THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers
by Dr. Liu Chun Lung (Kelvin)

Appendix 1: Formal Definition of Limits

1.1 Limits of sequences

Definition 1

For a sequence $(a_n)_{n\in\mathbb{Z}^+}$, we say that $\lim\limits_{n\to\infty} a_n = L$ if

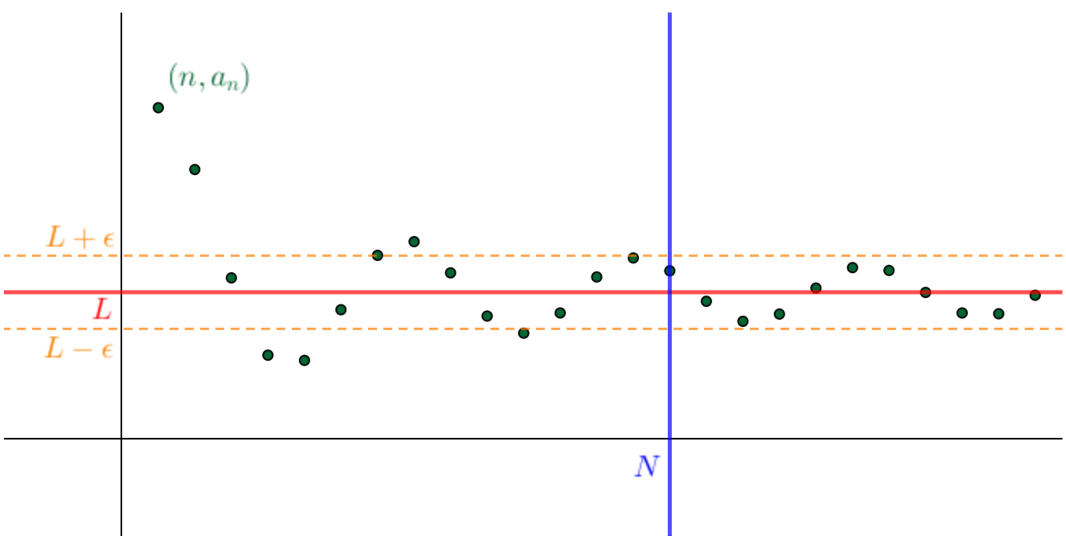
$$\forall \epsilon > 0, \exists N \in \mathbb{Z}^+, \forall n > N, \quad |a_n - L| < \epsilon$$

We say that $\lim\limits_{n\to\infty} a_n = \infty$ if

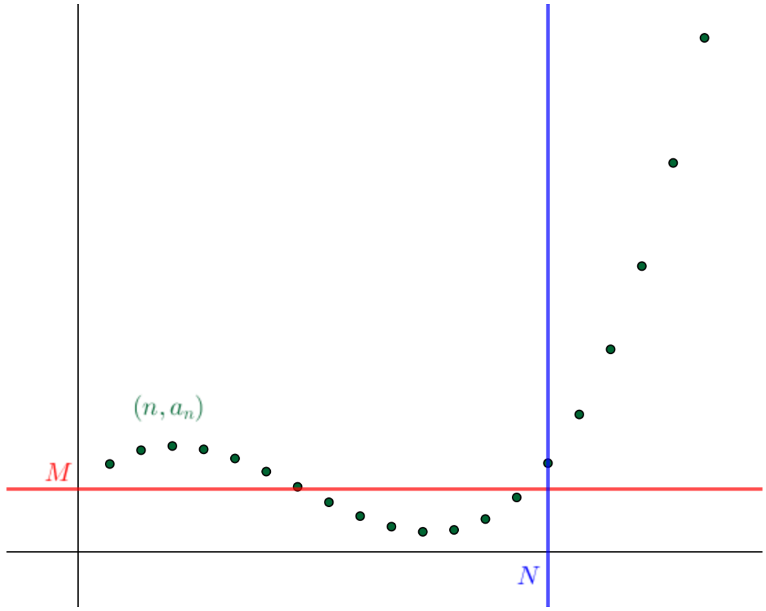
$$\forall M \in \mathbb{R}, \exists N \in \mathbb{Z}^+, \forall n > N, \quad a_n > M$$

