THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2024-2025 Term 1
Suggested Solutions of WeBWork Coursework 9

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1. (1 point) Evaluate the indefinite integral.

$$\int \sin^3(4x)\cos^2(4x)dx$$

________________________ $+C$

Solution:

Let $u = 4x$, then

$$\begin{aligned}\int \sin^3(4x)\cos^2(4x)dx &= \frac{1}{4}\int \sin^3(u)\cos^2(u)du \\ &= -\frac{1}{4}\int \sin^2(u)\cos^2(u)d(\cos(u)) \\ &= \frac{1}{4}\int (\cos^2(u)-1)\cos^2(u)d(\cos(u)).\end{aligned}$$

Let $v = \cos(u)$, then

$$\begin{aligned}\frac{1}{4}\int (\cos^2(u)-1)\cos^2(u)d(\cos(u)) &= \frac{1}{4}\int (v^2-1)v^2dv \\ &= \frac{1}{20}v^5 - \frac{1}{12}v^3 + C \\ &= \frac{1}{20}\cos^5(4x) - \frac{1}{12}\cos^3(4x) + C\end{aligned}$$

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2. (1 point)

Evaluate the integral

$$\int \sin^4(x)\,dx.$$

Note: Use an upper-case "C" for the constant of integration.

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Solution:

Set $v = \cos(2x)$. As $\cos(2x) = 1 - 2\sin^2(x) = 2\cos^2(x) - 1$, then

$$\begin{aligned}\int \sin^4(x)\ dx &= \frac{1}{4}\int (1-\cos(2x))^2\ dx\\ &= \frac{1}{8}\int (3 - 4\cos(2x) + \cos(4x))\ dx\\ &= \frac{1}{32}(12x - 8\sin(2x) + \sin(4x)) + C\end{aligned}$$

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3. (1 point)

Suppose that $f(1) = -9$, $f(4) = 8$, $f'(1) = 2$, $f'(4) = -9$, and f'' is continuous. Find the value of $\int_1^4 xf''(x)\,dx$.

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Solution:

Method 1:

$$\begin{aligned}\int_1^4 xf''(x)\ dx &= \int_1^4 xf''(x) + f'(x)\ dx - \int_1^4 f'(x)\ dx\\ &= \int_1^4 (xf''(x) + f'(x))\ dx - (f(4) - f(1))\\ &= \int_1^4 (xf'(x))'\ dx - f(4) + f(1)\\ &= 4f'(4) - f'(1) - f(4) + f(1)\\ &= -55\end{aligned}$$

Method 2: By integration by parts

$$\begin{aligned}\int_1^4 xf''(x)\ dx &= \int_1^4 x\ d(f'(x))\\ &= 4f'(4) - f'(1) - \int_1^4 f'(x)fx\\ &= 4f'(4) - f'(1) - f(4) + f(1)\\ &= -55.\end{aligned}$$

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4. (1 point)

A rumor is spread in a school. For $0 < a < 1$ and $b > 0$, the time t at which a fraction p of the school population has heard the rumor is given by

$$t(p) = \int_a^p \frac{b}{x(1-x)}\,dx.$$

(a) Evaluate the integral to find an explicit formula for $t(p)$. Write your answer so it has only one ln term.
$\int_a^p \frac{b}{x(1-x)}\,dx =$ ____________________

(b) At time $t = 0$, four percent of the school population ($p = 0.04$) has heard the rumor. What is a?
$a =$ _____

(c) At time $t = 1$, fifty-nine percent of the school population ($p = 0.59$) has heard the rumor. What is b?
$b =$ ____________

(d) At what time has ninety-one percent of the population ($p = 0.91$) heard the rumor?
$t =$ ____________

Solution:

(a) We integrate to find

$$\int \frac{b}{x(1-x)}\,dx = b\int\left(\frac{1}{x}+\frac{1}{1-x}\right)dx = b(\ln|x| - \ln|1-x|) + C = b\ln|\frac{x}{1-x}| + C,$$

so

$$t(p) = \int_a^p \frac{b}{x(1-x)}\,dx = b\ln\left(\frac{p}{1-p}\right) - b\ln\left(\frac{a}{1-a}\right) = b\ln\left(\frac{p(1-a)}{a(1-p)}\right).$$

(b) We know that $t(0.04) = 0$, so

$$0 = b\ln\left(\frac{0.04(1-a)}{0.96a}\right).$$

But $b > 0$ and $\ln x = 0$ means $x = 1$, so

$$\frac{0.04(1-a)}{0.96a} = 1, \qquad \text{or} \qquad 0.04(1-a) = 0.96a.$$

Solving, $a = 0.04$.

