**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**MATH1010 UNIVERSITY MATHEMATICS 2024-2025 Term 1**
**Suggested Solutions of WeBWork Coursework 7**

If you find any errors or typos, please email us at
math1010@math.cuhk.edu.hk

1. (1 point) Fine the maximum area of a triangle formed in the first quadrant by the $x$-axis, $y$-axis and a tangent line to the graph of $f = (x+7)^{-2}$.

   Area =_________

**Solution:** Let $P\left(t, \frac{1}{(t+7)^2}\right)$ be a point on the graph of the curve $y = \frac{1}{(x+7)^2}$ in the first quadrant. The tangent line to the curve at $P$ is

$$L(x) = \frac{1}{(t+7)^2} - \frac{2(x-t)}{(t+7)^3},$$

which has $x$-intercept $a = \frac{3t+7}{2}$ and $y$-intercept $b = \frac{3t+7}{(t+7)^3}$. The area of the triangle in question is

$$A(t) = \frac{1}{2}ab = \frac{(3t+7)^2}{4(t+7)^3}.$$

Solve

$$A'(t) = \frac{(3t+7)(3 \cdot 7 - 3t)}{4(t+7)^4} = 0$$

for $0 \leq t$ to obtain $t = 7$. Because $A(0) = \frac{1}{4 \cdot 7}$, $A(7) = \frac{1}{2 \cdot 7}$ and $A(t) \to 0$ as $t \to \infty$, it follows that the maximum area is $A(7) \approx 0.0714286$.

2. (2 points) Find the point $(x, y)$ of $x^2 + 14xy + 49y^2 = 100$ that is closest to the origin and lies in the first quadrant.

   $x =$_________

   $y =$_________

**Solution:** The distance of point $(x, y)$ to the origin is $dist(x, y) = \sqrt{x^2 + y^2}$. Then from the equation we can find that

$$x^2 + 14xy + 49y^2 = 100 = (x+7y)^2.$$

Therefore, $x + 7y = 10$ in the first quadrant, and

$$\begin{aligned} dist(x,y)^2 &= (10-7y)^2 + y^2 = 50y^2 - 140y + 100 \\ &= 50(y^2 - \frac{14}{5}y + \frac{49}{25}) + 2 \\ &= 50(y - \frac{7}{5})^2 + 2 \end{aligned}$$

This means that the minimum of distance take value when $y = \frac{7}{5}$, and $x = \frac{1}{5}$.

3. (3 points) Use L'Hôpital's Rule (possibly more than once) to evaluate the following limit

$$\lim_{x\to\infty}(\frac{6x^3+4x^2}{3x^3-2}) = \underline{\hspace{3cm}}$$

If the answer equals $\infty$ or $-\infty$, write INF or -INF in the blank.

**Solution:**

$$\begin{aligned} \lim_{x\to\infty}\frac{6x^3+4x^2}{3x^3-2} &= \lim_{x\to\infty}\frac{(6x^3+4x^2)'}{(3x^3-2)'} = \lim_{x\to\infty}\frac{18x^2+8x}{9x^2} \\ &= \lim_{x\to\infty}\frac{(18x^2+8x)'}{(9x^2)'} = \lim_{x\to\infty}\frac{36x+8}{18x} \\ &= \lim_{x\to\infty}\frac{(36x+8)'}{(18x)'} = \lim_{x\to\infty}\frac{36}{18} \\ &= 2 \end{aligned}$$

4. (4 points) Compute

$$\lim_{x\to 0}\frac{e^x - e^{-x}}{2\sin x} = \underline{\hspace{4cm}}$$

**Solution:**

$$\lim_{x\to 0}\frac{e^x - e^{-x}}{2\sin x} = \lim_{x\to 0}\frac{(e^x - e^{-x})'}{(2\sin x)'} = \lim_{x\to 0}\frac{e^x + e^{-x}}{2\cos x} = \frac{e^0+e^0}{2\cos 0} = 1$$

5. (5 points) Apply L'Hôpital's Rule to evaluate the following limit. It may be necessary to apply it more than once.

$$\lim_{x\to 0^+}(\sin x)^x = \underline{\hspace{3cm}}$$

**Solution:** We will use the identity $(\sin x)^x = e^{\ln((\sin x)^x)}$, and then compute

$$\begin{aligned}\lim_{x\to 0+} \ln\left((\sin x)^x\right) &= \lim_{x\to 0+} x\ln(\sin x) = \lim_{x\to 0+} \frac{\ln(\sin x)}{x^{-1}} = \lim_{x\to 0+} \frac{\left(\frac{1}{\sin x}\right)\cos x}{-x^{-2}} \\ &= \lim_{x\to 0+} -\frac{\cot x}{x^{-2}} = \lim_{x\to 0+} -\frac{x^2}{\tan x} = \lim_{x\to 0+} -\frac{2x}{\sec^2 x} = 0.\end{aligned}$$

Hence, $\lim_{x\to 0+} (\sin x)^x = \lim_{x\to 0^+} e^{\ln((\sin x)^x)} = e^0 = 1$

6. (6 points) Evaluate

$$\lim_{x\to 0} \frac{\cos(x) - 1 + \frac{x^2}{2}}{7x^4}.$$

Limit =_________

**Solution:**

$$\begin{aligned}\lim_{x\to 0} \frac{\cos(x) - 1 + \frac{x^2}{2}}{7x^4} &= \lim_{x\to 0} \frac{-\sin(x) + x}{28x^3} \\ &= \lim_{x\to 0} \frac{-\cos(x) + 1}{28 \times 3x^2} \\ &= \lim_{x\to 0} \frac{\sin(x)}{28 \times 6x} \\ &= \frac{1}{168}\end{aligned}$$

7. (7 points) Evaluate

$$\lim_{x\to 0} \frac{\ln(1-x) + x + \frac{x^2}{2}}{7x^3}.$$

Limit =_________

**Solution:**

$$\begin{aligned}\lim_{x\to 0} \frac{\ln(1-x) + x + \frac{x^2}{2}}{7x^3} &= \lim_{x\to 0} \frac{-1/(1-x) + 1 + x}{7 \times 3x^2} \\ &= \lim_{x\to 0} \frac{-1/(1-x)^2 + 1}{7 \times 6x} \\ &= \lim_{x\to 0} \frac{-2/(1-x)^3}{7 \times 6} = -\frac{1}{21}\end{aligned}$$

8. (8 points) Find the first three **nonzero** terms of the Taylor series for the function $f(x) = \sqrt{4x - x^2}$ about the point $a = 2$.