We say that $\lim\limits_{n\to\infty} a_n = -\infty$ if

$$\forall M \in \mathbb{R}, \exists N \in \mathbb{Z}^+, \forall n > N, \quad a_n < M$$



$$\lim_{n\to\infty} a_n = L$$



$$\lim_{n\to\infty} a_n = \infty$$

Theorem 1: Squeeze Theorem (Sandwich Theorem)

Suppose $(a_n)_{n\in\mathbb{Z}^+}, (b_n)_{n\in\mathbb{Z}^+}, (c_n)_{n\in\mathbb{Z}^+}$ are three sequences such that

$$b_n \leq a_n \leq c_n \quad \text{for sufficiently large } n$$

(That is, $\exists N \in \mathbb{Z}^+, \forall n > N, \quad b_n \leq a_n \leq c_n$)
Then,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L \implies \lim_{n\to\infty} a_n = L$$

Similarly,

$$\lim_{n\to\infty} b_n = \infty \text{ (DNE)} \implies \lim_{n\to\infty} a_n = \infty \text{ (DNE)}$$

$$\lim_{n\to\infty} c_n = -\infty \text{ (DNE)} \implies \lim_{n\to\infty} a_n = -\infty \text{ (DNE)}$$

Proof

Let $N_0 \in \mathbb{Z}^+$ such that

$$\forall n > N_0, \quad b_n \leq a_n \leq c_n$$

By definition, given $\epsilon > 0$, there exist $N_1, N_2 \in \mathbb{Z}$ such that

$$\forall n_1 > N_1, n_2 > N_2, \quad |b_{n_1} - L|, |c_{n_2} - L| < \epsilon$$

Take $N = \max\{N_0, N_1, N_2\}$. Then, for all $n > N$, we have

$$|b_n - L|, |c_n - L| < \epsilon \quad \text{and} \quad b_n \leq a_n \leq c_n$$

$$\implies -\epsilon < b_n - L \leq a_n - L \leq c_n - L < \epsilon$$

$$\implies |a_n - L| < \epsilon$$

as desired. The result about $\pm\infty$ follow directly from definitions. $\square$

Proposition 1

For any sequence $(a_n)_{n\in\mathbb{Z}^+}$,

$$\lim_{n\to\infty} a_n = L \iff \lim_{n\to\infty}(a_n - L) = 0$$

Proof

Straightly from definition. $\square$

Proposition 2

Suppose $(a_n)_{n\in\mathbb{Z}^+}$ be a sequence such that $\lim\limits_{n\to\infty} a_n$ exists. $\forall c \in \mathbb{R}$,

$$\lim_{n\to\infty}(ca_n) = c\left(\lim_{n\to\infty} a_n\right)$$

Proof

Let $\lim\limits_{n\to\infty} a_n = L$. By Proposition 1, we may assume $L = 0$. The statement is trivial when $c = 0$.

Suppose $c \neq 0$. Given $\epsilon > 0$, there exists N such that

$$\forall n > N, \quad |a_n| < \frac{\epsilon}{|c|}$$

Hence, for any $n > N$,

$$|ca_n| \leq |c||a_n| < \epsilon$$

$\square$

Proposition 3

For any sequence $(a_n)_{n\in\mathbb{Z}^+}$,

$$\lim_{n\to\infty} a_n = 0 \iff \lim_{n\to\infty} |a_n| = 0$$

Proof

$\Longrightarrow$

Straightly from definition.

