THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2024-2025 Term 1
Suggested Solutions of WeBWork Coursework 6

If you find any errors or typos, please email us at
math1010@math.cuhk.edu.hk

If you find any errors or typos, please email us at
math1010@math.cuhk.edu.hk

1. (1 point)

$$f(x) = \frac{x}{x^2 + 12x + 32}$$

a) Give the domain of f (in interval notation) ______

b) Determine the intervals on which f is increasing and decreasing.
Your answer should either be a single interval, such as "(0,1)", a comma separated list of intervals, such as "(-inf, 2), (3,4)" , or the word "none".
f is increasing on: ______
f is decreasing on: ______

For each interval, do take care to consider whether end points should be included.

Solution:

Since $f = \dfrac{x}{x^2 + 12x + 32}$ is a rational function, its domain is all real numbers, excluding those at which the denominator is zero. The denominator factors:

$$x^2 + 12x + 32 = (x + 8)(x + 4),$$

so the domain is $(-\infty, -8) \cup (-8, -4) \cup (-4, \infty)$.

$f'(x) = \dfrac{-x^2 + 32}{(x^2 + 12x + 32)^2}$. Setting equal to zero and solving, there are two critical numbers, $x = \pm\sqrt{32}$.

Use the first derivative test, choosing sample points in each interval. Note, the intervals are determined by both critical numbers and the points excluded from the domain.

Interval	Sign of f' at sample	Conclusion
$(-\infty, -8)$	negative	decreasing
$(-8, -\sqrt{32})$	negative	decreasing
$(-\sqrt{32}, -4)$	positive	increasing
$(-4, \sqrt{32})$	positive	increasing
$(\sqrt{32}, \infty)$	negative	decreasing

Based on the signs in each interval there is a relative maximum at $x = \sqrt{32}$ and a relative minimum at $x = -\sqrt{32}$.

Correct Answers:

- $(-\infty, -8) \cup (-8, -4) \cup (-4, \infty)$
- $[-5.6568, -4), (-4, 5.6568]$
- $(-\infty, --8), (-8, -5.6568], [3.4641, \infty)$

---

2. (1 point) Let $f(x) = 8\sqrt{x} - 8x$ for $x > 0$. Find the intervals on which f is increasing (decreasing). Pay attention to endpoints!

1. f is increasing on the intervals __________
2. f is decreasing on the intervals __________

Notes: Your answer should either be a single interval, such as (0,1), a comma separated list of intervals, such as (-inf, 2), (3,4), or the word "none".
Solution:

$$f' = \frac{4}{\sqrt{x}} - 8$$

Let $f' \geq 0$,

$$x \in (0, 1/4]$$

Let $f' \leq 0$,

$$x \in [1/4, \infty)$$

Correct Answers:

- $(0, \frac{1}{4}]$
- $[\frac{1}{4}, \infty)$

---

3. (1 point)

Find the critical point and the interval on which the given function is increasing or decreasing, and apply the First Derivative Test to the critical point. Let $f(x) = x - 3\ln(3x), x > 0$.

Value of x such that critical point is achieved = ________

Is f a maximum or minumum at the critical point? $\boxed{?}$

The **open** interval on the left of the critical point is ________.
On this interval, f is $\boxed{?}$ while f' is $\boxed{?}$.

The **open** interval on the right of the critical point is ________.
On this interval, f is $\boxed{?}$ while f' is $\boxed{?}$.

Solution: First we need to calculate f', thus

$$f'(x) = 1 - \frac{3}{x}$$

Setting this equal to zero and solving for x leads to the critical point 3.

We know that f is the same sign on the intervals defined by the critical points. This evaluating f' and some point in each of these intervals, determines if f' is positive, which implies f is increasing or negative, which implies f is decreasing on that interval.

From the sign change of f' at a critical point, we can determine if it is a local maximum/minimum. From + to −, a local maximum. From − to +, a local minimum.