**Solution:**

$$f(2) = 2$$
$$f'(x) = \frac{4-2x}{2\sqrt{4x-x^2}} = \frac{2-x}{\sqrt{4x-x^2}}$$
$$f'(2) = 0$$
$$f''(x) = -\frac{4x-x^2}{(4x-x^2)^{\frac{3}{2}}} - \frac{(2-x)(4-2x)}{2(4x-x^2)^{\frac{3}{2}}} = -\frac{4}{(4x-x^2)^{\frac{3}{2}}}$$
$$f''(2) = -\frac{1}{2}$$
$$f^{(3)}(x) = \frac{12(2-x)}{(4x-x^2)^{\frac{5}{2}}}$$
$$f^{(3)}(2) = 0$$
$$f^{(4)}(x) = -\frac{12(4x-x^2)}{(4x-x^2)^{\frac{7}{2}}} - \frac{12 \times 5(2-x)(4-2x)}{2(4x-x^2)^{\frac{7}{2}}} = -\frac{12(4x^2-16x+20)}{(4x-x^2)^{\frac{7}{2}}}$$
$$f^{(4)}(2) = -\frac{3}{8}$$

Then the Taylor expansion of $f(x)$ is

$$\begin{aligned}\sqrt{4x-x^2} &= 2 + \frac{1}{2!}(-\frac{1}{2})(x-2)^2 + \frac{1}{4!}(-\frac{3}{8})(x-2)^4 + ... \\ &= 2 - \frac{1}{4}(x-2)^2 - \frac{1}{64}(x-2)^4 + ...\end{aligned}$$

9. (9 points) Compute $T_2(x)$ at $x = 0.4$ for $y = e^x$ and use a calculator to compute the error $|e^x - T_2(x)|$ at $x = 1.3$.

$T_2(x) =$ ______________

$|e^x - T_2(x)| =$ ______________

**Solution:** Recall the general formula for the Taylor polynomial centered at $x = a$:

$$T(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^n(a)}{n!}(x-a)^n,$$
$$f'(x) = e^x, f''(x) = e^x.$$

So, in this case,

$$T_2(x) = e^{0.4} + e^{0.4}(x-0.4) + \frac{e^{0.4}}{2}(x-0.4)^2,$$

$$|e^x - T_2(x)| = \left| e^{0.4} + e^{0.4} \cdot 0.9 + \frac{e^{0.4}}{2} \cdot 0.81 - e^{1.3} \right| = 0.230641.$$

10. (10 points) Write the Taylor series for $f(x) = \ln(\sec(x))$ at $x = 0$ as $\sum_{n=0}^{\infty} c_n x^n$. Find the first five coefficients.

**Solution:**

$$f'(x) = \frac{\sin(x)}{\cos(x)} = \tan(x), f^{(2)}(x) = (\tan(x))' = \sec^2(x)$$

$$f^{(3)}(x) = 2\sec(x)\tan(x), f^{(4)}(x) = 2\sec^3(x) + 2\sec(x)\tan^2(x).$$

Taking value at $x = 0$, we get the first five coefficients:

$$c_0 = 0,\ c_1 = 0,\ c_2 = \frac{1}{2},\ c_3 = 0,\ c_4 = \frac{1}{12}.$$

11. (11 points) Write the Taylor series for $f(x) = \sin(x)$ at $x = \frac{\pi}{2}$ as $\displaystyle\sum_{n=0}^{\infty} c_n \left(x - \frac{\pi}{2}\right)^n$.

**Solution:** Since $f'(x) = \cos(x), f^{(2)}(x) = -\sin(x)$, it's easy to find that

$$f^{(4n)} = \sin(x),\ f^{(4n+1)} = \cos(x),\ f^{(4n+2)} = -\sin(x),\ f^{(4n+3)} = -\cos(x)$$

Therefore the Taylor series take value at $x = \frac{\pi}{2}$ is

$$1 + \sum_{n=1}^{\infty} \frac{1}{(4n)!} \left(x - \frac{\pi}{2}\right)^{4n} - \sum_{n=0}^{\infty} \frac{1}{(4n+2)!} \left(x - \frac{\pi}{2}\right)^{4n+2}.$$

12. (12 points) Suppose that $f(x)$ and $g(x)$ are given by the power series

$$f(x) = 3 + 4x + 5x^2 + 2x^3 + \cdots$$

and

$$g(x) = 3 + 9x + 4x^2 + 3x^3 + \cdots$$

Find the first few terms of the series for

$$h(x) = f(x) \cdot g(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

**Solution:** Denote the coefficients of series $f$ and $g$ by $a_n$ and $b_n$. Then their product have coefficients $c_k = \sum_{i+j=k, i,j\geq 0} a_i b_j$. Therefore

$$c_0 = 9, c_1 = 27 + 12 = 39, c_2 = 12 + 36 + 15 = 63, c_3 = 9 + 16 + 45 + 6 = 76.$$