$\Longleftarrow$

Since

$$\forall n \in \mathbb{Z}^+, \quad -|a_n| \leq a_n \leq |a_n|$$

and

$$\lim_{n\to\infty}(-|a_n|) = -\lim_{n\to\infty}|a_n| = 0 = \lim_{n\to\infty}|a_n|,$$

the result follows from the squeeze theorem. $\square$

Theorem 2
Suppose $(a_n)_{n\in\mathbb{Z}^+}$ and $(b_n)_{n\in\mathbb{Z}^+}$ are two sequences such that $\lim\limits_{n\to\infty} a_n$, $\lim\limits_{n\to\infty} b_n$ both exist.

$$\begin{aligned}
\lim_{n\to\infty}(a_n \pm b_n) &= \left(\lim_{n\to\infty} a_n\right) \pm \left(\lim_{n\to\infty} b_n\right) \\
\lim_{n\to\infty}(a_n \cdot b_n) &= \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right) \\
\lim_{n\to\infty}\frac{a_n}{b_n} &= \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} \quad \text{provided that } \lim_{n\to\infty} b_n \neq 0
\end{aligned}$$

$$a_n \leq b_n \quad \text{for sufficiently large } n \implies \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$$

Proof

Let $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = M$.

$\lim\limits_{n\to\infty}(a_n + b_n) = L + M$

By Proposition 1, we may assume $L = M = 0$. Given $\epsilon > 0$, there exist N_1, N_2 such that

$$\forall n_1 > N_1, n_2 > N_2, \quad |a_{n_1} - 0|, |b_{n_2} - 0| < \frac{\epsilon}{2}$$

By taking $N = \max\{N_1, N_2\}$, for any $n > N$

$$|a_n + b_n| \leq |a_n| + |b_n| < \epsilon$$

$\lim\limits_{n\to\infty}(a_n - b_n) = L - M$

As before, we may assume $L = M = 0$.

$$\lim_{n\to\infty}(-b_n) = 0 \iff \lim_{n\to\infty} |-b_n| = 0 \iff \lim_{n\to\infty} b_n = 0$$

Therefore,

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty}(a_n + (-b_n)) = 0 + 0$$

$\lim\limits_{n\to\infty}(a_n b_n) = LM$

By definition, there exist N such that

$$\forall n > N, \quad |b_n - M| < 1 \implies |b_n| < |M| + 1$$

Thus, for any $n > N$,

$$\begin{aligned}
0 &\leq |a_n b_n - LM| \\
&= |a_n b_n - L b_n + L b_n - LM| \\
&\leq |a_n - L||b_n| + |L||b_n - M| \\
&\leq |a_n - L|(|M| + 1) + |L||b_n - M|
\end{aligned}$$

and the result follows from squeeze theorem.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ provided that $M \neq 0$

Obviously, it's enough to show $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{M}$ if $M \neq 0$. Also, we may assume $M = 1$. By definition, there exists N such that

$$\forall n > N, \quad |b_n - 1| < \frac{1}{2} \implies \frac{1}{2} < b_n < \frac{3}{2} \implies \frac{1}{|b_n|} < 2$$

Therefore, for any $n > N$,

$$0 \leq \left|\frac{1}{b_n} - 1\right| = \frac{1}{|b_n|}|b_n - 1| < 2|b_n - 1|$$

and the result follows from squeeze theorem.

$a_n \leq b_n \quad$ for sufficiently large $n \implies \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$

Let $c_n = b_n - a_n$. It suffices to show $\lim_{n\to\infty} c_n = L - M \geq 0$. Assume not.

$$\exists N \in \mathbb{Z}^+, \forall n > N, \quad |c_n - (L - M)| < \frac{|L - M|}{2}$$

So, for some sufficiently large n,

$$0 \leq c_n < (L - M) + \frac{|L - M|}{2} = -\frac{|L - M|}{2} < 0$$

which is a contradiction. □

Theorem 3: Monotone Convergence Theorem

Suppose $(a_n)_{n\in\mathbb{Z}^+}$ is a sequence which is either

increasing and bounded above for sufficiently large n

or

decreasing and bounded below for sufficiently large n

Then, $\lim_{n\to\infty} a_n$ exists.