(c) We know that $t(0.59) = 1$ so

$$1 = b\ln\left(\frac{0.59\cdot 0.96}{0.41\cdot 0.04}\right) = b\ln\frac{0.5664}{0.0164},$$

so $b = \frac{1}{\ln\frac{0.5664}{0.0164}}$.

(d) We have

$$t(0.91) = \int_{0.04}^{0.91} \frac{b}{x(1-x)}\,dx = \frac{1}{\ln\frac{0.5664}{0.0164}}\ln\left(\frac{0.91(1-0.04)}{0.04(1-0.91)}\right) \approx 1.5504.$$

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5. (1 point) Given

$$f(x) = \int_0^x \frac{t^2-4}{1+\cos^2(t)}dt$$

At what value of x does the local max of $f(x)$ occur?

$x =$ __________

Solution:
First, note that we don't need to do any computation to compute the first derivative,

which we will use to check for local maxima and minima. By applying the Fundamental Theorem of Calculus, we see that:

$$f'(x) = \frac{x^2 - 4}{1 + \cos^2(x)}$$

Now, we can use this derivative to find the critical points of the function. We set this to zero and solve for x to get:

$$\frac{x^2 - 4}{1 + \cos^2(x)} = 0$$

$$x^2 - 4 = 0$$

$$(x + 2)(x - 2) = 0$$

$$x = 2 \text{ or } x = -2$$

Checking on either side of these two points shows that -2 is the local maximum for which we are looking.

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6. (1 point) Let $F(x) = \displaystyle\int_0^x \frac{2 - t}{t^2 + 5}\, dt$ for $-\infty < x < +\infty$

(a) Find the value of x where F obtains its maximum value.

$x =$ _____

(b) Find the intervals over which F is only increasing or decreasing. Use interval notation using U for union and enter "none" if no interval.

Intervals where F is increasing: __________
Intervals where F is decreasing: __________

(c) Find open intervals over which F is only concave up or concave down. Use interval notation using U for union and enter "none" if no interval.

Intervals where F is concave up: __________
Intervals where F is concave down: __________

Solution:

SOLUTION

(a) $F'(x) = \frac{d}{dx} \int_0^x \frac{2-t}{t^2+5}\, dt = \frac{2-x}{x^2+5} = 0$ when $x = 2$ which is the only critical point. From sign analysis of F' we see this is a maximum.

(b) F is increasing on $(-\infty, 2]$ and decreasing on $[2, +\infty)$.

(b) $F''(x) = \frac{d}{dx}\left[\frac{2-x}{x^2+5}\right] = \frac{x^2-4x-5}{(x^2+5)^2} = \frac{(x-5)(x+1)}{(x^2+5)^2} = 0$, when $x = -1, 5$. Sign analysis of F'' shows that F is concave up on $(-\infty, -1)$ and $(5, +\infty)$ and concave down on $(-1, 5)$.

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7. (1 point)

Evaluate the integral

$$\int_{-1}^{2} (-4x + 6|x|)\, dx$$

Integral = ____________

Solution:

$$\begin{aligned}\int_{-1}^{2}(-4x+6|x|)\,dx &= \int_{-1}^{0}(-4x+6|x|)\,dx+\int_{0}^{2}(-4x+6|x|)\,dx\\ &= \int_{-1}^{0}-10x\,dx+\int_{0}^{2}2x\,dx\\ &= 5+4=9\end{aligned}$$

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8. (1 point)

Evaluate the definite integral (if it exists)

$$\int_0^{\pi} -5\sec^2(t/4)\,dt$$

If the integral does not exist, type "DNE".