Correct Answers:

- 3
- `Local Min`
- `(0,3)`

- `decreasing`
- `Negative`
- $(3, \infty)$
- `Inreasing`
- `Positive`

---

4. (1 point)

Determine the intervals on which the given function is concave up or down and find the point of inflection. Let

$$f(x) = x\big(x - 4\sqrt{x}\big)$$

The x-coordinate of the point of inflection is ______

The **open** interval on the left of the inflection point is ______, and on this interval f is $\boxed{?}$.
The **open** interval on the right is ______, and on this interval f is $\boxed{?}$.

Solution:
One can compute f' using the product rule, but it is easier to re-write
$f(x) = x(x - 4\sqrt{x}) = x^2 - 4x^{3/2}$
Then $f'(x) = 2x - 6\sqrt{x}$ and $f''(x) = 2 - \frac{3}{\sqrt{x}}$.

Now, f is Concave Down for $0 < x < 2.25$ since $f''(x) < 0$ there. Moreover, f is Concave Up for $x > 2.25$ since $f''(x) > 0$ there.

Finally, because $f''(x)$ changes sign at $x = 2.25$, $f(x)$ has a point of inflection at $x = 2.25$.

Correct Answers:

- $\frac{9}{4}$
- `(0,2.25)`
- `Concave Down`
- $(2.25, \infty)$
- `Concave Up`

---

5. (1 point) Suppose that

$$f(x) = \frac{7e^x}{7e^x + 7}.$$

(A) Find all critical values of f. If there are no critical values, enter *None* . If there are more than one, enter them separated by commas.

Critical value(s) = ______

(B) Use **interval notation** to indicate where $f(x)$ is concave up.

Concave up: ______

(C) Use **interval notation** to indicate where $f(x)$ is concave down.

Concave down: ______

(D) Find all inflection points of f. If there are no inflection points, enter *None* . If there are more than one, enter them separated by commas.
Inflection point(s) at $x =$ ______

Solution:

$$f(x) = \frac{7e^x}{7e^x + 7}$$

$$f'(x) = \frac{e^x}{(e^x+1)^2} > 0$$

so f is increasing on R and there is no critical value.

$$f^{(2)}(x) = \frac{e^x(1-e^x)}{(e^x+1)^3}$$

So f is concave up when $x < 0$ and concave down when $x > 0$. The inflection point is $x = 0$.

Correct Answers:

- `None`
- $(-\infty, 0)$
- $(0, \infty)$
- 0

---

6. (1 point) Find the extreme values of the function f on the interval $[1.5, 6]$. If an extreme value does not exist, enter **DNE** .

$$f(x) = x^8 + \frac{8}{x}$$

Absolute minimum value: ______

Absolute maximum value: ______

Solution:

Set the derivative equal to zero to locate all critical numbers.

$$\begin{aligned} f'(x) = 8x^7 - \frac{8}{x^2} &= 0 \\ x^7 &= \frac{1}{x^2} \\ x^9 &= 1 \\ x &= 1 \end{aligned}$$

The only critical numbers is $x = 1$ which is out of the interval $[1.5, 6]$. There is no critical points on the interval $[1.5, 6]$. So for finding the extreme values of the function f on the interval $[1.5, 6]$, we find the value of f at the endpoints:

$$\begin{aligned} f(1.5) &= 30.962239583333 \\ f(6) &= 1679617.3333 \end{aligned}$$

The absolute minimum value is 30.962239583333, and the absolute maximum value is 1679617.3333.

Correct Answers:

- 30.962239583333
- 1679617.3333

7. (1 point)

Let $f(x) = \frac{(x+7)^2}{(x-7)^2}$.

Answer the following questions (for multiple answers enter each separated by commas e.g (a) 0,2 or (c) (-2,3),(0,-4) if no value enter "none".

(a) Vertical Asymptotes $x =$ _________

(b) Horizontal Asymptotes $y =$ _________

(c) Points where the graph crosses a horizontal asymptote $(x, y) =$ _________

(d) Critical Points $(x, y) =$ _________

(e) Inflection Points $(x, y) =$ _________

SOLUTION

(a) $f(x) = \frac{(x+7)^2}{(x-7)^2}$ has zero denominator and hence a vertical asymptote when $x = 7$.