Proof

Suppose $(a_n)_{n\in\mathbb{Z}^+}$ is a sequence which is increasing and bounded above when $n > N$. Let $S = \{a_n \mid n > N\}$. Since S is bounded above,

$$\sup S = L$$

for some $L \in \mathbb{R}$. Given $\epsilon > 0$, assume $S \cap (L - \epsilon, L] = \emptyset$. Then,

$$S \text{ is bounded above by } L - \epsilon \implies L = \sup S \leq L - \epsilon,$$

which is absurd. Thus, we know that $a_m \in S \cap (L - \epsilon, L]$ for some $m > N$.

As a_n is increasing when $n > N$, for all $n > m$,

$$L - \epsilon < a_m \leq a_n \leq \sup S = L \implies |a_n - L| < \epsilon$$

and we are done.

If $(a_n)_{n\in\mathbb{Z}^+}$ is decreasing and bounded below for sufficiently large n, then $(-a_n)_{n\in\mathbb{Z}^+}$ is increasing and bounded above for sufficiently large n. $\square$

1.2 Limits of functions

Definition 2 (limit at infinity)

For a function $f(x)$, we say that $\lim_{x\to\infty} f(x) = L$ ($\lim_{x\to-\infty} f(x) = L$) if

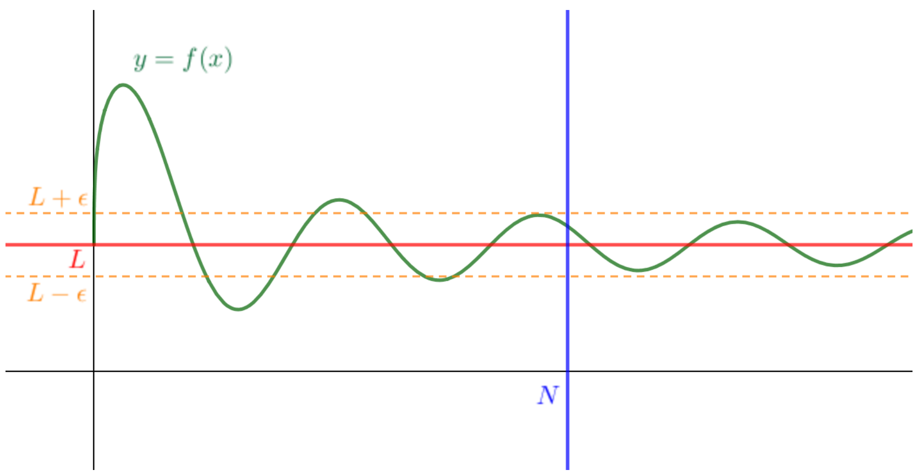
$$\forall \epsilon > 0, \exists N \in \mathbb{R}, \forall x > N (x < N), \quad |f(x) - L| < \epsilon$$

We say that $\lim_{x\to\infty} f(x) = \infty$ ($\lim_{x\to-\infty} f(x) = \infty$) if

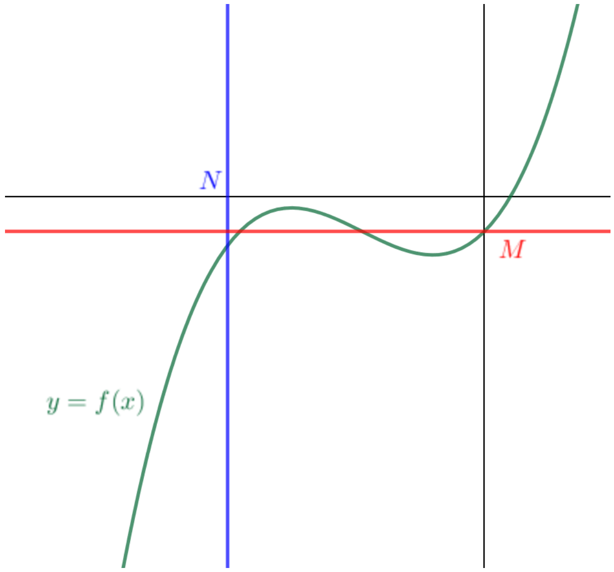
$$\forall M \in \mathbb{R}, \exists N \in \mathbb{R}, \forall x > N (x < N), \quad f(x) > M$$

