(a)$, then $g \circ f$ is continuous at a

(c) Constant functions and $f(x) = x$ are continuous over $\mathbb{R}$.

Proof

(a) Since f, g are continuous at a, we have

$$\lim_{x\to a} f(x) = f(a) \quad \text{and} \quad \lim_{x\to a} g(x) = g(a)$$

Therefore,

$$\lim_{x\to a}(f(x)+g(x)) = \left(\lim_{x\to a} f(x)\right)+\left(\lim_{x\to a} g(x)\right) = f(a)+g(a) \implies f+g \text{ is continuous at } a$$

The proofs for the other operations are similar.

(b) Given a sequence $(a_n)_{n\in Z^+}$ such that $\lim_{n\to\infty} a_n = a$, by the sequential criterion of continuity,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} f(a_n) = f(a) = b$$

Similarly, by consider the sequence $(b_n)_{n\in Z^+}$ that approaches b, we also have

$$\lim_{n\to\infty} g(b_n) = g(b)$$

Hence,

$$\lim_{n\to\infty} (g\circ f)(a_n) = \lim_{n\to\infty} g(b_n) = g(b) = (g\circ f)(a) \implies g\circ f \text{ is continuous at } a$$

(c) Let $g(x)$ be a constant function. Given $a \in \mathbb{R}$,

$$\forall \epsilon > 0, \text{ take } \delta = 1, \forall x \text{ such that } 0 < |x-a| < \delta, \quad |g(x) - g(a)| = 0 < \epsilon$$

So, $\lim_{x\to a} g(x) = g(a)$ and g is continuous at a.

Let $f(x) = x$. Given $a \in \mathbb{R}$,

$$\forall \epsilon > 0, \text{ take } \delta = \epsilon, \forall x \text{ such that } 0 < |x-a| < \delta, \quad |f(x) - f(a)| = |x-a| < \epsilon$$

So, $\lim_{x\to a} f(x) = f(a)$ and f is continuous at a. $\square$

Theorem 2: Extreme Value Theorem (EVT)

Suppose f is continuous on $[a, b]$. Then, f attains its maximum and minimum, i.e., there exist $c, d \in [a, b]$

$$f(c) \le f(x) \le f(d)$$

for all $x \in [a, b]$.

Proof

Let $M = \sup\{f(x) \mid x \in [a,b]\}$. Then, $\forall x \in [a,b], \quad f(x) \le M$.

Assume f doesn't attain M.

$$\forall n \in \mathbb{Z}^+, \exists x_n \in [a,b], \quad |f(x_n) - M| < \frac{1}{n}$$

Since $[a, b]$ is closed and bounded, by Bolzano-Weierstrass theorem, $(x_n)_{n\in\mathbb{Z}^+}$ has a subsequence $(x_{n_k})_{k\in\mathbb{Z}^+}$ such that

$$\lim_{k\to\infty} x_{n_k} = x_0$$

for some $x_0 \in [a, b]$. By squeeze theorem and continuity,

$$\lim_{k\to\infty} |f(x_{n_k}) - M| = 0 \implies f(x_0) = \lim_{k\to\infty} f(x_{n_k}) = M$$

which contradicts with our assumption. Hence, f attains its maximum.

f also attains its minimum because $-f$ attains its maximum. □

Theorem 3: Intermediate Value Theorem (IVT)

If f is continuous on $[a, b]$, then, for any $v \in [f(a), f(b)]$ (or $[f(b), f(a)]$),

$$f(c) = v \quad \text{for some } c \in [a, b].$$

Proof

The statement is trivial if $v = f(a)$ or $f(b)$. Otherwise, say, $f(a) < v < f(b)$. By subtracting v, we may assume $f(a) < 0, f(b) > 0$ and $v = 0$.

Assume $\forall x \in [a, b], \quad f(x) \neq 0$.