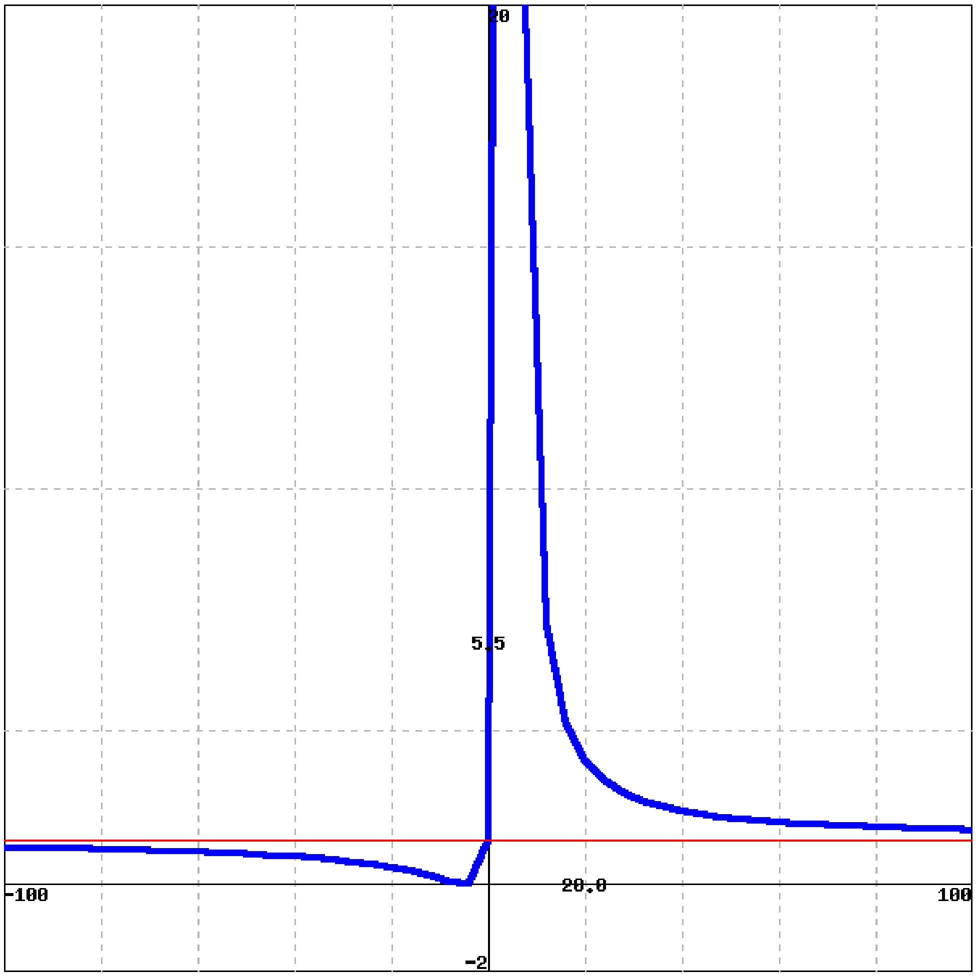
(b) $\lim_{x \to \pm\infty} \dfrac{(x+7)^2}{(x-7)^2} = 1$ so there is a horizontal asymptote at $y = 1$.

(c) $\frac{(x+7)^2}{(x-7)^2} = 1$ only when $x = 0$ so that $(0, 1)$ is the only point where the graph crosses the horizontal asymptote.

(d) $f'(x) = -\frac{28(x+7)}{(x-7)^3} = 0$ when $x = -7$ gives the only Stationary Point $(-7, 0)$.

(e) $f''(x) = \frac{56(x+14)}{(x-7)^4} = 0$ when $x = -14$ and $f''(x)$ changes sign so $(-14, \frac{1}{9})$ is an Inflection Point.

(click on image to enlarge)



$y = f(x)$

Correct Answers:

- 7
- 1
- $(0, 1)$
- $(-7, 0)$
- $(-14, \frac{1}{9})$

---

8. (1 point)

Consider the functions $f(x) = e^{x-1} - 1$ and $g(x) = x - 1$. These are continuous and differentiable for $x > 0$. In this problem we use the Racetrack Principle to show that one of these functions is greater than the other, except at one point where they are equal.