We say that $\lim_{x\to\infty} f(x) = -\infty$ ($\lim_{x\to-\infty} f(x) = -\infty$) if

$$\forall M \in \mathbb{R}, \exists N \in \mathbb{R}, \forall x > N (x < N), \quad f(x) < M$$



$$\lim_{x\to\infty} f(x) = L$$



$$\lim_{x \to -\infty} f(x) = -\infty$$

Definition 3 (limit at a point)
For a function $f(x)$ and a point $a \in \mathbb{R}$, we say that $\lim_{x \to a} f(x) = L$ if

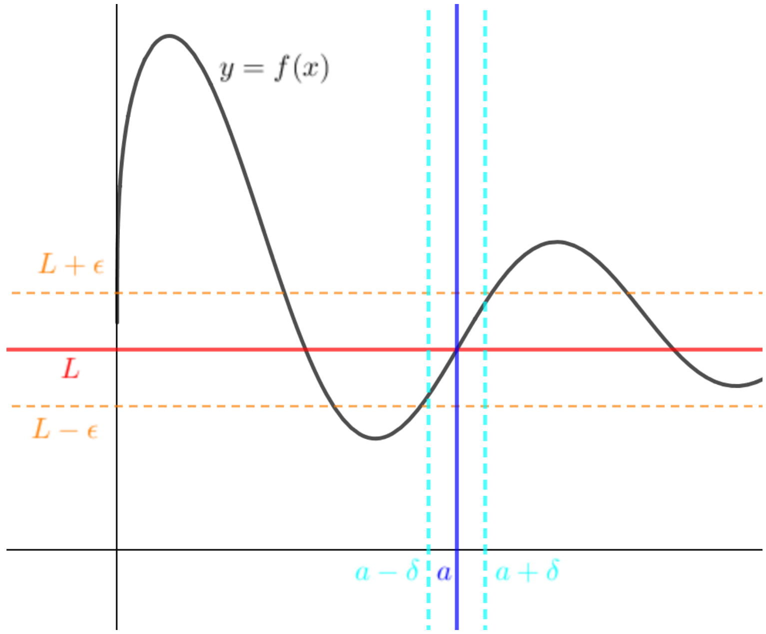
$$\forall \epsilon > 0, \exists \delta > 0, \forall x \text{ such that } 0 < |x - a| < \delta, \quad |f(x) - L| < \epsilon$$

We say that $\lim_{x \to a} f(x) = \infty$ if

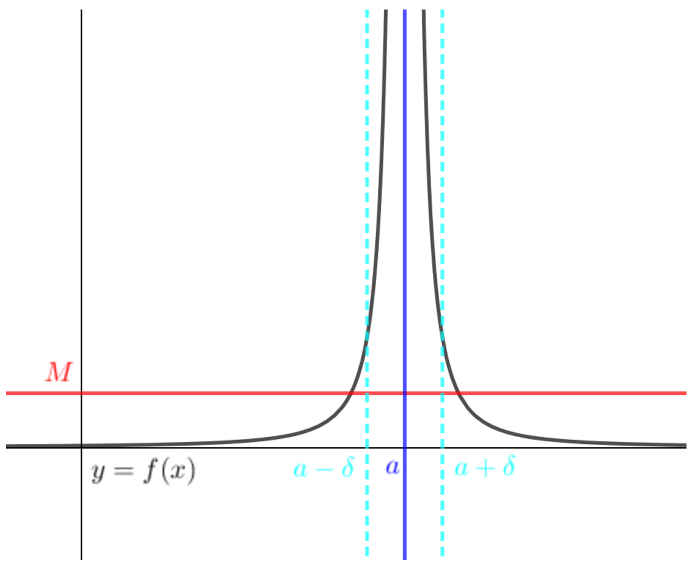
$$\forall M \in \mathbb{R}, \exists \delta > 0, \forall x \text{ such that } 0 < |x - a| < \delta, \quad f(x) > M$$