Let $a_0 = a, b_0 = b$ and $x_0 = \dfrac{a_0 + b_0}{2}$. Then, $f(x_0) \neq 0$. Let

$$(a_1, b_1) = \begin{cases} (a_0, x_0) & \text{if } f(x_0) > 0 \\ (x_0, b_0) & \text{if } f(x_0) < 0 \end{cases}$$

Either way, we have

$$f(a_1) < 0, \quad f(b_1) > 0, \quad |b_1 - a_1| = \frac{b - a}{2}$$

Similarly, we define $x_1 = \dfrac{a_1 + b_1}{2}$ and

$$(a_2, b_2) = \begin{cases} (a_1, x_1) & \text{if } f(x_1) > 0 \\ (x_1, b_1) & \text{if } f(x_1) < 0 \end{cases}$$

Again, we have

$$f(a_2) < 0, \quad f(b_2) > 0, \quad |b_2 - a_2| = \frac{b_1 - a_1}{2} = \frac{b - a}{2^2}$$

In this manner, we construct an increasing sequence $(a_n)_{n\in\mathbb{Z}^+}$ and a decreasing sequence $(b_n)_{n\in\mathbb{Z}^+}$ such that

$$\forall n \in \mathbb{Z}^+, \quad a_n < b_n, \quad f(a_n) < 0, \quad f(b_n) > 0, \quad |b_n - a_n| = \frac{b - a}{2^n}$$

Since $a_n < b_n < b_1$ and $a_1 < a_n < b_n$ for all n, a_n is bounded above and b_n is bounded below. By Monotone convergence theorem,

$$\lim_{n\to\infty} a_n = \overline{a} \leq \overline{b} = \lim_{n\to\infty} b_n$$

<u>Claim 1: $\quad \lim_{n\to\infty} \frac{1}{2^n} = 0$</u>

Given $\epsilon > 0$, take $N = \left\lfloor \frac{1}{\epsilon} \right\rfloor + 1$ ($\lfloor * \rfloor$ is the round-down function). Then,

$$\forall n > N, \qquad n > \frac{1}{\epsilon} \implies 2^n > n > \frac{1}{\epsilon} \implies \left| \frac{1}{2^n} \right| < \epsilon \qquad \triangle$$

By claim 1,

$$|b_n - a_n| = \frac{b-a}{2^n} \implies \lim_{n\to\infty} |b_n - a_n| = 0 \implies \overline{a} = \overline{b}$$

Notice that f is continuous at $\overline{a} = \overline{b}$. Thus,

$$0 \geq \lim_{n\to\infty} f(a_n) = f(\overline{a}) = f(\overline{b}) = \lim_{n\to\infty} f(b_n) \geq 0$$

Hence, $f(\overline{a}) = 0$, contradicting our assumption. $\square$

<u>Theorem 4: Bolzano's Theorem</u>

Suppose f is continuous on $[a, b]$. If $f(a), f(b)$ have opposite signs, then

$$f(c) = 0 \quad \text{for some } c \in (a, b).$$

<u>Proof</u>

By putting $v = 0$ in IVT. $\square$

2.2 Differentiability

<u>Proposition 2</u>

$$f(x) \text{ is differentiable at } a \iff Lf'(a), Rf'(a) \text{ both exist and are equal}$$

<u>Proof</u>

By the corresponding result about limits. $\square$

<u>Theorem 5</u>

$$f(x) \text{ is differentiable at } a \implies f(x) \text{ is continuous at } a$$

Proof

Since $g(x) = x - a$ is continuous over $\mathbb{R}$,

$$\lim_{x\to a}(f(x) - f(a)) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}(x-a) = f'(a)g(a) = 0 \implies \lim_{x\to a} f(x) = f(a)$$

$\square$

Proposition 3

(a) If f, g are differentiable at a, then

$f \pm g,\ f \cdot g$ and $\dfrac{f}{g}$ (if $g(a) \neq 0$) are all differentiable at a with

$$(f \pm g)'(a) = f'(a) \pm g'(a)$$

$$(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$$

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$$

(b) If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a with

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$$

(c) $f(x) = c \in \mathbb{R}$ and $g(x) = x$ are differentiable over $\mathbb{R}$ with $f'(x) = 0$ and $g'(x) = 1$.