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Solution: Set $x=\frac{t}{4}$, then

$$\begin{aligned}\int_0^{\pi} -5\sec^2(t/4)\,dt &= \int_0^{\frac{\pi}{4}} -20\sec^2(x)\,dx\\ &= -20(\tan(\frac{\pi}{4})-\tan(0)) = -20\end{aligned}$$

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9. (1 point) Let

$$I=\int_0^{\pi/8}\tan^6(2x)\sec(2x)\ dx.$$

Express the value of

$$\int_0^{\pi/8}\tan^8(2x)\sec(2x)\ dx$$

in terms of I.

$\int_0^{\pi/8}\tan^8(2x)\sec(2x)\ dx =$ ________.

Solution: Let

$$J=\int_0^{\pi/8}\tan^8(2x)\sec(2x)\ dx.$$

Factorizing the integrand as displayed below, and then integrating by parts, gives

$$\begin{aligned}J &= \int_0^{\pi/8}\tan^7(2x)\sec(2x)\tan(2x)\,dx=\frac{1}{2}\int_0^{\pi/8}\tan^7(2x)d(\sec(2x))\\ &= \tfrac{1}{2}\tan^7(2x)\sec(2x)\Big|_0^{\pi/8}-7\int_0^{\pi/8}\tan^6(2x)\sec^3(2x)\,dx.\end{aligned}$$

Now

$$\tfrac{1}{2}\tan^7(2x)\sec(2x)\Bigg|_0^{\pi/8} = \tfrac{1}{2}\tan^7\left(\tfrac{1}{4}\pi\right)\sec\left(\tfrac{1}{4}\pi\right) = \tfrac{1}{2}\left(1\right)^7\left(\sqrt{2}\right)$$
$$= \tfrac{1}{2}\sqrt{2},$$

and from $\sec^3(2x) = \sec^2(2x)\sec(2x) = \big(\tan^2(2x)+1\big)\sec(2x)$, it follows that

$$\int_0^{\pi/8} \tan^6(2x)\sec^3(2x)\,dx = \int_0^{\pi/8} \tan^6(2x)\big(\tan^2(2x)+1\big)\sec(2x)\,dx$$
$$= J + I.$$

Therefore,

$$J = \tfrac{1}{2}\sqrt{2} - 7J - 7I, \qquad \text{or} \qquad 8J = \tfrac{1}{2}\sqrt{2} - 7I,$$

and so

$$\int_0^{\pi/8} \tan^8(2x)\sec(2x)\ dx = J = \tfrac{1}{16}\sqrt{2} - \tfrac{7}{8}I.$$

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10. (1 point)

Evaluate $\displaystyle\int_{-\pi}^{\pi} f(x)\,dx$, where

$$f(x) = \begin{cases} 8x^3, & -\pi \le x < 0 \\ 9\sin(x), & 0 \le x \le \pi. \end{cases}$$

$\displaystyle\int_{-\pi}^{\pi} f(x)\,dx =$ __________

Solution:

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\,dx &= \textstyle\int_{-\pi}^{0} 8x^3\,dx + \int_0^{\pi} 9\sin(x)\,dx \\ &= \Big[2x^4\Big]_{-\pi}^{0} + \Big[-9\cos x\Big]_0^{\pi} \\ &= \big(2\cdot 0^4 - 2(-\pi)^4\big) + ((-9\cos\pi + 9\cos 0)) \\ &= -2\pi^4 + 18 \end{aligned}$$

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11. (1 point) Evaluate the integral.

$\displaystyle\int e^x\sqrt{16-e^{2x}}\ dx =$ ____________________ $+C$

Solution:

For $e^x = 4\sin(t)\,,\ dx = \frac{\cos(t)}{\sin(t)}\ dt$

$$\int e^x\sqrt{16-e^{2x}}\ dx = \int 16\cos^2(t)\ \ dt = 16\left[\frac{t}{2} + \frac{1}{2}\sin t\cos t\right] + C = 8\sin^{-1}\left(\frac{e^x}{4}\right) + \frac{e^x\sqrt{16-e^{2x}}}{2} + C$$

12. (1 point) If $f(x) = \int_{x}^{x^4} t^3 dt$

then
$f'(x) =$ ______________

Solution:

By the Fundamental Theorem of Calculus, we have

$f(x) = \int_{0}^{x^4} t^3 dt \; - \int_{0}^{x} t^3 dt$, thus $f'(x) = (x^4)^3(4x^3) - x^3 = 4x^{15} - x^3$.