We say that $\lim_{x \to a} f(x) = -\infty$ if

$$\forall M \in \mathbb{R}, \exists \delta > 0, \forall x \text{ such that } 0 < |x - a| < \delta, \quad f(x) < M$$



$$\lim_{x \to a} f(x) = L$$



$$\lim_{x \to a} f(x) = \infty$$

Definition 4 (one-sided limit)
For a function $f(x)$ and a point $a \in \mathbb{R}$, we say that $\lim_{x\to a^+} f(x) = L$ ($\lim_{x\to a^-} f(x) = L$) if

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \text{ such that } a < x < a + \delta (a - \delta < x < a), \quad |f(x) - L| < \epsilon$$

We say that $\lim_{x\to a^+} f(x) = \infty$ ($\lim_{x\to a^-} f(x) = \infty$) if

$$\forall M \in \mathbb{R}, \exists \delta > 0, \forall x \text{ such that } a < x < a + \delta (a - \delta < x < a), \quad f(x) > M$$

We say that $\lim_{x\to a^+} f(x) = -\infty$ ($\lim_{x\to a^-} f(x) = -\infty$) if

$$\forall M \in \mathbb{R}, \exists \delta > 0, \forall x \text{ such that } a < x < a + \delta (a - \delta < x < a), \quad f(x) < M$$

Proposition 4
For any function $f(x)$,

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$$

The same hold if L is replaced by ∞ or $-\infty$.

Proof

Straightly from the definitions. □

Theorem 4: Sequential criterion
For any function $f(x)$,

$$\lim_{x\to a} f(x) = L \iff \forall (a_n)_{n\in\mathbb{Z}^+} \text{ such that } a_n \neq a \text{ and } \lim_{n\to\infty} a_n = a, \quad \lim_{n\to\infty} f(a_n) = L$$

The same hold if a or L is replaced by ∞ or $-\infty$. Moreover,

$$\lim_{x\to a^+} f(x) = L \iff$$

$$\forall \text{ strictly decreasing } (a_n)_{n\in\mathbb{Z}^+} \text{ such that } \lim_{n\to\infty} a_n = a, \quad \lim_{n\to\infty} f(a_n) = L,$$

$$\lim_{x\to a^-} f(x) = L \iff$$

$$\forall \text{strictly increasing } (a_n)_{n\in\mathbb{Z}^+} \text{ such that } \lim_{n\to\infty} a_n = a, \quad \lim_{n\to\infty} f(a_n) = L$$

The same hold if L is replaced by ∞ or $-\infty$.

Proof

For $\lim_{x\to a} f(x) = L$:

$\Longrightarrow$

Given $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$

Since $\lim_{n\to\infty} a_n = a$,

$$\exists N \in \mathbb{Z}^+, \forall n > N, \quad |a_n - a| < \delta$$

As $a_n \neq a$,

$$0 < |a_n - a| < \delta \implies |f(a_n) - L| < \epsilon$$

By definition, $\lim_{n\to\infty} f(a_n) = L$.

$\Longleftarrow$

Assume the contrary:

$$\exists \epsilon_0 > 0, \forall \delta > 0, \exists x \text{ such that } 0 < |x - a| < \delta, \quad |f(x) - L| \geq \epsilon_0$$

Then, for each $n \in \mathbb{Z}^+$, we can pick a_n such that

$$0 < |a_n - a| < \frac{1}{n} \quad \text{and} \quad |f(a_n) - L| \geq \epsilon_0$$

to form a sequence $(a_n)_{n\in\mathbb{Z}^+}$. By construction, $a_n \neq a$. In addition, $\lim_{n\to\infty} a_n = a$ by squeeze theorem. By the premise, we should have $\lim_{n\to\infty} f(a_n) = L$.