(a) Find a point c such that $f(c) = g(c)$. $c =$ ______

(b) Find the equation of the tangent line to $f(x) = e^{x-1} - 1$ at $x = c$ for the value of c that you found in (a).
$y =$ ______________

(c) Based on your work in (a) and (b), what can you say about the derivatives of f and g?
$f'(x)$ [?/</=/>] $g'(x)$ for $0 < x < c$, and
$f'(x)$ [?/</=/>] $g'(x)$ for $c < x < \infty$.

(d) Therefore, the Racetrack Principle gives
$f(x)$ [?/<=/=/>=] $g(x)$ for $x \leq c$, and
$f(x)$ [?/<=/=/>=] $g(x)$ for $x \geq c$.

Solution:

Note that at $c = 1$ we have $f(c) = g(c) = 0$. Then, at $x = 1$ $f'(x) = e^{x-1}$, so that $f'(1) = 1$, and the equation of the tangent line is $y = x - 1$.

Then we note that for $x \leq 1$ we have $f'(x) \leq g'(x) = 1$ and for $x \geq 1$, that $f'(x) \geq g'(x) = 1$. Therefore, by the Racetrack Principle, we know that $f(x) \geq g(x)$ at every x.

Correct Answers:

- 1
- $x - 1$
- $<$
- $>$
- $\geqq$
- $\leqq$

---

9. (1 point)

Consider the function $f(x) = x^2 - 4x + 9$ on the interval $[0, 4]$. Verify that this function satisfies the three hypotheses of Rolle's Theorem on the inverval.

$f(x)$ is ________ on $[0, 4]$;
$f(x)$ is ________ on $(0, 4)$;
$f(0) = f(4) =$ ________.

Then by Rolle's theorem, there exists a c such that $f'(c) = 0$.
Find the value c.
$c =$ ____

Solution:

$$f'(x) = 2x - 4$$

So $c = 2$.

Correct Answers:

- `continuous`
- `differentiable`
- `9`
- `2`

---

10. (1 point) Find the absolute maximum and absolute minimum values of $f(x) = \dfrac{x^2 - 9}{x^2 + 9}$ on the interval $[-5, 5]$.

---

1. Find the absolute maximum of f on the interval.
Answer: ________

2. Find the absolute minimum of f on the interval.
Answer: ________

Solution:

$$f'(x) = \frac{36x}{(x^2+9)^2}$$

So $f'(x) < 0$, when $x < 0$ and $f'(x) > 0$,when $x > 0$. $f(-5) = f(5) = \frac{8}{17}$, $f(0) = -1$. Then the absolute maximum of f is $\frac{8}{17}$ and the absolute minimum is -1.

Correct Answers:

- $\frac{8}{17}$
- `-1`

---

11. (1 point)

Answer the following True-False quiz. (Enter "T" or "F".)

___1. If $f'(c) = 0$ and $f''(c) > 0$, then $f(x)$ has a local minimum at c.
___2. If $f'(x) < 0$ for all x in $(0, 1)$, then $f(x)$ is decreasing on $(0, 1)$.
___3. $(f(x) + g(x))' = f'(x) + g'(x)$.
___4. A continuous function on a closed interval always attains a maximum and a minimum value.
___5. If $f(x)$ and $g(x)$ are increasing on an interval I, then $f(x) + g(x)$ is increasing on I.
___6. If $f'(c) = 0$, then c is either a local maximum or a local minimum.
___7. Differentiable functions are always continuous.

Solution:

1. Function is concave up at c and local minimum.

2. Function is decreasing when $f'(x) < 0$.

3. The statement is trivially true.

4. By Extreme value theorem, which state that a continuous function on a closed interval must attain both maximum and minimum values.

5. Sum of two increasing function is also increasing.

6. It is a critical point and may not be an extreme point.

7. Differentiability implies continuity.

Correct Answers:

- T
- T
- T
- T
- T
- F
- T