Proof

(a) By definition,

$$\lim_{x\to a}\frac{(f(x) \pm g(x)) - (f(a) \pm g(a))}{x-a} = \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a} \pm \frac{g(x)-g(a)}{x-a}\right) = f'(a) \pm g'(a)$$

Notice that f, g are differentiable at a implies that f, g are continuous at a.

$$\begin{aligned}
(f \cdot g)'(a) &= \lim_{x\to a}\frac{f(x)g(x) - f(a)g(a)}{x-a} \\
&= \lim_{x\to a}\frac{f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)}{x-a} \\
&= \lim_{x\to a}\left(g(x)\frac{f(x)-f(a)}{x-a} + f(a)\frac{g(x)-g(a)}{x-a}\right) \\
&= g(a)f'(a) + f(a)g'(a)
\end{aligned}$$

$$
\begin{aligned}
\left(\frac{f}{g}\right)'(a) &= \lim_{x\to a}\frac{\frac{f(x)}{g(x)}-\frac{f(a)}{g(a)}}{x-a}\\
&= \lim_{x\to a}\frac{f(x)g(a)-f(a)g(x)}{(x-a)g(x)g(a)}\\
&= \lim_{x\to a}\frac{f(x)g(a)-f(a)g(a)+f(a)g(a)-f(a)g(x)}{(x-a)g(x)g(a)}\\
&= \lim_{x\to a}\frac{1}{g(x)g(a)}\left(g(a)\frac{f(x)-f(a)}{x-a}-f(a)\frac{g(x)-g(a)}{x-a}\right)\\
&= \frac{f'(a)g(a)-f(a)g'(a)}{g(a)^2}
\end{aligned}
$$

(b) Given $(x_n)_{n\in\mathbb{Z}^+}$ such that $x_n \neq a$ and $\lim_{n\to\infty} x_n = a$, let $y_n = f(x_n)$ and $b = f(a)$. Since f is differentiable at a,

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} f(x_n) = f(a) = b$$

$$\lim_{n\to\infty}\frac{y_n-b}{x_n-a} = \lim_{n\to\infty}\frac{f(x_n)-f(a)}{x_n-a} = f'(a)$$

Assume that $y_n \neq b$ for sufficiently large n. Then,

$$\lim_{n\to\infty}\frac{g(f(x_n))-g(f(a))}{x_n-a} = \lim_{n\to\infty}\frac{g(y_n)-g(b)}{y_n-b}\cdot\frac{y_n-b}{x_n-a} = g'(b)f'(a)$$

Otherwise, there exists a subsequence $(x_{n_k})_{k\in\mathbb{Z}^+}$ such that $y_{n_k} = b$ for all $k \in \mathbb{Z}^+$. In this case,

$$f'(a) = \lim_{k\to\infty}\frac{f(x_{n_k})-f(a)}{x_{n_k}-a} = 0$$

Notice that

$$\frac{g(f(x_n))-g(f(a))}{x_n-a} = \begin{cases} \dfrac{g(y_n)-g(b)}{x_n-a} = 0 & \text{if } y_n = b \\ \dfrac{g(y_n)-g(b)}{y_n-b}\cdot\dfrac{y_n-b}{x_n-a} \to g'(b)f'(a) = 0 & \text{if } y_n \neq b \end{cases}$$

That means

$$\forall\epsilon > 0, \exists N \in \mathbb{Z}^+, \forall n > N, \qquad \left|\frac{g(f(x_n))-g(f(a))}{x_n-a}\right| < \epsilon$$

Therefore,

$$\lim_{n\to\infty}\frac{g(f(x_n))-g(f(a))}{x_n-a} = 0 = g'(b)f'(a)$$

Hence, by sequential criterion,

$$(g\circ f)'(a) = g'(b)f'(a) = g'(f(a))\cdot f'(a)$$

(c) Given $a \in \mathbb{R}$, since constant functions are continuous,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} 0 = 0$$

$$g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = \lim_{x \to a} 1 = 1$$

□

Theorem 6: Rolle's Theorem

Suppose $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b).
If $f(a) = f(b)$, then

$$f'(c) = 0 \quad \text{for some } c \in (a, b).$$

Proof

If $\forall x \in (a, b), \quad f(x) = f(a)$, then we might take $c = \dfrac{a+b}{2}$:

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = 0$$

Assume $f(d) > f(a)$ for some $d \in (a, b)$. By EVT,

$$\exists c \in [a, b], \quad M = f(c) = \max\{f(x) \mid x \in [a, b]\}$$

$$f(c) \geq f(d) > f(a) \implies c \in (a, b)$$

$$0 \geq \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = Rf'(c) = Lf'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0$$

Therefore, $f'(c) = 0$ as desired. The case when $f(d) < f(a)$ for some $d \in (a, b)$ can be handled by considering $-f(x)$. □

Theorem 7: Mean Value Theorem (Lagrange)

Suppose $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Then,

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{for some } c \in (a, b).$$

Proof

Let

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Then, $g(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Moreover,

$$g(a) = f(a) = f(b) - \frac{f(b) - f(a)}{b - a}(b - a) = g(b)$$

By Rolle's Theorem,

$$\exists c \in (a, b), \quad 0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

and we are done.

$\square$

Theorem 8

(a) If f is continuous and strictly increasing (or strictly decreasing) over an open interval I, then f^{-1} exists and is continuous over $f(I)$.

(b) If f is differentiable and $f' > 0$ (or $f' < 0$) over an open interval I, then f^{-1} exists and is differentiable over $f(I)$ with

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

Proof

(a) Suppose f is strictly increasing over I. By definition, f with codomain $f(I)$ is surjective. Moreover,

$$x_1 \neq x_2 \implies x_1 < x_2 \text{ or } x_1 > x_2 \implies f(x_1) < f(x_2) \text{ or } f(x_1) > f(x_2) \implies f(x_1) \neq f(x_2)$$

So, f is also injective and thus, f^{-1} exists.

Given $b \in f(I)$, let $a = f^{-1}(b)$. For any strictly increasing sequence $(y_n)_{n\in\mathbb{Z}^+}$ such that $\lim_{n\to\infty} y_n = b$, we consider the sequence $x_n = f^{-1}(y_n)$.

$$x_n \geq a \implies y_n = f(x_n) \geq f(a) = b \quad \text{(contradiction)}$$

So, $x_n < a$. By similar arguments, we can show that x_n is a strictly increasing sequence. By monotone convergence theorem,

$$\lim_{n\to\infty} x_n = a' \leq a$$

Since f is continuous,

$$\lim_{n\to\infty} x_n = a' \implies \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} y_n = f(a') \implies f(a') = b \implies a' = a$$

In other words,

$$\lim_{n\to\infty} f^{-1}(y_n) = \lim_{n\to\infty} x_n = a = f^{-1}(b)$$

By sequential criterion, we can conclude that

$$\lim_{y\to b^-} f^{-1}(y) = f^{-1}(b)$$

By symmetry, we also have $\lim_{y\to b^+} f^{-1}(y) = f^{-1}(b)$. Hence, f^{-1} is continuous at any $b \in f(I)$.

If f is strictly decreasing, then $-f$ is strictly increasing and $g(x) = (-f)^{-1}(-x)$ will be the inverse of $f(x)$.

(b) Suppose $f' > 0$ over I. For any $x_1 < x_2$, f is differentiable over $[x_1, x_2]$. By MVT,

$$\exists c \in (x_1, x_2), \qquad \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0 \implies f(x_2) > f(x_1)$$

So, f is strictly increasing over I. By part (a), f^{-1} exists and is continuous over $f(I)$. Given $b \in f(I)$, let $a = f^{-1}(b)$. For any sequence $(y_n)_{n \in \mathbb{Z}^+}$ such that $y_n \neq b$ and $\lim_{n\to\infty} y_n = b$, we consider the sequence $x_n = f^{-1}(y_n)$. Then, $x_n \neq a$. Since f^{-1} is continuous,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} f^{-1}(y_n) = f^{-1}(b) = a$$

Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(b)}{y_n - b} &= \lim_{n\to\infty} \frac{x_n - a}{f(x_n) - f(a)} \\
&= \frac{1}{\lim_{n\to\infty} \dfrac{f(x_n) - f(a)}{x_n - a}} \\
&= \frac{1}{f'(a)} \qquad (f'(a) \neq 0)
\end{aligned}$$

By sequential criterion,

$$(f^{-1})'(b) = \lim_{y\to b} \frac{f^{-1}(y) - f^{-1}(b)}{y - b} = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))} \qquad \square$$