By taking $\epsilon = \dfrac{\epsilon_0}{2}$,

$$\exists N \in \mathbb{Z}^+, \forall n > N, \quad |f(a_n) - L| < \frac{\epsilon_0}{2}$$

In particular,

$$\epsilon_0 \leq |f(a_{N+1}) - L| < \frac{\epsilon_0}{2}$$

which is impossible.

The arguments for

$$\lim_{x\to a^+} f(x) = L, \quad \lim_{x\to a^-} f(x) = L, \quad \lim_{x\to\infty} f(x) = L, \quad \lim_{x\to-\infty} f(x) = L$$

are similar. The sequences constructed in "$\Longleftarrow$" would be given by

$$0 < a_n - a < \min\left\{\frac{1}{n}, a_{n-1} - a\right\}, 0 < a - a_n < \min\left\{\frac{1}{n}, a - a_{n-1}\right\}, a_n > n, a_n < -n$$

respectively.

For $\lim_{x\to a} f(x) = \infty$:

$\Longrightarrow$

Given $M \in \mathbb{R}$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \implies f(x) > M$$

Since $\lim_{n\to\infty} a_n = a$,

$$\exists N \in \mathbb{Z}^+, \forall n > N, \quad |a_n - a| < \delta$$

As $a_n \neq a$,

$$0 < |a_n - a| < \delta \implies f(a_n) > M$$

By definition, $\lim_{n\to\infty} f(a_n) = \infty$.

$\Longleftarrow$

Assume the contrary:

$$\exists M_0 \in \mathbb{R}, \forall \delta > 0, \exists x \text{ such that } 0 < |x - a| < \delta, \quad f(x) \leq M_0$$

Then, for each $n \in \mathbb{Z}^+$, we can pick a_n such that

$$0 < |a_n - a| < \frac{1}{n} \quad \text{and} \quad f(a_n) \leq M_0$$

to form a sequence $(a_n)_{n\in\mathbb{Z}^+}$. By construction, $a_n \neq a$. In addition, $\lim_{n\to\infty} a_n = a$ by squeeze theorem. By the premise, we should have $\lim_{n\to\infty} f(a_n) = \infty$.

By taking $M = M_0 + 1$,

$$\exists N \in \mathbb{Z}^+, \forall n > N, \quad f(a_n) > M_0 + 1$$

In particular,

$$M_0 \geq f(a_{N+1}) > M_0 + 1$$

which is impossible.

The arguments for

$$\lim_{x\to a^+} f(x) = \infty, \quad \lim_{x\to a^-} f(x) = \infty, \quad \lim_{x\to\infty} f(x) = \infty, \quad \lim_{x\to-\infty} f(x) = \infty$$

are similar. The sequences constructed in "$\Longleftarrow$" would be given by

$$0 < a_n - a < \min\left\{\frac{1}{n}, a_{n-1} - a\right\}, 0 < a - a_n < \min\left\{\frac{1}{n}, a - a_{n-1}\right\}, a_n > n, a_n < -n$$

respectively.

Observe that

$$\begin{aligned}
\lim_{x\to a} f(x) = -\infty \quad &\Longleftrightarrow \quad \lim_{x\to a}(-f(x)) = \infty \quad \text{(clear from definitions)} \\
&\Longleftrightarrow \quad \forall (a_n)_{n\in\mathbb{Z}^+} \text{ such that } a_n \neq a \text{ and } \lim_{n\to\infty} a_n = a, \quad \lim_{n\to\infty}(-f(a_n)) = \infty \\
&\Longleftrightarrow \quad \forall (a_n)_{n\in\mathbb{Z}^+} \text{ such that } a_n \neq a \text{ and } \lim_{n\to\infty} a_n = a, \quad \lim_{n\to\infty} f(a_n) = -\infty
\end{aligned}$$

Similarly, the equivalence regarding

$$\lim_{x\to a^+} f(x) = -\infty, \quad \lim_{x\to a^-} f(x) = -\infty, \quad \lim_{x\to\infty} f(x) = -\infty, \quad \lim_{x\to-\infty} f(x) = -\infty$$

also hold. $\square$

Theorem 5

Suppose $f(x)$ and $g(x)$ are two functions such that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.
$\forall c \in \mathbb{R}$,

$$\begin{aligned}
\lim_{x\to a}(f(x) \pm g(x)) &= \left(\lim_{x\to a} f(x)\right) \pm \left(\lim_{x\to a} g(x)\right) \\
\lim_{x\to a}(f(x) \cdot g(x)) &= \left(\lim_{x\to a} f(x)\right) \cdot \left(\lim_{x\to a} g(x)\right) \\
\lim_{x\to a} \frac{f(x)}{g(x)} &= \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{provided that } \lim_{x\to a} g(x) \neq 0 \\
\lim_{x\to a}(cf(x)) &= c\left(\lim_{x\to a} f(x)\right)
\end{aligned}$$

$$f(x) \leq g(x) \quad \text{around } a \text{ (not necessarily at } a) \implies \lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$$

(That is, $\exists \delta > 0, \forall x$ such that $0 < |x - a| < \delta, \quad f(x) \leq g(x)$)
The same hold if a is replaced by a^+, a^-, ∞ or $-\infty$.

Proof

Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Given sequence $(a_n)_{n\in\mathbb{Z}^+}$ such that $a_n \neq a$ and $\lim_{n\to\infty} a_n = a$, by sequential criterion,

$$\lim_{n\to\infty} f(a_n) = L \quad \text{and} \quad \lim_{n\to\infty} g(a_n) = M \implies \lim_{n\to\infty} (f(a_n) + g(a_n)) = L + M$$

Hence, by sequential criterion,

$$\lim_{x\to a}(f(x) + g(x)) = L + M$$

The others can be handled similarly. □

Proposition 5

For any function $f(x)$,

(a) $\lim_{x\to a} f(x) = L \iff \lim_{x\to a}(f(x) - L) = 0$

(b) $\lim_{x\to a} f(x) = 0 \iff \lim_{x\to a} |f(x)| = 0$

The same hold if a is replaced by a^+, a^-, ∞ or $-\infty$.

Proof

By sequential criterion. □

Theorem 6: Squeeze Theorem (Sandwich Theorem)

Suppose

$$g(x) \leq f(x) \leq h(x) \quad \text{around } x = a \text{ (not necessarily at } a\text{)}$$

Then,

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L \implies \lim_{x\to a} f(x) = L$$

Similarly,

$$\lim_{x\to a} g(x) = \infty \text{ (DNE)} \implies \lim_{x\to a} f(x) = \infty \text{ (DNE)}$$

$$\lim_{x\to a} h(x) = -\infty \text{ (DNE)} \implies \lim_{x\to a} f(x) = -\infty \text{ (DNE)}$$

The same hold if a is replaced by a^+, a^-, ∞ or $-\infty$.

Proof

Suppose

$$\forall x \text{ such that } 0 < |x-a| < \epsilon_0, \quad g(x) \leq f(x) \leq h(x)$$

Given sequence $(a_n)_{n\in\mathbb{Z}^+}$ such that $a_n \neq a$ and $\lim_{n\to\infty} a_n = a$,

$$\exists N \in \mathbb{Z}^+, \forall n > N, \quad 0 < |a_n - a| < \epsilon_0 \implies g(a_n) \leq f(a_n) \leq h(a_n)$$

by sequential criterion and squeeze theorem for sequences,

$$\lim_{n\to\infty} g(a_n) = \lim_{n\to\infty} h(a_n) = L \implies \lim_{n\to\infty} f(a_n) = L$$

Hence, by sequential criterion,

$$\lim_{x\to a} f(x) = L$$

The others can be handled similarly. $\square$